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General Fourier coefficients and problems of summability almost everywhere

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Abstract. S. Banach proved that for any L_2 function, there exists an orthonormal system such that the Fourier series of this function is not Cesàro summable a.e. In this paper, we present sufficient conditions that must be satisfied by functions of an orthonormal system so that the Fourier coefficients of any function of bounded variation satisfy the conditions of the Menshov–Kaczmarz theorem. The results obtained are the best possible in a certain sense. We also prove that any orthonormal system contains a subsystem for which the Fourier series of functions of bounded variation are Cesàro summable a.e. These results generalize those of L. Gogoladze and V. Tsagareishvili [Studia Sci. Math. Hungar. 52 (2015), 511–536].

1. Auxiliary notations and theorems. We denote by V the class of functions of bounded variation on $[0,1]$ and by $V(f)$ the total variation of a function f on $[0,1]$. Let A be the space of absolutely continuous functions on $[0,1]$ with the norm

$$\|f\|_A = \|f\|_C + \int_0^1 \left| \frac{d}{dx} f(x) \right| dx.$$

Let us also adopt a few more definitions:

If (φ_n) is an ONS on $[0,1]$, then the numbers

$$C_n(f) = \int_0^1 f(x)\varphi_n(x)\, dx$$

are the *Fourier coefficients* of $f \in L_2$.

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Suppose that

$$P_n(a,x)=\sum_{k=1}^{n} a_k \log(\log(k+2))\varphi_k(x), \tag{1.1}$$

where $a=(a_n)\in \ell_2$ is a sequence of real numbers, and set

$$H_n(a)=\max_{1\le i\le n}\Big|\int_0^{i/n} P_n(a,x)\,dx\Big|. \tag{1.2}$$

Definition 1.1. The series

$$\sum_{n=1}^{\infty} a_n\varphi_n(x)$$

is *summable by the Cesàro* (C,α) *method* to a function f if a.e. on $[0,1]$,

$$\lim_{m\to\infty}\sigma_m^{\alpha}(a,x)=f(x),$$

where

$$\sigma_m^{\alpha}(a,x)=\frac{1}{A_m^{\alpha}}\sum_{n=0}^{m} A_{m-n}^{\alpha-1}S_n(a,x),$$

$$S_n(a,x)=\sum_{k=1}^{n} a_k\varphi_k(x),$$

$$A_n^{\alpha}=\frac{(1+\alpha)(2+\alpha)\cdots(n+\alpha)}{n!}.$$

Definition 1.2. Write $(d_n)\in B$ if

$$d_n=O(1)\frac{\sqrt{n}}{\log(\log(n+2))}.$$

Lemma 1.3. *For any* $i=1,\dots,n$, *and a sequence* $(a_n)\in\ell_2$,

$$\int_{(i-1)/n}^{i/n} |P_n(a,x)|\,dx=O(1).$$

Proof. Using the Hölder inequality we get $(i=1,\dots,n)$

$$\begin{aligned}\int_{(i-1)/n}^{i/n} |P_n(a,x)|\,dx &\le \frac{1}{\sqrt{n}}\Big(\int_0^1\Big(\sum_{k=1}^{n} a_k\log(\log(k+2))\varphi_k(x)\Big)^2dx\Big)^{1/2}\\ &=\frac{1}{\sqrt{n}}\Big(\sum_{k=1}^{n} a_k^2(\log(\log(k+2)))^2\Big)^{1/2}\\ &\le \frac{\log(\log(n+2))}{\sqrt{n}}\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2}=O(1). \ \blacksquare\end{aligned}$$

LEMMA 1.4. *Suppose* (φ_n) *is an ONS on* $[0,1]$ *and* (a_n) *is a sequence of numbers. Then*

$$\Big|\int_0^1 \sum_{k=1}^n a_k \log(\log(k+2))\varphi_k(x)\,dx\Big| \le \Big(\sum_{k=1}^n a_k^2\Big)^{1/2}\Big(\sum_{k=1}^n \Big(\int_0^1 \varphi_k(x)\,dx\Big)^2 (\log(\log(k+2)))^2\Big)^{1/2}.$$

