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Maximum of exponential random variables, Hurwitz's zeta function, and the partition function

by

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Abstract. A natural problem in the context of the coupon collector's problem is the behavior of the maximum of independent geometrically distributed random variables (with distinct parameters). This question has been addressed by Brennan et al. [British J. Math. Computer Sci. 8 (2015), 330–336]. Here we provide explicit asymptotic expressions for the moments of that maximum, as well as of the maximum of exponential random variables with corresponding parameters. We also deal with the probability of each of the variables being the maximal one.

The calculations lead to expressions involving Hurwitz's zeta function at certain special points. We find here explicitly the values of the function at these points. Also, the distribution function of the maximum we deal with is closely related to the generating function of the partition function. Thus, our results (and proofs) rely on classical results pertaining to the partition function.

1. Introduction. Suppose that a company distributes a commercial product and that each package contains a single coupon. There are n types of coupons, and a customer wants to collect at least one of each. We want to know how many packages need to be bought on the average until getting all coupons. This is referred to as the *coupon collector problem.* The problem goes back at least as far as de Moivre [24], who mentioned it in a collection of problems regarding various games of chance.

The expected number of drawings is calculated in a straightforward manner. (Note, though, that if one does not take the right approach, the problem may become quite intricate; see [23].) After exactly j distinct coupons have been seen, the probability of drawing an as yet unseen coupon is $(n-j)/n$. Hence, the number D_j of drawings until we see such a new coupon

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is $G(1-j/n)$-distributed. The total number of drawings is the sum of all these D_j's. The expected number of drawings is therefore nH_n, where H_n is the nth harmonic number:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \qquad n = 1, 2, 3, \ldots .$$

Asymptotically, this expectation is $n(\ln n + \gamma) + O(1)$, where $\gamma = 0.577\ldots$ is the Euler–Mascheroni constant.

The problem, and various extensions thereof, have drawn much attention for many years. Laplace [21], and also Erdős and Rényi [9], found the asymptotic distribution of the number of drawings. Newman and Shepp [27] considered yet another generalization of the problem, when one wants to collect at least m copies of each coupon. They calculated the asymptotic expected number of required drawings, Erdős and Rényi [9] found the limiting distribution of this quantity, and Flatto [14] provided an estimate on the tail of the distribution. Schelling [31, 30] and Flajolet et al. [12] considered the case where various coupons may show up with distinct probabilities. For other related questions, we refer to [15, 17, 20, 25, 26, 4]. For more on the history of the problem, see [11].

Our starting point here is Brennan et al. [5]. In that paper, the maximum waiting time was considered. That is, let $D_{(n)}$ be the maximum of the variables $D_0, D_1, \ldots, D_{n-1}$, defined above. In [5], the expectation, and indeed all moments, of $D_{(n)}$ have been calculated asymptotically in terms of certain infinite series. For example,

$$E(D_{(n)}) = n \sum_{j=1}^{\infty} (-1)^{j+1} \left(\frac{2}{j(3j-1)} + \frac{2}{j(3j+1)} \right) + o(n) \approx 1.255n, \tag{1}$$

and similar formulas hold for all moments.

Our first result is an explicit expression for these infinite series in terms of the values of Hurwitz's zeta function (see, for example, [3, 18, 1]) at certain special points. To understand these expressions better, we have calculated these values of Hurwitz's zeta function in terms of the values of the Riemann zeta function at integer points, and eventually as rational polynomials in $\sqrt{3}$ and π. In fact, this issue has attracted quite some attention. (See, for example, [6, 7, 8].) The values we need have been calculated in [29], but here we develop less cumbersome expressions. (See Example 5 below.)

Following the results of Brennan et al. [5], one may ask: what is the probability of each of the waiting times D_j to be the longest? Obviously, the last waiting time, D_{n-1}, has the best chance of being the maximal, the second last, D_{n-2}, has a smaller chance, etc. The fact that $E(D_{(n)}) \approx 1.255n$, which is not much larger than $E(D_{n-1})$ $(= n)$, hints that the last waiting time has a non-negligible probability of being larger than all its predecessors.

Here we find how the asymptotic probability of D_{n-m} being the maximal one decays as a function of m for large n.

Brennan et al. [5] note that the limiting distribution of $D_{(n)}/n$ is the distribution of an infinite series of independent exponential random variables with parameters $1, 2, 3, \ldots$. In this vein, we also consider a continuous version of a coupon collector. In this version, the interarrival times between coupons are exponentially distributed instead of being fixed. This model has been considered by Boneh and Hofri in a different context [4].

In Section 2 we define the continuous analogue of the coupon collector problem, and then state our main results in both the discrete and the continuous settings. We also state our results regarding Hurwitz's zeta function. Section 3 presents the calculations leading to explicit expressions for the values of Hurwitz's zeta function at some special points, Section 4 the proof of the asymptotic results for the moments, and Section 5 the proof of the asymptotic results concerning the identity of the maximal waiting time.

2. Main results. Recall that the number of coupons one needs to get, after having already acquired j distinct coupons, until gettting a new one, is a geometric variable with parameter $1 - j/n$. We may approximate this geometric variable by an exponential variable with the same parameter. Thus, we define a continuous analogue for the coupon collector problem as follows. There are n types of coupons, arriving with independent Exp(1)-distributed interarrival times. Each coupon has the same probability $1/n$ of being of each of the types. The basic question is now about the expected time until all coupon types are obtained. Note that, in the continuous version (unlike the discrete one), we may alternatively assume that there are n independent flows of coupons, and the interarrival times of each type are $\mathrm{Exp}(1/n)$-distributed. In the process, the collector gets new coupons n times. Denote the times these new coupons arrive, for both the discrete model and the continuous model, by $T_1 < \cdots < T_n$. We set $T_0 = 0$. The waiting times $T_{j+1} - T_j$, $0 \le j \le n-1$, between new coupons will be denoted by D_j for the discrete model and by W_j for the continuous model. Thus, $W_j \sim \mathrm{Exp}\left(\frac{n-j}{n}\right)$ and, as mentioned above, $D_j \sim G\left(\frac{n-j}{n}\right)$, $0 \le j \le n-1$.

The main quantity that has been studied in detail is the total time $T_n = \sum_{j=0}^{n-1} D_j$ until the collection is complete (under the discrete model). In this paper, though, our main interest is in $D_{(n)} = \max_{0 \le j \le n-1} D_j$ and $W_{(n)} = \max_{0 \le j \le n-1} W_j$.

We start with the continuous model. Put $X_i = W_{n-i}/n$, so that $X_i \sim \mathrm{Exp}(i)$, $1 \le i \le n$. We have $W_{(n)} = n \cdot \max\{X_1, \ldots, X_n\}$. We want to understand the asymptotic behavior of $W_{(n)}$. The advantage of passing to the variables X_i is that we may once and for all take an infinite sequence

$(X_i)_{i=1}^{\infty}$ of independent variables $X_i \sim \mathrm{Exp}(i)$. Put $M = \max_{1\le i<\infty} X_i$. (Note that, by Borel–Cantelli's Lemma, M is well defined.) We will start with the moments of M.

Theorem 1. *The expectation and variance of the maximum waiting time are*

$$E(M) = \frac{4\sqrt{3}}{3}\pi - 6 \approx 1.255, \qquad V(M) = -\frac{28}{3}\pi^2 - 16\sqrt{3}\pi + 180 \approx 0.821.$$

The theorem is a special case of Theorem 2 (or Theorem 6) below, which gives all moments of M. It will be instructive, though, to calculate $E(M)$ separately from $E(M^k)$. We will omit the calculation of $V(M)$.

To express the higher moments of M, we need Hurwitz's zeta function $\zeta(s,a)$, defined by

$$\zeta(s,a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}, \qquad s,a\in\mathbb{C},\ \mathrm{Re}\, s>1,\ \mathrm{Re}\, a>0. \tag{2}$$

For more information about Hurwitz's zeta function, we refer the reader to [3, 18, 1, 28].

Theorem 2. *For* $k = 1, 2, \ldots,$

$$\begin{aligned} E(M^k) = k!(-1)^k \Bigg(& -6^k\left(\frac{2\pi}{3\sqrt{3}} - 1\right)\binom{2k-2}{k-1} \\ & + \sum_{j=1}^{\lfloor k/2\rfloor} 2^{k+1}\binom{2k-2j-1}{k-1} 3^{k-2j}\zeta(2j)(1-2^{1-2j}) \\ & + \sum_{j=2}^{k}\binom{2k-j-1}{k-1}\cdot 6^{k-j}\Big((-1)^j\big(\zeta(j,\tfrac{1}{3}) - \zeta(j,\tfrac{5}{6})\big) \\ & \qquad + \zeta(j,\tfrac{2}{3}) - \zeta(j,\tfrac{1}{6}) + 6^j\Big)\Bigg). \end{aligned} \tag{3}$$

The right-hand side of (3) involves the values of Hurwitz's zeta function $\zeta(s,a)$ for positive integers s and $a = 1/6, 1/3, 2/3, 5/6$. More specifically, for positive even values of s, we need the sums $\zeta(s,1/3)+\zeta(s,2/3)$ and $\zeta(s,1/6)+\zeta(s,5/6)$, which are easy to calculate (using, say, [29]). However, for odd s we need the differences $\zeta(s,1/3)-\zeta(s,2/3)$ and $\zeta(s,1/6)-\zeta(s,5/6)$. Thus, to write the right-hand side of (3) in a more elementary way, we need to find the values of $\zeta(s,a)$ for odd integers and $a = 1/6, 1/3, 2/3, 5/6$. These values were given in [29] as quite cumbersome expressions. Our next result provides more convenient expressions for these values (which, of course, should yield the same results). For completeness, we deal with $a = 1/4$, $1/2, 3/4$ as well, although these are not needed for simplifying Theorem 2.

