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Rosenthal's space revisited

by

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Dedicated to Professor Evgeniy Semenov
on the occasion of his 80th birthday

Abstract. Let E be a rearrangement invariant (r.i.) function space on $[0,1]$, and let Z_E consist of all measurable functions f on $(0,\infty)$ such that $f^*\chi_{[0,1]} \in E$ and $f^*\chi_{[1,\infty)} \in L^2$. We reveal close connections between properties of the generalized Rosenthal space, corresponding to the space Z_E, and the behaviour of independent symmetrically distributed random variables in E. The results obtained are applied to the problem of existence of isomorphisms between r.i. spaces on $[0,1]$ and $(0,\infty)$. Exploiting particular properties of disjoint sequences, we identify a rather wide new class of r.i. spaces on $[0,1]$, "close" to L^∞, which fail to be isomorphic to r.i. spaces on $(0,\infty)$. In particular, this property is shared by the Lorentz spaces $\Lambda_2(\log^{-\alpha}(e/u))$ with $0<\alpha\le 1$.

1. Introduction. Let $p>2$. Given any sequence $w=(w_n)_{n=1}^\infty$ of positive scalars such that

$$(1)\qquad \sum_{n=1}^\infty w_n^{2p/(p-2)} = \infty \quad\text{and}\quad \lim_{n\to\infty} w_n = 0,$$

we define $X_{p,w}$ to be the space of all sequences $(a_n)_{n=1}^\infty$ of scalars which satisfy

$$\sum_{n=1}^\infty |a_n|^p < \infty \quad\text{and}\quad \sum_{n=1}^\infty |a_n|^2 w_n^2 < \infty,$$

under the norm

$$\|(a_n)_{n=1}^\infty\| := \max\{\|(a_n)_{n=1}^\infty\|_p, \|(a_n w_n)_{n=1}^\infty\|_2\},$$

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where $\|(a_n)_{n=1}^\infty\|_r = (\sum_{n=1}^\infty |a_n|^r)^{1/r}$ for $1 \le r < \infty$. Note that, up to isomorphism, the definition of the space $X_{p,w}$ does not depend on the sequence w, i.e., $X_{p,w} \approx X_{p,w'}$ as long as both w and w' satisfy (1) [27, Theorem 13]. Hence, we can denote $X_{p,w}$ simply by X_p.

The space X_p, introduced by Rosenthal in 1970 (see [27]), became a new separable $\mathcal{L}_p$-space, which was considerably different from previously known spaces of this type. On the other hand, X_p turned out to be very useful when studying the geometric structure of L^p-spaces. Specifically, X_p is isomorphic to the complemented subspace of L^p spanned by a certain sequence of independent 3-valued symmetrically distributed random variables (r.v.'s) [27, pp. 282–283]. Moreover, for each $p > 2$ and an arbitrary sequence $\{f_n\}_{n=1}^\infty \subseteq L^p[0,1]$ of mean zero independent r.v.'s, the mapping $T\colon X_p \to L^p$ defined by

$$T(a_n) := \sum_{n=1}^\infty a_n f_n$$

is an isomorphic embedding [27, Theorem 3 and p. 280].

Later on, Johnson, Maurey, Schechtman, and Tzafriri [16, p. 218] introduced the following generalized space of Rosenthal type. Let Y be an arbitrary rearrangement invariant (r.i.) space on $(0,\infty)$. Suppose that $\{A_n\}_{n=1}^\infty$ is a sequence of disjoint measurable subsets of $(0,\infty)$ of positive measure such that

$$m(A_n) \le 1, \quad m(A_n) \to 0 \ (n \to \infty), \quad \sum_{n=1}^\infty m(A_n) = \infty \tag{2}$$

(m is the Lebesgue measure). Then $\widetilde{\mathcal{U}}_Y$ is defined as a Banach space which is isomorphic to the closed linear span of the sequence $\{\chi_{A_n}\}_{n=1}^\infty$ in Y. It is worth noting that, up to isomorphism, the latter span does not depend on the particular choice of the sequence $\{A_n\}_{n=1}^\infty$ which satisfies conditions (2) [16, Lemma 8.7]. The sequence $\{\|\chi_{A_n}\|_Y^{-1}\chi_{A_n}\}_{n=1}^\infty$ is clearly equivalent to an unconditional basis in $\widetilde{\mathcal{U}}_Y$. Moreover, if the space $Y(0,1)$ is not equal to $L^\infty(0,1)$ up to an equivalent renorming, $\widetilde{\mathcal{U}}_Y$ is isomorphic to a complemented subspace of Y.

To establish a link between the concepts so far introduced, recall a further important definition from [16] (see also [23, §2f]). Given a r.i. space E on $[0,1]$, we define the r.i. space Z_E on $(0,\infty)$ to consist of all measurable functions f on $(0,\infty)$ such that

$$\|f\|_{Z_E} := \|f^*\chi_{[0,1]}\|_E + \|f^*\chi_{[1,\infty)}\|_{L^2} < \infty, \tag{3}$$

where f^* is the nonincreasing, left-continuous rearrangement of $|f|$ (observe that $\|\cdot\|_{Z_E}$ is a quasinorm which is equivalent to a norm [23, Theorem 2.f.1]).

Then, denoting $\mathcal{U}_E := \widetilde{\mathcal{U}}_{Z_E}$, it can be checked that Rosenthal's space X_p coincides, up to equivalence of norms, with the space $\mathcal{U}_{L^p[0,1]}$ (in particular, we choose $w_n = m(A_n)^{1/2-1/p}$, see details in [16, p. 221]).

The main aim of this paper is to reveal close connections between properties of $\mathcal{U}_E$ and the behaviour of independent r.v.'s in the corresponding r.i. space E.

Let E be a r.i. space on $[0,1]$. According to [17, Theorem 1], if $L^q[0,1] \subseteq E$ for some $q < \infty$, then there is a constant $C = C(q) > 0$ such that for every sequence $\{x_n\}_{n=1}^\infty$ of independent symmetrically distributed r.v.'s from E we have

$$\Big\|\sum_{n=1}^\infty x_n\Big\|_E \le C\Big\|\sum_{n=1}^\infty \overline{x}_n\Big\|_{Z_E}, \tag{4}$$

where the sequence $\{\overline{x}_n\}_{n=1}^\infty$ consists of pairwise disjoint functions defined on $(0,\infty)$ such that $\overline{x}_n$ and x_n are equimeasurable for each $n = 1, 2, \dots$ (it is worth mentioning that the opposite inequality holds in every r.i. space E). More recently, in [9] (for a simpler proof see [10, Theorem 25]), the latter result was sharpened; it was proved that inequality (4) holds in every r.i. space E that has the so-called Kruglov property (for definitions see the next section). Observe, for instance, that the exponential Orlicz space $\operatorname{Exp} L^p$, generated by an Orlicz function equivalent to the function e^{u^p} for large $u > 0$, has the Kruglov property if and only if $0 < p \le 1$ (clearly, $\operatorname{Exp} L^p$ does not contain L^q for any $q < \infty$).

