**STUDIA MATHEMATICA**

Online First version

# Interpolation and the John–Nirenberg inequality on symmetric spaces of noncommutative martingales

by

Turdebek N. Bekjan (Urumqi), Zeqian Chen (Wuhan), Madi Raikhan (Nur-Sultan) and Mu Sun (Wuhan)

**Abstract.** We prove various John–Nirenberg inequalities on symmetric spaces of noncommutative martingales, including the crude and fine versions, which extend the corresponding results of Junge and Musat (2007) and Hong and Mei (2012) in the $L_p$-case. As an application, we provide the atomic decomposition of a noncommutative martingale Hardy space $\mathsf{h}_1$ using symmetric atoms as building blocks, and give the boundedness of paraproducts on symmetric spaces of noncommutative martingales.

**1. Introduction.** The John–Nirenberg inequality plays a permanent role in harmonic analysis and martingale theory, which provides a subtle characterization of BMO space. Based on the fundamental work of Pisier and Xu [PX97] on noncommutative martingale Hardy and BMO spaces, a noncommutative version of the John–Nirenberg inequality was established by Junge and Musat [JM07]. Later, Hong and Mei [HM12] proved several versions of the John–Nirenberg inequality for noncommutative martingales and provided an application to the atomic decomposition of noncommutative martingale Hardy space $\mathsf{h}_1$ using $q$-atoms as building blocks, extending the work of [BCPY10].

The aim of this paper is to prove various John–Nirenberg inequalities on symmetric spaces of noncommutative martingales, extending the results obtained in [JM07, HM12] in the $L_p$-case. This follows the current line of investigations in noncommutative martingale theory. Thanks to its interactions with operator space theory and free probability, noncommutative martingale theory is now a steadily developing field in noncommutative analysis;

2020 *Mathematics Subject Classification*: Primary 46L52; Secondary 47L05.

*Key words and phrases*: noncommutative martingale, John–Nirenberg inequality, noncommutative symmetric spaces, interpolation, Hardy space, BMO-space, atomic decomposition.

Received 5 May 2020; revised 1 December 2020.

Published online 2 September 2021.

DOI: 10.4064/sm200508-11-12

see among others [Bek15, BC12, BCO17, CRX20, DPPS11, JSZZ17, Jun02, JP14, JX03, JX08, LS08, PR06, Per09, Ran02, Ran07, RW15, RWX19, RWZ21, RX16] and references therein. In turn, this theory has important applications to operator spaces and quantum probability. We refer to [Jun05, JX10, PS02, Xu06] for some illustrations of applications to operator space theory.

The paper is organized as follows. In Section 2, we include preliminaries and notations on symmetric quasi-Banach function spaces, noncommutative symmetric spaces and martingale spaces. Section 3 is devoted to extending the interpolation theorem of [BCPY10] to the case of semifinite von Neumann algebras, which will be used in what follows. In Section 4, we prove various John–Nirenberg inequalities on symmetric spaces of noncommutative martingales, including crude and fine versions. Finally, in Section 5, we apply the John–Nirenberg inequalities obtained to atomic decomposition and paraproducts on symmetric spaces of noncommutative martingales.

## 2. Preliminaries

**2.1. Symmetric quasi-Banach function spaces.** A quasi-normed space is a vector space over $\mathbb{R}$ or $\mathbb{C}$ equipped with a quasi-norm $\|\cdot\|$, that is, (1) $\|x\| = 0$ if and only if $x = 0$; (2) $\|\alpha x\| = |\alpha|\,\|x\|$ for all scalars $\alpha$ and vectors $x$; and (3) there exists a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\|+\|y\|)$ for all vectors $x$ and $y$. The Aoki–Rolewicz theorem says that a quasi-normed space $X$ is always $p$-normed for some $0 < p \leq 1$, namely there is an equivalent $p$-subadditive quasi-norm $\|\cdot\|_*$ such that $\|x+y\|_*^p \leq \|x\|_*^p + \|y\|_*^p$ for all vectors $x$ and $y$. In this case, $d(x,y) = \|x-y\|_*^p$ is a metric on $X$, and $X$ is called a *quasi-Banach space* if $X$ is complete for this metric. We refer to [Kal03] for the details on quasi-Banach spaces.

This subsection will focus on quasi-Banach spaces of measurable functions on $\Omega = (0,\infty)$ equipped with the usual Lebesgue measure $\mu$. We denote by $L_0(0,\infty)$ the space of $\mu$-measurable real-valued functions $f$ on $(0,\infty)$ such that $\mu(\{\omega \in (0,\infty) : |f(\omega)| > s\}) < \infty$ for some $s$. The decreasing rearrangement function $f^* : [0,\infty) \to [0,\infty]$ for $f \in L_0(0,\infty)$ is defined by

$$f^*(t) = \inf\{s > 0 : \mu(\{\omega \in \Omega : |f(\omega)| > s\}) \leq t\}$$

for $t \geq 0$. If $f, g \in L_0(0,\infty)$ are such that $\int_0^t f^*(s)\,ds \leq \int_0^t g^*(s)\,ds$ for all $t \geq 0$, then $f$ is said to be *majorized* by $g$, denoted by $f \preccurlyeq g$. Let $E$ be a quasi-Banach subspace of $L_0(0,\infty)$, simply called a quasi-Banach function space on $(0,\infty)$ in what follows. $E$ is said to be *symmetric* if, for $f \in E$ and $g \in L_0(0,\infty)$ such that $g^*(t) \leq f^*(t)$ for all $t \geq 0$, one has $g \in E$ and $\|g\|_E \leq \|f\|_E$; $E$ is *fully symmetric* if, for $f \in L_0(0,\infty)$ and $g \in E$ such that $f \preccurlyeq g$, we have $f \in E$ and $\|f\|_E \leq \|g\|_E$; and $E$ is said to have the *Fatou* *property* if, for every net $(f_i)_{i\in I}$ in $E$ satisfying $0 \le f_i \uparrow$ and $\sup_{i\in I}\|f_i\|_E < \infty$, the supremum $f = \sup_{i\in I} f_i$ exists in $E$ and $\|f_i\|_E \uparrow \|f\|_E$. We say that $E$ has *order continuous* quasi-norm if for every net $(f_i)$ in $E$ such that $f_i \downarrow 0$ we have $\|f_i\|_E \downarrow 0$. A symmetric quasi-Banach function space is called *rearrangement invariant* if it has order continuous quasi-norm or the Fatou property. The *Köthe dual* of $E$ is given by

$$E^\times = \Big\{ f \in L_0(0,\infty) : \sup_{\|g\|_E\le 1} \int_0^\infty |f(t)g(t)|\,dt < \infty \Big\}$$

with the quasi-norm $\|f\|_{E^\times} = \sup_{\|g\|_E\le 1}\int_0^\infty |f(t)g(t)|\,dt$. If $E$ is a symmetric Banach function space on $(0,\infty)$, then $E^\times$ is fully symmetric and has the Fatou property.

For any $s > 0$, the *dilation operator* $D_s$ on $L_0(0,\infty)$ is defined by $(D_s f)(t) = f(t/s)$ for all $t \in (0,\infty)$. For a quasi-Banach function space $E$ on $(0,\infty)$, the *lower* and *upper Boyd indices* $p_E$ and $q_E$ of $E$ are respectively defined by

$$p_E = \sup_{s>1} \frac{\log s}{\log \|D_s\|_E} \quad \text{and} \quad q_E = \inf_{0<s<1} \frac{\log s}{\log \|D_s\|_E}.$$

For a symmetric quasi-Banach function space $E$ on $(0,\infty)$, $D_s$ is a bounded linear operator on $E$ for every $s > 0$ and $0 < p_E \le q_E \le \infty$ (see [Dir15, Lemma 2.2]). If $E$ is a symmetric Banach function space, then $1 \le p_E \le q_E \le \infty$ (see [LT79, Proposition 2.b.2]).

Let $0 < p, q \le \infty$. A symmetric quasi-Banach function space $E$ is said to be $p$-convex (resp., $q$-concave) if there exists a constant $C > 0$ such that for any finite sequence $(f_n)_{n\ge 1}$ in $E$ we have

$$\Big\| \Big( \sum_{n\ge 1} |f_n|^p \Big)^{1/p} \Big\|_E \le C \Big( \sum_{n\ge 1} \|f_n\|_E^p \Big)^{1/p}$$

$$\Big(\text{resp., } \Big( \sum_{n\ge 1} \|f_n\|_E^q \Big)^{1/q} \le C \Big\| \Big( \sum_{n\ge 1} |f_n|^q \Big)^{1/q} \Big\|_E \Big)$$

when $0 < p < \infty$ (resp., $0 < q < \infty$), or,

$$\Big\| \max_n |f_n| \Big\|_E \le C \max_n \|f_n\|_E$$

$$\Big(\text{resp., } \max_n \|f_n\|_E \le C \Big\| \max_n |f_n| \Big\|_E \Big)$$

when $p = \infty$ (resp., $q = \infty$).

Given a quasi-Banach function space $E$ on $\Omega = (0,\infty)$, for $0 < p < \infty$, $E^{(p)}$ denotes the quasi-Banach function space on $(0,\infty)$ defined by

$$E^{(p)} = \{ f \in L_0(\Omega) : |f|^p \in E \},$$

equipped with the quasi-norm $\|f\|_{E^{(p)}} = \||f|^p\|_E^{1/p}$. Note that if $0 < p$, $q < \infty$, then $(E^{(p)})^{(q)} = E^{(pq)}$. If $E$ is a Banach function space on $(0,\infty)$ and $p > 1$, then $E^{(p)}$ is a Banach function space on $(0,\infty)$, which is usually called the *p-convexification* of $E$.

Let $E_i$ be a quasi-Banach function space on $\Omega = (0,\infty)$ for every $i = 1, 2$. The pointwise product space $E_1 \odot E_2$ is defined by

$$E_1 \odot E_2 = \{f \in L_0(\Omega) : f = f_1 f_2,\ f_i \in E_i,\ i = 1, 2\} \tag{2.1}$$

with a functional $\|f\|_{E_1 \odot E_2}$ defined by

$$\|f\|_{E_1 \odot E_2} = \inf\{\|f_1\|_{E_1}\|f_2\|_{E_2} : f = f_1 f_2,\ f_i \in E_i,\ i = 1, 2\}.$$

If $E_1$ and $E_2$ are both symmetric quasi-Banach function spaces on $(0,\infty)$, then there is an equivalent quasi norm $\|\cdot\|$ such that $(E_1 \odot E_2, \|\cdot\|)$ is a symmetric quasi-Banach function space on $(0,\infty)$ (see [BO21, Corollary 1]). By a Hölder type inequality (see [LT79, Proposition 1.d.2] or [Xu91, (3.1)]), we have

$$E^{(1/2)} = E \odot E \tag{2.2}$$

whenever $E$ is a symmetric Banach function space on $(0,\infty)$.

We need the following results (see [KLM14, Theorem 1(ii), Corollary 2] and [Loz69, Theorem 6]).

THEOREM 2.1. *Let $E$ and $F$ be two symmetric quasi-Banach function spaces on* $(0,\infty)$.

(i) *If* $0 < p < \infty$, *then* $(E \odot F)^{(p)} = E^{(p)} \odot F^{(p)}$.
(ii) $L_1(0,\infty) = E \odot E^{\times}$.

Our main references for general interpolation theory are [BL76, KM03]. Recall that for a compatible quasi-Banach space couple $(X, Y)$, a quasi-Banach space $Z$ is called an *interpolation space* for $(X, Y)$ if $X \cap Y \subset Z \subset X + Y$, and whenever $T$ is a bounded linear operator on $X + Y$ such that $T[X] \subset X$ and $T[Y] \subset Y$, we have $T[Z] \subset Z$ and $\|T\|_{Z\to Z} \le C(\|T\|_{X\to X} + \|T\|_{Y\to Y})$ for some $C > 0$. In this case, we write $Z \in \mathrm{Int}(X, Y)$. Note that if $E$ is a symmetric quasi-Banach function space on $\Omega = (0,\infty)$ which either has order continuous quasi-norm or has the Fatou property and $0 < p < p_E \le q_E < q < \infty$, then $E \in \mathrm{Int}(L_p(\Omega), L_q(\Omega))$ (see [Mon96, Theorem 3]). If $E$ is a symmetric Banach function space on $(0,\infty)$, then $E$ is fully symmetric if and only if $E \in \mathrm{Int}(L_1(\Omega), L_\infty(\Omega))$.

LEMMA 2.2. *Let $E$ be a symmetric quasi-Banach function space on* $(0,\infty)$ *such that* $E \in \mathrm{Int}(L_1, L_\infty)$. *Then $E$ can be renormed as a symmetric Banach function space.*

*Proof.* By [DPPS11, Theorem 3.4], there is a Banach function space $F$ on $(0,\infty)$ such that for any $f \in E$, $t \mapsto K(t,f,L_1,L_\infty) \in F$ and

$$\|t \mapsto K(t,f,L_1,L_\infty)\|_F \approx \|f\|_E.$$

Let $\|f\|'_E = \|t \mapsto K(t,f,L_1,L_\infty)\|_F$ for $f \in E$. Since $K(t,f,L_1,L_\infty) = \int_0^t f^*(s)\,ds$, we find that $\|\cdot\|'_E$ is an equivalent norm on $E$ and $(E,\|\cdot\|'_E)$ is a symmetric Banach function space on $(0,\infty)$. ∎

We refer to [BS88, DPPS11, KPS82, LT79, Suk14, Xu91] for the details on symmetric quasi-Banach function spaces.

In what follows, unless otherwise specified, we always denote by $E$ a symmetric quasi-Banach function space on $(0,\infty)$. For two nonnegative quantities $A$ and $B$, by writing $A \lesssim B$ we mean that there exists an absolute constant $C > 0$ such that $A \le CB$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. Also, $A \lesssim_{p,k,\dots} B$ denotes the inequality $A \le C_{p,k,\dots}B$ for some constant $C_{p,k,\dots} > 0$ depending only on $p, k, \dots$.

**2.2. Noncommutative symmetric spaces.** Throughout this paper, $\mathcal{M}$ always denotes a semifinite von Neumann algebra with a faithful normal semifinite trace $\tau$. The set of all $\tau$-measurable operators is denoted by $L_0(\mathcal{M})$. For $x \in L_0(\mathcal{M})$, the *distribution function* $\lambda(x)$ of $x$ is defined by $\lambda_t(x) = \tau(e_{(t,\infty)}(|x|))$ for $t > 0$, where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ in the interval $(t,\infty)$, and the *rearrangement function* $\mu(x)$ of $x$ by $\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \le t\}$ for $t > 0$. Given a symmetric quasi-Banach function space $E$ on $(0,\infty)$, the space

$$E(\mathcal{M},\tau) = \{x \in L_0(\mathcal{M}) : \|\mu(x)\|_E < \infty\}$$

is a quasi-Banach space under the quasi-norm $\|x\|_{E(\mathcal{M})} = \|\mu(x)\|_E$ (see [Suk14, Xu91]), denoted by $E(\mathcal{M})$ for convenience. If $0 < p < \infty$ and $E = L_p(0,\infty)$, then $E(\mathcal{M}) = L_p(\mathcal{M})$, which are the usual noncommutative $L_p$-spaces associated with $(\mathcal{M},\tau)$. We refer to [FK86, PX03] for the details on the theory of noncommutative $L_p$-spaces.