THEOREM 3. *For any $m \geq 1$ we have*

$$\begin{aligned}
\zeta(k, 1/2) &= (2^k - 1)\zeta(k), \qquad k = 2m, 2m+1,\\
\zeta(2m+1, 1/3) &= \frac{3^{2m+1} - 1}{2}\zeta(2m+1) + \frac{K_m(3)}{\sqrt{3}},\\
\zeta(2m+1, 2/3) &= \frac{3^{2m+1} - 1}{2}\zeta(2m+1) - \frac{K_m(3)}{\sqrt{3}},\\
\zeta(2m+1, 1/4) &= [2^{2m}(2^{2m+1} - 1)]\zeta(2m+1) + \tfrac{1}{2}K_m(4),\\
\zeta(2m+1, 3/4) &= [2^{2m}(2^{2m+1} - 1)]\zeta(2m+1) - \tfrac{1}{2}K_m(4),\\
\zeta(2m+1, 1/6) &= \frac{(2^{2m+1} - 1)(3^{2m+1} - 1)}{2}\zeta(2m+1) + \frac{K_m(6) - K_m(3)}{\sqrt{3}},\\
\zeta(2m+1, 5/6) &= \frac{(2^{2m+1} - 1)(3^{2m+1} - 1)}{2}\zeta(2m+1) - \frac{K_m(6) - K_m(3)}{\sqrt{3}},
\end{aligned}$$

where, for $q \in \{3, 4, 6\}$,

$$K_m(q) = \frac{(2\pi)^{2m+1}(-1)^m}{2(2m)!}\left[\frac{q}{2} - \sum_{j=0}^{m} B_{2j}\frac{(2m)! \cdot q^{2j}}{(2m-2j+1)!(2j)!}\right]. \tag{4}$$

REMARK 4. $K_m(q)$ in the theorem coincides with $C_m(a)$, to be used in the proof for $a = 1/q$. We have written it here this way to make the expression simpler.

EXAMPLE 5. The second addend in the first non-trivial value in the theorem, $\zeta(2m+1, 1/3)$, is $K_m(3)/\sqrt{3}$, whereas in [29] it is

$$\frac{\sqrt{3}}{2\pi}\Big((2m + 2 + 3^{2m+2})\zeta(2m+2) - 2\sum_{j=0}^{m-1} 3^{2m-2j}\zeta(2m-2j)\zeta(2j+2)\Big).$$

Employing Theorem 3, we are able to rewrite Theorem 2 without reference to Hurwitz's zeta function or, indeed, even Riemann's zeta function (at odd integers). We do use, however, the Bernoulli numbers. We first define the function $B_m(x)$ implicitly by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m(x)}{m!} z^m, \qquad |z| < 2\pi.$$

The numbers $B_m(0)$ are the *Bernoulli numbers*, and are denoted by B_m. The Bernoulli numbers are a very classical object in analytic number theory. For further details we refer the reader to [3, 19].

Theorem 6. *For $k = 1, 2, \ldots,$*

$$E(M^k) = k!(-6)^k \cdot \left(\left(\binom{2k-1}{k} - \frac{2\pi}{3\sqrt{3}} \binom{2k-2}{k-1} \right) + \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{(-1)^j \cdot \pi^{2j} B_{2j}}{(2j)!} (2^{2j-1} - 1) \left(1 - \frac{3}{3^{2j}}\right) \binom{2k-2j-1}{k-1} - \frac{2}{\sqrt{3}} \sum_{j=1}^{\lfloor (k-1)/2 \rfloor} \left(\frac{\pi}{3}\right)^{2j+1} (-1)^j \binom{2k-2j-2}{k-1} \cdot \left(\frac{3j+1}{(2j+1)!} - 3 \sum_{\ell=1}^{j} \frac{6^{2\ell-1} B_{2\ell}}{(2j-2\ell+1)!(2\ell)!} \right) \right),$$

where B_m is the mth Bernoulli number.

In Table 1 we present some results concerning the first five moments of M. For each k, we present the kth moment of M as obtained by Theorem 6 and as calculated numerically by Brennan et al. [5]. One checks easily that the two indeed coincide.

Table 1. Moments of the maximum waiting time

k	$E(M^k)$ by Theorem 6	$E(M^k)$ (numerical)
1	$\frac{4\sqrt{3}}{3}\pi - 6$	1.255
2	$-4\pi^2 - 32\sqrt{3}\,\pi + 216$	2.397
3	$\frac{80\sqrt{3}}{3}\pi^3 + 216\pi^2 + 1728\sqrt{3}\,\pi - 12960$	6.689
4	$-\frac{1456}{5}\pi^4 - 2560\sqrt{3}\,\pi^3 - 17280\pi^2 - 138240\sqrt{3}\pi + 1088640$	25.453
5	$\frac{5440\sqrt{3}}{3}\pi^5 + 43680\pi^4 + 288000\sqrt{3}\,\pi^3 + 1814400\pi^2 + 14515200\sqrt{3}\,\pi - 117573120$	123.705

The following proposition shows that once we know the moments of M, we get good estimates for the moments of the maximal waiting time in both finite models.

Proposition 7. *For every fixed k, as $n \to \infty$:*

(a) $E(W_{(n)}^k) = n^k E(M^k) + O(1)$.
(b) $E(D_{(n)}^k) = n^k E(M^k) + o(n^k)$.

Remark 8. In fact, the error terms in the proposition are much smaller, as one can show employing the proof techniques of Theorem 9 below.

Next, we consider the probability of each X_m in the sequence $(X_i)_{i=1}^{\infty}$ being the maximum.

THEOREM 9. *As* $m \to \infty$,

$$P(X_m = M) = \pi\sqrt{2m} \cdot e^{-\pi\sqrt{2/3}\cdot\sqrt{m}} \cdot (1 + o(1)). \tag{5}$$

The right-hand side of (5) is very reminiscent of the asymptotic expression for the partition function. Recall that the partition function $p(m)$ counts the various possibilities of representing a positive integer m as a sum of positive integers. It is a classical object in number theory [3, Chapter 14]. Hardy and Ramanujan [16] showed that, asymptotically as $m \to \infty$,

$$p(m) = \frac{1}{4m\sqrt{3}} \cdot e^{\pi\sqrt{2/3}\cdot\sqrt{m}}(1 + o(1)).$$

Thus, by Theorem 9, $P(X_m = M)$ is asymptotically the same as $1/p(m)$ up to a relatively small factor of $\Theta(\sqrt{m})$. This fact, which may seem coincidental, is not surprising once we notice that the distribution function of M is intimately related to the generating function of the partition function. (This will be explained later on at the beginning of Section 5.) The asymptotic behavior of this generating function was investigated by Hardy and Ramanujan [16], and is the key to the proof of Theorem 9. It would be interesting to explain intuitively the proximity of the two quantities $P(X_m = M)$ and $1/p(m)$.

In Table 2 we present some numerical results relating to Theorem 9. For several values of m, we present three quantities:

(a) The exact value of $P(X_m = M)$, given by the integral on the right-hand side of (54) below, calculated numerically by Mathematica.
(b) The main term on the right-hand side of (5).
(c) The value of the integral

$$\int_0^1 \frac{mx^{m-1}}{1-x^m}\sqrt{\frac{2\pi}{1-x}}\,e^{-\frac{\pi^2}{6(1-x)}+\frac{\pi^2}{12}}\,dx, \tag{6}$$

in which the correct integrand (see the right-hand side of (54) below) has been replaced by the estimate of Hardy and Ramanujan. The integral has been calculated numerically by Mathematica. It turns out that although the asymptotics holds only near the point 1, the values in (a) and (c) are pretty close. Note that, for $m = 1$, the substitution $y = \sqrt{1/(1-x)}$ leads to an explicit value for the integral (6),

$$\int_0^1 \frac{1}{1-x}\sqrt{\frac{2\pi}{1-x}}\,e^{-\frac{\pi^2}{6(1-x)}+\frac{\pi^2}{12}}\,dx = \int_1^{\infty} 2\sqrt{2\pi}\,e^{-\frac{\pi^2}{6}y^2+\frac{\pi^2}{12}}\,dy$$
$$= 4\sqrt{3}\,e^{\frac{\pi^2}{12}}(1 - \Phi(\pi\sqrt{3}/3)),$$

but it seems like the integral cannot be calculated explicitly for $m \geq 2$.

Table 2. The probability of each variable to be the maximum

m	$P(X_m = M)$	$\pi\sqrt{2m}\cdot e^{-\pi\sqrt{\frac{2}{3}}\cdot\sqrt{m}}$	$\int_0^1 \frac{mx^{m-1}}{1-x^m}\sqrt{\frac{2\pi}{1-x}}\cdot e^{-\frac{\pi^2}{6(1-x)}+\frac{\pi^2}{12}}\,dx$
1	0.516	0.342	0.550
2	0.213	0.167	0.225
3	0.107	0.091	0.112
4	$5.98\cdot 10^{-2}$	$5.26\cdot 10^{-2}$	$6.23\cdot 10^{-2}$
5	$3.55\cdot 10^{-2}$	$3.21\cdot 10^{-2}$	$3.68\cdot 10^{-2}$
10	$4.41\cdot 10^{-3}$	$4.22\cdot 10^{-3}$	$4.53\cdot 10^{-3}$
50	$4.20\cdot 10^{-7}$	$4.17\cdot 10^{-7}$	$4.26\cdot 10^{-7}$
100	$3.24\cdot 10^{-10}$	$3.22\cdot 10^{-10}$	$3.27\cdot 10^{-10}$

Similarly to Proposition 7, we may use the results for an infinite sequence of random variables to estimate the analogous probabilities in the finite versions. (Again, as in Proposition 7, the error can actually be reduced.)

PROPOSITION 10. *For fixed $m \geq 1$, as $n \to \infty$:*

(a) $P(W_{n-m} = W_{(n)}) = P(X_m = M) + O(1/n)$.
(b) $P(D_{n-m} = D_{(n)}) = P(X_m = M) + o(1)$.

3. Hurwitz's zeta function at special points. Now we want to prove some formulas for Hurwitz's zeta function $z(s,a)$, defined in (2). Not all formulas in this section are new, but we bring them here to make the calculations as self-contained as possible. It is well-known that, similarly to the Riemann zeta function, $z(s,a)$ is analytic in the entire complex plane with one simple pole at $s = 1$.