Proof. According to the Cauchy inequality, we have

$$\begin{aligned}\Big|\int_0^1 \sum_{k=1}^n a_k \log(\log(k+2))\varphi_k(x)\,dx\Big| &= \Big|\sum_{k=1}^n a_k \log(\log(k+2)) \int_0^1 \varphi_k(x)\,dx\Big| \\ &\le \Big(\sum_{k=1}^n a_k^2\Big)^{1/2}\Big(\sum_{k=1}^n \Big(\int_0^1 \varphi_k(x)\,dx\Big)^2 (\log(\log(k+2)))^2\Big)^{1/2}. \ \blacksquare\end{aligned}$$

MENSHOV–KACZMARZ THEOREM ([1, Ch. II, #8, p. 131]). *If* (φ_n) *is an ONS on* $[0,1]$ *and*

$$\sum_{n=1}^{\infty} a_n^2 (\log(\log(n+2)))^2 < +\infty,$$

then the series $\sum_{n=1}^{\infty} a_n\varphi_n(x)$ *is* $(C,\alpha>0)$ *summable a.e. on* $[0,1]$.

BANACH THEOREM ([2]). *Let* $f \in L_2[0,1]$ $(f \nsim 0)$. *Then there exists an ONS* (φ_n) *on* $[0,1]$ *such that*

$$\varlimsup_{m\to\infty} |\sigma_m(f,x)| = +\infty$$

a.e. on $[0,1]$, *where* $\sigma_m(f,x) = \sigma_m(a,x)$ *and* $a_k = C_k(f)$, $k=1,2,\dots,$ *are the Fourier coefficients of* f.

2. Main problem. It is evident from the Banach theorem that for general ONS even the Fourier series of the function $f(x)=1$ is not summable. Despite this, the Menshov–Kaczmarz theorem gives us conditions that must be satisfied by the Fourier coefficients for the Fourier Series to be $(C,\alpha>0)$ summable. We formulate the problem like this: since even the best differential properties of functions do not guarantee that their Fourier series are summable, we look for conditions for the ONS functions that will ensure that the Fourier coefficients of any function of bounded variation will satisfy the condition of the Menshov–Kaczmarz theorem. Similar problems are explored in [4, 6, 7]. In our paper, we find the aforementioned conditions and generalize the following two theorems (see [5]).

Theorem A. *Let* (φ_n) *be an ONS on* $[0,1]$ *and suppose* $\int_0^1 \varphi_n(x)\,dx = 0$ $(n = 1, 2, \dots)$. *If for any sequence* $(a_n) \in \ell_2$,

$$B_n(a) = O(e_n(a)),$$

where

$$B_n(a) = \max_{1\le i\le n} \Big| \int_0^{i/n} M_n(a,x)\,dx \Big|,$$

$$M_n(a,x) = \sum_{k=1}^{n} a_k (\log(\log(k+2)))^2 \varphi_k(x),$$

$$e_n(a) = \Big(\sum_{k=1}^{n} a_k^2 (\log(\log(k+2)))^2\Big)^{1/2},$$

then the Fourier coefficients of any function $f \in V$ *satisfy*

$$\sum_{n=1}^{\infty} C_n^2(f)(\log(\log(n+2)))^2 < +\infty.$$

Theorem B. *Suppose that for some sequence* $(b_n) \in \ell_2$,

$$\overline{\lim_{n\to\infty}} \frac{B_n(b)}{e_n(b)} = +\infty.$$

Then there exists a function $g \in A$ *such that*

$$\sum_{n=1}^{\infty} C_n^2(g)(\log(\log(n+2)))^2 = +\infty.$$

In this paper, we generalize Theorems A and B.

3. Main results

Theorem 3.1. *Let* (φ_n) *be an ONS on* $[0,1]$ *and*

$$\sum_{n=1}^{\infty} \Big(\int_0^1 \varphi_n(x)\,dx\Big)^2 (\log(\log(n+2)))^2 < +\infty.$$

If for any sequence $(a_n) \in \ell_2$,

$$H_n(a) = O(1), \tag{3.1}$$

then the Fourier coefficients of any function $f \in V$ *satisfy*

$$\sum_{n=1}^{\infty} C_n^2(f)(\log(\log(n+2)))^2 < +\infty.$$

Proof. If $(a_n) \in \ell_2$, then (see (1.1))

$$\sum_{k=1}^{n} C_k(f)\log(\log(k+2))a_k = \int_0^1 f(x)\sum_{k=1}^{n} a_k \log(\log(k+2))\varphi_k(x)\,dx = \int_0^1 f(x)P_n(a,x)\,dx. \tag{3.2}$$

We have (see [7])

$$\int_0^1 f(x)F(x)\,dx = \sum_{i=1}^{n-1}\left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right)\int_0^{i/n} F(x)\,dx + \sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left(f(x) - f\left(\frac{i}{n}\right)\right)F(x)\,dx + f(1)\int_0^1 F(x)\,dx, \tag{3.3}$$

where $f, F \in L_2$ and f takes finite values at every point of $[0,1]$.