In the first part of the paper we show that inequality (4) is fulfilled for the class of independent symmetrically distributed r.v.'s in a r.i. space E with the Fatou property whenever a similar estimate holds for the subspace $\mathcal{U}_E$ of Z_E. More precisely, if $\{A_n\}_{n=1}^\infty$ is a sequence of disjoint measurable subsets of $(0,\infty)$ which satisfy (2), then inequality (4) is a consequence of the following much weaker condition: there is a constant $C > 0$ such that for every set $S \subseteq \mathbb{N}$ with $\sum_{n\in S} m(A_n) \le 1$, and all $a_n \in \mathbb{R}$,

$$\Big\|\sum_{n\in S} a_n u_n\Big\|_E \le C\Big\|\sum_{n\in S} a_n \chi_{A_n}\Big\|_{Z_E}, \tag{5}$$

where u_n are independent symmetrically distributed functions, equimeasurable with the characteristic functions χ_{A_n} (see Theorem 1). Moreover, we prove in Theorem 2 that estimate (5) combined with a certain geometrical property of the subspace $[u_n]$ of a r.i. space E ensures that $E \approx Z_E$.

Next, we apply the results obtained to consider the problem of existence of isomorphisms between r.i. spaces on $[0,1]$ and $(0,\infty)$, first posed by Mityagin [24]. This and other closely related questions were intensively studied in [16] (see also [23]), by using the approach based on a construction of the stochastic integral with respect to a symmetrized Poisson process. In particular, it was shown that a r.i. space E is isomorphic to the space Z_E whenever $0 < \alpha_E \leq \beta_E < 1$, where α_E and β_E are the Boyd indices of E (see [16, Theorem 8.6] or [23, Theorem 2.f.1]). Later on, in [5], this result was improved: it turned out that nontriviality of the Boyd indices of E can be replaced with the weaker condition that both E and its Köthe dual E' have the Kruglov property.

However, there exist r.i. spaces on $[0, 1]$ which are not isomorphic to r.i. spaces on $(0, \infty)$. Roughly speaking, this property is shared by some r.i. spaces which are "very close" to the extreme r.i. spaces on $[0, 1]$, L^1 and L^∞. Assume that E is a r.i. space on $[0, 1]$ such that the only r.i. space on $(0, \infty)$ which can be isomorphic to E is the space Z_E. Then, since both spaces $\mathcal{U}_E$ and Z_E, clearly, contain sequences equivalent to the unit vector ℓ^2-basis, the fact that E does not contain such sequences clearly implies that E fails to be isomorphic to any r.i. space on $(0, \infty)$. If E is separable the same result holds also if the dual E^* does not contain sequences equivalent to the unit vector ℓ^2-basis. Indeed, if we assume that $E \approx Z_E$, then it would imply by duality that $E^* \approx (Z_E)^* = Z_{E^*}$ (see Lemma 1), which is impossible because Z_{E^*} contains sequences equivalent to the unit vector ℓ^2-basis but E^* does not. For instance, this holds for the Orlicz space L_{F_α}, $0 < \alpha < 1/2$, where $F_\alpha(u)$ is an Orlicz function equivalent to the function $u \log^\alpha u$ for large $u > 0$ [16, p. 235]. To see that, observe first that the only r.i. space on $(0, \infty)$ which can be isomorphic to L_{F_α} is the space $Z_{L_{F_\alpha}}$ (see [16, Corollary 8.15 and subsequent remarks]). Moreover, the exponential Orlicz space $(L_{F_\alpha})^* = \mathrm{Exp}L^{1/\alpha}$, $0 < \alpha < 1/2$, contains no sequences equivalent to the unit vector ℓ^2-basis (for instance, this is a consequence of Proposition 4 with its proof combined with the well-known fact that any disjoint sequence in $\mathrm{Exp}\, L^r$ contains a subsequence equivalent to the unit vector c_0-basis; see e.g. [28]).

Here, we present more nontrivial examples of r.i. spaces E of that sort, showing that even the existence of complemented subspaces isomorphic to ℓ^2 does not guarantee that $\mathcal{U}_E$ is isomorphically embedded into E. Specifically, exploiting particular properties of disjoint sequences, we identify a rather wide new class of r.i. spaces on $[0, 1]$, "close" to L^∞, which fail to be isomorphic to r.i. spaces on $(0, \infty)$ (see Theorems 3–5). Furthermore, in Corollary 2, we provide examples of Lorentz spaces $\Lambda_2(\varphi)$ containing plenty of complemented subspaces isomorphic to ℓ^2, but without subspaces isomorphic to the corresponding Rosenthal spaces and not isomorphic to r.i. spaces on $(0, \infty)$. In particular, these properties are shared by the Lorentz spaces $\Lambda_2(\log^{-\alpha}(e/u))$ with $0 < \alpha \leq 1$ (see Corollary 3).

In the concluding part of the paper, in Theorem 6, we prove a partial result related to the problem of whether the Kruglov property of a r.i. space

E is a necessary condition for the existence of an isomorphic embedding $T\colon \mathcal{U}_E \to E$. We consider the case when T sends the basis functions χ_{A_n}, $n = 1, 2, \dots$, of Z_E to some independent symmetrically distributed r.v.'s in E.

2. Preliminaries

2.1. Rearrangement invariant spaces. For a detailed account of basic properties of rearrangement invariant spaces, we refer to [11, 21, 23].

Let $I = [0,1]$ or $(0,\infty)$. A Banach lattice E on I is said to be a *rearrangement invariant* (in brief, r.i.) (or *symmetric*) space if from the conditions that functions $x(t)$ and $y(t)$ are *equimeasurable*, i.e.,

$$m\{t \in I : |x(t)| > \tau\} = m\{t \in I : |y(t)| > \tau\} \quad \text{for all } \tau > 0$$

and $y \in E$ it follows that $x \in E$ and $\|x\|_E = \|y\|_E$ (throughout, m denotes the Lebesgue measure).

In particular, every measurable function $x(t)$ on I is equimeasurable with the nonincreasing, left-continuous rearrangement of $|x(t)|$ given by

$$x^*(t) := \inf\{\tau > 0 : m(\{s \in I : |x(s)| > \tau\}) < t\}, \qquad t > 0.$$

We note that for any r.i. space E on $[0,1]$ we have $L^\infty[0,1] \subseteq E \subseteq L^1[0,1]$. Denote by E_0 the closure of $L^\infty[0,1]$ in the r.i. space E on $[0,1]$ (the *separable part* of E). The space E_0 is r.i., and it is separable if $E \neq L^\infty$. The *fundamental function* ϕ_E of a r.i. space E is defined by $\phi_E(t) := \|\chi_{[0,t]}\|_E$, $t > 0$. In what follows, χ_A is the characteristic function of a set A. The function ϕ_E is *quasi-concave*, that is, it is nonnegative and increases, $\phi_E(0) = 0$, and $\phi_E(t)/t$ decreases. Without loss of generality, we will assume that $\phi_E(1) = \|\chi_{[0,1]}\|_E = 1$ for every r.i. space E.

It is well known that the dilation operator $\sigma_\tau x(t) := x(t/\tau)\chi_{[0,\min(1,\tau)]}(t)$, $0 \le t \le 1$, is bounded on every r.i. space E on $[0,1]$ and $\|\sigma_\tau\|_{E\to E} \le \max(1,\tau)$ (see e.g. [21, Ch. II, §4.3]). The numbers α_E and β_E given by

$$\alpha_E := \lim_{\tau\to 0} \frac{\ln\|\sigma_\tau\|_{E\to E}}{\ln\tau}, \qquad \beta_E := \lim_{\tau\to\infty} \frac{\ln\|\sigma_\tau\|_{E\to E}}{\ln\tau}$$

satisfy $0 \le \alpha_E \le \beta_E \le 1$ and are called the *Boyd indices* of E.