Define sequences of polynomials $(p_j)_{j=1}^{\infty}$ and $(q_j)_{j=1}^{\infty}$ inductively by $p_1(x) = \frac{x-x^2}{2}$ and

$$q_j(x) = \int_0^x p_j(t)\,dt, \qquad p_{j+1}(x) = q_j(x) - xq_j(1) \tag{7}$$

for $j \geq 1$.

LEMMA 11. *Hurwitz's zeta function has the following properties:*

(a) $\zeta(0,a) = 1/2 - a$.
(b) $\zeta(-1,a) = \dfrac{a-a^2}{2} - \dfrac{1}{12}$.
(c) *For every s and $k = 0,1,2,\ldots,$*

$$\zeta(s,a) = \frac{a^{1-s}}{s-1} + \frac{1}{2}a^{-s} + \sum_{j=1}^{k+1} s(s+1)\cdots(s+j-1)q_j(1)a^{-s-j} + s(s+1)\cdots(s+k+2)\int_0^1 p_{k+2}(x)\zeta(s+k+3,x+a)\,dx. \tag{8}$$

(d) $\zeta(-k,a) = -\dfrac{a^{k+1}}{k+1} + \tfrac{1}{2}a^k + \sum_{j=1}^{k}(-1)^j q_j(1)\dfrac{k!}{(k-j)!}a^{k-j},\ k \geq 2.$

(e) *For every* s *and* $k = 0, 1, 2, \ldots,$

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{j=1}^{k+1} s(s+1)\cdots(s+j-1)q_j(1) + s(s+1)\cdots(s+k+2)\int_0^1 p_{k+2}(x)\zeta(s+k+3, x+1)\,dx. \tag{9}$$

(f) *For* $k = 0, 1, 2, \ldots,$

$$\zeta'(-k) = -\frac{1}{(k+1)^2} + \sum_{j=1}^{k}(-1)^{j+1}\frac{k!}{(k-j)!}q_j(1)[H_k - H_{k-j}] + (-1)^k k! \cdot \int_0^1 p_{k+1}(x)\zeta(2, x+1)\,dx. \tag{10}$$

(g) $\zeta'(0) = -1 + \int_0^1 p_1(x)\zeta(2, x+1)\,dx.$

Proof. (a) Assume first that $\operatorname{Re} s > 1$. Using Stieltjes integration, we obtain

$$\begin{aligned} \zeta(s,a) &= \int_{0^-}^{\infty} \frac{d[x]}{(x+a)^s} = \int_0^{\infty} \frac{dx}{(x+a)^s} - \int_{0^-}^{\infty} \frac{d\{x\}}{(x+a)^s} \\ &= \frac{(x+a)^{1-s}}{1-s}\Big|_{x=0^-}^{\infty} - \frac{\{x\}}{(x+a)^s}\Big|_{x=0^-}^{\infty} - s\int_0^{\infty} \frac{\{x\}\,dx}{(x+a)^{s+1}} \\ &= \frac{a^{1-s}}{s-1} + a^{-s} - s\int_0^1 x\zeta(s+1, x+a)\,dx. \end{aligned} \tag{11}$$

Since the right-hand side is analytic for $\operatorname{Re} s > 0$, this defines $\zeta(s,a)$ for $\operatorname{Re} s > 0$. Multiplying both sides of (11) by $s-1$ and taking the limit as $s \to 1$, we obtain $\lim_{s\to 1}\zeta(s,a)(s-1) = 1$ for any $a > 0$. Passing to the limit as $s \to 0^+$, we get

$$\zeta(0,a) = 1 - a - \int_0^1 x\,dx = \frac{1}{2} - a.$$

(b) We have

$$\int_0^1 x\zeta(s+1, x+a)\,dx = \sum_{n=0}^{\infty}\int_0^1 \frac{x\,dx}{(n+x+a)^{s+1}} = \sum_{n=0}^{\infty}\int_0^1 \frac{dx^2/2}{(n+x+a)^{s+1}}$$

$$= \sum_{n=0}^{\infty} \left[\frac{x^2}{2(n+x+a)^{s+1}} \bigg|_{x=0}^{1} + (s+1) \int_0^1 \frac{x^2\, dx}{2(n+x+a)^{s+2}} \right]$$

$$= \frac{1}{2}\zeta(s+1, a+1) + \frac{s+1}{2} \int_0^1 x^2 \zeta(s+2, x+a)\, dx.$$

Use (11) with $s+1$ instead of s to replace $\zeta(s+1, a+1)$, and the equality

$$\zeta(s+1, a+1) = \zeta(s+1, a) - a^{-s-1},$$

to obtain

$$\zeta(s,a) = \frac{a^{1-s}}{s-1} + a^{-s} - \frac{s}{2}\left[\frac{a^{-s}}{s} + a^{-s-1} - a^{-s-1} - (s+1)\int_0^1 x\zeta(s+2, x+a)\, dx \right] - \frac{s(s+1)}{2} \int_0^1 x^2 \zeta(s+2, x+a)\, dx$$

$$= \frac{a^{1-s}}{s-1} + a^{-s} - \frac{1}{2}a^{-s} + s(s+1)\int_0^1 \frac{x - x^2}{2} \zeta(s+2, x+a)\, dx.$$

We may write the above formula in the form

$$\zeta(s,a) = \frac{a^{1-s}}{s-1} + \frac{1}{2}a^{-s} + s(s+1)\int_0^1 p_1(x)\zeta(s+2, x+a)\, dx. \tag{12}$$

Using the equality $\lim_{s\to -1}(s+1)\zeta(s+2, x+a) = 1$, and taking the limit as $s \to -1$, we obtain

$$\zeta(-1, a) = -\frac{a^2}{2} + \frac{a}{2} - \int_0^1 p_1(x)\, dx = -\frac{a^2}{2} + \frac{a}{2} - \frac{1}{12}.$$

(c) We prove (8) by induction:

$$\zeta(s,a) = \frac{a^{1-s}}{s-1} + \frac{1}{2}a^{-s} + \sum_{j=1}^{k+1} s(s+1)\cdots(s+j-1) q_j(1) a^{-s-j} + s(s+1)\cdots(s+k+2)\int_0^1 p_{k+2}(x)\zeta(s+k+3, x+a)\, dx. \tag{13}$$

For $k = 0$, by (12),

$$\zeta(s,a) = \frac{a^{1-s}}{s-1} + \frac{1}{2}a^{-s} + s(s+1)\sum_{n=0}^{\infty}\left[\frac{q_1(x)}{(n+x+a)^{s+2}} \bigg|_{x=0}^{1} + (s+2)\int_0^1 q_1(x) \frac{dx}{(n+x+a)^{s+3}} \right]$$

$$= \frac{a^{1-s}}{s-1} + \frac{1}{2}a^{-s} + s(s+1)q_1(1)[\zeta(s+2,a) - a^{-s-2}] + s(s+1)(s+2)\int_0^1 q_1(x)\zeta(s+3,a+x)\,dx.$$

Using (11) with $s+2$ instead of s, we obtain

$$\begin{aligned}\zeta(s,a) &= \frac{a^{1-s}}{s-1} + \frac{1}{2}a^{-s} \\ &\quad + s(s+1)q_1(1)\left[\frac{a^{-s-1}}{s+1} + a^{-s-2} - a^{-s-2} - (s+2)\int_0^1 x\zeta(s+3,x+a)\,dx\right] \\ &\quad + s(s+1)(s+2)\int_0^1 q_1(x)\zeta(s+3,x+a)\,dx \\ &= \frac{a^{1-s}}{s-1} + \frac{1}{2}a^{-s} + sq_1(1)a^{-s-1} \\ &\quad + s(s+1)(s+2)\int_0^1 [q_1(x) - xq_1(1)]\zeta(s+3,x+a)\,dx.\end{aligned}$$

Since $q_1(x) - xq_1(1) = p_2(x)$, this proves the formula for $k=0$.

Now assume the formula is true for some $k \ge 0$ and prove it for $k+1$. Just as above, we have

$$\begin{aligned}\int_0^1 p_{k+2}(x)\zeta(s+k+3,x+a)\,dx &= \sum_{n=0}^{\infty}\int_0^1 \frac{dq_{k+2}(x)}{(n+x+a)^{s+k+3}} \\ &= \sum_{n=0}^{\infty}\left[\frac{q_{k+2}(x)}{(n+x+2)^{s+k+3}}\Big|_{x=0}^{1}\right] \\ &\quad + (s+k+3)\int_0^1 q_{k+2}(x)\zeta(s+k+4,x+a)\,dx \\ &= q_{k+2}(1)[\zeta(s+k+3,a) - a^{-s-k-3}] \\ &\quad + (s+k+3)\int_0^1 q_{k+2}(x)\zeta(s+k+4,x+a)\,dx.\end{aligned} \tag{14}$$

Using (11) with $s+k+3$ instead of s, we see that the right-hand side of (14) becomes

$$q_{k+2}(1)\left[\frac{a^{-s-k-2}}{s+k+2}-(s+k+3)\int_0^1 x\zeta(s+k+4,x+a)\,dx\right]$$
$$+(s+k+3)\int_0^1 q_{k+2}(x)\zeta(s+k+4,x+a)\,dx$$
$$=q_{k+2}(1)\frac{a^{-s-k-2}}{s+k+2}+(s+k+3)\int_0^1 p_{k+3}(x)\zeta(s+k+4,x+a)\,dx.$$

Substituting in (13), we obtain

$$\zeta(s,a)=\frac{a^{1-s}}{s-1}+\frac{a^{-s}}{2}+\sum_{j=1}^{k+2}s(s+1)\cdots(s+j+1)q_j(1)a^{-s-j}$$
$$+s(s+1)\cdots(s+k+3)\int_0^1 p_{k+3}(x)\zeta(x+k+4,x+a)\,dx. \tag{15}$$

This completes the proof of (8).