In (3.3) suppose that $f \in V$ and $F(x) = P_n(a,x)$. Then

$$\int_0^1 f(x)P_n(a,x)\,dx = \sum_{i=1}^{n-1}\left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right)\int_0^{i/n} P_n(a,x)\,dx + \sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left(f(x) - f\left(\frac{i}{n}\right)\right)P_n(a,x)\,dx + f(1)\int_0^1 P_n(a,x)\,dx = I_1 + I_2 + I_3. \tag{3.4}$$

Now let us evaluate I_1, I_2 and I_3.

Because $f \in V$ and by (3.1), we have (see (1.2))

$$|I_1| \le \sum_{i=1}^{n-1}\left|f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right|\left|\int_0^{i/n} P_n(a,x)\,dx\right| \le V(f)\max_{1\le i\le n}\left|\int_0^{i/n} P_n(a,x)\,dx\right| = O(1)H_n(a) = O(1). \tag{3.5}$$

Next, according to Lemma 1.3,

$$|I_2| = \left|\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left(f(x) - f\left(\frac{i}{n}\right)\right)P_n(a,x)\,dx\right| \le \sum_{i=1}^{n}\sup_{x\in[(i-1)/n,i/n]}\left|f(x) - f\left(\frac{i}{n}\right)\right|\int_{(i-1)/n}^{i/n}|P_n(a,x)|\,dx \le V(f)O(1) = O(1). \tag{3.6}$$

Using Lemma 1.4 we get

$$\begin{aligned}|I_3| &= |f(1)|\Big|\int_0^1 P_n(a,x)\,dx\Big| \\ &= O(1)\Big(\sum_{k=1}^n a_k^2\Big)^{1/2}\Big(\sum_{k=1}^n \Big(\int_0^1 \varphi_k(x)\,dx\Big)^2(\log(\log(k+2)))^2\Big)^{1/2} \\ &= O(1).\end{aligned}$$

Taking into account the last equality in (3.4), along with (3.5) and (3.6), we conclude

$$\Big|\int_0^1 f(x)P_n(a,x)\,dx\Big| = O(1).$$

Thus from (3.2) we get

$$\Big|\sum_{k=1}^n C_k(f)\log(\log(k+2))a_k\Big| = O(1),$$

from which we conclude that for any $(a_n)\in \ell_2$,

$$\sum_{k=1}^\infty C_k(f)\log(\log(k+2))a_k.$$

Consequently, from the known theorem (see [8, Ch. II, p. 40]) we find that $(C_k(f)\log(\log(k+2)))\in \ell_2$. So, for any $f\in V$,

$$\sum_{n=1}^\infty C_n^2(f)(\log(\log(n+2)))^2 < +\infty. \ \blacksquare$$

THEOREM 3.2. *Let* (φ_n) *be an ONS on* $[0,1]$ *and suppose*

$$\sum_{k=1}^\infty \Big(\int_0^1 \varphi_k(x)\,dx\Big)^2(\log(\log(k+2)))^2 < +\infty.$$

If for any sequence $(a_n)\in \ell_2$,

$$H_n(a) = O(1),$$

then for any $f\in V$ *the series*

$$\sum_{k=1}^\infty C_k(f)\varphi_k(x)$$

is $(C,\alpha>0)$ *summable a.e. on* $[0,1]$.

Theorem 3.2 follows from Theorem 3.1 and the Menshov–Kaczmarz theorem.

THEOREM 3.3. *Let* (φ_n) *be an ONS on* $[0,1]$. *If for some* $(b_n) \in \ell_2$,

$$\overline{\lim_{n\to\infty}} H_n(b) = +\infty,$$

then there exists a function $h \in A$ *such that*

$$\sum_{n=1}^{\infty} C_n^2(h)(\log(\log(n+2)))^2 = +\infty.$$

Proof. Suppose

$$H_n(b) = \max_{1\le i\le n}\Big|\int_0^{i/n} P_n(b,x)\,dx\Big| = \Big|\int_0^{i_n/n} P_n(b,x)\,dx\Big|, \tag{3.7}$$

where $1 \le i_n \le n$.