The *Köthe dual* E' of a r.i. space E on I consists of all measurable functions y such that

$$\|y\|_{E'} := \sup\Big\{\int_I |x(t)y(t)|\,dt : x \in E,\ \|x\|_E \le 1\Big\} < \infty.$$

If E^* denotes the Banach dual of E, then $E' \subset E^*$ and $E' = E^*$ if and only if E is separable. A r.i. space E on I is said to have the *Fatou property* if whenever $\{x_n\}_{n=1}^\infty \subseteq E$ and x measurable on $[0,1]$ satisfy $x_n \to x$ a.e. on I and $\sup_{n=1,2,\dots}\|x_n\|_E < \infty$, it follows that $x \in E$ and $\|x\|_E \le \liminf_{n\to\infty}\|x_n\|_E$.

It is well known that a r.i. space E has the Fatou property if and only if the natural embedding of E into its Köthe bidual E'' is a surjective isometry (see e.g. [19, Theorem 6.1.7]).

Important examples of r.i. spaces are Orlicz spaces. Let Φ be an *Orlicz function*, i.e., a continuous increasing convex function on $[0,\infty)$ such that $\Phi(0)=0$. Then the *Orlicz space* $L_\Phi := L_\Phi(I)$ consists of all measurable functions x on I such that the Luxemburg–Nakano norm

$$\|x\|_{L_\Phi} := \inf\Big\{\lambda>0\colon \int_I \Phi(|x(t)|/\lambda)\,dt \le 1\Big\}$$

is finite (see e.g. [20]). In particular, if $\Phi(u)=u^p$, $1\le p<\infty$, then $L_\Phi = L^p$. If $\Phi(u)$ is equivalent for large $u>0$ to the function e^{u^p}, $p>0$, we obtain the exponential Orlicz space $\mathrm{Exp}\,L^p[0,1]$.

Let $1\le q<\infty$. Every increasing concave function $\varphi(t)$ on $[0,1]$ with $\varphi(0)=0$ generates the *Lorentz* space $\Lambda_q(\varphi)$ endowed with the norm

$$\|x\|_{\Lambda_q(\varphi)} := \Big(\int_0^1 x^*(t)^q\,d\varphi(t)\Big)^{1/q}.$$

2.2. The Kruglov property and comparison of sums of independent functions and their disjoint copies in r.i. spaces. Let f be a measurable function on $[0,1]$. Denote by $\pi(f)$ the random variable (briefly, r.v.) $\sum_{i=1}^N f_i$, where f_i are independent copies of f (that is, independent r.v.'s equidistributed with f) and N is a r.v. independent of the sequence $\{f_i\}$ and having the Poisson distribution with parameter 1. The following property has its origin in Kruglov's paper [22] and was actively studied and used by Braverman [12]. We say that a r.i. space E on $[0,1]$ has the *Kruglov property* if the relation $f\in E$ implies that $\pi(f)\in E$.

Roughly speaking, a r.i. space E has the Kruglov property if it is sufficiently "far away" from L^∞. In particular, if E contains L^p with some $p<\infty$, then E has the Kruglov property. However, the latter condition is not necessary; for instance, the exponential Orlicz space $\mathrm{Exp}\,L^p$ has the Kruglov property if and only if $0<p\le 1$ (see [12, §2.4], [8]), but clearly $\mathrm{Exp}\,L^p$ does not contain L^q with any $p>0$ and $1\le q<\infty$.

The Kruglov property is closely related to the famous Rosenthal inequality [27] and more generally to the problem of comparison of sums of independent functions and their disjoint copies in r.i. spaces.

Let E be a r.i. space on $[0,1]$. As already mentioned in Section 1, by [17, Theorem 1], if $L^q[0,1]\subseteq E$ for some $q<\infty$, then inequality (4) holds for some constant $C=C(q)>0$ and for each sequence $\{x_n\}_{n=1}^\infty\subset E$ of independent symmetrically distributed functions. Here, $\bar{x}_n$ are disjoint copies of x_n defined on the semi-axis $[0,\infty)$ (for instance, we may take $\bar{x}_n(t) = x_n(t-n+1)\chi_{[n-1,n)}(t)$, $n=1,2,\dots$). We will refer to such a sequence $\{\bar{x}_n\}$ as a *disjointification* of $\{x_n\}$. Using an operator approach initiated in [8] (see also [10]), Astashkin and Sukochev have shown that inequality (4) holds for a wider class of r.i. spaces with the above-defined Kruglov property.

It is easy to check that the above r.v. $\pi(f)$ is equidistributed with the sum

$$Kf(t) := \sum_{n=1}^{\infty} \sum_{i=1}^{n} f_{n,i}(t)\chi_{E_n}(t),\ 0 \leq t \leq 1,$$

where E_n are disjoint subsets of $[0,1]$, $m(E_n) = 1/(en!)$, $n = 1, 2, \dots$, and $f_{n,i}$ are functions identically distributed with f, $i = 1, \dots, n$, $n = 1, 2, \dots$, such that $f_{n,1}, \dots, f_{n,n}, \chi_{E_n}$ are independent for each positive integer n. It turns out that the above mapping K can be treated as a linear operator defined on suitable r.i. spaces (see [10, p. 1029]). Moreover, a r.i. space E on $[0,1]$ has the Kruglov property if and only if the operator K is bounded in E. For this reason, we refer to K as the *Kruglov operator*.

We will say that subsets F_n of $[0,1]$, $n = 1, 2, \dots$, are *independent* if the characteristic functions χ_{F_n}, $n = 1, 2, \dots$, are independent on $[0,1]$.

Standard Banach space notation is used throughout. In particular, $X \approx Y$, where X and Y are Banach spaces, means that X and Y are isomorphic. We will write $Y \underset{\sim}{\subsetneq} X$ if there is an isomorphic embedding $T\colon Y \to X$. The notation $f \asymp g$ will mean that there exists a constant $C > 0$ not depending on the arguments of the quantities (norms) f and g such that $C^{-1} \cdot f \leq g \leq C \cdot f$. Finally, in what follows, C, c etc. denote constants whose value may change from line to line.

3. Rosenthal's space $\mathcal{U}_E$ and comparison of sums of independent functions and their disjoint copies in r.i. spaces. Let $\{A_n\}_{n=1}^{\infty}$ be an arbitrary (fixed) sequence of disjoint measurable subsets of $(0, \infty)$ which satisfy conditions (2). Denote by u_n independent symmetrically distributed r.v.'s supported on $[0,1]$ and equimeasurable with the characteristic functions χ_{A_n}, $n = 1, 2, \dots$. As mentioned in Section 1, if a r.i. space E has the Kruglov property (see Section 2.2), then there is a constant $C > 0$ such that for any sequence $\{x_n\}_{n=1}^{\infty}$ of independent symmetrically distributed r.v.'s from E inequality (4) holds. Clearly, the above r.v.'s u_n, $n = 1, 2, \dots$, then satisfy condition (5). In this section, assuming that a r.i. space E has the Fatou property, we prove the converse nontrivial implication: (5) implies (4). Moreover, starting with this result we will show that estimate (5) combined with a geometrical property of the closed linear span $[u_n]$ in E implies that $E \approx Z_E$.

First, we consider independent r.v.'s v_n, $n = 1, 2, \dots$, which are identically distributed with χ_{A_n}, $n = 1, 2, \dots$.

Proposition 1. *Let E be a r.i. space on $[0,1]$. Suppose that there exists $C > 0$ such that for every set $S \subseteq \mathbb{N}$ such that $\sum_{n\in S} m(A_n) \leq 1$ and all*

$a_n \in \mathbb{R}$, $n \in S$, we have

$$\Big\|\sum_{n\in S} a_n v_n\Big\|_E \le C\Big\|\sum_{n\in S} a_n \chi_{A_n}\Big\|_{Z_E}. \tag{6}$$

Then the Kruglov operator K is bounded from E into E''.