(d) The preceding part provides a formula for the analytic continuation of $\zeta(s,a)$ for $\operatorname{Re} s>-k-2$. Putting $s=-k$, we obtain

$$\zeta(-k,a)=-\frac{a^{k+1}}{k+1}+\frac{1}{2}a^k+\sum_{j=1}^{k}(-1)^j q_j(1)a^{k-j}\frac{k!}{(k-j)!}. \tag{16}$$

(e) follows from (15) by taking $a=1$; (f) follows from (9) for $s=-k$; and (g) follows from (10) for $k=0$. ■

COROLLARY 12. *Let B_n be the nth Bernoulli number. Then $B_0=1$, $B_1=-1/2$, and for any $n\geq 2$ we have*

$$B_n=(-1)^n q_{n-1}(1)n!. \tag{17}$$

Proof. By [3, Thms. 12.12 and 12.13],

$$\zeta(-n,a)=-\frac{B_{n+1}(a)}{n+1}=-\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\cdot B_k\cdot a^{n+1-k}.$$

Comparing this with the formula for $\zeta(-n,a)$ in Lemma 11(c), we get

$$-\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\cdot B_k\cdot a^{n+1-k}$$
$$=-\frac{a^{n+1}}{n+1}+\frac{1}{2}a^n+\sum_{k=1}^{n}(-1)^k q_k(1)\frac{n!}{(n-k)!}a^{n-k}.$$

Since the two sides, considered as polynomials in a, are identical, the coefficients of the corresponding terms must be equal. Equating the coefficients of a^{n+1} we obtain $-B_0/(n+1) = -1/(n+1)$, thus, $B_0 = 1$. Equating the coefficients of a^n we obtain $-B_1 = 1/2$, and thus $B_1 = -1/2$. Equating the coefficients of a^{n+1-k} for $k \geq 2$ we obtain $-n!/(k!(n+1-k)!) \cdot B_k = (-1)^{k-1} q_{k-1}(1) n!/(n+1-k)!$, and therefore $B_k = (-1)^k q_{k-1}(1) \cdot n!$. ∎

Proof of Theorem 3. Using the functional equation for Hurwitz's zeta function with $s = k+1$ and $k = 2m$, $m \geq 1$, we obtain

$$\begin{aligned}\zeta(-2m, a) &= \frac{\Gamma(2m+1)}{(2\pi)^{2m+1}} \left[e^{-\frac{\pi i(2m+1)}{2}} \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^{2m+1}} + e^{\frac{\pi i(2m+1)}{2}} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n a}}{n^{2m+1}} \right] \\ &= \frac{2(2m)!(-1)^m}{(2\pi)^{2m+1}} \sum_{n=1}^{\infty} \frac{\sin(2\pi n a)}{n^{2m+1}}.\end{aligned}$$

For a reduced fraction $a = p/q$, define

$$C_m(a) = \frac{(2\pi q)^{2m+1}(-1)^m}{2(2m)!} \zeta(-2m, a).$$

Thus

$$\begin{aligned}C_m(a) &= \sum_{c=1}^{q-1} \sin(2\pi c a) \sum_{n=0}^{\infty} \frac{q^{2m+1}}{(nq+c)^{2m+1}} \\ &= \sum_{c=1}^{q-1} \sin(2\pi c a) \zeta\left(2m+1, \frac{c}{q}\right).\end{aligned} \tag{18}$$

Taking $a = 1/q, 2/q, \ldots, (q-1)/q$, we get $q-1$ equations with $q-1$ unknowns $\zeta(2m+1, c/q)$ $(c = 1, \ldots, q-1)$ but the coefficient matrix has rank at most $(q-1)/2$.

We have one more equation for every q:

$$\sum_{c=1}^{q} \sum_{n=0}^{\infty} \frac{1}{(nq+c)^s} = \zeta(s),$$

and therefore

$$\begin{aligned}\sum_{c=1}^{q-1} \zeta\left(s, \frac{c}{q}\right) &= (q^s - 1)\zeta(s), \\ \sum_{c=1}^{q-1} \zeta\left(2m+1, \frac{c}{q}\right) &= (q^{2m+1} - 1)\zeta(2m+1).\end{aligned} \tag{19}$$

Thus, for $q = 2$ we get

$$\zeta(k, 1/2) = (2^k - 1)\zeta(k), \qquad k \geq 2. \tag{20}$$

For $q=3$ we get, by (18),

$$C_m(1/3)=\frac{\sqrt{3}}{2}\zeta(2m+1,1/3)-\frac{\sqrt{3}}{2}\zeta(2m+1,2/3),$$

and by (19),

$$\begin{aligned}C_m(1/3)&=\frac{\sqrt{3}}{2}(3^{2m+1}-1)\zeta(2m+1)\\&=\zeta(2m+1,1/3)+\zeta(2m+1,2/3)\end{aligned}$$

and

$$\begin{aligned}\zeta(2m+1,1/3)&=\frac{C_m(1/3)}{\sqrt{3}}+\frac{3^{2m+1}-1}{2}\zeta(2m+1),\\ \zeta(2m+1,2/3)&=\frac{3^{2m+1}-1}{2}\zeta(2m+1)-\frac{C_m(1/3)}{\sqrt{3}}.\end{aligned}\tag{21}$$

For $q=4$, we get again two equations with two unknowns: By (18),

$$C_m(1/4)=\zeta(2m+1,1/4)-\zeta(2m+1,3/4),$$

and by (19)–(20),

$$\begin{aligned}\zeta(2m+1,1/4)+\zeta(2m+1,3/4)&=(4^{2m+1}-1)\zeta(2m+1)-\zeta(2m+1,1/2)\\&=(4^{2m+1}-2^{2m+1})\zeta(2m+1).\end{aligned}$$

Hence

$$\zeta(2m+1,1/4)=\tfrac{1}{2}C_m(1/4)+\tfrac{1}{2}(4^{2m+1}-2^{2m+1})\zeta(2m+1)$$

and

$$\zeta(2m+1,3/4)=\tfrac{1}{2}(4^{2m+1}-2^{2m+1})\zeta(2m+1)-\tfrac{1}{2}C_m(1/4).$$

For $q=6$, we get two equations with two unknowns (as $\zeta(2m+1,1/3)$ and $\zeta(2m+1,1/2)$ and $\zeta(2m+1,2/3)$ are known by (20) and (21)):

$$C_m(1/6)=\sum_{c=1}^{5}\sin\frac{\pi c}{3}\zeta(2m+1,c/6)$$

and

$$\sum_{c=1}^{5}\zeta(2m+1,c/6)=(6^{2m+1}-1)\zeta(2m+1),$$

i.e.,

$$\begin{aligned}\tfrac{1}{2}[\zeta(2m+1,1/6)+\zeta(2m+1,1/3)-\zeta(2m+1,2/3)-\zeta(2m+1,5/6)]\\=\frac{C_m(1/6)}{\sqrt{3}}\end{aligned}\tag{22}$$

and

$$\frac{1}{2}\sum_{c=1}^{5}\zeta(2m+1,c/6)=\frac{(6^{2m+1}-1)}{2}\zeta(2m+1). \tag{23}$$

Adding (22) and (23) by sides, we get

$$\frac{6^{2m+1}-1}{2}\zeta(2m+1)+\frac{C_m(1/6)}{\sqrt{3}} \\ =\zeta(2m+1,1/6)+\zeta(2m+1,1/3)+\tfrac{1}{2}\zeta(2m+1,1/2),$$

i.e.,

$$\zeta(2m+1,1/6)=\frac{C_m(1/6)}{\sqrt{3}} \\ +\left[\frac{6^{2m+1}-1}{2}-\frac{3^{2m+1}-1}{2}-\frac{2^{2m+1}-1}{2}\right]\zeta(2m+1)-\frac{C_m(1/3)}{\sqrt{3}}. \tag{24}$$

Subtracting (22) from (23), we obtain

$$\zeta(2m+1,2/3)+\zeta(2m+1,5/6)+\tfrac{1}{2}\zeta(2m+1,1/2) \\ =\frac{6^{2m+1}-1}{2}\zeta(2m+1)-\frac{C_m(1/6)}{\sqrt{3}},$$

i.e.,

$$\zeta(2m+1,5/6)=\left[\frac{6^{2m+1}-1}{2}-\frac{3^{2m+1}-1}{2}-\frac{2^{2m+1}-1}{2}\right]\zeta(2m+1) \\ -\frac{C_m(1/6)}{\sqrt{3}}+\frac{C_m(1/3)}{\sqrt{3}}. \ \blacksquare \tag{25}$$

4. The moments of the maximum

Proof of Theorem 1. Denote by F_M the distribution function of M. Clearly,

$$F_M(t)=P\Big(\max_{1\le m<\infty}X_m\le t\Big)=\prod_{m=1}^{\infty}(1-e^{-mt}),\qquad t\ge 0.$$

As M is positive,

$$E(M)=\int_0^\infty(1-F_M(t))\,dt=\int_0^\infty\Big(1-\prod_{m=1}^{\infty}(1-e^{-mt})\Big)\,dt \\ =\int_0^1\Big(1-\prod_{m=1}^{\infty}(1-x^m)\Big)\,\frac{dx}{x}. \tag{26}$$

As noted by Brennan et al. [5], Euler's Pentagonal Theorem (see, for example, [2, Cor. 1.7]) enables us to rewrite the infinite product on the right-hand side of (26) as a power series:

$$\prod_{m=1}^{\infty}(1-x^m) = 1+\sum_{m=1}^{\infty}(-1)^m(x^{m(3m-1)/2}+x^{m(3m+1)/2}), \qquad |x|<1. \tag{27}$$

Thus, similarly to [5, Thm. 2.1],

$$\begin{aligned} E(M) &= \int_0^1 \sum_{m=1}^{\infty}(-1)^{m+1}(x^{m(3m-1)/2}+x^{m(3m+1)/2})\,\frac{dx}{x} \\ &= \sum_{m=1}^{\infty}(-1)^{m+1}\left(\frac{2}{m(3m-1)}+\frac{2}{m(3m+1)}\right) \\ &= 6\sum_{m=1}^{\infty}(-1)^{m+1}\left(\frac{1}{3m-1}-\frac{1}{3m+1}\right). \end{aligned}$$

Denote

$$f(x) = \sum_{m=1}^{\infty}\left(\frac{x^{3m-1}}{3m-1}-\frac{x^{3m+1}}{3m+1}\right). \tag{28}$$

Note that $E(M) = 6f(-1)$. Now,

$$\begin{aligned} f'(x) &= \sum_{m=1}^{\infty}(x^{3m-2}-x^{3m}) = (x-x^3)(1+x^3+x^6+\cdots) \\ &= \frac{x-x^3}{1-x^3} = \frac{1-x^3}{1-x^3}-\frac{1-x}{1-x^3} = 1-\frac{1}{1+x+x^2}. \end{aligned}$$

A simple integration yields

$$f(x) = x-\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C.$$

Using the fact that $f(0)=0$, we get $C=\pi/(3\sqrt{3})$. Thus

$$f(-1) = -1-\frac{2}{\sqrt{3}}\arctan\left(\frac{-1}{\sqrt{3}}\right)+\frac{\pi}{3\sqrt{3}} = \frac{2\sqrt{3}\pi}{9}-1. \tag{29}$$

Altogether $E(M) = 4\sqrt{3}\,\pi/3-6$. ∎

The calculation of the infinite series that will arise in the proof of Theorem 2 requires the partial fraction decomposition of several rational functions.