Let us introduce the sequence of functions

$$f_n(x) = \begin{cases} 0 & \text{when } x \in [0, i_n/n], \\ 1 & \text{when } x \in [(i_n+1)/n, 1], \\ \text{linear and continuous} & \text{when } x \in [i_n/n, (i_n+1)/n]. \end{cases} \tag{3.8}$$

In (3.2) assume that $f = f_n(x)$ and $F(x) = P_n(b,x)$. We obtain

$$\begin{aligned} \int_0^1 f_n(x)P_n(b,x)\,dx &= \sum_{i=1}^{n-1}\Big(f_n\Big(\frac{i}{n}\Big) - f_n\Big(\frac{i+1}{n}\Big)\Big)\int_0^{i/n} P_n(b,x)\,dx \\ &\quad + \sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\Big(f_n(x) - f_n\Big(\frac{i}{n}\Big)\Big)P_n(b,x)\,dx \\ &\quad + f_n(1)\int_0^1 P_n(b,x)\,dx \\ &= J_1 + J_2 + J_3. \end{aligned} \tag{3.9}$$

Using (3.7) and (3.8) we get

$$\begin{aligned} |J_1| &= \Big|\sum_{i=1}^{n-1}\Big(f_n\Big(\frac{i}{n}\Big) - f_n\Big(\frac{i+1}{n}\Big)\Big)\int_0^{i/n} P_n(b,x)\,dx\Big| \\ &= \Big|\int_0^{i_n/n} P_n(b,x)\,dx\Big| = H_n(b). \end{aligned} \tag{3.10}$$

On the other hand, by virtue of (3.8) and Lemma 1.3,

$$|J_2| = \Big|\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\Big(f_n(x)-f_n\Big(\frac{i}{n}\Big)\Big)P_n(b,x)\,dx\Big| \le \int_{i_n/n}^{(i_n+1)/n}|P_n(b,x)|\,dx = O(1). \tag{3.11}$$

Next, using Lemma 1.4,

$$|J_3| = |f_n(1)|\Big|\int_0^1 P_n(b,x)\,dx\Big| = O(1)\Big(\sum_{k=1}^{n} b_k^2\Big)^{1/2}\Big(\sum_{k=1}^{n}\Big(\int_0^1\varphi_k(x)\,dx\Big)^2(\log(\log(k+2)))^2\Big)^{1/2}. \tag{3.12}$$

Now if

$$\sum_{k=1}^{\infty}\Big(\int_0^1\varphi_k(x)\,dx\Big)^2(\log(\log(k+2)))^2 = +\infty,$$

then Theorem 3.3 is proved, because $\int_0^1\varphi_n(x)\,dx$ are the Fourier coefficients of the function $f(x)\equiv 1$.

In the case when

$$\sum_{k=1}^{\infty}\Big(\int_0^1\varphi_k(x)\,dx\Big)^2(\log(\log(k+2)))^2 < +\infty,$$

since $(b_n)\in\ell_2$, we obtain

$$|J_3| = O(1).$$

Taking this result into account in (3.9), along with (3.10) and (3.12), we obtain

$$\Big|\int_0^1 f_n(x)P_n(b,x)\,dx\Big| \ge H_n(b) - O(1).$$

Hence, according to the assumption of Theorem 3.3, we have

$$\varlimsup_{n\to\infty}\Big|\int_0^1 f_n(x)P_n(b,x)\,dx\Big| = +\infty. \tag{3.13}$$

Now consider the following sequence of continuous linear functionals on A:

$$U_n(f) = \int_0^1 f(x)P_n(b,x)\,dx.$$

According to (3.13),

$$\overline{\lim_{n\to\infty}} |U_n(f_n)| = +\infty. \tag{3.14}$$

Since (see (3.8))

$$\|f_n\|_A = \|f_n\|_C + \int_0^1 |f_n'(x)|\,dx \le 2,$$

by the Banach–Steinhaus theorem (see (3.14)) there exists a function $h \in A$ such that

$$\overline{\lim_{n\to\infty}} |U_n(h)| = +\infty.$$

Thus

$$\overline{\lim_{n\to\infty}} \Big|\int_0^1 h(x)P_n(b,x)\,dx\Big| = +\infty. \tag{3.15}$$