We refer to [Xu07] for interpolation theory on noncommutative $L_p$-spaces. For complex interpolation, if $1 \le p_0 \ne p_1 \le \infty$, then

$$L_p(\mathcal{M}) = (L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))_\theta \tag{2.3}$$

with equivalent norms for $0 < \theta < 1$ and $1 \le p < \infty$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. For real interpolation, if $0 < p_0 \ne p_1 < \infty$ and $0 < q_0, q_1, q \le \infty$, then

$$L_{p,q}(\mathcal{M}) = (L_{p_0,q_0}(\mathcal{M}), L_{p_1,q_1}(\mathcal{M}))_{\theta,q} \tag{2.4}$$

with equivalent quasi-norms for $0 < \theta < 1$ and $0 < p < \infty$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Recall that for $0 < p < \infty$ and $0 < q \le \infty$, the noncommutative Lorentz space is

$$L_{p,q}(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \|x\|_{p,q} < \infty\} \tag{2.5}$$

where

$$\|x\|_{p,q} = \begin{cases} \left(\int_0^\infty [t^{1/p}\mu_t(x)]^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{1/p}\mu_t(x), & q = \infty. \end{cases}$$

Equipped with $\|\cdot\|_{p,q}$, $L_{p,q}(\mathcal{M})$ is a quasi-Banach space which can be renormed into a Banach space when $p > 1$ and $q \geq 1$. Usually, $L_{p,\infty}(\mathcal{M})$ is called a noncommutative weak $L_p$-space. We refer to [Ran03, Xu90] for more details on the noncommutative Lorentz spaces.

We will repeatedly use the following fact which follows from [Dir15, Theorem 4.8].

LEMMA 2.3. *Let $E$ be a symmetric quasi-Banach function space on $(0,\infty)$ which is $s$-convex for some $0 < s < \infty$. Let $\mathcal{M}$ and $\mathcal{N}$ be semifinite von Neumann algebras. Given $0 < p < q \leq \infty$, let $T : L_p(\mathcal{M}) + L_q(\mathcal{M}) \to L_p(\mathcal{N}) + L_q(\mathcal{N})$ be a linear operator such that $T : L_p(\mathcal{M}) \to L_p(\mathcal{N})$ and $T : L_q(\mathcal{M}) \to L_q(\mathcal{N})$ are bounded. If $p < p_E \leq q_E < q < \infty$, or $p < p_E$ and $q = \infty$, then*

$$\|T : E(\mathcal{M}) \to E(\mathcal{N})\| \\ \leq C \max\{\|T : L_p(\mathcal{M}) \to L_p(\mathcal{N})\|, \|T : L_q(\mathcal{M}) \to L_q(\mathcal{N})\|\},$$

*where $C > 0$ is a constant depending only on $p$ and $q$.*

We define $E(\mathcal{M}, \ell_c^2)$ to be the space of all sequences $a = (a_n)_{n\geq 1}$ in $E(\mathcal{M})$ such that

$$\|a\|_{E(\mathcal{M},\ell_c^2)} = \Big\|\Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2}\Big\|_{E(\mathcal{M})} < \infty,$$

which is a quasi-Banach space under the quasi-norm $\|\cdot\|_{E(\mathcal{M},\ell_c^2)}$ (see [RW15, RWX19]). The conditioned version of $E(\mathcal{M}, \ell_c^2)$, denoted by $E^{\rm cond}(\mathcal{M}, \ell_c^2)$, is defined to be the completion of the space of finite sequences $a = (a_n)_{n\geq 1}$ in $E(\mathcal{M}) \cap \mathcal{M}$ under the quasi-norm (see [Jun02, RWX19])

$$\|a\|_{E^{\rm cond}(\mathcal{M},\ell_c^2)} = \Big\|\Big(\sum_{n\geq 1} \mathcal{E}_{n-1}[a_n^* a_n]\Big)^{1/2}\Big\|_{E(\mathcal{M})},$$

which is a quasi-Banach space under $\|\cdot\|_{E^{\rm cond}(\mathcal{M},\ell_c^2)}$. Also, we define $E(\mathcal{M}, \ell_r^2)$ to be the space of all sequences $a = (a_n)_{n\geq 1}$ in $E(\mathcal{M})$ such that

$$\|a\|_{E(\mathcal{M},\ell_r^2)} = \|(a_n^*)_{n\geq 1}\|_{E(\mathcal{M},\ell_c^2)} < \infty,$$

and $E^{\rm cond}(\mathcal{M}, \ell_r^2)$ to be the completion of the space of finite sequences $a = (a_n)_{n\geq 1}$ in $E(\mathcal{M}) \cap \mathcal{M}$ under the quasi-norm $\|a\|_{E^{\rm cond}(\mathcal{M},\ell_r^2)} = \|a^*\|_{E^{\rm cond}(\mathcal{M},\ell_c^2)}$. Both $E(\mathcal{M}, \ell_r^2)$ and $E^{\rm cond}(\mathcal{M}, \ell_r^2)$ are quasi-Banach spaces under the corresponding quasi-norms. Note that for $0 < p < \infty$ and $E = L_p(0,\infty)$, $E(\mathcal{M}, \ell_c^2) = L_p(\mathcal{M}, \ell_c^2)$, $E(\mathcal{M}, \ell_r^2) = L_p(\mathcal{M}, \ell_r^2)$, $E^{\rm cond}(\mathcal{M}, \ell_c^2) = L_p^{\rm cond}(\mathcal{M}, \ell_c^2)$ and $E^{\rm cond}(\mathcal{M}, \ell_r^2) = L_p^{\rm cond}(\mathcal{M}, \ell_r^2)$.

**2.3. Noncommutative martingale spaces.** In what follows, we always denote by $(\mathcal{M}_n)_{n\geq 1}$ an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ whose union generates $\mathcal{M}$ in the $w^*$-topology. For every $n \geq 1$, the restriction $\tau|_{M_n}$ remains semifinite, still denoted by $\tau$, and we assume that there exists a trace preserving conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$. In this case, $(\mathcal{M}_n)_{n\geq 1}$ is called a *filtration* of $\mathcal{M}$. Note that $\mathcal{E}_n$ extends to a contractive projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}_n)$ for all $1 \leq p \leq \infty$. A *noncommutative martingale* with respect to $(\mathcal{M}_n)_{n\geq 1}$ is a sequence $x = (x_n)_{n\geq 1}$ in $L_1(\mathcal{M}) + \mathcal{M}$ such that $\mathcal{E}_n(x_{n+1}) = x_n$ for any $n \geq 1$. If in addition, all $x_n$'s are in $E(\mathcal{M})$ then $x$ is called an $E(\mathcal{M})$-*martingale.* In this case, if $\|x\|_E = \sup_{n\geq 1} \|x_n\|_{E(\mathcal{M})} < \infty$, then $x$ is said to be a *bounded* $E(\mathcal{M})$-martingale. The *martingale difference sequence* $dx = (dx_n)_{n\geq 1}$ of $x$ is defined by $dx_n = x_n - x_{n-1}$ for $n \geq 1$. Here and in the following, we set $x_0 = 0$ and $\mathcal{E}_0 = \mathcal{E}_1$ for the sake of convenience. If there exists $N$ such that $dx_n = 0$ for all $n \geq N$, then $x$ is called a *finite* martingale.

For a finite $E(\mathcal{M})$-martingale $x = (x_n)_{n\geq 1}$, we set

$$\|x\|_{\mathcal{H}_E^c} = \|S_c(x)\|_{E(\mathcal{M})} = \|dx\|_{E(\mathcal{M},\ell_c^2)}$$

and

$$\|x\|_{\mathcal{H}_E^r} = \|S_r(x)\|_{E(\mathcal{M})} = \|dx\|_{E(\mathcal{M},\ell_r^2)},$$

where $S_c(x)$ and $S_r(x)$ are the column and row *square functions* of $x$, defined respectively by

$$S_c(x) = \Big(\sum_{n\geq 1} |dx_n|^2\Big)^{1/2} \quad \text{and} \quad S_r(x) = \Big(\sum_{n\geq 1} |dx_n^*|^2\Big)^{1/2}.$$

Let $\mathcal{H}_E^c(\mathcal{M})$ and $\mathcal{H}_E^r(\mathcal{M})$ be the corresponding completions under the quasi-norms $\|x\|_{\mathcal{H}_E^c}$ and $\|x\|_{\mathcal{H}_E^c}$, respectively. Then $\mathcal{H}_E^c(\mathcal{M})$ and $\mathcal{H}_E^r(\mathcal{M})$ are both quasi-Banach spaces, respectively called the *column* and *row symmetric Hardy spaces* of noncommutative martingales. For $0 < p < \infty$ and $E = L_p(0,\infty)$, $\mathcal{H}_E^c(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M})$ and $\mathcal{H}_E^r(\mathcal{M}) = \mathcal{H}_p^r(\mathcal{M})$, the *column* and *row* $\mathcal{H}_p$*-spaces* of noncommutative martingales (see [PX97]). The *mixture* $\mathcal{H}_p$*-space* $\mathcal{H}_p(\mathcal{M})$ of noncommutative martingales is defined as follows: For $2 \leq p < \infty$, $\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M})$, equipped with the norm $\|x\|_{\mathcal{H}_p} = \|x\|_{\mathcal{H}_p^c} + \|x\|_{\mathcal{H}_p^r}$; for $0 < p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \{x = y + z : y \in \mathcal{H}_p^c(\mathcal{M}),\ z \in \mathcal{H}_p^r(\mathcal{M})\},$$

equipped with the quasi-norm $\|x\|_{\mathcal{H}_p} = \inf_{x=y+z}\{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\}$, where the infimum is taken over all decompositions $x = y + z$ with $y \in \mathcal{H}_p^c(\mathcal{M})$ and $z \in \mathcal{H}_p^r(\mathcal{M})$. For $0 < p < 1$, the quasi-norm $\|x\|_{\mathcal{H}_p}$ is a $p$-norm, while it is actually a norm for $p \geq 1$.

The *column* BMO-*space* of noncommutative martingales is defined by

$$\mathrm{BMO}^c(\mathcal{M}) = \Big\{x \in L_1(\mathcal{M}) + \mathcal{M} : \sup_{n\geq 1} \|\mathcal{E}_n(|x - x_{n-1}|^2)\|_\infty < \infty\Big\},$$

equipped with the norm $\|x\|_{\mathrm{BMO}^c} = \sup_{n\geq 1} \|\mathcal{E}_n(|x - x_{n-1}|^2)\|_\infty^{1/2}$, and the *row* BMO-*space* of noncommutative martingales by

$$\mathrm{BMO}^r(\mathcal{M}) = \{x \in L_1(\mathcal{M}) + \mathcal{M} : x^* \in \mathrm{BMO}^c(\mathcal{M})\},$$

equipped with the norm $\|x\|_{\mathrm{BMO}^r} = \|x^*\|_{\mathrm{BMO}^c}$. The *mixture* BMO-*space* of noncommutative martingales is defined by $\mathrm{BMO}(\mathcal{M}) = \mathrm{BMO}^c(\mathcal{M}) \cap \mathrm{BMO}^r(\mathcal{M})$, equipped with the norm $\|x\|_{\mathrm{BMO}} = \|x\|_{\mathrm{BMO}^c} + \|x\|_{\mathrm{BMO}^r}$. Then $\mathrm{BMO}^c(\mathcal{M}) = \mathcal{H}_1^c(\mathcal{M})^*$, $\mathrm{BMO}^r(\mathcal{M}) = \mathcal{H}_1^r(\mathcal{M})^*$, and $\mathrm{BMO}(\mathcal{M}) = \mathcal{H}_1(\mathcal{M})^*$ (see [PX97, JX03]).

The column and row *conditioned Hardy spaces* $\mathsf{h}_E^c(\mathcal{M})$ and $\mathsf{h}_E^r(\mathcal{M})$ of noncommutative martingales are respectively defined to be the completions of the space of all finite martingales $x$ in $E(\mathcal{M}) \cap \mathcal{M}$ under the associated quasi-norms (see [Jun02, RW15])

$$\|x\|_{\mathsf{h}_E^c} = \|s_c(x)\|_{E(\mathcal{M})} = \|dx\|_{E^{\mathrm{cond}}(\mathcal{M},\ell_c^2)}$$

and

$$\|x\|_{\mathsf{h}_E^r} = \|s_r(x)\|_{E(\mathcal{M})} = \|dx\|_{E^{\mathrm{cond}}(\mathcal{M},\ell_r^2)},$$

where $s_c(x)$ and $s_r(x)$ are the column and row *conditioned square functions* of $x$, defined by

$$s_c(x) = \Big(\sum_{n\geq 1} \mathcal{E}_{n-1}[|dx_n|^2]\Big)^{1/2} \quad \text{and} \quad s_r(x) = \Big(\sum_{n\geq 1} \mathcal{E}_{n-1}[|dx_n^*|^2]\Big)^{1/2}.$$

The *diagonal Hardy space* $\mathsf{h}_E^d(\mathcal{M})$ of noncommutative martingales is defined to be the completion of the space of all finite $E(\mathcal{M})$-martingales $x$ under the quasi-norm $\|x\|_{\mathsf{h}_E^d} = \|(dx_n)\|_{E(\mathcal{M}\bar{\otimes}\ell_\infty)}$.

In particular, if we let $E = L_{p,q}(\mathcal{M})$, then we have the column conditioned Lorentz–Hardy space $\mathsf{h}_{L_{p,q}}^c(\mathcal{M})$, simply denoted by $\mathsf{h}_{p,q}^c(\mathcal{M})$. Similarly, we have the row conditioned Lorentz–Hardy space $\mathsf{h}_{p,q}^r(\mathcal{M})$, martingale Lorentz–Hardy spaces $\mathcal{H}_{p,q}^c(\mathcal{M})$ and $\mathcal{H}_{p,q}^r(\mathcal{M})$.

In general, we have no explicit description of elements in $\mathsf{h}_E^c(\mathcal{M})$ or $\mathsf{h}_E^r(\mathcal{M})$. However, if $E = L_p(0,\infty)$ for $0 < p < \infty$, then $\mathsf{h}_E^c(\mathcal{M}) = \mathsf{h}_p^c(\mathcal{M})$ and $\mathsf{h}_E^r(\mathcal{M}) = \mathsf{h}_p^r(\mathcal{M})$, called the column and row conditioned $\mathcal{H}_p$-spaces of noncommutative martingales, and there exist isometric embeddings of $\mathsf{h}_p^c(\mathcal{M})$ and $\mathsf{h}_p^r(\mathcal{M})$ into noncommutative $L_p$-spaces, as shown in [Jun02]. More precisely, for $0 < p \leq \infty$ there exists an isometry

$$U : L_p^{\mathrm{cond}}(\mathcal{M},\ell_c^2) \to L_p(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathbb{N}^2))) \tag{2.6}$$

such that if $a = (a_n) \in L_p^{\rm cond}(\mathcal{M}, \ell_c^2)$ and $b = (b_n) \in L_p^{\rm cond}(\mathcal{M}, \ell_c^2)$ with $1/p + 1/q \le 1$, then

$$U(a)^*U(b) = \Big(\sum_{n\ge1} \mathcal{E}_{n-1}[a_n^* b_n]\Big) \otimes e_{1,1} \otimes e_{1,1},$$

where $(e_{i,j})_{i,j\ge1}$ denotes the natural basis in $\mathcal{B}(\ell_2(\mathbb{N}^2))$. Let $D_c : \mathsf{h}_p^c(\mathcal{M}) \to L_p^{\rm cond}(\mathcal{M}, \ell_c^2)$ be the extension of the embedding map $x \mapsto (dx_n)$ and denote $W_c = UD_c$. One then has the isometric embedding

$$W_c : \mathsf{h}_p^c(\mathcal{M}) \to L_p(\mathcal{M} \mathbin{\bar\otimes} \mathcal{B}(\ell_2(\mathbb{N}^2))) \tag{2.7}$$

with the property that if $x \in \mathsf{h}_p^c(\mathcal{M})$ and $y \in \mathsf{h}_q^c(\mathcal{M})$ with $1/p + 1/q \le 1$, then

$$W_c(x)^*W_c(y) = \Big(\sum_{n\ge1} \mathcal{E}_{n-1}[dx_n^* dy_n]\Big) \otimes e_{1,1} \otimes e_{1,1}.$$

The same results hold for $\mathsf{h}_p^r(\mathcal{M})$.