LEMMA 13. *For every $k \geq 1$ and $a \in \mathbb{R}$,*

$$\frac{1}{x^k(1+ax)^k} = \sum_{j=1}^{k}\binom{2k-j-1}{k-1}\frac{(-a)^{k-j}}{x^j}+\sum_{j=1}^{k}\binom{2k-j-1}{k-1}\frac{(-a)^k}{(1+ax)^j}.$$

Proof. By partial fraction decomposition,

$$\frac{1}{x^k(1+ax)^k} = \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_k}{x^k} + \frac{d_1}{1+ax} + \frac{d_2}{(1+ax)^2} + \cdots + \frac{d_k}{(1+ax)^k}$$

for suitable constants $c_1, \ldots, c_k, d_1, \ldots, d_k$. Consider any c_j. By [22], it is the coefficient of x^{k-j} in the Laurent expansion of $(1+ax)^{-k}$:

$$(1+ax)^{-k} = \sum_{i=0}^{\infty} \binom{-k}{i} (ax)^i = \sum_{i=0}^{\infty} \binom{k+i-1}{k-1} (-a)^i x^i.$$

Thus,

$$c_j = \binom{2k-j-1}{k-1} (-a)^{k-j}, \qquad 1 \le j \le k. \tag{30}$$

Similarly, d_j is the coefficient of $(1+ax)^{k-j}$ in the Laurent expansion of x^{-k} at $-1/a$. Let $y = 1+ax$, so that $x = (1-y)/(-a)$. Then

$$\frac{1}{x^k} = (-a)^k (1-y)^{-k} = (-a)^k \sum_{i=0}^{\infty} \binom{-k}{i} (-y)^i = (-a)^k \sum_{i=0}^{\infty} \binom{k+i-1}{i} y^i,$$

and therefore

$$d_j = (-a)^k \binom{2k-j-1}{k-1}, \qquad 1 \le j \le k. \ \blacksquare \tag{31}$$

Proof of Theorem 2. Using the same notations as in the proof of Theorem 1, by (27) and [10, p. 150, (6.3)],

$$E(M^k) = \int_0^{\infty} kt^{k-1}(1 - F_M(t))\, dt = \sum_{m=1}^{\infty} (-1)^{m+1} \int_0^{\infty} kt^{k-1} (e^{-tm(3m-1)/2} + e^{-tm(3m+1)/2})\, dt.$$

As in the proof of [5, Thm. 4.1],

$$E(M^k) = k!2^k \sum_{m=1}^{\infty} (-1)^{m+1} \left(\frac{1}{m^k(3m-1)^k} + \frac{1}{m^k(3m+1)^k} \right). \tag{32}$$

Denote

$$g(k) = \sum_{m=1}^{\infty} (-1)^{m+1} \left(\frac{1}{m^k(3m-1)^k} + \frac{1}{m^k(3m+1)^k} \right). \tag{33}$$

By Lemma 13,

$$g(k)=\sum_{m=1}^{\infty}(-1)^{m+1}\sum_{j=1}^{k}\Big(\frac{(-1)^k c_j}{m^j}+\frac{c'_j}{m^j}+\frac{(-1)^k d_j}{(1-3m)^j}+\frac{d'_j}{(1+3m)^j}\Big),$$

where

$$\begin{aligned} c_j &= \binom{2k-j-1}{k-1}3^{k-j}, \qquad c'_j=\binom{2k-j-1}{k-1}(-3)^{k-j},\\ d_j &= \binom{2k-j-1}{k-1}3^{k}, \qquad\quad d'_j=\binom{2k-j-1}{k-1}(-3)^{k}. \end{aligned} \tag{34}$$

Thus

$$\begin{aligned} g(k) &= \sum_{j=1}^{k}((-1)^k c_j+c'_j)\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^j}\\ &\quad+\sum_{j=1}^{k}(-1)^{k-j}d_j\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(3m-1)^j}+\sum_{j=1}^{k}(-1)^k d_j\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(3m+1)^j}\\ &= \sum_{j=1}^{\lfloor k/2\rfloor}2(-1)^k c_{2j}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^{2j}}\\ &\quad+\sum_{j=1}^{k}\Bigg((-1)^k d_j\bigg((-1)^j\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(3m-1)^j}+\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(3m+1)^j}\bigg)\Bigg). \end{aligned} \tag{35}$$

For $j\ge 1$, by [3, Thm. 13.11],

$$\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^{2j}}=\zeta(2j)(1-2^{1-2j}). \tag{36}$$

Consider the second addend on the right-hand side of (35). For $j=1$, by (28) and (29),

$$\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{3m-1}-\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{3m+1}=\frac{1}{9}(2\sqrt{3}\,\pi-9). \tag{37}$$

For $j\ge 2$,

$$\begin{aligned} \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(3m-1)^j} &= \frac{1}{2^j}-\frac{1}{5^j}+\frac{1}{8^j}-\frac{1}{11^j}+\cdots=\frac{1}{6^j}\big(\zeta(j,\tfrac{1}{3})-\zeta(j,\tfrac{5}{6})\big),\\ \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(3m+1)^j} &= \frac{1}{4^j}-\frac{1}{7^j}+\frac{1}{10^j}-\frac{1}{13^j}+\cdots=\frac{1}{6^j}\big(\zeta(j,\tfrac{2}{3})-\zeta(j,\tfrac{7}{6})\big)\\ &= \frac{1}{6^j}\big(\zeta(j,\tfrac{2}{3})-\zeta(j,\tfrac{1}{6})+6^j\big). \end{aligned} \tag{38}$$

Altogether, by (35)–(38),

$$g(k) = \sum_{j=1}^{\lfloor k/2 \rfloor} 2(-1)^k c_{2j}\zeta(2j)(1-2^{1-2j}) - (-1)^k d_1 \cdot \tfrac{1}{9}(2\sqrt{3}\,\pi - 9) + \sum_{j=2}^{k} \frac{(-1)^k d_j}{6^j}\big((-1)^j\big(\zeta(j,\tfrac{1}{3}) - \zeta(j,\tfrac{5}{6})\big) + \zeta(j,\tfrac{2}{3}) - \zeta(j,\tfrac{1}{6}) + 6^j\big). \tag{39}$$

Now, (32)–(34) and (39) complete the proof. ■

Proof of Theorem 6. Consider the third addend on the right-hand side of (3). By (19) and (20),

$$\begin{aligned} \zeta(s,\tfrac{1}{2}) &= (2^s-1)\zeta(s) = 6^s\left(\frac{1}{3^s} - \frac{1}{6^s}\right)\zeta(s),\\ \zeta(s,\tfrac{1}{3}) + \zeta(s,\tfrac{2}{3}) &= (3^s-1)\zeta(s) = 6^s\left(\frac{1}{2^s} - \frac{1}{6^s}\right)\zeta(s). \end{aligned} \tag{40}$$

We also have

$$\begin{aligned} &\zeta(s,\tfrac{1}{6}) + \zeta(s,\tfrac{5}{6})\\ &= \sum_{i=1}^{5}\zeta(s,\tfrac{i}{6}) - \big(\zeta(s,\tfrac{1}{3}) + \zeta(s,\tfrac{2}{3})\big) - \zeta(s,\tfrac{1}{2})\\ &= 6^s\left(1 - \frac{1}{6^s}\right)\zeta(s) - 6^s\left(\frac{1}{2^s} - \frac{1}{6^s}\right)\zeta(s) - 6^s\left(\frac{1}{3^s} - \frac{1}{6^s}\right)\zeta(s)\\ &= 6^s\left(1 + \frac{1}{6^s} - \frac{1}{2^s} - \frac{1}{3^s}\right)\zeta(s). \end{aligned} \tag{41}$$

Thus, for even j, by (40) and (41),

$$\frac{1}{6^j}\big(\zeta(j,\tfrac{1}{3}) - \zeta(j,\tfrac{5}{6}) + \zeta(j,\tfrac{2}{3}) - \zeta(j,\tfrac{1}{6}) + 6^j\big) = 1 + \zeta(j)\left(\frac{1}{2^{j-1}} + \frac{1}{3^j} - \frac{2}{6^j} - 1\right). \tag{42}$$

For odd j, by Theorem 3,

$$-\zeta(j,\tfrac{1}{3}) + \zeta(j,\tfrac{2}{3}) + \zeta(j,\tfrac{5}{6}) - \zeta(j,\tfrac{1}{6}) = -\frac{2K_{(j-1)/2}(6)}{\sqrt{3}}, \tag{43}$$

where $K_m(q)$ is defined in (4). Using the same notation as in the proof of

Theorem 2, by (39) we have

$$g(k) = (-1)^k \Bigg(\sum_{j=1}^{\lfloor k/2 \rfloor} 2c_{2j}\zeta(2j)(1-2^{1-2j}) - d_1 \cdot \frac{2\sqrt{3}\,\pi}{9} + \sum_{j=1}^{k} d_j + \sum_{j=1}^{\lfloor k/2 \rfloor} d_{2j} \cdot \zeta(2j) \left(\frac{2}{2^{2j}} + \frac{1}{3^{2j}} - \frac{2}{6^{2j}} - 1 \right) - \sum_{j=1}^{\lfloor (k-1)/2 \rfloor} \frac{d_{2j+1}}{6^{2j+1}} \frac{2K_j(6)}{\sqrt{3}} \Bigg). \tag{44}$$