Finally, since $(b_n) \in \ell_2$, using the Cauchy inequality we obtain

$$\begin{aligned}\Big|\int_0^1 h(x)P_n(b,x)\,dx\Big| &= \Big|\sum_{k=1}^n b_k \log(\log(k+2)) \int_0^1 h(x)\varphi_k(x)\,dx\Big| \\ &= \Big|\sum_{k=1}^n b_k \log(\log(k+2)) C_k(h)\Big| \\ &\le \Big(\sum_{k=1}^n b_k^2\Big)^{1/2} \Big(\sum_{k=1}^n C_k^2(h)(\log(\log(k+2)))^2\Big)^{1/2} \\ &= O(1)\Big(\sum_{k=1}^n C_k^2(h)(\log(\log(k+2)))^2\Big)^{1/2}.\end{aligned}$$

Hence, according to (3.15), we get

$$\lim_{n\to\infty} \sum_{k=1}^n C_k^2(h)(\log(\log(k+2)))^2 = \lim_{n\to\infty} \Big|\int_0^1 h(x)P_n(b,x)\,dx\Big| = +\infty. \quad \blacksquare$$

THEOREM 3.4. *Any ONS (φ_n) on $[0,1]$ contains a subsystem (φ_{n_k}) for which any sequence $(d_n) \in B$ and any $f \in V$ satisfy the condition*

$$\sum_{k=1}^\infty d_k^2 C_{n_k}^2(f)(\log(\log(k+2)))^2 < +\infty. \tag{3.16}$$

Proof. Assume that the system (φ_n) is complete. Then by the Parseval equality for any $x \in [0,1]$ we have

$$\sum_{n=1}^\infty \Big(\int_0^x \varphi_n(u)\,du\Big)^2 = x.$$

Hence, according to the Dini Theorem, there exists an increasing sequence (n_k) of natural numbers for which, uniformly for $x \in [0,1]$,

$$\sum_{n=n_k}^{\infty} \Big(\int_0^x \varphi_n(u)\,du\Big)^2 < k^{-3}.$$

Thus, uniformly for $x \in [0,1]$,

$$\Big|\int_0^x \varphi_{n_k}(u)\,du\Big| < k^{-3/2}. \tag{3.17}$$

It is evident that (3.17) still holds even if the system is not complete.

Consider the ONS (φ_{n_k}) and let $f \in V$. Then

$$\begin{aligned}\sum_{k=1}^{m} d_k C_{n_k}(f)\log(\log(k+2))a_k \\ = \int_0^1 f(x)\sum_{k=1}^{m} d_k a_k \log(\log(k+2))\varphi_{n_k}(x)\,dx = \int_0^1 f(x)P_m(d,a,x)\,dx,\end{aligned} \tag{3.18}$$

where $(d_k) \in B$ and $(a_n) \in \ell_2$ are arbitrary and

$$P_m(d,a,x) = \sum_{k=1}^{m} d_k a_k \log(\log(k+2))\varphi_{n_k}(x).$$

If in (3.2) we suppose that $f \in V$, $F(x) = P_m(d,a,x)$ and $n = m$, we obtain

$$\begin{aligned}\int_0^1 f(x)P_m(d,a,x) &= \sum_{i=1}^{m-1}\Big(f\Big(\frac{i}{m}\Big) - f\Big(\frac{i+1}{m}\Big)\Big)\int_0^{i/m} P_m(d,a,x)\,dx \\ &\quad + \sum_{i=1}^{m}\int_{(i-1)/m}^{i/m}\Big(f(x) - f\Big(\frac{i}{m}\Big)\Big)P_m(d,a,x)\,dx \\ &\quad + f_m(1)\int_0^1 P_m(d,a,x)\,dx \\ &= S_1 + S_2 + S_3.\end{aligned} \tag{3.19}$$