Remark 1. Clearly, from $\sum_{n\in S} m(A_n) \le 1$ and definition of the norm in Z_E (see (3)) it follows that (6) can be equivalently rewritten as

$$\Big\|\sum_{n\in S} a_n v_n\Big\|_E \le C\Big\|\sum_{n\in S} a_n \chi_{A'_n}\Big\|_{E}, \tag{6$'$}$$

where the sets $A'_n \subseteq [0,1]$ are pairwise disjoint and $m(A'_n) = m(A_n)$, $n = 1,2,\dots$.

Proof of Proposition 1. According to [10, Theorem 22(i)], it suffices to prove that there is a constant $C' > 0$ such that for every sequence $\{x_n\}_{n=1}^l \subseteq E$ of independent functions with $\sum_{n=1}^l m(\{t : x_n(t) \neq 0\}) \le 1$, we have

$$\Big\|\sum_{n=1}^{l} x_n\Big\|_E \le C'\Big\|\sum_{n=1}^{l} \overline{x}_n\Big\|_{E}, \tag{7}$$

where $\{\overline{x}_n\}_{n=1}^l$ is a disjointification of $\{x_n\}_{n=1}^l$ (we may and will assume that all the functions $\overline{x}_n$ are supported on $[0,1]$). Moreover, without loss of generality, we suppose that $x_n \ge 0$, $n = 1,\dots,l$. For arbitrary $\varepsilon > 0$ and $k \in \mathbb{N}$ we set

$$G_n^k := \{t : \varepsilon(k-1) < x_n(t) \le \varepsilon k\}, \qquad F_n^k := \{t : \varepsilon(k-1) < \overline{x}_n(t) \le \varepsilon k\}.$$

Observe that, for every $n = 1,\dots,l$, the sets G_n^k, $k = 1,2,\dots$ (resp. F_n^k, $k = 1,2,\dots$, $n = 1,\dots,l$) are pairwise disjoint. Due to properties (2), for each $n = 1,\dots,l$ and all $k \in \mathbb{N}$, we can find pairwise disjoint sets $S_n^k \subseteq \mathbb{N}$ such that

$$m(G_n^k) = m(F_n^k) = \sum_{i\in S_n^k} m(A_i). \tag{8}$$

Define now the step functions

$$y_n := \sum_{k=1}^{\infty} \varepsilon k \cdot \chi_{G_n^k} \quad \text{and} \quad z_n := \sum_{k=1}^{\infty} \varepsilon k \cdot \chi_{F_n^k}, \quad n = 1,\dots,l.$$

Clearly, the functions y_n, $n = 1,\dots,l$, are independent and

$$x_n \le y_n, \quad n = 1,\dots,l. \tag{9}$$

Fix $n = 1,\dots,l$. Since the sets G_n^k, $k \in \mathbb{N}$, are pairwise disjoint, they can be represented, thanks to (8), in the form

$$G_n^k = \bigcup_{i\in S_n^k} G_n^{k,i}, \qquad k \in \mathbb{N},$$

where $G_n^{k,i} \subseteq [0,1]$ are pairwise disjoint for all $i \in S_n^k$, $k \in \mathbb{N}$, and $m(G_n^{k,i}) = m(A_i)$, $i \in S_n^k$. Furthermore, we see that

$$y_n = \sum_{k=1}^{\infty} \varepsilon k \sum_{i \in S_n^k} \chi_{G_n^{k,i}}, \qquad n = 1, \dots, l.$$

Next, denote by $v_n^{k,i}$ independent copies of the characteristic functions $\chi_{G_n^{k,i}}$, $i \in S_n^k$, $k \in \mathbb{N}$, $n = 1, \dots, l$. Then, for each $n = 1, \dots, l$, the sequence $\{\varepsilon k \cdot \chi_{G_n^{k,i}}\}_{i \in S_n^k, k \in \mathbb{N}}$ is a disjointification of $\{\varepsilon k \cdot v_n^{k,i}\}_{i \in S_n^k,\, k \in \mathbb{N}}$ (see Section 2.2). Therefore, if

$$f_n := \sum_{k=1}^{\infty} \varepsilon k \sum_{i \in S_n^k} v_n^{k,i}, \qquad n = 1, \dots, l,$$

then, by [15, Proposition 1] (see also [10, Proposition 7]), we have

$$\begin{aligned} (10) \qquad m(\{t : y_n(t) > \tau\}) &\le 2m(\{t : \sup_{k \in \mathbb{N},\, i \in S_n^k} \varepsilon k \cdot v_n^{k,i}(t) > \tau\}) \\ &\le 2m(\{t : f_n(t) > \tau\}). \end{aligned}$$

Since y_n, $n = 1, \dots, l$, (respectively, f_n, $n = 1, \dots, l$) are nonnegative independent r.v.'s, the sequence $\{y_n\}_{n=1}^l$ (resp. $\{f_n\}_{n=1}^l$) has the same distribution as $\{y_n^*(t_n)\}_{n=1}^l$ (resp. $\{f_n^*(t_n)\}_{n=1}^l$), which is defined on the probability space $([0,1]^l, \prod_{n=1}^l m_n)$ (each m_n is the Lebesgue measure on $[0,1]$). Furthermore, from (10) and the definition of the rearrangement of a measurable function it follows that

$$(11) \qquad \sigma_{1/2}(y_n^*)(t_n) = y_n^*(2t_n) \le f_n^*(t_n), \qquad 0 \le t_n \le 1/2.$$

It can easily be checked that the functions $\sigma_{1/2} y_n$, $n = 1, \dots, l$, are independent on $[0, 1/2]$. Indeed, for arbitrary intervals $I_1, \dots, I_l$ of $\mathbb{R}$ we have

$$\begin{aligned} &m(\{t \in [0,1/2] : (\sigma_{1/2} y_j)(t) \in I_j,\ j = 1, \dots, l\}) \\ &\qquad = m(\{t \in [0,1/2] : y_j(2t) \in I_j,\ j = 1, \dots, l\}) \\ &\qquad = \frac{1}{2} m(\{t \in [0,1] : y_j(t) \in I_j,\ j = 1, \dots, l\}) \\ &\qquad = \frac{1}{2} \prod_{j=1}^{l} m(\{t \in [0,1] : y_j(t) \in I_j\}) \\ &\qquad = \frac{1}{2^{l+1}} \prod_{j=1}^{l} m(\{t \in [0,1/2] : y_j(2t) \in I_j\}) \\ &\qquad = \frac{1}{2^{l+1}} \prod_{j=1}^{l} m(\{t \in [0,1/2] : (\sigma_{1/2} y_j)(t) \in I_j\}). \end{aligned}$$

Hence, from (11), we have

$$\Big\|\sigma_{1/2}\Big(\sum_{n=1}^{l} y_n\Big)\Big\|_E = \Big\|\sum_{n=1}^{l} \sigma_{1/2}(y_n)\Big\|_E = \Big\|\sum_{n=1}^{l}(\sigma_{1/2}y_n)^*(t_n)\Big\|_{E([0,1]^l)}$$
$$\le \Big\|\sum_{n=1}^{l} f_n^*(t_n)\Big\|_{E([0,1]^l)} = \Big\|\sum_{n=1}^{l} f_n\Big\|_E.$$