The sum of the first and fourth addends on the right-hand side of (44) yields $\sum_{j=1}^{\lfloor k/2 \rfloor} g_j(k)$, where

$$\begin{aligned} g_j(k) &= \zeta(2j)\left(2c_{2j}\left(1-\frac{2}{2^{2j}}\right) - d_{2j}\left(1-\frac{2}{2^{2j}}\right)\left(1-\frac{1}{3^{2j}}\right)\right) \\ &= \zeta(2j)\left(1-\frac{2}{2^{2j}}\right)\left(2\binom{2k-2j-1}{k-1}3^{k-2j} - \binom{2k-2j-1}{k-1}3^k\left(1-\frac{1}{3^{2j}}\right)\right) \\ &= 3^k\zeta(2j)\left(1-\frac{2}{2^{2j}}\right)\left(\frac{3}{3^{2j}}-1\right)\binom{2k-2j-1}{k-1}. \end{aligned}$$

By, e.g., [3, p. 266], for $j \geq 1$ we have

$$\zeta(2j) = \frac{(-1)^{j+1} \cdot 2^{2j-1}\pi^{2j}}{(2j)!} B_{2j},$$

where B_m is the mth Bernoulli number. Thus

$$\begin{aligned} g_j(k) &= 3^k \frac{(-1)^{j+1} \cdot 2^{2j-1}\pi^{2j}}{(2j)!} B_{2j}\left(1-\frac{2}{2^{2j}}\right)\left(\frac{3}{3^{2j}}-1\right)\binom{2k-2j-1}{k-1} \\ &= 3^k \frac{(-1)^{j} \cdot \pi^{2j}}{(2j)!} B_{2j}(2^{2j-1}-1)\left(1-\frac{3}{3^{2j}}\right)\binom{2k-2j-1}{k-1}. \end{aligned} \tag{45}$$

The second addend on the right-hand side of (44) is

$$d_1 \frac{2\pi}{3\sqrt{3}} = \frac{2\pi}{3\sqrt{3}} 3^k \binom{2k-1-1}{k-1} = 3^k \frac{2\pi}{3\sqrt{3}} \binom{2k-2}{k-1}, \tag{46}$$

and the third is

$$\sum_{j=1}^{k} d_j = \sum_{j=1}^{k} \binom{2k-j-1}{k-j} 3^k = 3^k \cdot \frac{2k-1}{k}\binom{2k-2}{k-1} = 3^k \binom{2k-1}{k}. \tag{47}$$

The jth term in the last addend on the right-hand side of (44) is the product of two terms. The first, for $j \geq 1$, is

$$\frac{d_{2j+1}}{6^{2j+1}} = \frac{1}{6^{2j+1}}\binom{2k-(2j+1)-1}{k-1}3^k = \frac{3^k}{6^{2j+1}}\binom{2k-2j-2}{k-1}, \tag{48}$$

and the second is

$$\begin{aligned} \frac{2K_j(6)}{\sqrt{3}} &\\ &= \frac{2}{\sqrt{3}}\,\frac{(2\pi)^{2j+1}(-1)^j}{2(2j)!}\Big(-\frac{1}{2j+1}+\frac{6}{2}-\sum_{\ell=1}^{j}\frac{(2j)!6^{2\ell}B_{2\ell}}{(2j-2\ell+1)!(2\ell)!}\Big) \\ &= \frac{2(2\pi)^{2j+1}(-1)^j}{\sqrt{3}}\Big(\frac{3j+1}{(2j+1)!}-3\sum_{\ell=1}^{j}\frac{6^{2\ell-1}B_{2\ell}}{(2j-2\ell+1)!(2\ell)!}\Big). \end{aligned} \tag{49}$$

Therefore, by (48) and (49), the last addend on the right-hand side of (44) is

$$\begin{aligned} &\frac{3^k 2}{\sqrt{3}}\sum_{j=1}^{\lfloor (k-1)/2\rfloor}\Big(\frac{\pi}{3}\Big)^{2j+1}(-1)^j\binom{2k-2j-2}{k-1} \\ &\qquad\cdot\Big(\frac{3j+1}{(2j+1)!}-3\sum_{\ell=1}^{j}\frac{6^{2\ell-1}B_{2\ell}}{(2j-2\ell+1)!(2\ell)!}\Big). \end{aligned} \tag{50}$$

Altogether, by (44)–(50), we get our claim. ∎

Proof of Proposition 7. (a) The idea is to bound the difference between the distribution functions of $W_{(n)}/n$ and M. Since $F_{W_{(n)}/n}(t) = \prod_{j=1}^{n}(1-e^{-jt})$,

$$F_{W_{(n)}/n}(t) - F_M(t) = \prod_{j=1}^{n}(1-e^{-jt})\Big(1-\prod_{j=n+1}^{\infty}(1-e^{-jt})\Big).$$

At $t=0$ the difference vanishes. For $t>0$,

$$\prod_{j=n+1}^{\infty}(1-e^{-jt}) \geq 1-\sum_{j=n+1}^{\infty}e^{-jt} = 1-e^{-(n+1)t}\cdot\frac{1}{1-e^{-t}},$$

and therefore

$$\begin{aligned} F_{W_{(n)}/n}(t) - F_M(t) &\leq \prod_{j=1}^{n}(1-e^{-jt})\cdot\frac{e^{-(n+1)t}}{1-e^{-t}} \\ &= \prod_{j=2}^{n}(1-e^{-jt})\cdot e^{-(n+1)t}. \end{aligned} \tag{51}$$

Thus

$$F_{W_{(n)}/n}(t) - F_M(t) \leq e^{-nt}, \qquad t \geq 0. \tag{52}$$

Consider the kth moment of $W_{(n)}$. As $W_{(n)}/n$ is non-negative, by [10, p. 150, (6.3)] and (52) we have

$$E\left(\frac{W_{(n)}^k}{n^k}\right) = \int_0^\infty kt^{k-1}(1 - F_{X_{(n)}}(t))\,dt = \int_0^\infty kt^{k-1}(1 - F_M(t) + O(e^{-nt}))\,dt = E(M^k) + k\int_0^\infty t^{k-1}\cdot O(e^{-nt})\,dt.$$

Thus, using [3, p. 250, (1)],

$$E(W_{(n)}^k) = n^k E(M^k) + n^k \cdot O\left(\frac{k\Gamma(k)}{n^k}\right) = n^k E(M^k) + O(1). \tag{53}$$

(b) By [5, Thm. 4], for every k we have

$$\lim_{n\to\infty} \frac{E(D_{(n)}^k)}{n^k} = k!2^k \sum_{m=1}^\infty (-1)^{m+1}\left(\frac{1}{m^k(3m-1)^k} + \frac{1}{m^k(3m+1)^k}\right).$$

Thus, (32) implies that $E(D_{(n)}^k) = n^k E(M^k) + o(n^k)$. ■

5. The asymptotic distribution of the maximal waiting time. In this section we prove Theorem 9. We have to estimate the probability $P(X_m = M)$, which we will denote for brevity by p_m. By independence,

$$\begin{aligned} p_m &= P(\max\{X_1,\dots,X_{m-1},X_{m+1},\dots\} \le X_m) \\ &= \int_0^\infty me^{-my}\prod_{j\ne m}^\infty (1-e^{-jy})\,dy = \int_0^\infty me^{-my}\frac{1}{1-e^{-my}}\prod_{j=1}^\infty (1-e^{-jy})\,dy \\ &= \int_1^0 \frac{mx^m}{1-x^m}\prod_{j=1}^\infty (1-x^j)(-x^{-1})\,dx = \int_0^1 \frac{mx^{m-1}}{1-x^m}\prod_{j=1}^\infty (1-x^j)\,dx \\ &= \int_0^1 mx^{m-1}\frac{f(x)}{1-x^m}\,dx, \end{aligned} \tag{54}$$

where $f : [0,1] \to \mathbb{R}$ is Euler's function, given by

$$f(x) = \prod_{j=1}^\infty (1-x^j), \quad 0 \le x \le 1.$$

We will estimate the right-hand side of (54) by splitting the interval $[0,1]$ into five subintervals, $[x_{j-1}, x_j]$, $1 \le j \le 5$, and estimating the integral on each subinterval separately. The endpoints of these intervals are

$$
\begin{aligned}
&x_0 = 0, && x_3 = 1 - b/\sqrt{m} + m^{-2/3},\\
&x_1 = 1 - c_1/\sqrt{m}, && x_4 = 1 - c_2/\sqrt{m},\\
&x_2 = 1 - b/\sqrt{m} - m^{-2/3}, && x_5 = 1,
\end{aligned} \tag{55}
$$

where $b = \pi/\sqrt{6}$ and c_1, c_2 are constants with $c_1 > 2b$ and $0 < c_2 < b/6$. We estimate the first integral trivially:

(56)

$$\int_0^{1-c_1/\sqrt{m}} m x^{m-1} \frac{f(x)}{1-x^m}\, dx \le \int_0^{1-c_1/\sqrt{m}} m x^{m-1}\, dx = \left(1 - \frac{c_1}{\sqrt{m}}\right)^m \le e^{-c_1\sqrt{m}}.$$

The other four integrals are trickier. In these integrals we will replace f by an estimate thereof at 1^-. As it turns out, the function f is a classical object in number theory, as its inverse

$$\frac{1}{f(x)} = \prod_{m=1}^{\infty} \frac{1}{1-x^m} = \sum_{m=0}^{\infty} p(m) x^m$$

is the generating function of $p(m)$. In their study of the partition function, Hardy and Ramanujan [16] gave the following estimate, as $x \to 1^-$ (sea also [13, p. 576, (68)]):

$$f(x) = \sqrt{\frac{2\pi}{1-x}}\, e^{-\frac{\pi^2}{6(1-x)} + \frac{\pi^2}{12}} \cdot (1 + o(1)). \tag{57}$$

In the proof of Theorem 9 we shall need the next lemma:

Lemma 14. *For any fixed positive integer m, let $J_m : [0,1] \to \mathbb{R}$ be given by*

$$J_m(y) = \begin{cases} 0, & y = 0,\\ \dfrac{(1-y)^{m-1}}{\sqrt{y}} e^{-b/y}, & 0 < y \le 1,\ b > 0. \end{cases} \tag{58}$$

(a) *J_m attains its maximum at the point*

$$y_0(m) = \frac{-1/2 - b + \sqrt{4bm + b^2 - 5b + 1/4}}{2m - 3}. \tag{59}$$

In particular, $y_0(m) = \sqrt{b/m} + O(1/m)$ as $m \to \infty$.