If $E$ is a symmetric quasi-Banach function space on $(0,\infty)$ which is $s$-convex for some $0 < s < \infty$ and $p_E > p > 0$, then $\mathsf{h}_E^c(\mathcal{M})$ coincides with the space of all martingales $x = (x_n)$ in $\mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_\infty^c(\mathcal{M})$ for which $W_c(x) \in E(\mathcal{M} \mathbin{\bar\otimes} \mathcal{B}(\ell_2(\mathbb{N}^2)))$, and $\|x\|_{\mathsf{h}_E^c} \equiv \|W_c(x)\|_{E(\mathcal{M}\bar\otimes\mathcal{B}(\ell_2(\mathbb{N}^2)))}$. This can be shown by using Lemma 2.3. Note that if $E \subset L_2 + L_\infty$, then $\mathsf{h}_E^c(\mathcal{M})$ coincides with the space of all martingales $x$ such that $s_c(x) \in E(\mathcal{M})$, and $\|x\|_{\mathsf{h}_E^c} = \|s_c(x)\|_{E(\mathcal{M})}$ for all $x \in \mathsf{h}_E^c(\mathcal{M})$. The same results hold for $\mathsf{h}_E^r(\mathcal{M})$. As remarked in [RWX19], if $E$ is a symmetric Banach function space on $(0,\infty)$ which has the Fatou property and $E \in \mathrm{Int}(L_p, L_q)$ for some $1 < p < q < \infty$, then every element of $\mathsf{h}_E^c(\mathcal{M})$ and $\mathsf{h}_E^r(\mathcal{M})$ can be represented by a martingale.

Recall that the mixture conditioned $\mathcal{H}_p$-spaces of noncommutative martingales are defined as follows: For $2 \le p < \infty$, $\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M}) \cap \mathsf{h}_p^d(\mathcal{M})$, equipped with the norm $\|x\|_{\mathsf{h}_p} = \|x\|_{\mathsf{h}_p^c} + \|x\|_{\mathsf{h}_p^r} + \|x\|_{\mathsf{h}_p^d}$; for $0 < p < 2$,

$$\mathsf{h}_p(\mathcal{M}) = \{x = y + z + w : y \in \mathsf{h}_p^c(\mathcal{M}),\ z \in \mathsf{h}_p^r(\mathcal{M}),\ w \in \mathsf{h}_p^d(\mathcal{M})\}$$

equipped with the quasi-norm $\|x\|_{\mathsf{h}_p} = \inf_{x=y+z+w}\{\|y\|_{\mathsf{h}_p^c} + \|z\|_{\mathsf{h}_p^r} + \|w\|_{\mathsf{h}_p^d}\}$, where the infimum is taken over all decompositions $x = y + z + w$ with $y \in \mathsf{h}_p^c(\mathcal{M})$, $z \in \mathsf{h}_p^r(\mathcal{M})$, and $w \in \mathsf{h}_p^d(\mathcal{M})$. For $0 < p < 1$, the quasi-norm $\|x\|_{\mathsf{h}_p}$ is a $p$-norm, while for $p \ge 1$, it is actually a norm.

The *column little* BMO-*space* of noncommutative martingales is

$$\mathrm{bmo}^c(\mathcal{M}) = \{x \in L_1(\mathcal{M}) + \mathcal{M} : \|x\|_{\mathrm{bmo}^c} < \infty\},$$

where $\|x\|_{\mathrm{bmo}^c} = \|\mathcal{E}_1(x)\|_\infty + \sup_{n\ge1} \|\mathcal{E}_n(|x - x_n|^2)\|_\infty^{1/2}$, and the *row little* BMO-*space* of noncommutative martingales is

$$\mathrm{bmo}^r(\mathcal{M}) = \{x \in L_1(\mathcal{M}) + \mathcal{M} : x^* \in \mathrm{bmo}^c(\mathcal{M})\},$$

equipped with the norm $\|x\|_{\mathrm{bmo}^r} = \|x^*\|_{\mathrm{bmo}^c}$. For any sequence $x = (x_n)_{n\geq 1}$ in $\mathcal{M}$, we set

$$\|x\|_{\ell_\infty(L_\infty(\mathcal{M}))} = \sup_{n\geq 1} \|x_n\|_\infty.$$

Let $\mathrm{bmo}^d(\mathcal{M})$ be the subspace of $\ell_\infty(L_\infty(\mathcal{M}))$ consisting of all martingale difference sequences with the same norm. The *mixture little* BMO-*space* of noncommutative martingales is defined by $\mathrm{bmo}(\mathcal{M}) = \mathrm{bmo}^c(\mathcal{M}) \cap \mathrm{bmo}^r(\mathcal{M}) \cap \mathrm{bmo}^d(\mathcal{M})$ equipped with the norm $\|x\|_{\mathrm{bmo}} = \|x\|_{\mathrm{bmo}^c} + \|x\|_{\mathrm{bmo}^r} + \|x\|_{\mathrm{bmo}^d}$. Then $\mathrm{bmo}^c(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M})^*$, $\mathrm{bmo}^r(\mathcal{M}) = \mathsf{h}_1^r(\mathcal{M})^*$, and $\mathrm{bmo}(\mathcal{M}) = \mathsf{h}_1(\mathcal{M})^*$. We refer to [JX03, JX08, RW15] for more information on these spaces.

Using [PX97, Theorem 2.3], [JX03, Lemma 6.4], and [LS08, Proposition 2.1], we obtain the following result.

PROPOSITION 2.4. *Let $E$ be a symmetric Banach function space on $(0,\infty)$ such that $E \in \mathrm{Int}(L_p, L_q)$ for some $1 < p \leq q < \infty$.*

(i) *Both $\mathsf{h}_E^c(\mathcal{M})$ and $\mathsf{h}_E^r(\mathcal{M})$ are complemented in $E(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathbb{N}^2)))$.*
(ii) *Both $\mathcal{H}_E^c(\mathcal{M})$ and $\mathcal{H}_E^r(\mathcal{M})$ are complemented in $E(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))$.*

By [DDP93, Theorem 5.6] and Proposition 2.4, we have

THEOREM 2.5. *Let $E$ be a separable symmetric Banach function space on $(0,\infty)$ with $1 < p_E \leq q_E < \infty$.*

(i) *$\mathsf{h}_E^c(\mathcal{M})^* = \mathsf{h}_{E^\times}^c(\mathcal{M})$ and $\mathsf{h}_E^r(\mathcal{M})^* = \mathsf{h}_{E^\times}^r(\mathcal{M})$ with equivalent norms.*
(ii) *$\mathcal{H}_E^c(\mathcal{M})^* = \mathcal{H}_{E^\times}^c(\mathcal{M})$ and $\mathcal{H}_E^r(\mathcal{M})^* = \mathcal{H}_{E^\times}^r(\mathcal{M})$ with equivalent norms.*

**3. Interpolation.** Musat [Mus03] proved that for $1 \leq q < p < \infty$,

$$(\mathrm{BMO}^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M}))_{q/p} = \mathcal{H}_p^c(\mathcal{M}) \tag{3.1}$$

with equivalent norms for a filtration of a finite von Neumann algebra $\mathcal{M}$ (see [JM07] for a different proof with better constants), which extended the interpolation theorem of Janson and Jones [JJ82] to the setting of noncommutative martingales. The conditioned case of (3.1) was obtained in [BCPY10]: For $1 \leq q < p < \infty$,

$$(\mathrm{bmo}^c(\mathcal{M}), \mathsf{h}_q^c(\mathcal{M}))_{q/p} = \mathsf{h}_p^c(\mathcal{M}) \tag{3.2}$$

with equivalent norms for a filtration of a finite von Neumann algebra $\mathcal{M}$, which indeed implies (3.1) by using noncommutative Davis decomposition [Per09].

This section is devoted to showing that both (3.1) and (3.2) also hold for a filtration of a semifinite von Neumann algebra $\mathcal{M}$. To this end, we need several lemmas.

LEMMA 3.1. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Then there exists a net $(e_i)_{i\in I}$ of $\tau$-finite projections in $\mathcal{M}_1$ such that*

$$\lim_I \|e_i x e_i - x\|_{\mathsf{h}_p^c} = 0 \tag{3.3}$$

*for every $x \in \mathsf{h}_p^c(\mathcal{M})$, $1 \leq p < \infty$.*

*Proof.* Since $\mathcal{M}_1$ is semifinite, we can choose a net $\{e_i\}_{i\in I}$ of $\tau$-finite projections in $\mathcal{M}_1$ such that $e_i \to \mathbf{1}$ strongly, where $\mathbf{1}$ is the unit element of $\mathcal{M}$ (see e.g. [Sak71, Theorem 2.5.6]). For each $i$, we consider the finite von Neumann algebra $(e_i\mathcal{M}e_i, \tau|_{e_i\mathcal{M}e_i})$ with the filtration $(e_i\mathcal{M}_n e_i)_{n\geq 1}$. We denote by $\mathcal{E}_{n,i}$ the trace preserving conditional expectation from $e_i\mathcal{M}e_i$ onto $e_i\mathcal{M}_n e_i$. Since $e_i \in \mathcal{M}_1$, we get $\mathcal{E}_{n,i} = \mathcal{E}_n|_{e_i\mathcal{M}e_i}$.

Let $x$ be a finite $L_\infty$-martingale. For $1 \leq p < \infty$, since $e_i \to \mathbf{1}$ strongly and $\mathcal{E}_{n-1}$ is a contraction in $L_p(\mathcal{M})$, we get

$$\lim_I \|e_i dx_n - dx_n\|_{\mathsf{h}_p^c} \leq \lim_I \|e_i dx_n - dx_n\|_p = 0$$

and

$$\lim_I \|dx_n e_i - dx_n\|_{\mathsf{h}_p^c} \leq \lim_I \|dx_n e_i - dx_n\|_p = \lim_I \|e_i dx_n^* - dx_n^*\|_p = 0$$

(see [Jun02, Lemma 2.3]). Then

$$\lim_I \|e_i dx_n e_i - dx_n\|_{\mathsf{h}_p^c} \leq \lim_I \|dx_n e_i - dx_n\|_{\mathsf{h}_p^c} + \lim_I \|(e_i dx_n - dx_n)e_i\|_{\mathsf{h}_p^c} = 0,$$

and so $\lim_I \|e_i x e_i - x\|_{\mathsf{h}_p^c} = 0$. Since the set of finite $L_\infty$-martingales is dense in $\mathsf{h}_p^c(\mathcal{M})$, we obtain (3.3) for all $x \in \mathsf{h}_p^c(\mathcal{M})$. ∎

LEMMA 3.2. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Then*

$$(\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta = \mathsf{h}_q^c(\mathcal{M}) \tag{3.4}$$

*with equivalent norms for $1 < p < \infty$ and $0 < \theta < 1$, where $q = \frac{p}{\theta + p(1-\theta)}$.*

*Proof.* As $\mathsf{h}_p^c(\mathcal{M})$ can be identified with a subspace of $L_p(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2(\mathbb{N}^2)))$ (see (2.7)), by interpolation between noncommutative $L_p$-spaces (2.3) we have $(\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta \subset \mathsf{h}_q^c(\mathcal{M})$ and

$$\|x\|_{\mathsf{h}_q^c} \lesssim \|x\|_{(\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta} \tag{3.5}$$

for all $x \in (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta$. Because $e_i\mathcal{M}e_i$ is a finite von Neumann algebra for each $i \in I$, where $(e_i)_{i\in I}$ is a net of $\tau$-finite projections in $\mathcal{M}_1$ appearing in Lemma 3.1, by [BCPY10, Lemma 4.3] we have

$$(\mathsf{h}_1^c(e_i\mathcal{M}e_i), \mathsf{h}_p^c(e_i\mathcal{M}e_i))_\theta = \mathsf{h}_q^c(e_i\mathcal{M}e_i)$$

with equivalent norms for each $i \in I$. Hence, $\mathsf{h}_q^c(e_i\mathcal{M}e_i) \subset (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta$ for each $i \in I$, and

$$\|a\|_{(\mathsf{h}_1^c(\mathcal{M}),\mathsf{h}_p^c(\mathcal{M}))_\theta} \lesssim \|a\|_{\mathsf{h}_q^c} \tag{3.6}$$

for all $a \in \mathsf{h}_q^c(e_i\mathcal{M}e_i))$. For a given $x \in \mathsf{h}_q^c(\mathcal{M})$, from (3.6) we deduce that

$$\|e_ixe_i - e_jxe_j\|_{(\mathsf{h}_1^c(\mathcal{M}),\mathsf{h}_p^c(\mathcal{M}))_\theta} \lesssim \|e_ixe_i - e_jxe_j\|_{\mathsf{h}_q^c}$$

for all $i, j \in I$. By Lemma 3.1, $\{e_ixe_i\}_{i\in I}$ is a Cauchy net in $(\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta$. Therefore, there exists $y \in (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta$ such that

$$\lim_I \|e_ixe_i - y\|_{(\mathsf{h}_1^c(\mathcal{M}),\mathsf{h}_p^c(\mathcal{M}))_\theta} = 0.$$

By Lemma 3.1 again, $\lim_I \|e_ixe_i - x\|_{\mathsf{h}_p^c} = 0$. Therefore, $y = x$. Thus by (3.6),

$$\|x\|_{(\mathsf{h}_1^c(\mathcal{M}),\mathsf{h}_p^c(\mathcal{M}))_\theta} \lesssim \|x\|_{\mathsf{h}_q^c}$$

for all $x \in \mathsf{h}_q^c(\mathcal{M})$. This completes the proof. ■

Lemma 3.3. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Then*

$$(\mathrm{bmo}^c(\mathcal{M}), \mathsf{h}_q^c(\mathcal{M}))_{q/p} = \mathsf{h}_p^c(\mathcal{M}) \tag{3.7}$$

*with equivalent norms for $1 < q < p < \infty$.*

*Proof.* By Lemma 3.2 and the duality theorem [BL76, Theorem 4.5.1] we immediately obtain (3.7). ■

We will need Wolff's interpolation theorem [Wol81], which states that

- For complex interpolation: given compatible Banach spaces $E_1, E_2, E_3$, and $E_4$ such that $E_1 \cap E_4$ is a dense subspace of $E_2$ and of $E_3$, and

  $$E_2 = (E_1, E_3)_\theta, \qquad E_3 = (E_2, E_4)_\vartheta$$

  for some $0 < \theta, \vartheta < 1$, one has

  $$E_2 = (E_1, E_4)_\xi, \qquad E_3 = (E_1, E_4)_\eta, \tag{3.8}$$

  where $\xi = \theta\vartheta/(1 - \theta + \theta\vartheta)$ and $\eta = \vartheta/(1 - \theta + \theta\vartheta)$.
- For real interpolation: if $E_1, E_2, E_3$, and $E_4$ are compatible quasi-Banach spaces such that $E_1 \cap E_4 \subset E_2 \cap E_3$, and

  $$E_2 = (E_1, E_3)_{\theta,p}, \qquad E_3 = (E_2, E_4)_{\vartheta,q}$$

  for some $0 < \theta, \vartheta < 1$ and $0 < p, q \leq \infty$, then

  $$E_2 = (E_1, E_4)_{\xi,p}, \qquad E_3 = (E_1, E_4)_{\eta,q}, \tag{3.9}$$

  where $\xi = \theta\vartheta/(1 - \theta + \theta\vartheta)$ and $\eta = \vartheta/(1 - \theta + \theta\vartheta)$.

Combining Lemmas 3.2 and 3.3 with Wolff's interpolation theorem (3.8) yields (3.2) in the case of semifinite von Neumann algebras:

THEOREM 3.4. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Then*

$$(\mathrm{bmo}^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_{p/q} = \mathsf{h}_q^c(\mathcal{M}) \tag{3.10}$$

*with equivalent norms for $1 \leq p < q < \infty$. Similarly,*

$$(\mathrm{bmo}^r(\mathcal{M}), \mathsf{h}_p^r(\mathcal{M}))_{p/q} = \mathsf{h}_q^r(\mathcal{M}), \tag{3.11}$$

*and in particular*

$$(\mathrm{bmo}(\mathcal{M}), \mathsf{h}_p(\mathcal{M}))_{p/q} = \mathsf{h}_q(\mathcal{M}) \tag{3.12}$$

*with equivalent norms for $1 \leq p < q < \infty$.*

By Theorem 3.4 and noncommutative Davis decomposition of [RWX19], we can generalize (3.1) to the case of semifinite von Neumann algebras using the argument in [BCPY10, Remark 4.9].