Since $\|\sigma_\tau\|_{E\to E} \le \max(1,\tau)$ (see Section 2.1 or [21, Ch. II, §4.3]), from the above inequality it follows that

$$(12)\qquad \Big\|\sum_{n=1}^{l} y_n\Big\|_E = \Big\|\sigma_2\Big(\sigma_{1/2}\Big(\sum_{n=1}^{l} y_n\Big)\Big)\Big\|_E$$
$$\le 2\Big\|\sigma_{1/2}\Big(\sum_{n=1}^{l} y_n\Big)\Big\|_E \le 2\Big\|\sum_{n=1}^{l} f_n\Big\|_E.$$

Therefore, combining this with (9), we have

$$(13)\qquad \Big\|\sum_{n=1}^{l} x_n\Big\|_E \le 2\,\Big\|\sum_{n=1}^{l} f_n\Big\|_E.$$

On the other hand, from (8) it follows that there are pairwise disjoint sets $F_n^{k,i} \subseteq [0,1]$ such that $m(F_n^{k,i}) = m(A_i)$, $i \in S_n^k$, $k \in \mathbb{N}$, $n = 1,\dots,l$, and

$$F_n^k = \bigcup_{i\in S_n^k} F_n^{k,i}, \qquad k \in \mathbb{N},\, n = 1,\dots,l.$$

Moreover, by the above definitions, $v_n^{k,i}$ are independent copies of the characteristic functions χ_{A_i}, $i \in S_n^k$, $k \in \mathbb{N}$, $n = 1,\dots,l$. Since the sets A_i (resp. $F_n^{k,i}$), $i \in S_n^k$, $k \in \mathbb{N}$, $n = 1,\dots,l$, are pairwise disjoint and

$$\sum_{n=1}^{l}\sum_{k=1}^{\infty}\sum_{i\in S_n^k} m(A_i) \le \sum_{n=1}^{l} m(\{t : x_n(t) \ne 0\}) \le 1,$$

by the hypothesis of the proposition (see also Remark 1) we have

$$(14)\qquad \Big\|\sum_{n=1}^{l} f_n\Big\|_E \le C\Big\|\sum_{n=1}^{l}\sum_{k=1}^{\infty} \varepsilon k \sum_{i\in S_n^k} \chi_{A_i}\Big\|_{Z_E}$$
$$= C\Big\|\sum_{n=1}^{l}\sum_{k=1}^{\infty} \varepsilon k \sum_{i\in S_n^k} \chi_{F_n^{k,i}}\Big\|_E = C\Big\|\sum_{n=1}^{l} z_n\Big\|_E,$$

where $z_n := \sum_{k=1}^{\infty} \varepsilon k \cdot \chi_{F_n^k}$, $n = 1,\dots,l$.

Further, for every $n = 1, \dots, l$, $k = 2, 3, \dots$ and all $t \in F_n^k$ we have

$$\overline{x}_n(t) > \varepsilon(k-1) \geq \tfrac{1}{2}\varepsilon k = \tfrac{1}{2} z_n(t).$$

Hence, taking into account the disjointness of the sets F_n^k, $k \in \mathbb{N}$, $n = 1, \dots, l$, we obtain

$$\Big\|\sum_{n=1}^{l} \overline{x}_n\Big\|_E \geq \frac{1}{2}\Big\|\sum_{n=1}^{l} z_n \sum_{k=2}^{\infty} \chi_{F_n^k}\Big\|_E.$$

Additionally, since the sets $F_n^1 \subseteq [0,1]$, $n = 1, \dots, l$, are pairwise disjoint, we have

$$\Big\|\sum_{n=1}^{l} z_n \chi_{F_n^1}\Big\|_E \leq \varepsilon \, \|\chi_{0,1]}\|_E = \varepsilon.$$

As a result, from inequalities (13) and (14) we get

$$\Big\|\sum_{n=1}^{l} x_n\Big\|_E \leq 2C\Big\|\sum_{n=1}^{l} z_n\Big\|_E \leq 2C\Big(\Big\|\sum_{n=1}^{l} z_n \sum_{k=2}^{\infty} \chi_{F_n^k}\Big\|_E + \Big\|\sum_{n=1}^{l} z_n \chi_{F_n^1}\Big\|_E\Big)$$

$$\leq 4C\Big(\varepsilon + \Big\|\sum_{n=1}^{l} \overline{x}_n\Big\|_E\Big).$$

Letting $\varepsilon \to 0$, we obtain (7) with $C' = 4C$. ∎

Next, we proceed with comparing $\{v_i\}$ with a sequence $\{u_i\}$ of independent symmetrically distributed r.v.'s equimeasurable with the characteristic functions χ_{A_i}, $i = 1, 2, \dots$.

PROPOSITION 2. *Let E be a r.i. space on $[0,1]$. Then, for every $S \subseteq \mathbb{N}$ such that $\sum_{i \in S} m(A_i) \leq 1$ and all $a_i \in \mathbb{R}$, $i \in S$, we have*

$$\Big\|\sum_{i \in S} a_i v_i\Big\|_E \leq 16e \cdot \Big\|\sum_{i \in S} a_i u_i\Big\|_E. \tag{15}$$

Proof. First, since u_i, $i = 1, 2, \dots$, are independent symmetrically distributed r.v.'s, the sequence $\{u_n\}_{n=1}^{\infty}$ is 1-unconditional in E (see, e.g., [12, Proposition 1.14]). Therefore, we may (and will) assume that the coefficients a_i, $i \in S$, are nonnegative.

For each $i \in S$, recalling that $m(A_i) > 0$, we denote by α_i the least root of the equation

$$2t(1-t) = \tfrac{1}{4} m(A_i).$$

Straightforward calculations show that

$$\tfrac{1}{8} m(A_i) < \alpha_i < \tfrac{1}{2} m(A_i), \quad i \in S. \tag{16}$$

Let $\{G_i, H_i\}_{i \in S}$ be a family of independent subsets of $[0,1]$ such that $m(G_i) = m(H_i) = \alpha_i$, $i \in S$. Then, clearly, $h_i := \chi_{H_i} - \chi_{G_i}$, $i \in S$, are independent symmetrically distributed r.v.'s. Moreover, since $m(\{t:|u_i(t)|=1\}) = m(A_i)$ for each $i\in S$, and, due to independence,

$$m(\{t:|h_i(t)|=1\})=2\alpha_i(1-\alpha_i)=\tfrac{1}{4}m(A_i),\qquad i\in S,$$

we have

$$m(\{t:|h_i(t)|>\tau\})\le m(\{t:|u_i(t)|>\tau\}),\quad \tau>0.$$

Hence, by the well-known Kwapień–Rychlik inequality (see e.g. [29, Ch. V, Theorem 4.4]), for all $a_i\ge 0$ and $\tau>0$ we get

$$m\Big(\Big\{t:\Big|\sum_{i\in S}a_ih_i(t)\Big|>\tau\Big\}\Big)\le 2m\Big(\Big\{t:\Big|\sum_{i\in S}a_iu_i(t)\Big|>\tau\Big\}\Big). \tag{17}$$

Next, denoting $h:=\sum_{i\in S}a_ih_i$, we represent $h=h'-h''$, where

$$h':=\sum_{i\in S}a_i\chi_{H_i},\qquad h'':=\sum_{i\in S}a_i\chi_{G_i}.$$

Since h' and h'' are independent, for each $\tau>0$ it follows that

$$\begin{aligned} m(\{t:|h(t)|>\tau\}) &\ge m(\{t:|h'(t)|>\tau\}\cap\{t:h''(t)=0\})\\ &= m(\{t:|h'(t)|>\tau\})\cdot m(\{t:h''(t)=0\}). \end{aligned}\tag{18}$$