Since $f \in V$ and $(d_n) \in B$, using (3.17) and the Cauchy inequality we have

$$\begin{aligned}|S_1| &= \Big|\sum_{i=1}^{m-1}\Big(f\Big(\frac{i}{m}\Big) - f\Big(\frac{i+1}{m}\Big)\Big)\int_0^{i/m} P_m(d,a,x)\,dx\Big| \\ &\le V(f)\max_{1\le i\le m}\Big|\int_0^{i/m}\sum_{k=1}^{m} d_k a_k \log(\log(k+2))\varphi_{n_k}(x)\,dx\Big|\end{aligned} \tag{3.20}$$

$$= O(1) \max_{1\le i\le m} \Big|\sum_{k=1}^{m} d_k a_k \log(\log(k+2)) \int_0^{i/m} \varphi_{n_k}(x)\,dx\Big|$$
$$= O(1) \sum_{k=1}^{m} d_k |a_k| k^{-3/2} \log(\log(k+2))$$
$$= O(1)\Big(\sum_{k=1}^{m} a_k^2\Big)^{1/2}\Big(\sum_{k=1}^{m} d_k^2 k^{-3}(\log(\log(k+2)))^2\Big)^{1/2}$$
$$= O(1)\Big(\sum_{k=1}^{m} \frac{k\cdot M^2}{(\log(\log(k+2)))^2}\,k^{-3}(\log(\log(k+2)))^2\Big)^{1/2}$$
$$= O(1)\cdot M\cdot\Big(\sum_{k=1}^{m} k^{-2}\Big)^{1/2} = O(1).$$

Next, since $(d_n) \in B$ and $f \in V$, the Hölder inequality implies

$$|S_2| = \Big|\sum_{i=1}^{m} \int_{(i-1)/m}^{i/m} \Big(f(x) - f\Big(\frac{i}{m}\Big)\Big) P_m(d,a,x)\,dx\Big| \tag{3.21}$$
$$\le \sum_{i=1}^{m} \sup_{x\in[(i-1)/m,\,i/m]} \Big|f(x) - f\Big(\frac{i}{m}\Big)\Big| \int_{(i-1)/m}^{i/m} |P_m(d,a,x)|\,dx$$
$$\le V(f) \max_{1\le i\le m} \int_{(i-1)/m}^{i/m} |P_m(d,a,x)|\,dx$$
$$= \frac{O(1)}{\sqrt{m}}\Big(\int_0^1\Big(\sum_{k=1}^{m} d_k a_k \log(\log(k+2))\varphi_{n_k}(x)\Big)^2 dx\Big)^{1/2}$$
$$= \frac{O(1)}{\sqrt{m}}\Big(\sum_{k=1}^{m} d_k^2 a_k^2 (\log(\log(k+2)))^2\Big)^{1/2}$$
$$\le O(1)\frac{d_m \log(\log(m+2))}{\sqrt{m}}\Big(\sum_{k=1}^{m} a_k^2\Big)^{1/2}$$
$$= O(1)\cdot M = O(1).$$

Using (3.17) and the Cauchy inequality we have ($(a_n) \in \ell_2$, $(d_n) \in B$)

$$|S_3| = \Big|\int_0^1 \sum_{k=1}^{m} d_k a_k \varphi_{n_k}(x) \log(\log(k+2))\;dx\Big|$$
$$= O(1) \sum_{k=1}^{m} d_k |a_k| \Big|\int_0^1 \varphi_{n_k}(x)\,dx\Big| \log(\log(k+2))$$

$$= O(1) \sum_{k=1}^{m} \frac{|a_k|\sqrt{k}}{\log(\log(k+2))} \, k^{-3/2} \log(\log(k+2))$$
$$= O(1) \Big(\sum_{k=1}^{m} a_k^2 \Big)^{1/2} \Big(\sum_{k=1}^{m} k^{-2} \Big)^{1/2} = O(1).$$

If we use the last condition, (3.20) and (3.21) in (3.19), we conclude that for any $(a_n) \in \ell_2$,

$$\int_0^1 f(x) P_m(d, a, x) = O(1).$$

Consequently, from (3.18) we deduce that for every $(a_n) \in \ell_2$, the series

$$\sum_{k=1}^{\infty} d_k C_{n_k}(f) \log(\log(k+2)) a_k$$

is convergent, i.e., for any $f \in V$,

$$\sum_{k=1}^{\infty} d_k^2 C_{n_k}^2(f) (\log(\log(k+2)))^2 < +\infty. \ \blacksquare$$

THEOREM 3.5. *Any ONS* (φ_n) *on* $[0,1]$ *contains a subsystem* (φ_{n_k}) *such that for any sequence* $(d_n) \in B$ *and any* $f \in V$*, the series*

$$\sum_{k=1}^{\infty} d_k C_{n_k}(f) \varphi_{n_k}(x)$$

is $(C, \alpha > 0)$ *summable a.e. on* $[0,1]$.

Theorem 3.5 follows from Theorem 3.4 and the Menshov–Kaczmarz theorem.