COROLLARY 3.5. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Then*

$$(\mathrm{BMO}^c(\mathcal{M}), \mathcal{H}_p^c(\mathcal{M}))_{p/q} = \mathcal{H}_q^c(\mathcal{M})$$

*with equivalent norms for $1 \leq p < q < \infty$. Similarly,*

$$(\mathrm{BMO}^r(\mathcal{M}), \mathcal{H}_p^r(\mathcal{M}))_{p/q} = \mathcal{H}_q^r(\mathcal{M})$$

*and*

$$(\mathrm{BMO}(\mathcal{M}), \mathcal{H}_p(\mathcal{M}))_{p/q} = \mathcal{H}_q(\mathcal{M})$$

*with equivalent norms for $1 \leq p < q < \infty$.*

Next, we turn to real interpolation.

LEMMA 3.6. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. For $1 \leq q \leq \infty$, $1 < p_0 \neq p_1 < \infty$ and $0 < \eta < 1$, one has*

$$(\mathsf{h}_{p_0}^c(\mathcal{M}), \mathsf{h}_{p_1}^c(\mathcal{M}))_{\eta,q} = \mathsf{h}_{p,q}^c(\mathcal{M}) \tag{3.13}$$

*with equivalent norms, where $p = \frac{p_0 p_1}{\eta p_0 + (1-\eta)p_1}$. Similarly,*

$$(\mathsf{h}_{p_0}^r(\mathcal{M}), \mathsf{h}_{p_1}^r(\mathcal{M}))_{\eta,q} = \mathsf{h}_{p,q}^r(\mathcal{M}) \tag{3.14}$$

*with equivalent norms.*

*Proof.* This can be shown as in [BCPY10, proof of (4.11)]. ∎

COROLLARY 3.7. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. For $1 \leq p' < p < \infty$ and $1 \leq q \leq \infty$,*

$$(\mathrm{bmo}^c(\mathcal{M}), \mathsf{h}_{p'}^c(\mathcal{M}))_{p'/p,q} = \mathsf{h}_{p,q}^c(\mathcal{M}) \tag{3.15}$$

*with equivalent norms.*

*Proof.* The argument is the same as in [BCPY10, proof of Theorem 4.8]. We include it for convenience. Let $p' < p < \infty$. We choose $p' < p_0 < p < p_1 < \infty$. Then there exists $0 < \eta < 1$ such that

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}.$$

By [BL76, Theorem 4.7.2] we obtain

$$\begin{aligned}(\mathrm{bmo}^c(\mathcal{M}), \mathsf{h}^c_{p'}(\mathcal{M}))_{p'/p,q} \\ &= ((\mathrm{bmo}^c(\mathcal{M}), \mathsf{h}^c_{p'}(\mathcal{M}))_{p'/p_0}, (\mathrm{bmo}^c(\mathcal{M}), \mathsf{h}^c_{p'}(\mathcal{M}))_{p'/p_1})_{\eta,q}.\end{aligned}$$

Then (3.10) yields

$$(\mathrm{bmo}^c(\mathcal{M}), \mathsf{h}^c_{p'}(\mathcal{M}))_{p'/p,q} = (\mathsf{h}^c_{p_0}(\mathcal{M}), \mathsf{h}^c_{p_1}(\mathcal{M}))_{\eta,q}.$$

An application of (3.13) gives (3.15). ∎

Using the argument above, we can get real interpolation between $\mathcal{H}_p$ and BMO spaces:

COROLLARY 3.8. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. For $1 \leq p' < p < \infty$ and $1 \leq q \leq \infty$,*

$$(\mathrm{BMO}^c(\mathcal{M}), \mathcal{H}^c_{p'}(\mathcal{M}))_{p'/p,q} = \mathcal{H}^c_{p,q}(\mathcal{M}) \tag{3.16}$$

*with equivalent norms. Similarly,*

$$(\mathrm{BMO}^r(\mathcal{M}), \mathcal{H}^r_{p'}(\mathcal{M}))_{p'/p,q} = \mathcal{H}^r_{p,q}(\mathcal{M})$$

*and*

$$(\mathrm{BMO}(\mathcal{M}), \mathcal{H}_{p'}(\mathcal{M}))_{p'/p,q} = \mathcal{H}_{p,q}(\mathcal{M})$$

*with equivalent norms for $1 \leq p' < p < \infty$ and $1 \leq q \leq \infty$.*

**4. John–Nirenberg inequality.** In this section, we present the John–Nirenberg inequality on symmetric spaces of noncommutative martingales in the setting of semifinite von Neumann algebras, generalizing the results obtained in [HM12, JM07].

**4.1. Crude version.** We first give a crude version of the John–Nirenberg inequality on symmetric spaces of noncommutative martingales.

DEFINITION 4.1. Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. For a symmetric Banach function space $E$ on $(0,\infty)$, we define

(i) $\mathrm{BMO}^c_E(\mathcal{M}) = \{x \in L_1(\mathcal{M}) + \mathcal{M} : \|x\|_{\mathrm{BMO}^c_E} < \infty\}$, where

$$\|x\|_{\mathrm{BMO}^c_E} = \sup_{n\geq 1}\{\|(x - x_{n-1})a\|_{\mathcal{H}^c_E} : \forall a \in E(\mathcal{M}_n),\ \|a\|_{E(\mathcal{M}_n)} \leq 1\},$$

which is a norm on $\mathrm{BMO}^c_E(\mathcal{M})$;

(ii) $\mathrm{BMO}_E^r(\mathcal{M}) = \{x \in L_1(\mathcal{M}) + \mathcal{M} : x^* \in \mathrm{BMO}_E^c(\mathcal{M})\}$, equipped with the norm $\|x\|_{\mathrm{BMO}_E^r} = \|x^*\|_{\mathrm{BMO}_E^c}$;

(iii) $\mathrm{BMO}_E(\mathcal{M}) = \mathrm{BMO}_E^c(\mathcal{M}) \cap \mathrm{BMO}_E^r(\mathcal{M})$, equipped with the intersection norm
$$\|x\|_{\mathrm{BMO}_E} = \|x\|_{\mathrm{BMO}_E^c} + \|x\|_{\mathrm{BMO}_E^r}.$$

REMARK 4.2. For $E = L_p(0,\infty)$, these are $\mathrm{BMO}_p^c(\mathcal{M}), \mathrm{BMO}_p^r(\mathcal{M})$ and $\mathrm{BMO}_p(\mathcal{M})$ defined in [JM07, HM12]. In particular, for $E = L_2(0,\infty)$, they are exactly the spaces $\mathrm{BMO}^c(\mathcal{M}), \mathrm{BMO}^r(\mathcal{M})$ and $\mathrm{BMO}(\mathcal{M})$.

LEMMA 4.3. *Let $E$ be a symmetric Banach function space on $(0,\infty)$ with the Fatou property. If $E \in \mathrm{Int}(L_2, L_q)$ for some $2 < q < \infty$, then*
$$\|b\|_\infty \approx \sup_{a \in E(\mathcal{M}),\, \|a\|_E \le 1} \|ba\|_{\mathcal{H}_E^c}$$
*for $b \in L_\infty(\mathcal{M})$.*

*Proof.* By noncommutative Davis decomposition (see [RWX19, Corollary 3.10(ii) and Theorem 4.1(ii)]), we have $E(\mathcal{M}) \subset \mathcal{H}_E^c(\mathcal{M})$ and $\|ba\|_{\mathcal{H}_E^c} \lesssim \|ba\|_E$. Hence,
$$\sup_{a \in E(\mathcal{M}),\, \|a\|_E \le 1} \|ba\|_{\mathcal{H}_E^c} \lesssim \sup_{a \in E(\mathcal{M}),\, \|a\|_E \le 1} \|ba\|_E = \|b\|_\infty.$$
Conversely, since $\|x\|_{E(\mathcal{M})} \lesssim \|x\|_{\mathcal{H}_E^c}$ for self-adjoint $x \in E(\mathcal{M})$ (see [RWX19, Proposition 4.3(ii)]), one has
$$\begin{aligned}
\|b^*\|_\infty &= \sup_{x \in E^{(2)}(\mathcal{M}),\, \|x\|_{E^{(2)}} \le 1} \|xb^*\|_{E^{(2)}} = \sup_{x \in E^{(2)}(\mathcal{M}),\, \|x\|_{E^{(2)}} \le 1} \|bx^*xb^*\|_E^{1/2} \\
&\lesssim \sup_{x \in E^{(2)}(\mathcal{M}),\, \|x\|_{E^{(2)}} \le 1} \|bx^*xb^*\|_{\mathcal{H}_E^c}^{1/2} \le \sup_{a \in E(\mathcal{M}),\, \|a\|_E \le \|b\|_\infty} \|ba\|_{\mathcal{H}_E^c}^{1/2} \\
&= \sup_{a \in E(\mathcal{M}),\, \|a\|_E \le 1} \|b\|_\infty^{1/2} \|ba\|_{\mathcal{H}_E^c}^{1/2}.
\end{aligned}$$
This yields $\|b\|_\infty \lesssim \sup_{a \in E(\mathcal{M}),\, \|a\|_E \le 1} \|ba\|_{\mathcal{H}_E^c}$. ■

LEMMA 4.4. *Let $E$, $E_1$, $E_2$ be symmetric Banach function spaces on $(0,\infty)$ such that $E = E_1 \odot E_2$. If $x \in E(\mathcal{M})^+$, then for $\varepsilon > 0$, there exist $a_1 \in E_1^{(2)}(\mathcal{M})^+$ and $a_2 \in E_2(\mathcal{M})^+$ such that $x = a_1 a_2 a_1$ and*
$$\|x\|_E \le \|a_1\|_{E_1^{(2)}}^2 \|a_2\|_{E_2} < \|x\|_E + \varepsilon.$$

*Proof.* Let $\mathcal{N}$ be the commutative von Neumann subalgebra of $\mathcal{M}$ generated by the spectral projection of $x$. Then $\mathcal{N}$ is isometrically isomorphic to $L_\infty(\Omega, \Sigma, \mu)$, where $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space. Hence, $x \in E(\mathcal{N}) = E(\Omega, \mu)$. Since $E = E_1 \odot E_2$, for every $\varepsilon > 0$ there are $x_1 \in E_1(\Omega,\mu)^+ = E_1(\mathcal{N})^+$ and $x_2 \in E_2(\Omega,\mu)^+ = E_2(\mathcal{N})^+$ such that $x = x_1 x_2$ and $\|x\|_{E(\mathcal{N})} + \varepsilon > \|x_1\|_{E_1(\mathcal{N})} \|x_2\|_{E_2(\mathcal{N})}$. Set $a_1 = x_1^{1/2}$ and $a_2 = x_2$.

Then $a_1 \in E_1^{(2)}(\mathcal{N})^+ \subset E_1^{(2)}(\mathcal{M})^+$ and $a_2 \in E_2(\mathcal{N})^+ \subset E_2(\mathcal{M})^+$ satisfy $x = a_1a_2a_1$ and $\|a_1\|_{E_1^{(2)}} = \|x_1\|_{E_1}^{1/2}$. Thus $\|a_1\|^2_{E_1^{(2)}}\|a_2\|_{E_2} < \|x\|_E + \varepsilon$. ∎

THEOREM 4.5. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric Banach function space on $(0,\infty)$ with the Fatou property. If $E \in \mathrm{Int}(L_2, L_q)$ for some $2 < q < \infty$, then*

$$\mathrm{BMO}^c_E(\mathcal{M}) = \mathrm{BMO}^c(\mathcal{M})$$

*with equivalent norms. Similarly, $\mathrm{BMO}^r_E(\mathcal{M}) = \mathrm{BMO}^r(\mathcal{M})$ with equivalent norms. Consequently,*

$$\mathrm{BMO}_E(\mathcal{M}) = \mathrm{BMO}(\mathcal{M})$$

*with equivalent norms.*

REMARK 4.6. The assumption $E \in \mathrm{Int}(L_2, L_q)$ for some $2 < q < \infty$ cannot be dropped: see [HM12, Remark 3.9].

*Proof of Theorem 4.5.* It suffices to prove the column case. By [DPPS11, Proposition 3.5], $E^{(1/2)} \in \mathrm{Int}(L_1, L_{q/2})$. Using Lemma 2.2, we know that $E^{(1/2)}$ can be renormed into a symmetric Banach space, and so we can assume $E^{(1/2)}$ is a symmetric Banach function space on $(0,\infty)$. Since

$$L_1(0,\infty) = E^{(1/2)} \odot (E^{(1/2)})^\times = E \odot (E^{(1/2)})^\times \odot E$$

(see Theorem 2.1(ii) and (2.2)), by Lemmas 4.3 and 4.4 we have

$$\begin{aligned}
&\|\mathcal{E}_n(|x - x_{n-1}|^2)\|_\infty \leq \sup_{b\in L_1(\mathcal{M}_n)^+,\, \|b\|_1\leq 1} \tau(|x-x_n|^2 b) + \|x_n - x_{n-1}\|^2_\infty \\
&\quad\lesssim \sup_{\substack{b_1\in E(\mathcal{M}_n)^+,\, \|b_1\|_E\leq 1;\\ b_2\in (E^{(1/2)})^\times(\mathcal{M}_n)^+,\, \|b_2\|_{(E^{(1/2)})^\times}\leq 1}} \tau\Big(\sum_{k\geq n+1} |dx_k b_1|^2 b_2\Big) \\
&\qquad + \sup_{a\in E(\mathcal{M}_n),\, \|a\|_E\leq 1} \|(x_n - x_{n-1})a\|^2_{\mathcal{H}^c_E} \\
&\quad\leq \sup_{\substack{b_1\in E(\mathcal{M}_n)^+,\, \|b_1\|_E\leq 1;\\ b_2\in (E^{(1/2)})^\times(\mathcal{M}_n)^+,\, \|b_2\|_{(E^{(1/2)})^\times}\leq 1}} \Big\|\sum_{k\geq n+1} |dx_k b_1|^2\Big\|_{E^{(1/2)}} \|b_2\|_{(E^{(1/2)})^\times} \\
&\qquad + \sup_{a\in E(\mathcal{M}_n),\, \|a\|_E\leq 1} \|(x_n - x_{n-1})a\|^2_{\mathcal{H}^c_E} \\
&\quad\leq \sup_{b_1\in E(\mathcal{M}_n)^+,\, \|b_1\|_E\leq 1} \|(x-x_n)b_1\|^2_{\mathcal{H}^c_E} + \sup_{a\in E(\mathcal{M}_n),\, \|a\|_E\leq 1} \|(x_n - x_{n-1})a\|^2_{\mathcal{H}^c_E}.
\end{aligned}$$

Since $\|\mathcal{E}_n x\|_{\mathcal{H}^c_E} \leq \|x\|_{\mathcal{H}^c_E}$, one has

$$\|x\|_{\mathrm{BMO}^c} \lesssim \sup_{n\geq 1} \sup_{a\in E(\mathcal{M}_n),\, \|a\|_E\leq 1} \|(x - x_{n-1})a\|_{\mathcal{H}^c_E} = \|x\|_{\mathrm{BMO}^c_E}.$$

Conversely, by Lemma 4.3 we obtain

$$\begin{aligned}\|x\|_{\mathrm{BMO}_E^c} &\le \sup_{n\ge 1}\sup_{a\in E(\mathcal{M}_n),\,\|a\|_{E(\mathcal{M}_n)}\le 1}\|(x-x_n)a\|_{\mathcal{H}_E^c}+\sup_{n\ge1}\|x_n-x_{n-1}\|_\infty\\ &\le \sup_{n\ge 1}\sup_{a\in E(\mathcal{M}_n),\,\|a\|_{E(\mathcal{M}_n)}\le 1}\|(dx_ka)_{k=n+1}^\infty\|_{E(\mathcal{M},\ell_c^2)}+\|x\|_{\mathrm{BMO}^c},\end{aligned}$$

where we have used the Kadison inequality $|\mathcal{E}_n(x-x_{n-1})|^2\le\mathcal{E}_n(|x-x_{n-1}|^2)$ to get the second term on the right hand side of the last inequality. It remains to estimate the first term on the right hand side. To this end, by the Hahn–Banach theorem and the duality of $\mathcal{H}_1^c(\mathcal{M})$ and $\mathrm{BMO}^c(\mathcal{M})$, there exists a sequence $(b_n)_{n=1}^\infty\in L_\infty(\mathcal{M},\ell_c^2)$ such that $\|(b_n)_{n=1}^\infty\|_{L_\infty(\mathcal{M},\ell_c^2)}=\|x\|_{\mathrm{BMO}^c}$ with $dx_k=\mathcal{E}_k(b_k)-\mathcal{E}_{k-1}(b_k)$ for $k\ge 1$. Using the noncommutative Stein inequality (see [Bek06, Lemma 2.2] and Proposition 2.4(ii)) and the fact that $E\odot L_\infty=E$, we obtain

$$\begin{aligned}&\sup_{a\in E(\mathcal{M}_n),\,\|a\|_{E(\mathcal{M}_n)}\le 1}\|(dx_ka)_{k=n+1}^\infty\|_{E(\mathcal{M},\ell_c^2)}\\ &\qquad\le \sup_{a\in E(\mathcal{M}_n),\,\|a\|_{E(\mathcal{M}_n)}\le 1}\|(\mathcal{E}_k[b_ka])_{k=n+1}^\infty\|_{E(\mathcal{M},\ell_c^2)}\\ &\qquad\quad+\sup_{a\in E(\mathcal{M}_n),\,\|a\|_{E(\mathcal{M}_n)}\le 1}\|(\mathcal{E}_k[b_{k+1}a])_{k=n}^\infty\|_{E(\mathcal{M},\ell_c^2)}\\ &\qquad\lesssim \sup_{a\in E(\mathcal{M}_n),\,\|a\|_{E(\mathcal{M}_n)}\le 1}\|(b_ka)_{k=n+1}^\infty\|_{E(\mathcal{M},\ell_c^2)}\\ &\qquad\le \Big\|\Big(\sum_{k=n+1}^\infty|b_k|^2\Big)^{1/2}\Big\|_\infty\le\|x\|_{\mathrm{BMO}^c}.\end{aligned}$$

This completes the proof. ∎

In what follows, we consider the conditioned version of the crude John–Nirenberg inequality on symmetric spaces of noncommutative martingales.