Further, since the G_i are independent, by (16) we have

$$\begin{aligned} m(\{t:h''(t)=0\}) &\ge m\Big(\bigcap_{i\in S}([0,1]\setminus G_i)\Big)=\prod_{i\in S}(1-m(G_i))\\ &=\prod_{i\in S}(1-\alpha_i)\ge\prod_{i\in S}\big(1-\tfrac{1}{2}m(A_i)\big). \end{aligned}\tag{19}$$

Finally, from the elementary inequality

$$\log(1-x)\ge-\frac{x}{1-x},\qquad 0\le x<1,$$

and the assumption that $\sum_{i\in S}m(A_i)\le 1$ it follows that

$$\begin{aligned} \log\Big(\prod_{i\in S}\big(1-\tfrac{1}{2}m(A_i)\big)\Big) &=\sum_{i\in S}\log\big(1-\tfrac{1}{2}m(A_i)\big)\ge-\frac{1}{2}\sum_{i\in S}\frac{m(A_i)}{1-\frac{1}{2}m(A_i)}\\ &\ge-\sum_{i\in S}m(A_i)\ge-1. \end{aligned}$$

Combining the latter inequality with (18) and (19), we obtain

$$m\Big(\Big\{t:\Big|\sum_{i\in S}a_ih_i(t)\Big|>\tau\Big\}\Big)\ge\frac{1}{e}m\Big(\Big\{t:\Big|\sum_{i\in S}a_i\chi_{H_i}(t)\Big|>\tau\Big\}\Big). \tag{20}$$

On the other hand, one can easily see that, by (16), for all $i\in S$,

$$m(\{t:v_i(t)>\tau\})\le 8m(\{t:\chi_{H_i}(t)(t)>\tau\}),\qquad \tau>0.$$

Therefore, by passing to the rearrangements of r.v.'s v_i and χ_{H_i}, $i \in S$, in the same way as in the proof of Proposition 1 when showing that (10) implies (12), we deduce that for all $\tau > 0$ and $a_i \geq 0$,

$$m\Big(\Big\{t : \Big|\sum_{i\in S} a_i\chi_{H_i}(t)\Big| > \tau\Big\}\Big) \geq \frac{1}{8} m\Big(\Big\{t : \Big|\sum_{i\in S} a_i v_i(t)\Big| > \tau\Big\}\Big).$$

Combining this inequality, (17) and (20), we arrive at the estimate

$$m\Big(\Big\{t : \Big|\sum_{i\in S} a_i v_i(t)\Big| > \tau\Big\}\Big) \leq 16e \cdot m\Big(\Big\{t : \Big|\sum_{i\in S} a_i u_i(t)\Big| > \tau\Big\}\Big), \qquad \tau > 0.$$

As a result, applying [21, Ch. II, §4.3, Corollary 2], we obtain (15). ∎

Summing up, we get the first main result of the paper.

THEOREM 1. *Let E be a r.i. space on $[0,1]$. Suppose there is a constant $C > 0$ such that for every set $S \subseteq \mathbb{N}$ with $\sum_{n\in S} m(A_n) \leq 1$, and all $a_n \in \mathbb{R}$, $n \in S$, we have (5), that is,*

$$\Big\|\sum_{n\in S} a_n u_n\Big\|_E \leq C\Big\|\sum_{n\in S} a_n \chi_{A_n}\Big\|_{Z_E},$$

where u_n are independent symmetrically distributed functions, equimeasurable with χ_{A_n}. Then the Kruglov operator K is bounded from E into E''.

Therefore, if E has the Fatou property, then it has the Kruglov property and hence there is a constant $C > 0$, depending only on E, such that for every sequence $\{x_n\}_{n=1}^\infty$ of independent symmetrically distributed r.v.'s from E inequality (4) holds, that is,

$$\Big\|\sum_{n=1}^\infty x_n\Big\|_E \leq C\Big\|\sum_{n=1}^\infty \overline{x}_n\Big\|_{Z_E},$$

where $\{\overline{x}_n\}_{n=1}^\infty$ is a disjointification of $\{x_n\}_{n=1}^\infty$.

If we additionally assume that for some constant $C > 0$ and every $S \subseteq \mathbb{N}$ such that $\sum_{n\in S} m(A_n) \leq 1$ and all $a_n \in \mathbb{R}$, $n \in S$,

$$\Big\|\sum_{n\in S} a_n u_n\Big\|_{E'} \leq C\Big\|\sum_{n\in S} a_n \chi_{A_n}\Big\|_{Z_{E'}}, \tag{21}$$

then the spaces E and Z_E are isomorphic.

Proof. First, from Propositions 1, 2 and hypothesis (5) it follows that the Kruglov operator K is bounded from E into E''. If E has the Fatou property, this implies that E has the Kruglov property. Consequently, by [9] (or [10, Theorem 25]), we conclude that inequality (4) holds for every sequence $\{x_n\}_{n=1}^\infty$ of independent symmetrically distributed r.v.'s from E.

Finally, when (21) is additionally fulfilled, we find similarly that E' has the Kruglov property and hence, by [5, Theorem 2.4] (see also Section 1), the spaces E and Z_E are isomorphic. ∎

Theorem 1 asserts that $E \approx Z_E$ under some conditions related to both spaces E and E'. Next, we prove a statement, showing that the same result holds provided that, along with inequality (5), the subspace $[u_n]$ of E has a certain geometric property.

We will repeatedly use the following auxiliary result.

LEMMA 1. *For every r.i. space E on $[0,1]$, we have $(Z_E)' = Z_{E'}$. Moreover, if E has the Fatou property (resp. is separable), then so does/is Z_E.*

Proof. Since Z_E is a r.i. space on $[0,\infty)$, we have

$$\|y\|_{(Z_E)'} = \sup_{\|x\|_{Z_E}\le 1} \int_0^\infty x^*(t)y^*(t)\,dt$$

(see, for instance, [21, Ch. II, §2.2, property 14^0]). Hence, by definition of the norm in Z_E, we have

$$\begin{aligned}\|y\|_{(Z_E)'} &\asymp \sup_{\|x\|_E\le 1} \int_0^1 x^*(t)y^*(t)\,dt + \sup_{\|(x^*(k))_{k=1}^\infty\|_{l_2}\le 1} \sum_{k=1}^\infty x^*(k)y^*(k)\\ &= \|y^*\chi_{[0,1]}\|_{E'} + \Big(\sum_{k=1}^\infty y^*(k)^2\Big)^{1/2} \asymp \|y\|_{Z^2_{E'}},\end{aligned}$$

and the first assertion of the lemma follows.

Next, suppose that E has the Fatou property. Let $\{x_n\}_{n=1}^\infty \subseteq Z_E$ satisfy the conditions $0 \le x_n \uparrow x$ and $\sup_n \|x_n\|_{Z_E} < \infty$. Observe that then $x_n^* \uparrow x^*$ a.e. on $[0,1]$ (see e.g. [21, Ch. II, §2.2, property 11^0]). Therefore, by the hypothesis and the inequality

$$\max\Big\{\sup_n \|x_n^*\chi_{[0,1]}\|_E, \sup_n \|x_n^*\chi_{[1,\infty)}\|_{L^2}\Big\} \le \sup_n \|x_n\|_{Z_E} < \infty,$$

we have $x^*\chi_{[0,1]} \in E$ and $x^*\chi_{[1,\infty)} \in L^2(0,\infty)$. As a result, $x \in Z_E$ and $\|x\|_{Z_E} = \lim_{n\to\infty} \|x_n\|_{Z_E}$. This means that Z_E has the Fatou property.