(b) *J_m is increasing in $[0, y_0(m)]$ and decreasing in $[y_0(m), 1]$.*

Proof. (a) One easily checks that the function is continuous on $[0,1]$. In the open interval we have

$$\ln J_m(y) = (m-1)\ln(1-y) - \tfrac{1}{2}\ln y - b/y, \qquad 0 < y < 1.$$

Thus

$$\frac{d}{dy}\ln J_m(y) = -\frac{m-1}{1-y} - \frac{1}{2y} + \frac{b}{y^2}. \tag{60}$$

Equating the right-hand side to 0, we get

$$y_{1,2} = \frac{-1/2 - b \pm \sqrt{4bm + b^2 - 5b + 1/4}}{2m-3}.$$

As $y_0(m) \in (0,1)$, we obtain (59).

(b) Since J_m is positive in $(0,1)$, and $J_m(0) = J_m(1) = 0$, the claim follows from the preceding part. ■

Proof of Theorem 9. Denote

$$\beta_j(m) = \int_{x_{j-1}}^{x_j} m x^{m-1} \frac{f(x)}{1-x^m}\, dx, \qquad 1 \le j \le 5,$$

where the x_j's are defined in (55). Thus, by (56) we have $\beta_1(m) \le e^{-c_1\sqrt{m}}$. Let J_m be as in Lemma 14. Denote

$$I_m(x) = m\sqrt{2\pi}\, e^{\pi^2/12} J_m(1-x) = m x^{m-1}\sqrt{\frac{2\pi}{1-x}}\, e^{-\frac{\pi^2}{6(1-x)}+\frac{\pi^2}{12}}$$

with b in (58) taken as π^2. Then

$$\begin{aligned}\beta_2(m) &= \int_{1-c_1/\sqrt{m}}^{1-b/\sqrt{m}-m^{-2/3}} \frac{m x^{m-1}}{1-x^m}\sqrt{\frac{2\pi}{1-x}}\, e^{-\frac{\pi^2}{6(1-x)}+\frac{\pi^2}{12}} \cdot (1+o(1))\, dx \\ &= \int_{1-c_1/\sqrt{m}}^{1-b/\sqrt{m}-m^{-2/3}} \frac{I_m(x)}{1-x^m} \cdot (1+o(1))\, dx.\end{aligned} \tag{61}$$

For $x \le 1 - b/\sqrt{m} - m^{-2/3}$,

$$1 - x^m \ge 1 - \left(1 - \frac{b}{\sqrt{m}} - \frac{1}{m^{-2/3}}\right)^m \ge 1 - e^{-b\sqrt{m}} = 1 - o(1). \tag{62}$$

By Lemma 14,

$$\beta_2(m) \le \frac{c_1 - b}{\sqrt{m}} \cdot I_m\left(1 - \frac{b}{\sqrt{m}} - \frac{1}{m^{2/3}}\right). \tag{63}$$

Now,

$$\text{(64)} \quad \ln I_m\left(1-\frac{b}{\sqrt{m}}-\frac{1}{m^{2/3}}\right)$$
$$= \ln m + (m-1)\ln\left(1-\frac{b}{\sqrt{m}}-\frac{1}{m^{2/3}}\right)+\frac{1}{2}\ln(2\pi)$$
$$-\frac{1}{2}\ln\left(1-\left(1-\frac{b}{\sqrt{m}}-\frac{1}{m^{2/3}}\right)\right)+\frac{\pi^2}{12}-\frac{\pi^2/6}{1-(1-b/\sqrt{m}-1/m^{2/3})}.$$

Let us estimate the (non-immediate) terms on the right-hand side of (64). For the second term we have

$$\text{(65)} \quad \ln\left(1-\frac{b}{\sqrt{m}}-\frac{1}{m^{2/3}}\right)$$
$$= -\frac{b}{\sqrt{m}}-\frac{1}{m^{2/3}}-\frac{1}{2}\left(\frac{b}{\sqrt{m}}+\frac{1}{m^{2/3}}\right)^2+O(m^{-3/2})$$
$$= -\frac{b}{\sqrt{m}}-\frac{1}{m^{2/3}}-\frac{b^2}{2m}+O(m^{-7/6}).$$

The fourth term is

$$\text{(66)} \quad \ln\left(\frac{b}{\sqrt{m}}+\frac{1}{m^{2/3}}\right)=\ln\frac{b}{\sqrt{m}}+\ln\left(1+\frac{1}{bm^{1/6}}\right)$$
$$= -\frac{1}{2}\ln m+\ln b+O(m^{-1/6}).$$

The last term is

$$\text{(67)} \quad \frac{b^2}{b/\sqrt{m}+1/m^{2/3}}=\frac{b\sqrt{m}}{1+1/(bm^{1/6})}$$
$$= b\sqrt{m}\left(1-\frac{1}{bm^{1/6}}+\frac{1}{b^2m^{1/3}}-\frac{1}{b^3\sqrt{m}}+O(m^{-2/3})\right)$$
$$= b\sqrt{m}-m^{1/3}+\frac{m^{1/6}}{b}-\frac{1}{b^2}+O(m^{-1/6}).$$

Altogether, by (64)–(67),

$$\text{(68)} \quad \ln I_m\left(1-\frac{b}{\sqrt{m}}-\frac{1}{m^{2/3}}\right)$$
$$= \ln m+(m-1)\left(-\frac{b}{\sqrt{m}}-\frac{1}{m^{2/3}}-\frac{b^2}{2m}+O(m^{-7/6})\right)$$
$$+\frac{1}{2}\ln(2\pi)-\frac{1}{2}\left(-\frac{1}{2}\ln m+\ln b+O(m^{-1/6})\right)+\frac{\pi^2}{12}$$
$$-\left(b\sqrt{m}-m^{1/3}+\frac{m^{1/6}}{b}-\frac{1}{b^2}+O(m^{-1/6})\right)$$
$$= \frac{5}{4}\ln m-2b\sqrt{m}-\frac{m^{1/6}}{b}-\frac{\ln b}{2}+\frac{\ln(2\pi)}{2}+\frac{1}{b^2}+O(m^{-1/6}).$$

By (63) and (68),

$$\beta_2(m) \le \frac{c_1 - b}{\sqrt{m}} \cdot e^{\frac{5}{4}\ln m - 2b\sqrt{m} - \frac{m^{1/6}}{b} + o(m^{1/6})} \tag{69}$$
$$= e^{-2b\sqrt{m} - \frac{m^{1/6}}{b} + o(m^{1/6})} = o(e^{-2b\sqrt{m}}).$$

Consider $\beta_3(m)$. Substituting $x = 1 - b/\sqrt{m} + s/m^{3/4}$ and using (62), we get

$$\beta_3(m) = \int_{-m^{1/12}}^{m^{1/12}} I_m\left(1 - \frac{b}{\sqrt{m}} + \frac{s}{m^{3/4}}\right) \cdot m^{-3/4} \cdot (1 + o(1))\, ds. \tag{70}$$

Denoting

$$\tilde{I}_m(s) = I_m\left(1 - \frac{b}{\sqrt{m}} + \frac{s}{m^{3/4}}\right), \qquad -m^{1/12} \le s \le m^{1/12},$$

we have

$$\ln \tilde{I}_m(s) = \ln m + (m-1)\ln\left(1 - \frac{b}{\sqrt{m}} + \frac{s}{m^{3/4}}\right) + \frac{1}{2}\ln(2\pi) \tag{71}$$
$$- \frac{1}{2}\ln\left(1 - \left(1 - \frac{b}{\sqrt{m}} + \frac{s}{m^{3/4}}\right)\right) + \frac{\pi^2}{12}$$
$$- \frac{b^2}{1 - (1 - b/\sqrt{m} + s/m^{3/4})}.$$

We estimate the terms on the right-hand side of (71). For the second term we have

$$\ln\left(1 - \frac{b}{\sqrt{m}} + \frac{s}{m^{3/4}}\right) \tag{72}$$
$$= -\frac{b}{\sqrt{m}} + \frac{s}{m^{3/4}} - \frac{1}{2}\left(\frac{b}{\sqrt{m}} - \frac{s}{m^{3/4}}\right)^2 + O(m^{-3/2})$$
$$= -\frac{b}{\sqrt{m}} + \frac{s}{m^{3/4}} - \frac{b^2}{2m} + O(m^{-7/6}).$$

The fourth term is

$$\ln\left(\frac{b}{\sqrt{m}} - \frac{s}{m^{3/4}}\right) = \ln\frac{b}{\sqrt{m}} + \ln\left(1 - \frac{s}{bm^{1/4}}\right) \tag{73}$$
$$= -\frac{1}{2}\ln m + \ln b + O(m^{-1/6}).$$

The last term is

$$\frac{b^2}{b/\sqrt{m} - s/m^{3/4}} = \frac{b\sqrt{m}}{1 - s/(bm^{1/4})} \tag{74}$$
$$= b\sqrt{m}\left(1 + \frac{s}{bm^{1/4}} + \frac{s^2}{b^2\sqrt{m}} + \frac{s^3}{b^3 m^{3/4}} + O(m^{-2/3})\right)$$
$$= b\sqrt{m} + sm^{1/4} + \frac{s^2}{b} + \frac{s^3}{b^2 m^{1/4}} + O(m^{-1/6}).$$