Definition 4.7. Let $(\mathcal{M}_n)_{n\ge1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric quasi-Banach function space on $(0,\infty)$ which is $s$-convex for some $0<s<\infty$ and $0<p_E\le q_E<\infty$. We define

(i) $\mathrm{bmo}_E^c(\mathcal{M})=\{x\in L_1(\mathcal{M})+\mathcal{M}:\|x\|_{\mathrm{bmo}_E^c}<\infty\}$, equipped with the quasi-norm

$$\|x\|_{\mathrm{bmo}_E^c}=\|\mathcal{E}_1(x)\|_\infty+\sup_{n\ge1}\sup_{a\in E(\mathcal{M}_n),\,\|a\|_{E(\mathcal{M}_n)}\le1}\|(x-x_n)a\|_{\mathsf{h}_E^c};$$

(ii) $\mathrm{bmo}_E^r(\mathcal{M})=\{x:x^*\in\mathrm{bmo}_E^c(\mathcal{M})\}$, equipped with the quasi-norm $\|x\|_{\mathrm{bmo}_E^r}=\|x^*\|_{\mathrm{bmo}_E^c}$; and

(iii) $\mathrm{bmo}_E(\mathcal{M})=\mathrm{bmo}_E^c(\mathcal{M})\cap\mathrm{bmo}_E^r(\mathcal{M})\cap\mathrm{bmo}_E^d(\mathcal{M})$, equipped with the intersection quasi-norm

$$\|x\|_{\mathrm{bmo}_E}=\|x\|_{\mathrm{bmo}_E^c}+\|x\|_{\mathrm{bmo}_E^r}+\|x\|_{\mathrm{bmo}_E^d}.$$

If $E = L_p(0,\infty)$, these spaces are $\mathrm{bmo}_p^c(\mathcal{M}), \mathrm{bmo}_p^r(\mathcal{M})$ and $\mathrm{bmo}_p(\mathcal{M})$ defined in [HM12] (cf. [JM07]) and in particular, for $E = L_2(0,\infty)$, they are exactly the spaces $\mathrm{bmo}^c(\mathcal{M}), \mathrm{bmo}^r(\mathcal{M})$ and $\mathrm{bmo}(\mathcal{M})$.

LEMMA 4.8. *For* $0 < p < \infty$, $\mathrm{bmo}_p^c = \mathrm{bmo}^c$ *with equivalent quasi-norms. Similarly,* $\mathrm{bmo}_p^r = \mathrm{bmo}^r$ *with equivalent quasi-norms.*

*Proof.* This can be shown as in [HM12, proof of Theorem 3.3] with the help of Theorem 3.4. We omit the details. ∎

THEOREM 4.9. *Let* $(\mathcal{M}_n)_{n\geq 1}$ *be a filtration of a semifinite von Neumann algebra* $\mathcal{M}$. *Let* $E$ *be a symmetric quasi-Banach function space on* $(0,\infty)$ *which is* $s$*-convex for some* $0 < s < \infty$. *If* $0 < p_E \leq q_E < \infty$, *then*

$$\mathrm{bmo}_E^c(\mathcal{M}) = \mathrm{bmo}^c(\mathcal{M})$$

*with equivalent quasi-norms. Similarly,* $\mathrm{bmo}_E^r(\mathcal{M}) = \mathrm{bmo}^r(\mathcal{M})$ *with equivalent quasi-norms.*

*Proof.* Fix some $p, q$ with $0 < p < p_E \leq q_E < q < \infty$. Using [DPPS11, Proposition 3.5], we know that $E^{(1/p)} \in \mathrm{Int}(L_1, L_{q/p})$. By Lemma 2.2, $E^{(1/p)}$ can be renormed into a symmetric Banach space, and so we assume that $E^{(1/p)}$ is a symmetric Banach function space on $(0,\infty)$. By Theorem 2.1(ii), we get $L_1(0,\infty) = E^{(1/p)} \odot (E^{(1/p)})^\times$. Let $F = ([E^{(1/p)}]^\times)^{(p)}$. Then by Theorem 2.1(i), $L_p(0,\infty) = E \odot F$. Given $a \in L_p(\mathcal{M}_n) \cap \mathcal{M}_n$ with $\|a\|_p \leq 1$, let $a = u|a|$ be the polar decomposition of $a$. Using the same method as in the proof of Lemma 4.4, we deduce that for any $\varepsilon > 0$, there exists a factorization $|a| = a_1' a_2$ such that $\|a_1'\|_E \leq 1$ and $\|a_2\|_F < 1 + \varepsilon$. Letting $a_1 = ua_1'$, we have $a = a_1 a_2$ and $\|a_1\|_E \leq 1$. Since $L_{p/2}(0,\infty) = F \odot E^{(1/2)} \odot F$,

$$\begin{aligned}\|(x - x_n)a\|_{\mathsf{h}_p^c} &= \|a_2^* a_1^* s_c^2(x - x_n) a_1 a_2\|_{p/2}^{1/2} \leq \|a_2\|_F \|a_1^* s_c^2(x - x_n) a_1\|_{E^{(1/2)}}^{1/2} \\ &\leq (1 + \varepsilon)\|(x - x_n)a_1\|_{\mathsf{h}_E^c}.\end{aligned}$$

Thus $\|x\|_{\mathrm{bmo}_p^c} \leq \|x\|_{\mathrm{bmo}_E^c}$. By Lemma 4.8, we get $\|x\|_{\mathrm{bmo}^c} \lesssim \|x\|_{\mathrm{bmo}_E^c}$.

Conversely, we define

$$T : L_p(\mathcal{M}_n) + L_q(\mathcal{M}_n) \to L_p(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathbb{N}^2))) + L_q(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathbb{N}^2)))$$

by $T(a) = W_c((x - x_n)a)$. By (2.7), one has

$$\|T : L_p(\mathcal{M}_n) \to L_p(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathbb{N}^2)))\| \leq \|x\|_{\mathrm{bmo}_p^c},$$
$$\|T : L_q(\mathcal{M}_n) \to L_q(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathbb{N}^2)))\| \leq \|x\|_{\mathrm{bmo}_q^c}.$$

By Lemmas 2.3 and 4.8, we obtain

$$\|T : E(\mathcal{M}_n) \to E(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathbb{N}^2)))\| \leq C\|x\|_{\mathrm{bmo}^c},$$

so $\|x\|_{\mathrm{bmo}_E^c} \lesssim \|x\|_{\mathrm{bmo}^c}$. This completes the proof. ∎

In the rest of this subsection, we turn to the John–Nirenberg inequality of Junge/Musat's type [JM07] on symmetric spaces of noncommutative martingales.

LEMMA 4.10. *For* $0<p<\infty$,

$$\|x\|_{\mathrm{bmo}} \approx \sup_{n\ge1}\|dx_n\|_\infty + \sup_{n\ge1}\sup_{a\in\mathcal{M}_n,\,\|a\|_p\le1}(\|(x-x_n)a\|_p+\|a(x-x_n)\|_p).$$

*Proof.* The proof is the same as that of [HM12, Theorem 3.10] with the use of Theorem 3.4. ■

THEOREM 4.11. *Let* $(\mathcal{M}_n)_{n\ge1}$ *be a filtration of a semifinite von Neumann algebra* $\mathcal{M}$. *Let* $E$ *be a symmetric quasi-Banach function space on* $(0,\infty)$ *which is* $s$*-convex for some* $0<s<\infty$. *If* $0<p_E\le q_E<\infty$, *then*

$$\|x\|_{\mathrm{bmo}} \approx \sup_{n\ge1}\|dx_n\|_\infty + \sup_{n\ge1}\sup_{a\in\mathcal{M}_n,\,\|a\|_{E(\mathcal{M}_n)}\le1}(\|(x-x_n)a\|_E+\|a(x-x_n)\|_E).$$

*Proof.* Denote by $b_E(x)$ the quantity on the right hand side. Denote by $S(\mathcal{M}_n)$ the linear span of all $x\in\mathcal{M}_n^+$ such that $\tau(s(x))<\infty$, and by $\mathcal{P}(\mathcal{M}_n)$ the lattice of projections of $\mathcal{M}_n$. Let $a\in S(\mathcal{M}_n)$ and $\|a\|_E\le1$. Then there exists $e\in\mathcal{P}(\mathcal{M}_n)$ such that $a\in e\mathcal{M}_ne$ and $\tau(e)<\infty$. From the proof of Theorem 4.9 and using the notations there, we know that for any $\varepsilon>0$, there are $a_1'\in E(e\mathcal{M}_ne)$ and $a_2'\in F(e\mathcal{M}_ne)^+$ such that $a=a_1'a_2'$, $\|a_1'\|_E\le1$ and $\|a_2'\|_F\le1+\varepsilon$. Set $a_2=a_2'+\varepsilon e$ and $a_1=a_1'a_2'(a_2'+\varepsilon e)^{-1}$. Then $a=a_1a_2$, $a_2$ is invertible with bounded inverse, $a_1\in\mathcal{M}_n\cap E(\mathcal{M}_n)$, $\|a_1\|_E\le1$ and $\|a_2\|_F\le K[1+\varepsilon(1+\|e\|_F)]$, where $K$ is the constant of the quasi-norm $\|\cdot\|_F$. Therefore, we have

$$\|(x-x_n)a\|_p\le\|(x-x_n)a_1\|_E\|a_2\|_E\le K[1+\varepsilon(1+\|e\|_F)]\|(x-x_n)a_1\|_E.$$

Since $S(\mathcal{M}_n)$ is dense in $L_p(\mathcal{M}_n)\cap\mathcal{M}_n$ under the $p$-norm, we obtain

$$\sup_{a\in\mathcal{M}_n,\,\|a\|_p\le1}\|(x-x_n)a\|_p\le K\sup_{a\in\mathcal{M}_n,\,\|a\|_{E(\mathcal{M}_n)}\le1}\|(x-x_n)a\|_E.$$

Similarly,

$$\sup_{a\in\mathcal{M}_n,\,\|a\|_p\le1}\|a(x-x_n)\|_p\le K\sup_{a\in\mathcal{M}_n,\,\|a\|_{E(\mathcal{M}_n)}\le1}\|a(x-x_n)\|_E.$$

Hence, by Lemma 4.10 we get $\|x\|_{\mathrm{bmo}}\lesssim b_E(x)$.

We can obtain the reverse inequality using the operators

$$T_1: L_p(\mathcal{M}_n)+L_q(\mathcal{M}_n)\to L_p(\mathcal{M})+L_q(\mathcal{M}),\quad a\mapsto(x-x_n)a,$$
$$T_2: L_p(\mathcal{M}_n)+L_q(\mathcal{M}_n)\to L_p(\mathcal{M})+L_q(\mathcal{M}),\quad a\mapsto a(x-x_n),$$

and the interpolation argument as in Theorem 4.9. ■

Also, we can obtain the BMO version of Theorem 4.11:

Lemma 4.12. *For* $0<p<\infty$,

$$\|x\|_{\mathrm{BMO}} \approx \sup_{n\geq 1} \sup_{a\in\mathcal{M}_n,\,\|a\|_p\leq 1} (\|(x-x_{n-1})a\|_p + \|a(x-x_{n-1})\|_p).$$

*Proof.* The proof is the same as that of [HM12, Corollary 3.13]. ∎

Theorem 4.13. *Let* $(\mathcal{M}_n)_{n\geq 1}$ *be a filtration of a semifinite von Neumann algebra* $\mathcal{M}$. *Let* $E$ *be a symmetric quasi-Banach function space on* $(0,\infty)$ *which is* $s$*-convex for some* $0<s<\infty$. *If* $0<p_E\leq q_E<\infty$, *then*

$$\|x\|_{\mathrm{BMO}} \approx \sup_{n\geq 1} \sup_{a\in\mathcal{M}_n,\,\|a\|_{E(\mathcal{M}_n)}\leq 1} (\|(x-x_{n-1})a\|_E + \|a(x-x_{n-1})\|_E).$$

*Proof.* This can be shown by using the interpolation method as in the proof of Theorem 4.9 and Lemma 4.12. We omit the details. ∎

**4.2. Fine version.** In this subsection, we give a fine version of the John–Nirenberg inequality on symmetric spaces of noncommutative martingales.

Definition 4.14. Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric quasi-Banach function space on $(0,\infty)$ which is $s$-convex for some $0<s<\infty$ and $0<p_E\leq q_E<\infty$. We define

(i) $\mathrm{bmo}^c_{E,\mathrm{pr}}(\mathcal{M}) = \{x\in L_1(\mathcal{M})+\mathcal{M} : \|x\|_{\mathrm{bmo}^c_{E,\mathrm{pr}}}<\infty\}$, where

$$\|x\|_{\mathrm{bmo}^c_{E,\mathrm{pr}}} = \|\mathcal{E}_1(x)\|_\infty + \sup_{n\geq 1}\sup_{e\in\mathcal{P}(\mathcal{M}_n)} \Big\|(x-x_n)\frac{e}{\|e\|_E}\Big\|_{\mathsf{h}^c_E};$$

(ii) $\mathrm{bmo}^r_{E,\mathrm{pr}}(\mathcal{M}) = \{x : x^*\in \mathrm{bmo}^c_{E,\mathrm{pr}}(\mathcal{M})\}$, equipped with $\|x\|_{\mathrm{bmo}^r_{E,\mathrm{pr}}} = \|x^*\|_{\mathrm{bmo}^c_{E,\mathrm{pr}}}$; and

(iii) $\mathrm{bmo}_{E,\mathrm{pr}}(\mathcal{M}) = \mathrm{bmo}^c_{E,\mathrm{pr}}(\mathcal{M})\cap \mathrm{bmo}^r_{E,\mathrm{pr}}(\mathcal{M})\cap \mathrm{bmo}^d(\mathcal{M})$, equipped with

$$\|x\|_{\mathrm{bmo}_{E,\mathrm{pr}}} = \|x\|_{\mathrm{bmo}^c_{E,\mathrm{pr}}} + \|x\|_{\mathrm{bmo}^r_{E,\mathrm{pr}}} + \|x\|_{\mathrm{bmo}^d}.$$

First, we need to generalize [HM12, Theorem 3.16] to the setting of a semifinite von Neumann algebra.