It remains to prove that Z_E is separable provided E is. To this end, in view of [21, Ch. II, §4.5, Theorem 4.8], it suffices to show that each nonnegative function $x \in Z_E$ can be approximated in Z_E by its truncations, i.e., we need to deduce that $\|x - x_n\|_{Z_E} \to 0$ and $\|x - x^n\|_{Z_E} \to 0$ as $n \to \infty$, where $x_n := x\chi_{[0,n]}$ and $x^n := \min(x,n)$, $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary. Since E and $L^2(0,\infty)$ are separable r.i. spaces, there is $\delta > 0$ such that

$$\max\{\|x^*\chi_{[0,\delta]}\|_E, \|x^*\chi_{[1,1+\delta]}\|_{L^2}\} < \varepsilon. \tag{22}$$

On the other hand, taking into account that $m\{t > 0 : x(t) > \varepsilon\} < \infty$ and $\|x^*\chi_{[n,\infty)}\|_{L^2(0,\infty)} \to 0$ as $n \to \infty$, we can find a positive integer N which satisfies

$$m(\{t > N : x(t) > \varepsilon\}) < \delta, \tag{23}$$

$$\|x^* \chi_{[N,\infty)}\|_{L^2[0,\infty)} < \varepsilon. \tag{24}$$

By definition of the rearrangement of a measurable function and inequality (23), for all $n \geq N$,

$$m(\{t > 0 : (x\chi_{[n,\infty)})^*(t) > \varepsilon\}) = m(\{t > n : x(t) > \varepsilon\}) < \delta.$$

Combining this inequality with (22), we obtain

$$\begin{aligned} \|(x\chi_{[n,\infty)})^* \chi_{[0,1]}\|_E &\leq \|x^* \chi_{[0,\delta]}\|_E + \|(x\chi_{[n,\infty)})^* \chi_{[\delta,1]}\|_E \\ &\leq \varepsilon(1 + \|\chi_{[0,1]}\|_E) = 2\varepsilon \end{aligned} \tag{25}$$

(because $\|\chi_{[0,1]}\|_E = 1$; see Section 2.1). Moreover, since

$$m(\{t > 0 : x(t)\chi_{[n,\infty)}(t) > x^*(N)\}) \to 0 \quad \text{as } n \to \infty,$$

there exists a positive integer $M > N$ such that for all $n \geq M$,

$$\begin{aligned} m(\{t > 0 &: (x\chi_{[n,\infty)})^*(t) > x^*(N)\}) \\ &= m(\{t > 0 : x(t)\chi_{[n,\infty)}(t) > x^*(N)\}) < \delta. \end{aligned}$$

Hence, from (22) it follows that

$$\|(x\chi_{[n,\infty)})^* \chi_{[1,\infty)} \chi_{\{(x\chi_{[n,\infty)})^* > x^*(N)\}}\|_{L^2} \leq \|x^* \chi_{[1,1+\delta]}\|_{L^2} < \varepsilon, \quad n \geq M.$$

On the other hand, in view of (24),

$$\begin{aligned} \|(x\chi_{[n,\infty)})^* \chi_{\{(x\chi_{[n,\infty)})^* \leq x^*(N)\}}\|_{L^2} &\leq \|x^* \chi_{\{x^* \leq x^*(N)\}}\|_{L^2} \\ &\leq \|x^* \chi_{[N,\infty)}\|_{L^2} < \varepsilon, \quad n \geq M. \end{aligned}$$

Summing up the last inequalities, we find that for all $n \geq M$,

$$\begin{aligned} \|(x\chi_{[n,\infty)})^* \chi_{[1,\infty)}\|_{L^2} &\leq \|(x\chi_{[n,\infty)})^* \chi_{[1,\infty)} \chi_{\{(x\chi_{[n,\infty)})^* > x^*(N)\}}\|_{L^2} \\ &\quad + \|(x\chi_{[n,\infty)})^* \chi_{\{(x\chi_{[n,\infty)})^* \leq x^*(N)\}}\|_{L^2} \leq 2\varepsilon. \end{aligned}$$

This inequality and (25) imply that $\|x\chi_{[n,\infty)}\|_{Z_E} \leq 4\varepsilon$ for all $n \geq M$. Since $\varepsilon > 0$ is arbitrary, this yields $\|x - x_n\|_{Z_E} \to 0$ as $n \to \infty$.

Finally, we prove a similar assertion for the upper truncations x^n, $n \in \mathbb{N}$. Suppose that, as above, $\delta > 0$ satisfies condition (22). Then, if a positive integer N' is sufficiently large, we have $m(\{t > 0 : x(t) > N'\}) < \delta$. Combining this inequality with (22), for all $n \geq N'$ we get

$$\|x - x^n\|_{Z_E} = \|x\chi_{\{x \geq n\}}\|_{Z_E} \leq \|x^* \chi_{[0,\delta]}\|_E < \varepsilon,$$

whence $\|x - x^n\|_{Z_E} \to 0$ as $n \to \infty$. ■

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint measurable subsets of $(0,\infty)$ which satisfy conditions (2). Moreover, let E be a r.i. space on $[0,1]$ and ϕ_E its fundamental function. Denoting by u_n, $n = 1, 2, \ldots$, independent symmetrically distributed r.v.'s supported on $[0,1]$, which are equimeasurable with the characteristic functions χ_{A_n}, $n = 1, 2, \ldots$, we set

$$
\begin{aligned}
f_n &:= \frac{\chi_{A_n}}{\phi_E(m(A_n))}, \qquad g_n := \frac{\chi_{A_n}}{\phi_{E'}(m(A_n))}, \\
\tilde{f}_n &:= \frac{u_n}{\phi_E(m(A_n))}, \qquad \tilde{g}_n := \frac{u_n}{\phi_{E'}(m(A_n))}, \qquad n = 1, 2, \ldots .
\end{aligned} \tag{26}
$$

Since $\phi_{E'}(t) = t/\phi_E(t)$, $0 < t \le 1$ [21, Ch. II, §4.6], $\{f_n, g_n\}$ and $\{\tilde{f}_n, \tilde{g}_n\}$ are biorthogonal systems in E. Also, we denote

$$
\langle f, g\rangle := \int_0^1 f(t)g(t)\,dt, \qquad f \in E,\ g \in E'.
$$

PROPOSITION 3. *Let E be a r.i. space on $[0,1]$, and let $S \subseteq \mathbb{N}$ be such that $\sum_{i\in S} m(A_i) \le 1$. Suppose that the mapping*

$$
Pf := \sum_{n\in S} \langle f, \tilde{g}_n\rangle \tilde{f}_n \tag{27}
$$

is a bounded projection on E. Then there is a constant $C > 0$, which depends only on E and $\|P\|$, such that for all $a_n \in \mathbb{R}$,

$$
\Big\|\sum_{n\in S} a_n u_n\Big\|_{E'} \le C \Big\|\sum_{n\in S} a_n \chi_{A_n}\Big\|_{Z_{E'}}. \tag{28}
$$

Proof. First, we estimate

$$
\begin{aligned}
\Big\|\sum_{n\in S} a_n \tilde{g}_n\Big\|_{E'} &= \sup\Big\{\Big\langle \sum_{n\in S} a_n \tilde{g}_n, f\Big\rangle : \|f\|_E \le 1\Big\} \\
&= \sup\Big\{\Big\langle \sum_{n\in S} a_n \tilde{g}_n, Pf\Big\rangle : \|f\|_E \le 1\Big\} \\
&\le \sup\Big\{\Big\langle \sum_{n\in S} a_n \tilde{g}_n, Pf\Big\rangle : \|Pf\|_E \le \|P\|\Big\}.
\end{aligned}
$$