REMARK 3.6. In case $d_n = 1$, Theorem 3.4 implies Theorem 4 of [5].

4. Problems of efficiency

THEOREM 4.1. *If the ONS* (φ_n) *is such that, uniformly for* $x \in [0,1]$,

$$\int_0^x \varphi_n(t)\, dt = O(1/n), \tag{4.1}$$

then this system satisfies the condition (3.1).

Proof. Suppose $(a_n) \in \ell_2$ is arbitrary. Taking into account (4.1) we have

$$H_n(a) = \max_{1 \le i \le n} \Big| \int_0^{i/n} P_n(a, x)\, dx \Big| = \max_{1 \le i \le n} \Big| \sum_{k=1}^{n} a_k \log(\log(k+2)) \int_0^{i/n} \varphi_k(x)\, dx \Big|$$

$$= O(1)\sum_{k=1}^{n}|a_k|\log(\log(k+2))\,\frac{1}{k}$$
$$= O(1)\Big(\sum_{k=1}^{n}a_k^2\Big)^{1/2}\Big(\sum_{k=1}^{n}(\log(\log(k+2)))^2\,\frac{1}{k^2}\Big)^{1/2} = O(1). \;\blacksquare$$

REMARK 4.2. From Theorem 4.1 we can deduce that since the trigonometric system (see [1, Ch. I]) and the Walsh system [3] satisfy the condition (4.1), they also satisfy the condition (3.1).

THEOREM 4.3. *The Haar system* (X_n) *satisfies the condition* (3.1).

Proof. According to the definition of the Haar system (see [9]), we find that if $m = 2^k + l$ $(1 \le l \le 2^k)$, then for any $x \in [0,1]$,

$$\Big|\int_0^x X_m(t)\,dt\Big| \le \begin{cases} 2^{-k/2} & \text{when } x \in ((l-1)/2^k, l/2^k), \\ 0 & \text{when } x \notin [(l-1)/2^k, l/2^k]. \end{cases}$$

Thus, if (a_m) is an arbitrary sequence of numbers, then for all $x \in [0,1]$,

$$\Big|\int_0^x \sum_{m=2^k}^{2^{k+1}-1} a_m X_m(t)\,dt\Big| \le 2^{-k/2}|a_{m(k)}|$$

with some $2^k \le m(k) < 2^{k+1}$.

Now assume that $n = 2^s - 1$ and $(a_n) \in \ell_2$ is arbitrary. If we use the last inequality along with the Cauchy inequality, we obtain $(i = 1, \dots, n)$

$$\Big|\int_0^{i/n}\sum_{k=1}^{n} a_k \log(\log(k+2))X_k(t)\,dt\Big|$$
$$= \Big|\int_0^{i/n}\sum_{m=1}^{2^s-1} a_m \log(\log(m+2))X_m(t)\,dt\Big|$$
$$= \Big|\int_0^{i/n}\sum_{k=0}^{s-1}\sum_{m=2^k}^{2^{k+1}-1} a_m \log(\log(m+2))X_m(t)\,dt\Big|$$
$$\le \sum_{k=0}^{s-1}\Big|\int_0^{i/n}\sum_{m=2^k}^{2^{k+1}-1} a_m \log(\log(m+2))X_m(t)\,dt\Big|$$
$$\le \sum_{k=0}^{s-1} 2^{-k/2}|a_{m(k)}|\log(\log(m(k))) = O(1)\sum_{k=0}^{s-1}\Big(\sum_{m=2^k}^{2^{k+1}-1} a_m^2\Big)^{1/2}\cdot k\cdot 2^{-k/2}$$
$$= O(1)\Big(\sum_{k=0}^{s-1}\sum_{m=2^k}^{2^{k+1}-1} a_m^2\Big)^{1/2}\Big(\sum_{k=0}^{s-1} k^2\cdot 2^{-k}\Big)^{1/2} = O(1).$$

An analogous inequality will hold if $n = 2^s + p$ $(1 \le p \le 2^s)$. $\blacksquare$

Theorem 4.4. *Let (φ_n) be an ONS on $[0,1]$ with*

$$\int_0^1 \varphi_n(x)\,dx = 0 \qquad (n = 1, 2, \dots).$$

Then for any sequence $(a_n) \in \ell_2$ the following two statements are equivalent:

(a) $B_n(a) = O(e_n(a))$ (*see Theorem A*), *and*
(b) $H_n(a) = O(1)$ (*see* (3.1)).