Lemma 4.15. *For* $0<p<\infty$,

$$\mathrm{bmo}^c_{p,\mathrm{pr}}(\mathcal{M}) = \mathrm{bmo}^c(\mathcal{M})$$

*with equivalent quasi-norms, and* $\mathrm{bmo}^r_{p,\mathrm{pr}}(\mathcal{M}) = \mathrm{bmo}^r(\mathcal{M})$ *with equivalent quasi-norms.*

*Proof.* The proof is similar to that of [HM12, Theorem 3.16] with the help of Lemma 4.12. ∎

Theorem 4.16. *Let* $(\mathcal{M}_n)_{n\geq 1}$ *be a filtration of a semifinite von Neumann algebra* $\mathcal{M}$. *Let* $E$ *be a symmetric quasi-Banach function space on* $(0,\infty)$ *which is* $s$*-convex for some* $0<s<\infty$ *and* $0<p_E\leq q_E<\infty$. *Then*

$$\mathrm{bmo}^c_{E,\mathrm{pr}}(\mathcal{M}) = \mathrm{bmo}^c(\mathcal{M})$$

*with equivalent quasi-norms. Similarly,*

$$\mathrm{bmo}^r_{E,\mathrm{pr}}(\mathcal{M}) = \mathrm{bmo}^r(\mathcal{M})$$

*with equivalent quasi-norms.*

*Proof.* By definition, we have $\|x\|_{\mathrm{bmo}^c_{E,\mathrm{pr}}} \le \|x\|_{\mathrm{bmo}^c_E}$. Then by Theorem 4.9, we have

$$\|x\|_{\mathrm{bmo}^c_{E,\mathrm{pr}}} \lesssim \|x\|_{\mathrm{bmo}^c}.$$

Conversely, as $L_1(0,\infty) = E^{(1/p)} \odot (E^{(1/p)})^\times$, we have $L_p(0,\infty) = E \odot F$ by Theorem 2.1(i), where $F = ([E^{(1/p)}]^\times)^{(p)}$, $0 < p < p_E$. For $e \in \mathcal{P}(\mathcal{M}_n)$ one has $\mu_t(e) = \chi_{[0,\tau(e)]}(t)$. By [KLM14, Theorem 2], we have

$$\tau(e) = \|e\|_1 = \|e\|_{E^{(1/p)}}\|e\|_{(E^{(1/p)})^\times}.$$

Hence,

$$\tau(e)^{1/p} = \|e\|_1^{1/p} = \|e\|_{E^{(1/p)}}^{1/p}\|e\|_{(E^{(1/p)})^\times}^{1/p} = \|e\|_E\|e\|_{((E^{(1/p)})^\times)^{(p)}} = \|e\|_E\|e\|_F.$$

Therefore, since $L_{p/2}(0,\infty) = E^{1/2} \odot F^{1/2} = F \odot E^{1/2} \odot F$, we have

$$\begin{aligned}\Big\|(x-x_n)\frac{e}{\tau(e)^{1/p}}\Big\|_{\mathsf{h}^c_p} &= \Big\|\frac{e}{\|e\|_F}\,\frac{e}{\|e\|_E}s_c^2(x-x_n)\frac{e}{\|e\|_E}\,\frac{e}{\|e\|_F}\Big\|_{p/2}^{1/2}\\ &\le \Big\|\frac{e}{\|e\|_F}\Big\|_F\Big\|\frac{e}{\|e\|_E}s_c^2(x-x_n)\frac{e}{\|e\|_E}\Big\|_{E^{(1/2)}}^{1/2}\\ &\le \Big\|(x-x_n)\frac{e}{\|e\|_E}\Big\|_{\mathsf{h}^c_E}.\end{aligned}$$

Thus $\|x\|_{\mathrm{bmo}^c_{p,\mathrm{pr}}} \le \|x\|_{\mathrm{bmo}^c_{E,\mathrm{pr}}}$. By Lemma 4.15, we get $\|x\|_{\mathrm{bmo}^c} \lesssim \|x\|_{\mathrm{bmo}^c_{E,\mathrm{pr}}}$. This completes the proof. ∎

Using Theorem 4.11 (resp. Theorem 4.13) and the argument in the proof of Theorem 4.16 for $\mathrm{bmo}(\mathcal{M})$ (resp. $\mathrm{BMO}(\mathcal{M})$), we obtain a fine version of the John–Nirenberg inequality of Junge/Musat's type on symmetric spaces of noncommutative martingales.

Corollary 4.17. *Let $(\mathcal{M}_n)_{n\ge1}$ be a filtration of a semi-finite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric quasi-Banach function space on $(0,\infty)$ which is $s$-convex for some $0 < s < \infty$ and $0 < p_E \le q_E < \infty$. Then*

$$\begin{aligned}\|x\|_{\mathrm{bmo}} \approx{}& \sup_{n\ge1}\|dx_n\|_\infty\\ &+\sup_{n\ge1}\sup_{e\in\mathcal{P}(\mathcal{M}_n)}\Big(\Big\|(x-x_n)\frac{e}{\|e\|_E}\Big\|_E + \Big\|\frac{e}{\|e\|_E}(x-x_n)\Big\|_E\Big)\end{aligned}$$

*and*

$$\|x\|_{\mathrm{BMO}} \approx \sup_{n\ge1}\sup_{e\in\mathcal{P}(\mathcal{M}_n)}\Big(\Big\|(x-x_{n-1})\frac{e}{\|e\|_E}\Big\|_E + \Big\|\frac{e}{\|e\|_E}(x-x_{n-1})\Big\|_E\Big).$$

Finally, we can get analogues of [HM12, Theorems 3.17 and 3.20] in the setting of a semifinite von Neumann algebra by arguments similar to the proof of [HM12, Theorem 3.17].

THEOREM 4.18. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$.*

(i) *There is an absolute constant $c > 0$ such that for any $x \in \mathrm{bmo}^c(\mathcal{M})$,*

$$\frac{1}{\tau(e)}\tau(e_{(\lambda,\infty)}(s^c((x-x_n)e))) \leq 2\exp\left(-\frac{c\lambda}{\|x\|_{\mathrm{bmo}^c}}\right)$$

*for all $e \in \mathcal{P}(\mathcal{M}_n)$, $n \geq 1$, and $\lambda > 0$.*

(ii) *There is an absolute constant $c > 0$ such that for any $x \in \mathrm{BMO}(\mathcal{M})$,*

$$\frac{1}{\tau(e)}\tau(e_{(\lambda,\infty)}(|(x-x_{n-1})e|)+e_{(\lambda,\infty)}(|e(x-x_{n-1})|)) \leq 4\exp\left(-\frac{c\lambda}{\|x\|_{\mathrm{BMO}}}\right)$$

*for all $e \in \mathcal{P}(\mathcal{M}_n)$, $n \geq 1$, and $\lambda > 0$.*

**5. Applications.** In this section, we will give applications of the results obtained in the previous sections to atomic decomposition and paraproducts on noncommutative martingales. We refer to [CRX20, RWZ21] for some recent results on the atomic decomposition of noncommutative martingales.

**5.1. Atomic decomposition.** We begin with the definition of crude symmetric atoms.

DEFINITION 5.1. Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric Banach function space on $(0,\infty)$ which is $s$-convex for some $0 < s < \infty$ and $1 < p_E \leq q_E < \infty$. An operator $a \in L_1(\mathcal{M})$ is said to be a $(1,E)_c$-*crude atom* with respect to $(\mathcal{M}_n)_{n\geq 1}$ if there exist $n \geq 1$ and a factorization $a = yb$ such that

(i) $\mathcal{E}_n(y) = 0$;
(ii) $b \in E^\times(\mathcal{M}_n)$ and $\|b\|_{E^\times} \leq 1$;
(iii) $\|y\|_{\mathsf{h}_E^c} \leq 1$.

Replacing the factorization above by $a = by$ and $\|y\|_{\mathsf{h}_E^c}$ by $\|y\|_{\mathsf{h}_E^r}$, we get the notion of a $(1,E)_r$-*crude atom.*

We have the following basic property.

LEMMA 5.2. *If $a$ is a $(1,E)_c$-crude atom, then*

$$\|a\|_{\mathsf{h}_1^c} \leq 1.$$

*The same estimate holds for $(1,E)_r$-crude atoms.*

*Proof.* Suppose $a$ is a $(1,E)_c$-crude atom with the associated factorization $a = yb$ satisfying the conditions (i)–(iii) of Definition 5.1. Since $\mathcal{E}_k[a] = 0$

for $1 \le k \le n$ and $b \in E^{\times}(\mathcal{M}_n)$,

$$s_c^2(a) = b^* \sum_{k \ge n+1} \mathcal{E}_{k-1}(|dy_k|^2) b = b^* s_c^2(y) b$$

and so $s_c(a) = |s_c(y)b|$. Then

$$\|a\|_{\mathsf{h}_1^c} = \|s_c(a)\|_1 = \|s_c(y)b\|_1 \le \|s_c(y)\|_E \|b\|_{E^{\times}} \le 1,$$

as required. ■

Then, we may consider the atomic $\mathsf{h}_1$-space based on crude symmetric atoms as building blocks.

DEFINITION 5.3. Let $(\mathcal{M}_n)_{n\ge 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric Banach function space on $(0,\infty)$ which is $s$-convex for some $0 < s < \infty$ and $1 < p_E \le q_E < \infty$. We define $\mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$ to be the Banach space of $x \in L_1(\mathcal{M})$ which admits a decomposition

$$x = \sum_k \lambda_k a_k$$

where for each $k$, $a_k$ is either a $(1,E)_c$-crude atom or an element in $L_1(\mathcal{M}_1)$ of norm $\le 1$, and the $\lambda_k \in \mathbb{C}$ satisfy $\sum_k |\lambda_k| < \infty$. We equip this space with the quasi-norm

$$\|x\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} = \inf \sum_k |\lambda_k|$$

where the infimum is taken over all decompositions of $x$ described above.

Similarly, we may define $\mathsf{h}^r_{1,E;\mathrm{crude}}(\mathcal{M})$ with the quasi-norm $\|\cdot\|_{\mathsf{h}^r_{1,E;\mathrm{crude}}}$.

We need the following result.

LEMMA 5.4. *Let $(\mathcal{M}_n)_{n\ge 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a separable symmetric Banach function space on* $(0,\infty)$ *which is $s$-convex for some $0 < s < \infty$ and $1 < p_E \le q_E < \infty$. Suppose $\{e_i\}_{i\in I}$ is a net of $\tau$-finite projections in $\mathcal{M}_1$ constructed in Lemma* 3.1.

(i) *If $a$ is a $(1,E)_c$-crude atom, then $ae_i$, $e_i a$ and $e_i a e_i$ are all $(1,E)_c$-crude atoms and*

$$\lim_I \|e_i a - a\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} = \lim_I \|a e_i - a\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} = \lim_I \|e_i a e_i - a\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} = 0.$$

(ii) *The set $\bigcup_{i\in I} \mathsf{h}^c_{1,E;\mathrm{crude}}(e_i \mathcal{M} e_i)$ is dense in $\mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$.*

(iii) *For a given $q_E < q_0 < \infty$ and $q = \max\{2, q_0\}$, the set $\bigcup_{i\in I} L_q(e_i \mathcal{M} e_i)$ is dense in $\mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$.*

*Proof.* (i) Since $L_1(\mathcal{M})\cap\mathcal{M}$ is dense in $E(\mathcal{M})$, using [Bek18a, Lemma 3.1] we get

$$\lim_I \|xe_i - x\|_E = \lim_I \|e_i x - x\|_E = 0 \tag{5.1}$$

for all $x \in E(\mathcal{M})$. Hence, by arguments similar to (3.3), we get

$$\lim_I \|e_i x - x\|_{\mathsf{h}_E^c} = \lim_I \|xe_i - x\|_{\mathsf{h}_E^c} = 0 \tag{5.2}$$

for all $x \in \mathsf{h}_E^c(\mathcal{M})$. For a $(1,E)_c$-crude atom $a$, $e_i a$, $ae_i$ and $e_i a e_i$ are all $(1,E)_c$-crude atoms because $e_i \in \mathcal{M}_1$ for all $i \in I$. Let $a = yb$ be a factorization associated with $a$. Since

$$e_i a - a = \|e_i y - y\|_{\mathsf{h}_E^c} \frac{e_i y - y}{\|e_i y - y\|_{\mathsf{h}_E^c}} b$$

we have $\|e_i a - a\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} \le \|e_i y - y\|_{\mathsf{h}_E^c}$, and so by (5.2) we deduce that $\lim_I \|e_i a - a\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} = 0$. Similarly, we have $\lim_I \|a - ae_i\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} = 0$, and consequently $\lim_I \|e_i a e_i - a\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} = 0$.

(ii) This follows from (i).

(iii) Let $i \in I$ and $x \in L_q(e_i\mathcal{M}e_i)$. We write

$$\begin{aligned} x = {}& \|e_i\|_{q'}\|\mathcal{E}_1(x)\|_q \frac{\mathcal{E}_1(x)}{\|e_i\|_{q'}\|\mathcal{E}_1(x)\|_q} \\ & + C_q C_E \|x - \mathcal{E}_1(x)\|_q \|e_i\|_{E^\times} \frac{x - \mathcal{E}_1(x)}{C_q C_E \|x - \mathcal{E}_1(x)\|_q \|e_i\|_{E^\times}}, \end{aligned}$$

where $q' = q/(q-1)$, $C_E$ is the constant in the inequality $\|x\|_E \le C_E\|x\|_q$ for all $x \in L_q(e_i\mathcal{M}e_i)$, and $C_q$ is the constant in the noncommutative Burkholder inequality $\|f\|_{\mathsf{h}_q^c} \le C_q\|f\|_q$ for all $f \in L_q(\mathcal{M})$ (see [JX08]). Then $\frac{\mathcal{E}_1(x)}{\|e_i\|_{q'}\|\mathcal{E}_1(x)\|_q} \in L_1(e_i\mathcal{M}_1 e_i)$ and $\big\|\frac{\mathcal{E}_1(x)}{\|e_i\|_{q'}\|\mathcal{E}_1(x)\|_q}\big\|_1 \le 1$. Also,

$$\frac{x - \mathcal{E}_1(x)}{C_q C_E \|x - \mathcal{E}_1(x)\|_q \|e_i\|_{E^\times}} = \frac{x - \mathcal{E}_1(x)}{C_q C_E \|x - \mathcal{E}_1(x)\|_q} \, \frac{e_i}{\|e_i\|_{E^\times}} = yb.$$

Since $\mathcal{E}_1(y) = 0$ and

$$\|y\|_{\mathsf{h}_E^c} = \bigg\| \frac{x - \mathcal{E}_1(x)}{C_q C_E \|x - \mathcal{E}_1(x)\|_q} \bigg\|_{\mathsf{h}_E^c} \le \frac{C_E \|x - \mathcal{E}_1(x)\|_{\mathsf{h}_q^c}}{C_q C_E \|x - \mathcal{E}_1(x)\|_q} \le 1.$$

Therefore,

$$\|x\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} \le (\|e_i\|_{q'} + 2C_q C_E \|e_i\|_{E^\times})\|x\|_q$$

and so $L_q(e_i\mathcal{M}e_i) \subset \mathsf{h}^c_{1,E;\mathrm{crude}}(e_i\mathcal{M}e_i)$. By (ii), $\bigcup_{i\in I} L_q(e_i\mathcal{M}e_i)$ is dense in $\mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$. ∎

PROPOSITION 5.5. *Let $(\mathcal{M}_n)_{n\ge1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a separable symmetric Banach function space* *on* $(0,\infty)$. *If* $1 < p_E \le q_E < \infty$, *then* $\mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})^* = \mathrm{bmo}^c_{E^\times}(\mathcal{M})$ *isometrically. More precisely:*

(i) *Every* $z \in \mathrm{bmo}^c_{E^\times}(\mathcal{M})$ *defines a continuous linear functional on* $\mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$ *by*

$$\varphi_z(x) = \tau(z^* x) \tag{5.3}$$

*for all* $x \in \mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$. *In this case,* $\|\varphi_z\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})^*} \le \|z\|_{\mathrm{bmo}^c_{E^\times}}$.