Altogether, by (71)–(74),

$$\ln \tilde{I}_m(s) \tag{75}$$
$$= \ln m + (m-1)\left(-\frac{b}{\sqrt{m}} + \frac{s}{m^{3/4}} - \frac{b^2}{2m} + O(m^{-7/6})\right) + \frac{1}{2}\ln(2\pi)$$
$$- \frac{1}{2}\left(-\frac{1}{2}\ln m + \ln b + O(m^{-1/6})\right) + \frac{\pi^2}{12}$$
$$- \left(b\sqrt{m} + sm^{1/4} + \frac{s^2}{b} + \frac{s^3}{b^2 m^{1/4}} + O(m^{-1/6})\right)$$
$$= -2b\sqrt{m} + \frac{5}{4}\ln m + \frac{1}{4}\ln 24 - \frac{s^2}{b} - \frac{s^3}{b^2 m^{1/4}} + O(m^{-1/6}).$$

By (70) and (75),

$$\beta_3(m) = \int_{-m^{1/12}}^{m^{1/12}} m^{-3/4} e^{\ln \tilde{I}_m(s)} \cdot (1 - o(1))\, ds$$
$$= \sqrt{m} e^{-2b\sqrt{m}} 24^{1/4} \int_{-m^{1/12}}^{m^{1/12}} e^{-\frac{s^2}{b} - \frac{s^3}{b^2 m^{1/4}}} \cdot (1 - o(1))\, ds.$$

Now,

$$\int_{-m^{1/12}}^{m^{1/12}} e^{-\frac{s^2}{b} - \frac{s^3}{b^2 m^{1/4}}}\, ds \xrightarrow[m\to\infty]{} \int_{-\infty}^{\infty} e^{-s^2/b}\, ds = \sqrt{b\pi} = \frac{\pi}{6^{1/4}}.$$

Thus,

$$\beta_3(m) = \sqrt{m}\, e^{-2b\sqrt{m}} 24^{1/4} \cdot \frac{\pi}{6^{1/4}} \cdot (1 + o(1)) \tag{76}$$
$$= \pi\sqrt{2m}\, e^{-2b\sqrt{m}} \cdot (1 + o(1)).$$

We bound $\beta_4(m)$ in the same way as we have bounded $\beta_2(m)$, to obtain

$$\beta_4(m) \le e^{-2b\sqrt{m} - \frac{m^{1/6}}{b} + o(m^{1/6})} = o(e^{-2b\sqrt{m}}). \tag{77}$$

For $\beta_5(m)$, similarly to (61), we have

$$\begin{aligned}\beta_5(m) &= \int\limits_{1-c_2/\sqrt{m}}^{1} mx^{m-1}\frac{f(x)}{1-x^m}\,dx\\ &= \int\limits_{1-c_2/\sqrt{m}}^{1} \frac{mx^{m-1}}{1-x^m}\sqrt{\frac{2\pi}{1-x}}\,e^{-\frac{\pi^2}{6(1-x)}+\frac{\pi^2}{12}}\cdot(1+o(1))\,dx\\ &\le \int\limits_{1-c_2/\sqrt{m}}^{1} \frac{m}{1-x}\sqrt{\frac{2\pi}{1-x}}\,e^{-\frac{\pi^2/12}{1-x}}\cdot(1+o(1))\,dx.\end{aligned}$$

Substituting $t = 1/(1-x)$ and recalling that $c_2 < \pi/(6\sqrt{6})$, we obtain

$$\begin{aligned}\beta_5(m) &\le \int\limits_{\sqrt{m}/c_2}^{\infty} \frac{m\sqrt{2\pi}\,t^{3/2}}{t^2}e^{-\frac{\pi^2}{12}t}\cdot(1+o(1))\,dt\\ &= m\sqrt{2\pi}\int\limits_{\sqrt{m}/c_2}^{\infty}\frac{e^{-\frac{\pi^2}{12}t}}{\sqrt{t}}\cdot(1+o(1))\,dt\\ &= m\sqrt{2\pi}\int\limits_{\sqrt{m}/c_2}^{\infty}\frac{e^{-\frac{\pi^2}{12}t}}{\sqrt{\sqrt{m}/c_2}}\cdot(1+o(1))\,dt\\ &\le O(m^{3/4})\cdot\frac{12}{\pi^2}e^{-\frac{\pi^2}{12c_2}\sqrt{m}}\cdot(1+o(1))\\ &\le O(m^{3/4})\cdot\frac{12}{\pi^2}e^{-\frac{3\pi}{\sqrt{6}}\sqrt{m}}\cdot(1+o(1)).\end{aligned}\tag{78}$$

Altogether, by (56), (69) and (76)–(78), we obtain our claim. ∎

Proof of Proposition 10. (a) Denote $p_{m,n} = P(W_{n-m} = W_{(n)})$. Similarly to the proof of Theorem 9,

$$\begin{aligned}p_{m,n} &= P(W_{n-m} = W_{(n)})\\ &= P(X_{n-m} = \max\{X_0,\dots,X_{n-m-1},X_{n-m+1},\dots,X_{n-1}\})\\ &= \int\limits_0^{\infty} me^{-my}\frac{1}{1-e^{-my}}\prod_{j=1}^{n}(1-e^{-jy})\,dy.\end{aligned}$$

Similarly to the proof of Proposition 7, we shall be interested in the difference between the distribution functions $F_{W_{(n)}/n}(t)$ and $F_M(t)$, which we denote by $r(t)$. We have

$$p_{m,n} = \int_0^\infty me^{-my}\frac{1}{1-e^{-my}}(F_M(y)+r(y))\,dy$$
$$= P(X_m = M) + \int_0^\infty me^{-my}\frac{1}{1-e^{-my}}r(y)\,dy.$$

Denote

$$\varepsilon_m = \int_0^\infty me^{-my}\frac{1}{1-e^{-my}}r(y)\,dy.$$

By (51), for $m > 1$,

$$\varepsilon_m \le \int_0^\infty me^{-my}\frac{1}{1-e^{-my}}\prod_{j=2}^{n}(1-e^{-jy})\cdot e^{-(n+1)y}\,dy$$
$$= \int_0^\infty me^{-my}\prod_{j=2,\,j\neq m}^{n}(1-e^{-jy})\cdot e^{-(n+1)y}\,dy$$
$$\le \int_0^\infty me^{-(m+n+1)y}\,dy = \frac{m}{m+n+1}.$$

For $m = 1$,

$$\varepsilon_1 \le \int_0^\infty e^{-y}\frac{1}{1-e^{-y}}\prod_{j=2}^{n}(1-e^{-jy})\cdot e^{-(n+1)y}\,dy$$
$$= \int_0^\infty e^{-y}(1+e^{y})\prod_{j=3}^{n}(1-e^{-jy})\cdot e^{-(n+1)y}\,dy$$
$$\le \int_0^\infty (e^{-(n+2)y}+e^{-(n+1)y})\,dy \le \frac{2}{n+1}.$$

(b) Denote

$$p'_{m,n} = P(D_{n-m} > \max\{D_0,\dots,D_{n-m-1},D_{n-m+1},\dots,D_{n-1}\}).$$

We have

$$p'_{m,n} = \sum_{j=1}^{\infty} P(D_{n-m} = j)$$
$$\cdot P(\max\{D_0,\dots,D_{n-m-1},D_{n-m+1},\dots,D_{n-1}\} < j)$$
$$= \sum_{j=1}^{\infty}\Big(1-\frac{m}{n}\Big)^{j-1}\frac{m}{n}\prod_{i=0,\,i\neq n-m}^{n-1} P(D_i < j)$$

$$= \sum_{j=1}^{\infty}\left(1-\frac{m}{n}\right)^{j-1}\frac{m}{n}\prod_{i=0,\, i\neq n-m}^{n-1}\left(1-\left(1-\frac{n-i}{n}\right)^{j-1}\right)$$

$$= \sum_{j=1}^{\infty}\left(1-\frac{m}{n}\right)^{j-1}\frac{m}{n}\prod_{i=1,\, i\neq m}^{n}\left(1-\left(1-\frac{i}{n}\right)^{j-1}\right)$$

$$= \frac{1}{n}\sum_{j=0}^{\infty} m\left(1-\frac{m}{n}\right)^{j}\prod_{i=1,\, i\neq m}^{n}\left(1-\left(1-\frac{i}{n}\right)^{j}\right)$$

$$= \lim_{\gamma\to\infty}\frac{1}{n}\sum_{j=0}^{\gamma n} m\Big(1-(m/n)\Big)^{j}\prod_{i=1,\, i\neq m}^{n}\left(1-\left(1-\frac{i}{n}\right)^{j}\right).$$

Now,

$$m\left(1-\frac{m}{n}\right)^{j}\prod_{i=1,\, i\neq m}^{n}\left(1-\left(1-\frac{i}{n}\right)^{j}\right) - me^{-m\cdot\frac{j}{n}}\prod_{i=1,\, i\neq m}^{n}\left(1-e^{-i\cdot\frac{j}{n}}\right)\xrightarrow[n\to\infty]{} 0,$$

where the convergence is uniform over $j\in[0,\gamma n]$. It follows that

$$\frac{1}{n}\sum_{j=0}^{\gamma n} m\left(1-\frac{m}{n}\right)^{j}\prod_{i=1,\, i\neq m}^{n}\left(1-\left(1-\frac{i}{n}\right)^{j}\right)\xrightarrow[n\to\infty]{}\int_0^{\gamma} me^{-mx}\prod_{i=1,\, i\neq m}^{\infty}\left(1-e^{-ix}\right)dx.$$

Therefore

$$\lim_{n\to\infty} p'_{m,n} = \int_0^{\infty} me^{-mx}\prod_{i=1,\, i\neq m}^{\infty}(1-e^{-ix})dx = P(X_m = M). \ \blacksquare$$

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