Proof. Suppose that (a) holds. Then, according to Theorem A, for any $f \in V$,

$$\sum_{n=1}^{\infty} C_n^2(f)(\log(\log(n+2)))^2 < +\infty. \tag{4.2}$$

For contradiction, suppose that (b) does not hold. Then by Theorem 3.3 there exists a function $h \in A$ such that

$$\sum_{n=1}^{\infty} C_n^2(h)(\log(\log(n+2)))^2 = +\infty,$$

which contradicts (4.2).

Similarly, when (b) holds true, so does (a). ∎

From Theorem 4.4 we conclude that Theorems 3.1 and 3.2 are more general than Theorems A and B, whereas Theorems 3.1 and A and Theorems 3.2 and B are equivalent when $\int_0^1 \varphi_n(x)\,dx = 0$ $(n = 1, 2, \dots)$. Thus in Theorems 3.1 and 3.2 we demanded weaker conditions than those of Theorems A and B:

$$\sum_{k=1}^{\infty} \Big(\int_0^1 \varphi_k(x)\,dx\Big)^2 (\log(\log(k+2)))^2 < +\infty.$$

Remark 4.5. If the condition of Theorem 3.3 is fulfilled,

$$\varlimsup_{n\to\infty} \Big|\int_0^{i_n/n} P_n(b,x)\,dx\Big| = +\infty \tag{4.3}$$

and

$$\sum_{k=1}^{\infty} \Big(\int_0^1 \varphi_k(x)\,dx\Big)^2 (\log(\log(k+2)))^2 < +\infty,$$

then $i_n \neq n-1$. Indeed, if we assume that $i_n = n-1$, we get

$$\int_0^{i_n/n} P_n(b,x)\,dx = \int_0^{1-1/n} P_n(b,x)\,dx = \int_0^1 P_n(b,x)\,dx - \int_{1-1/n}^1 P_n(b,x)\,dx. \tag{4.4}$$

According to Lemma 1.4 ($(b_n) \in \ell_2$),

$$\Big|\int_0^1 P_n(b,x)\,dx\Big| \le \Big(\sum_{k=1}^n b_k^2\Big)^{1/2}\Big(\sum_{k=1}^n\Big(\int_0^1 \varphi_k(x)\,dx\Big)^2(\log(\log(k+2)))^2\Big)^{1/2} = O(1). \tag{4.5}$$

However from Lemma 1.3 we conclude

$$\Big|\int_{1-1/n}^1 P_n(b,x)\,dx\Big| = O(1).$$

Consequently, from (4.4) and (4.5) we get

$$\Big|\int_0^{i_n/n} P_n(b,x)\,dx\Big| = O(1),$$

which contradicts (4.3).

Remark 4.6. Suppose (φ_n) is an ONS on $[0,1]$ and the functions φ_n are 1-periodic. Then if

$$\sum_{k=1}^\infty\Big(\int_0^1 \varphi_k(x)\,dx\Big)^2(\log(\log(k+2)))^2 = +\infty,$$

we deduce that Theorem 3.3 holds for the periodic function $h(x)=1$, as the numbers $\int_0^1 \varphi_n(x)\,dx$ are the Fourier coefficients of the aforementioned function. Now if

$$\sum_{k=1}^\infty\Big(\int_0^1 \varphi_k(x)\,dx\Big)^2(\log(\log(k+2)))^2 < +\infty,$$

then in the proof of Theorem 3.3 we can take the following 1-periodic function f_n:

$$f_n(x) = \begin{cases} 0 & \text{when } x\in[0,i_n/n] \text{ and } x=1,\\ 1 & \text{when } x\in[(i_n+1)/n, 1-1/n],\\ \text{linear and continuous} & \text{when } x\in[i_n/n,(i_n+1)/n]\cup[1-1/n,1]. \end{cases}$$

Using Remark 4.2 we get

$$\Big|f_n\Big(1-\frac{1}{n}\Big) - f_n(1)\Big|\Big|\int_0^{(n-1)/n} P_n(b,x)\,dx\Big| = O(1).$$

If we continue the proof analogously to that of Theorem 3.3, we come to the conclusion that there exists a 1-periodic function $h\in A$ which satisfies the condition

$$\sum_{n=1}^\infty C_n^2(h)(\log(\log(n+2)))^2 = +\infty.$$

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