(ii) *Conversely, each* $\varphi \in \mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})^*$ *is given as* (5.3) *by some* $z \in \mathrm{bmo}^c_{E^\times}(\mathcal{M})$ *and* $\|z\|_{\mathrm{bmo}^c_{E^\times}} \le 2\|\varphi\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})^*}$.

*Similarly,* $\mathsf{h}^r_{1,E;\mathrm{crude}}(\mathcal{M})^* = \mathrm{bmo}^r_{E^\times}(\mathcal{M})$ *isometrically.*

*Proof.* (i) Let $z \in \mathrm{bmo}^c_{E^\times}(\mathcal{M})$. For a $(1,E)_c$-crude atom $a = yb$, we have

$$\tau(z^* a) = \tau(\mathcal{E}_n((z^* - z_n^*)y)b) = \tau(((z - z_n)b^*)^* y).$$

Since $\mathsf{h}^c_E(\mathcal{M})^* = \mathsf{h}^c_{E^\times}(\mathcal{M})$ by Theorem 2.5(i), we have

$$|\tau(z^* a)| \le \|(z - z_n)b^*\|_{\mathsf{h}^c_{E^\times}} \|y\|_{\mathsf{h}^c_E} \le \|z\|_{\mathrm{bmo}^c_{E^\times}}.$$

On the other hand, for any $a \in L_1(\mathcal{M}_1)$ with $\|a\|_1 \le 1$ we have

$$|\tau(z^* a)| = |\tau(\mathcal{E}_1(z)^* a)| \le \|\mathcal{E}_1(z)\|_\infty \|a\|_1 \le \|z\|_{\mathrm{bmo}^c_{E^\times}}.$$

Hence, we deduce that

$$|\tau(z^* x)| \le \|z\|_{\mathrm{bmo}^c_{E^\times}} \|x\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}}$$

for all $x \in \mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$.

(ii) Let $\varphi \in \mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})^*$. Set $q = \max\{2, q_0\}$, where $q_E < q_0 < \infty$. By Lemma 5.4(iii), we can find $z \in L_0(\mathcal{M})$ such that $z|_{e_i} \in L_{q'}(e_i \mathcal{M} e_i)$ ($q'$ is the conjugate index of $q$) for every $i \in I$,

$$\varphi(x) = \tau(z^* x), \qquad \forall x \in \bigcup_{i \in I} L_q(e_i \mathcal{M} e_i),$$

and

$$\|\varphi\| = \sup_{\substack{x \in \bigcup_{i\in I} L_q(e_i \mathcal{M} e_i), \\ \|x\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} \le 1}} |\tau(z^* x)|.$$

Fix $n$ and take any $y \in \mathcal{M}_n$ with $\|y\|_{E^\times} \le 1$. Since $\mathsf{h}^c_E(\mathcal{M})^* = \mathsf{h}^c_{E^\times}(\mathcal{M})$ by Theorem 2.5(i), one has

$$\begin{aligned}
\|(z - z_n)y\|_{\mathsf{h}^c_{E^\times}} &= \sup_{\|x\|_{\mathsf{h}^c_E} \le 1} |\tau((z - z_n)yx^*)| = \sup_{\|x\|_{\mathsf{h}^c_E} \le 1} |\tau(y^*(z^* - z_n^*)x)| \\
&= \sup_{\|x\|_{\mathsf{h}^c_E} \le 1} |\tau(y^*(z^* - z_n^*)(x - x_n))| \\
&= \sup_{\|x\|_{\mathsf{h}^c_E} \le 1} |\tau(z^*(x - x_n)y^*)| \\
&\le \|\varphi\|,
\end{aligned}$$

since $(x-x_n)y^*$ is a $(1,E)_c$-crude atom. Also,

$$\|\mathcal{E}_1(z)\|_\infty = \sup_{\substack{x\in\bigcup_{i\in I} L_q(e_i\mathcal{M}_1e_i),\\ \|x\|_1\le 1}} |\tau(z^*x)| \le \|\varphi\|.$$

Thus $z\in \mathrm{bmo}^c_{E^\times}(\mathcal{M})$ and $\|z\|_{\mathrm{bmo}^c_{E^\times}} \le 2\|\varphi\|$. ∎

By Lemma 5.2, we know that the inclusion $\mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M}) \subset \mathsf{h}^c_1(\mathcal{M})$ is contractive. We show the converse holds true under a mild condition on $E$.

THEOREM 5.6. *Let $(\mathcal{M}_n)_{n\ge1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a separable symmetric Banach function space on $(0,\infty)$. If $1<p_E\le q_E<\infty$, then*

$$\mathsf{h}^c_1(\mathcal{M}) = \mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$$

*with equivalent norms. Similarly, $\mathsf{h}^r_1(\mathcal{M}) = h^r_{1,E;\mathrm{crude}}(\mathcal{M})$ with equivalent norms.*

*Proof.* Let $z\in \mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$. By Theorem 4.9 and Proposition 5.5, we have

$$\|z\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} = \sup_{\|y\|_{\mathrm{bmo}^c_{E^\times}}\le 1} |\tau(z^*y)| = \sup_{\|y\|_{\mathrm{bmo}^c}\le C_E} |\tau(z^*y)| \le C_E\|z\|_{\mathsf{h}^c_1}.$$

Since $\bigcup_{i\in I} L_q(e_i\mathcal{M}e_i)$ is dense in $\mathsf{h}^c_1(\mathcal{M})$, by Lemma 5.4(ii) we find that $\mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$ is dense in $\mathsf{h}^c_1(\mathcal{M})$. Hence,

$$\mathsf{h}^c_1(\mathcal{M}) = \mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$$

with equivalent norms. ∎

We can generalize this decomposition to the whole space $\mathsf{h}_1(\mathcal{M})$. To this end we need the following definition.

DEFINITION 5.7. Let $(\mathcal{M}_n)_{n\ge1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric Banach function space on $(0,\infty)$. We set

$$\mathsf{h}_{1,E;\mathrm{crude}}(\mathcal{M}) = \mathsf{h}^d_1(\mathcal{M}) + \mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M}) + \mathsf{h}^r_{1,E;\mathrm{crude}}(\mathcal{M}),$$

equipped with the sum norm

$$\|x\|_{\mathsf{h}_{1,E;\mathrm{crude}}} = \inf\{\|w\|_{\mathsf{h}^d_1} + \|y\|_{\mathsf{h}^c_{1,E;\mathrm{crude}}} + \|z\|_{\mathsf{h}^r_{1,E;\mathrm{crude}}}\},$$

where the infimum is taken over all $w\in \mathsf{h}^d_1(\mathcal{M})$, $y\in \mathsf{h}^c_{1,E;\mathrm{crude}}(\mathcal{M})$, and $z\in \mathsf{h}^r_{1,E;\mathrm{crude}}(\mathcal{M})$ such that $x=w+y+z$.

By Theorem 5.6, we obtain the crude atomic decomposition of $\mathsf{h}_1(\mathcal{M})$.

PROPOSITION 5.8. *Let $(\mathcal{M}_n)_{n\ge1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a separable symmetric Banach function space on $(0,\infty)$. If $1<p_E\le q_E<\infty$, then*

$$\mathsf{h}_1(\mathcal{M}) = \mathsf{h}_{1,E;\mathrm{crude}}(\mathcal{M})$$

*with equivalent norms.*

Next, we turn to the true symmetric atomic decomposition of $\mathsf{h}_1(\mathcal{M})$.

DEFINITION 5.9. Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric Banach function space on $(0,\infty)$ with $1 < p_E \leq q_E < \infty$. An element $a \in L_1(\mathcal{M})$ is called a $(1,E)_c$-*atom* with respect to $(\mathcal{M}_n)_{n\geq 1}$ if there exist $n \geq 1$ and a projection $e \in \mathcal{M}_n$ such that

(i) $\mathcal{E}_n(a) = 0$;
(ii) $r(a) \leq e$;
(iii) $\|a\|_{\mathsf{h}_E^c} \leq \|e\|_{E^\times}^{-1}$.

Replacing (ii) by (ii)′ $l(a) \leq e$, we get the notion of a $(1,E)_r$-*atom*.

Note that if $a$ is a $(1,E)_c$-atom with the associated projection $e \in \mathcal{M}_n$, then $a = (\|e\|_{E^\times} a)\frac{e}{\|e\|_{E^\times}} = yb$ is a $(1,E)_c$-crude atom.

DEFINITION 5.10. We define $\mathsf{h}_{1,E}^c(\mathcal{M})$ as the Banach space of $x \in L_1(\mathcal{M})$ which admits a decomposition

$$x = \sum_k \lambda_k a_k$$

with for each $k$, $a_k$ a $(1,E)_c$-atom or an element in the unit ball of $L_1(\mathcal{M}_1)$, and $\lambda_k \in \mathbb{C}$ satisfying $\sum_k |\lambda_k| < \infty$. We equip this space with the norm

$$\|x\|_{\mathsf{h}_{1,E}^c} = \inf \sum_k |\lambda_k|$$

where the infimum is taken over all decompositions of $x$ described above.

Similarly, we can define $\mathsf{h}_{1,E}^r(\mathcal{M})$ and $\|\cdot\|_{\mathsf{h}_{1,E}^r}$.

By an argument similar to Proposition 5.5, we obtain

PROPOSITION 5.11. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a separable symmetric Banach function space on $(0,\infty)$. If $1 < p_E \leq q_E < \infty$, then*

$$\mathsf{h}_{1,E}^c(\mathcal{M})^* = \mathrm{bmo}_{E^\times,\mathrm{pr}}^c(\mathcal{M})$$

*isometrically. Similarly, $\mathsf{h}_{1,E}^r(\mathcal{M})^* = \mathrm{bmo}_{E^\times,\mathrm{pr}}^r(\mathcal{M})$ isometrically.*

By Theorem 4.16 and Proposition 5.11, we have

THEOREM 5.12. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a separable symmetric Banach function space on $(0,\infty)$. If $1 < p_E \leq q_E < \infty$, then*

$$\mathsf{h}_1^c(\mathcal{M}) = \mathsf{h}_{1,E}^c(\mathcal{M})$$

*with equivalent norms. Similarly, $\mathsf{h}_1^r(\mathcal{M}) = \mathsf{h}_{1,E}^r(\mathcal{M})$ with equivalent norms.*

Definition 5.13. Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric Banach function space on $(0,\infty)$. We set

$$\mathsf{h}_{1,E}(\mathcal{M}) = \mathsf{h}_1^d(\mathcal{M}) + \mathsf{h}_{1,E}^c(\mathcal{M}) + \mathsf{h}_{1,E}^r(\mathcal{M}),$$

equipped with the sum norm

$$\|x\|_{\mathsf{h}_{1,E}} = \inf\{\|w\|_{\mathsf{h}_1^d} + \|y\|_{\mathsf{h}_{1,E}^c} + \|z\|_{\mathsf{h}_{1,E}^r}\},$$

where the infimum is taken over all $w \in \mathsf{h}_1^d(\mathcal{M})$, $y \in \mathsf{h}_{1,E}^c(\mathcal{M})$, and $z \in \mathsf{h}_{1,E}^r(\mathcal{M})$ such that $x = w + y + z$.

Finally, by Theorem 5.12, we obtain the symmetric atomic decomposition of $\mathsf{h}_1(\mathcal{M})$.

Proposition 5.14. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a separable symmetric Banach function space on $(0,\infty)$. If $1 < p_E \leq q_E < \infty$, then*

$$\mathsf{h}_1(\mathcal{M}) = \mathsf{h}_{1,E}(\mathcal{M})$$

*with equivalent norms.*

**5.2. Paraproducts for noncommutative martingales.** Recall that a filtration $(\mathcal{M}_n)_{n\geq 1}$ of a semifinite von Neumann algebra $\mathcal{M}$ is said to be *regular* if there exists a constant $C > 0$ such that for any $n$ and $a \in \mathcal{M}_n$, $a \geq 0$,

$$\|a\|_\infty \leq C\|\mathcal{E}_{n-1}(a)\|_\infty.$$

Given a regular filtration $(\mathcal{M}_n)_{n\geq 1}$ of $\mathcal{M}$, for a fixed $b \in L_2(\mathcal{M})$ we define paraproducts $\pi_b^l$ and $\pi_b^r$ as operators for bounded $L^p(\mathcal{M})$-martingales ($1 < p < \infty$) $x = (x_n)_{n\geq 1}$ by

$$\pi_b^l(x) = \sum_{n\geq 2} db_n x_{n-1}, \qquad \pi_b^r(x) = \sum_{n\geq 2} x_{n-1} db_n,$$

where $db_2 = \mathcal{E}_2(b)$, $db_n = \mathcal{E}_n(b) - \mathcal{E}_{n-1}(b)$, $n \geq 3$.

We refer to [Kat97, Mei06] for more information on paraproducts.

Lemma 5.15. *Let $(\mathcal{M}_n)_{n\geq 1}$ be a regular filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric quasi-Banach function space on $(0,\infty)$ which is $s$-convex for some $0 < s < \infty$. If $0 < p_E \leq q_E < \infty$, then*

$$\|x\|_{\mathrm{BMO}} \approx \sup_{n\geq 1} \sup_{a\in E(\mathcal{M}_n),\, \|a\|_E\leq 1} (\|(x - x_n)a\|_E + \|a(x - x_n)\|_E).$$

*Proof.* By the regularity of $(\mathcal{M}_n)_{n\geq 1}$, we have

$$\begin{aligned}\|x\|_{\mathrm{BMO}^c} &= \sup_{n\geq 1} \|\mathcal{E}_n(|x - x_{n-1}|^2)\|_\infty^{1/2} \\ &\leq \sup_{n\geq 1}[\|\,|dx_n|^2\|_\infty^{1/2} + \|\mathcal{E}_n(|x - x_n|^2)\|_\infty^{1/2}]\end{aligned}$$

$$\le \sup_{n\ge 1}[C\|\mathcal{E}_{n-1}(|dx_n|^2)\|_\infty^{1/2} + \|\mathcal{E}_n(|x-x_n|^2)\|_\infty^{1/2}]$$
$$\le \sup_{n\ge 1}[C\|\mathcal{E}_{n-1}(|x-x_{n-1}|^2)\|_\infty^{1/2} + \|\mathcal{E}_n(|x-x_n|^2)\|_\infty^{1/2}]$$
$$\le (C+1)\sup_{n\ge 1}\|\mathcal{E}_n(|x-x_n|^2)\|_\infty^{1/2}.$$

Hence,

$$\frac{1}{1+C}\|x\|_{\mathrm{BMO}^c} \le \sup_{n\ge 1}\|\mathcal{E}_n(|x-x_n|^2)\|_\infty^{1/2} \le \|x\|_{\mathrm{BMO}^c}. \tag{5.4}$$

The rest of the proof is the same as for Theorem 4.13. ∎

Corollary 5.16. *Let $(\mathcal{M}_n)_{n\ge 1}$ be a regular filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric Banach function space on $(0,\infty)$. If $1 < p_E \le q_E < \infty$, then*

$$\|b\|_{\mathrm{BMO}} \le C_E \max\{\|\pi_b^l\|_{E(\mathcal{M})\to E(\mathcal{M})}, \|\pi_b^r\|_{E(\mathcal{M})\to E(\mathcal{M})}\}.$$

*Proof.* Fix $b \in L_2(\mathcal{M})$. Let $a \in E(\mathcal{M}_n)$ and $\|a\|_E \le 1$. Then

$$(b-b_n)a = \pi_b^l(a) - \mathcal{E}_n(\pi_b^l(a)), \qquad a(b-b_n) = \pi_b^r(a) - \mathcal{E}(\pi_b^r(a)_n).$$

Hence,

$$\sup_{n\ge 1}\sup_{a\in E(\mathcal{M}_n),\,\|a\|_E\le 1}\|(b-b_n)a\|_E \le \sup_{n\ge 1}\sup_{a\in E(\mathcal{M}_n),\,\|a\|_E\le 1}\|\pi_b^l(a) - \mathcal{E}_n(\pi_b^l(a))\|_E$$

and

$$\sup_{n\ge 1}\sup_{a\in E(\mathcal{M}_n),\,\|a\|_E\le 1}\|a(b-b_n)\|_E \le \sup_{n\ge 1}\sup_{a\in E(\mathcal{M}_n),\,\|a\|_E\le 1}\|\pi_b^r(a) - \mathcal{E}_n(\pi_b^r(a))\|_E.$$

By Lemma 5.15, we obtain the desired result. ∎

Proposition 5.17. *Let $(\mathcal{M}_n)_{n\ge 1}$ be a regular filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric Banach function space on $(0,\infty)$. If $1 < p_E \le q_E < \infty$, then*

$$\max\{\|\pi_b^l\|_{\mathcal{M}\to \mathrm{BMO}(\mathcal{M})}, \|\pi_b^r\|_{\mathcal{M}\to \mathrm{BMO}(\mathcal{M})}\} \le C_E \max\{\|\pi_b^l\|_{E(\mathcal{M})\to E(\mathcal{M})}, \|\pi_b^r\|_{E(\mathcal{M})\to E(\mathcal{M})}\}.$$

*Proof.* Fix $x \in \mathcal{M}$ with $\|x\| \le 1$. Note that

$$\|\mathcal{E}_n(|\pi_b^l(x) - \mathcal{E}_n(\pi_b^l(x))|^2)\|_\infty = \sup\{|\tau(\mathcal{E}_n(|\pi_b^l(x)) - \mathcal{E}_n(\pi_b^l(x))|^2 a)| : a \in L_1(\mathcal{M}_n),\ \|a\|_1 \le 1\}.$$

For $a \in L_1(\mathcal{M}_n)^+$ and $\|a\|_1 \le 1$, by Theorem 2.1(iii), for $\varepsilon > 0$ there exist $a_1 \in E(\mathcal{M}_n)^+$ and $a_2 \in E^\times(\mathcal{M}_n)^+$ such that $a = a_1a_2$, $\|a_1\|_E \le 1$,

$\|a_2\|_{E^\times} < 1+\varepsilon$ (cf. Lemma 4.4). Then

$$\begin{aligned}|\tau(\mathcal{E}_n(|\pi_b^l(x)) - \mathcal{E}_n(\pi_b^l(x))|^2 a)| &= |\tau(a_2|\pi_b^l(x) - \mathcal{E}_n(\pi_b^l(x))|^2 a_1)| \\ &\le \|(\pi_b^l(x) - \mathcal{E}_n(\pi_b^l(x)))a_1\|_E \|(\pi_b^l(x) - \mathcal{E}_n(\pi_b^l(x)))a_2\|_{E^\times} \\ &= \Big\|\Big(\sum_{k>n} db_k \mathcal{E}_{k-1}(x)\Big)a_1\Big\|_E \|(\pi_b^l(x) - \mathcal{E}_n(\pi_b^l(x)))a_2\|_{E^\times} \\ &= \|\pi_b^l(xa_1) - \mathcal{E}_n(\pi_b^l(xa_1))\|_E \|(\pi_b^l(x) - \mathcal{E}_n(\pi_b^l(x)))a_2\|_{E^\times} \\ &\lesssim 2\|\pi_b^l\|_{E(\mathcal{M})\to E(\mathcal{M})}\|\pi_b^l(x)\|_{\mathrm{BMO}},\end{aligned}$$

where the last inequality follows from Lemma 5.15. Thus,

$$\|\pi_b^l(x)\|_{\mathrm{BMO}^c}^2 \le C\|\pi_b^l\|_{E(\mathcal{M})\to E(\mathcal{M})}\|\pi_b^l(x)\|_{\mathrm{BMO}}. \tag{5.5}$$

Since $\|\mathcal{E}_{k-1}(x)\| \le \|x\| \le 1$ for $k\ge 2$, we get $\mathcal{E}_{k-1}(x)\mathcal{E}_{k-1}(x)^* \le 1$. Hence,

$$\begin{aligned}\mathcal{E}_n(|\pi_b^l(x) - \mathcal{E}_n(\pi_b^l(x))|^2) &= \mathcal{E}_n\Big(\sum_{k>n}|db_k\mathcal{E}_{k-1}(x)|^2\Big) \\ &\le \mathcal{E}_n\Big(\sum_{k>n}|db_k|^2\Big) = \mathcal{E}_n(|b-\mathcal{E}_n(b)|^2).\end{aligned}$$

Therefore,

$$\|\pi_b^l(x)\|_{\mathrm{BMO}^r} \le \|b\|_{\mathrm{BMO}^r}. \tag{5.6}$$

Combining (5.5) and (5.6) yields

$$\|\pi_b^l(x)\|_{\mathrm{BMO}} \le \max\{C,1\}\max\{\|\pi_b^l\|_{E(\mathcal{M})\to E(\mathcal{M})}, \|b\|_{\mathrm{BMO}}\}.$$

Similarly,

$$\|\pi_b^r(x)\|_{\mathrm{BMO}} \le \max\{C,1\}\max\{\|\pi_b^r\|_{E(\mathcal{M})\to E(\mathcal{M})}, \|b\|_{\mathrm{BMO}}\}.$$

Applying Corollary 5.16, we obtain the desired result. ■

Using Corollary 3.5, Lemma 2.3 and Proposition 5.17, we deduce

COROLLARY 5.18. *Let $(\mathcal{M}_n)_{n\ge1}$ be a regular filtration of a semifinite von Neumann algebra $\mathcal{M}$. Let $E$ be a symmetric Banach function space on $(0,\infty)$ with $1 < p_E \le q_E < \infty$. If $\pi_b^l$ and $\pi_b^r$ are both bounded on $L_p(\mathcal{M})$ for some $1 \le p < p_E$, then they are both bounded on $E(\mathcal{M})$.*

REMARK 5.19. From the proof of Proposition 5.17, we know that the results also hold for the case $E = L_1(0,\alpha)$ for a fixed $\alpha > 0$.

**Acknowledgements.** The authors are grateful to the anonymous referee for giving many valuable suggestions, which have been incorporated into this version of the paper.

T. N. Bekjan is partially supported by NSFC grant No.11771372; Z. Chen is partially supported by NSFC grant No.11871468; M. Raikhan is partially supported by project AP05131557 of the Science Committee of Ministry of

Education and Science of the Republic of Kazakhstan; M. Sun is partially supported by NSFC grant No. 11801189.

### References

[Bek06] T. N. Bekjan, *Φ-inequalities of noncommutative martingales*, Rocky Mountain J. Math. 36 (2006), 401–412.

[Bek15] T. N. Bekjan, *Noncommutative symmetric Hardy spaces*, Integral Equations Operator Theory 81 (2015), 191–212.

[Bek18a] T. N. Bekjan, *On dominated convergence in noncommutative integration*, Houston J. Math. 44 (2018), 187–200.

[Bek18b] T. N. Bekjan, *Duality for symmetric Hardy spaces of noncommutative martingales*, Math. Z. 289 (2018), 787–802.

[BC12] T. N. Bekjan and Z. Chen, *Interpolation and Φ-moment inequalities of noncommutative martingales*, Probab. Theory Related Fields 152 (2012), 179–206.

[BCO17] T. N. Bekjan, Z. Chen and A. Osękowski, *Noncommutative maximal inequalities associated with convex functions*, Trans. Amer. Math. Soc. 369 (2017), 409–427.

[BCPY10] T. N. Bekjan, Z. Chen, M. Perrin and Z. Yin, *Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales*, J. Funct. Anal. 258 (2010), 2483–2505.

[BO21] T. N. Bekjan and M. N. Ospanov, *On products of noncommutative symmetric quasi Banach spaces and applications*, Positivity 25 (2021), 121–148.

[BS88] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, MA, 1988.

[BL76] J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Springer, New York, 1976.

[CRX20] Z. Chen, N. Randrianantoanina and Q. Xu, *Atomic decompositions for noncommutative martingales*, arXiv:2001.08775v1 (2020).

[Dir15] S. Dirksen, *Noncommutative Boyd interpolation theorems*, Trans. Amer. Math. Soc. 367 (2015), 4079–4110.

[DPPS11] S. Dirksen, B. de Pagter, D. Potapov and F. Sukochev, *Rosenthal inequalities in noncommutative symmetric spaces*, J. Funct. Anal. 261 (2011), 2890–2925.

[DDP93] P. G. Dodds, T. K. Dodds and B. de Pagter, *Noncommutative Köthe duality*, Trans. Amer. Math. Soc. 339 (1993), 717–750.

[FK86] T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators*, Pacific J. Math. 123 (1986), 269–300.

[HM12] G. Hong and T. Mei, *John–Nirenberg inequality and atomic decomposition for noncommutative martingales*, J. Funct. Anal. 263 (2012), 1064–1097.

[JJ82] S. Janson and P. Jones, *Interpolation between $H^p$-spaces: the complex method*, J. Funct. Anal. 48 (1982), 58–80.

[JSZZ17] Y. Jiao, F. Sukochev, D. Zanin and D. Zhou, *Johnson–Schechtman inequalities for noncommutative martingales*, J. Funct. Anal. 272 (2017), 976–1016.

[Jun02] M. Junge, *Doob's inequality for non-commutative martingales*, J. Reine Angew. Math. 549 (2002), 149–190.

[Jun05] M. Junge, *Embedding of the operator space OH and the logarithmic 'little Grothendieck inequality'*, Invent. Math. 161 (2005), 225–286.

[JM07] M. Junge and M. Musat, *A noncommutative version of the John–Nirenberg theorem*, Trans. Amer. Math. Soc. 359 (2007), 115–142.

[JP14] M. Junge and M. Perrin, *Theory of $H_p$-spaces for continuous filtrations in von Neumann algebras*, Astérisque 362 (2014), vi+134 pp.

[JX03] M. Junge and Q. Xu, *Noncommutative Burkholder/Rosenthal inequalities*, Ann. Probab. 31 (2003), 948–995.

[JX08] M. Junge and Q. Xu, *Noncommutative Burkholder/Rosenthal inequalities II: Applications*, Israel J. Math. 167 (2008), 227–282.

[JX10] M. Junge and Q. Xu, *Representation of certain homogeneous Hilbertian operator spaces and applications*, Invent. Math. 175 (2010), 75–118.

[Kal03] N. Kalton, *Quasi-Banach spaces*, in: Handbook of the Geometry of Banach Spaces, Vol. 2, North-Holland, Amsterdam, 2003, 1099–1130.

[KM03] N. Kalton and S. Montgomery-Smith, *Interpolation of Banach spaces*, in: Handbook of the Geometry of Banach Spaces, Vol. 2, North-Holland, Amsterdam, 2003, 1131–1175.

[Kat97] N. H. Katz, *Matrix valued paraproducts*, J. Fourier Anal. Appl. 3 (1997), 913–921.

[KLM14] P. Kolwicz, K. Leśnik and L. Maligranda, *Pointwise products of some Banach function spaces and factorization*, J. Funct. Anal. 266 (2014), 616–659.

[KPS82] S. G. Krein, J. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, Transl. Math. Monogr. 54, Amer. Math. Soc., 1982.

[LS08] C. Le Merdy and F. Sukochev, *Rademacher averages on noncommutative symmetric spaces*, J. Funct. Anal. 255 (2008), 3329–3355.

[LT79] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer, Berlin, 1979.

[Loz69] G. Y. Lozanovskiĭ, *On some Banach lattices*, Siberian Math. J. 10 (1969), 419–430.

[Mei06] T. Mei, *Notes on matrix valued paraproducts*, Indiana Univ. Math. J. 55 (2006), 747–760.

[Mon96] S. Montgomery-Smith, *The Hardy operator and Boyd indices*, in: Lecture Notes in Pure Appl. Math. 175, Dekker, New York, 1996, 359–364.

[Mus03] M. Musat, *Interpolation between noncommutative BMO and noncommutative $L_p$-spaces*, J. Funct. Anal. 202 (2003), 195–225.

[PR06] J. Parcet and N. Randrianantoanina, *Gundy's decomposition for non-commutative martingales and applications*, Proc. London Math. Soc. 93 (2006), 227–252.

[Per09] M. Perrin, *A noncommutative Davis' decomposition for martingales*, J. London Math. Soc. (2) 80 (2009), 627–648.

[PS02] G. Pisier and D. Shlyakhtenko, *Grothendieck's theorem for operator spaces*, Invent. Math. 150 (2002), 185–217.

[PX97] G. Pisier and Q. Xu, *Non-commutative martingale inequalities*, Comm. Math. Phys. 189 (1997), 667–698.

[PX03] G. Pisier and Q. Xu, *Non-commutative $L^p$-spaces*, in: Handbook of the Geometry of Banach Spaces, Vol. 2, North-Holland, Amsterdam, 2003, 1459–1517.

[Ran02] N. Randrianantoanina, *Non-commutative martingale transform*, J. Funct. Anal. 194 (2002), 181–212.

[Ran03] N. Randrianantoanina, *Embeddings of $\ell_p$ into non-commutative spaces*, J. Austral. Math. Soc. 74 (2003), 331–350.

[Ran07] N. Randrianantoanina, *Conditioned square functions for noncommutative martingales*, Ann. Probab. 35 (2007), 1039–1070.

[RW15] N. Randrianantoanina and L. Wu, *Martingale inequalities in noncommutative symmetric spaces*, J. Funct. Anal. 269 (2015), 2222–2253.

[RWX19] N. Randrianantoanina, L. Wu and Q. Xu, *Noncommutative Davis type decompositions and applications*, J. London Math. Soc. 99 (2019), 97–126.

[RWZ21] N. Randrianantoanina, L. Wu and D. Zhou, *Atomic decompositions and asymmetric Doob inequalities in noncommutative symmetric spaces*, J. Funct. Anal. 280 (2021), art. 108794, 64 pp.

[RX16] E. Ricard and Q. Xu, *A noncommutative martingale convexity inequality*, Ann. Probab. 44 (2016), 867–882.

[Sak71] S. Sakai, *$C^*$-Algebras and $W^*$-Algebras*, Springer, New York, 1971.

[Suk14] F. Sukochev, *Completeness of quasi-normed symmetric operator spaces*, Indag. Math. 25 (2014), 376–388.

[Wol81] T. Wolff, *A note on interpolation spaces*, in: Lecture Notes in Math. 908, Springer, Berlin, 1981, 199–204.

[Xu90] Q. Xu, *Applications du théorème de factorization pour les fonctions à valeurs opérateurs*, Studia Math. 95 (1990), 273–292.

[Xu91] Q. Xu, *Analytic functions with values in lattices and symmetric spaces of measurable operators*, Math. Proc. Cambridge Philos. Soc. 109 (1991), 541–563.

[Xu06] Q. Xu, *Operator-space Grothendieck inequalities for noncommutative $L_p$-spaces*, Duke Math. J. 131 (2006), 525–574.

[Xu07] Q. Xu, *Noncommutative $L_p$-Spaces and Martingale Inequalities*, book manuscript, 2007.

Turdebek N. Bekjan
College of Mathematics
and Systems Science
Xinjiang University
Urumqi 830046, China
E-mail: bekjant@yahoo.com
bek@xju.edu.cn

Zeqian Chen
Wuhan Institute of Physics and Mathematics
Chinese Academy of Sciences
30 West District
Xiao-Hong-Shan, Wuhan 430071, China
E-mail: zqchen@wipm.ac.cn

Madi Raikhan
Astana IT University
Nur-Sultan 010000, Kazakhstan
E-mail: mady1029@yandex.kz
madi.raikhan@astanait.edu.kz

Mu Sun
School of Mathematics and Statistics
Huazhong University of Science and Technology
Wuhan 430074, China
E-mail: musun@hust.edu.cn