Moreover,

$$
\Big\langle \sum_{n\in S} a_n \tilde{g}_n, Pf\Big\rangle = \sum_{n\in S} a_n \langle f, \tilde{g}_n\rangle = \int_0^\infty \Big(\sum_{n\in S} a_n g_n\Big) \cdot \Big(\sum_{m\in S} \langle f, \tilde{g}_m\rangle f_m\Big)\,dt,
$$

and since f_m are disjoint copies of the functions $\tilde{f}_m$, $m \in S$, by [17, Theorem 1] there is $C' > 0$, depending only on E, such that

$$
\Big\|\sum_{m\in S} \langle f, \tilde{g}_m\rangle f_m\Big\|_{Z_E} \le C' \Big\|\sum_{m\in S} \langle f, \tilde{g}_m\rangle \tilde{f}_m\Big\|_E = C'\|Pf\|_E.
$$

Hence,

$$\Big\|\sum_{n\in S} a_n\tilde g_n\Big\|_{E'} \le \sup\Big\{\int_0^\infty\Big(\sum_{n\in S} a_n g_n\Big)\cdot\Big(\sum_{m\in S}\langle f,\tilde g_m\rangle f_m\Big)\,dt : \Big\|\sum_{m\in S}\langle f,\tilde g_m\rangle f_m\Big\|_{Z_E}\le C'\|P\|\Big\}.$$

Since $(Z_E)' = Z_{E'}$ by Lemma 1, the latter inequality shows that for all $a_n\in\mathbb{R}$,

$$\Big\|\sum_{n\in S} a_n\tilde g_n\Big\|_{E'} \le C'\|P\|\Big\|\sum_{n\in S} a_n g_n\Big\|_{Z_{E'}},$$

which is equivalent to the desired estimate (28). ■

Theorem 1 and Proposition 3 yield

THEOREM 2. *Let E be a r.i. space on $[0,1]$ with the Fatou property. Suppose that there exists $C>0$ such that for every set $S\subseteq\mathbb{N}$ with $\sum_{n\in S} m(A_n)\le 1$, and all $a_n\in\mathbb{R}$, $n\in S$, inequality* (5) *holds and the projection P corresponding to S* (*see* (26) *and* (27)) *is bounded on E. Then $E\approx Z_E$.*

4. Existence of an isomorphic embedding $T\colon \mathcal{U}_E\to E$ when the functions $T(\chi_{A_n})$, $n=1,2,\ldots$, are "almost" disjoint. As mentioned in Section 1, if a r.i. space E and its Köthe dual E' have the Kruglov property, then the spaces E and Z_E are isomorphic (see [5]). In turn, according to Theorem 1, a r.i. space E with the Fatou property has the Kruglov property whenever there is an isomorphic embedding of Rosenthal's space $\mathcal{U}_E$ into E. Moreover, in the proof of the latter result the functions $T(\chi_{A_n})\,(=u_n)$, $n=1,2,\ldots$, were independent, symmetrically distributed and equimeasurable with the characteristic functions χ_{A_n}, $n=1,2,\ldots$. A natural question appears: Let T be an isomorphic embedding of Rosenthal's space $\mathcal{U}_E$ into E. What can we say about the functions $T(\chi_{A_n})$, $n=1,2,\ldots$? Further, we consider two different cases, when these functions are "almost" disjoint and independent. As a consequence, we will obtain new examples of r.i. spaces E such that $E\not\approx Z_E$.

We begin with an auxiliary result, which was proved earlier in the separable case by Raynaud (see [26, Proposition 1]). However, for the reader's convenience we provide here a simple alternative proof of this fact. Let G denote the separable part of the exponential Orlicz space $\operatorname{Exp} L^2$ (i.e., the closure of L^∞ in $\operatorname{Exp} L^2$).

PROPOSITION 4. *Let E be a r.i. space on $[0,1]$. Suppose that there exists a sequence $\{x_n\}_{n=1}^\infty\subseteq E$ with $\|x_n\|_E\asymp\|x_n\|_{L^1}$, $n=1,2,\ldots$, which is equivalent in E to the unit vector ℓ^2-basis. Then $E\supset G$.*

Proof. Clearly, it can be assumed that $E \neq L^1$. Since $\{x_n\}_{n=1}^\infty$ is equivalent in E to the unit ℓ^2-basis, we have $x_n \to 0$ weakly in E and so $x_n \to 0$ weakly in L^1. Hence, $\{x_n\}_{n=1}^\infty$ has no convergent subsequences in L^1. Applying then the well-known result by Aldous and Fremlin [2], we select a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that for some $c > 0$ and all $a_k \in \mathbb{R}$,

$$\Big\|\sum_{k=1}^\infty a_k x_{n_k}\Big\|_{L^1} \geq c\|(a_k)\|_2.$$

Combining this inequality with the assumptions and with the embedding $E \subseteq L^1$, we conclude that the norms of E and L^1 are equivalent on the infinite-dimensional subspace $[x_{n_k}]$ in E.

In other words, the canonical embedding $I\colon E \to L^1$ is not strictly singular. Assuming that $E \not\supseteq G$, by [7, Theorem 2], we infer that this embedding is not disjointly strictly singular. This means that there is a sequence $\{h_i\}_{i=1}^\infty$ of pairwise disjoint functions from E such that the norms of E and L^1 are equivalent on the closed linear span $[h_i]$. But this is a contradiction. Indeed, if the norms of E and L^1 were equivalent on the span $[h_i]$ of the pairwise disjoint functions h_i, $i = 1, 2, \ldots$, one can easily check that there exists $\delta > 0$ such that for every $i = 1, 2, \ldots$,

$$m(\{t \in [0,1] : |h_i(t)| > \delta\|h_i\|_E\}) > \delta$$

(see also [18, Theorem 1]). Clearly, the sets

$$U_i(\delta) := \{t \in [0,1] : |h_i(t)| > \delta\|h_i\|_E\}, \qquad i = 1, 2, \ldots,$$

are pairwise disjoint and $m(U_i(\delta)) > \delta$. Hence,

$$m\Big(\bigcup_{i=1}^\infty U_i(\delta)\Big) = \sum_{i=1}^\infty m(U_i(\delta)) = \infty,$$

which is not possible because the union $\bigcup_{i=1}^\infty U_i(\delta)$ is contained in $[0,1]$ (for other proofs of this and some close results, see [25] and [4, Corollary 3]). ∎

Let us recall a classical result, known as the *Kadec–Pełczyński alternative* [18] (see also [23, Proposition 1.c.8]): If $\{y_n\}_{n=1}^\infty$ is a normalized sequence in a separable r.i. space E on $[0,1]$, then either

- the sequence $(\|y_n\|_{L_1})$ is bounded away from zero, or
- there exist a subsequence $\{y_{n_j}\}$ and a disjoint sequence $\{z_j\}$ in E such that $\|y_{n_j} - z_j\|_E \to 0$ as $j \to \infty$.

Corollary 1. *Suppose E is a separable r.i. space on $[0,1]$ such that $E \not\supset G$. If E contains a sequence $\{x_n\}_{n=1}^\infty$ equivalent in E to the unit vector ℓ^2-basis, then there is a disjoint sequence $\{z_n\}_{n=1}^\infty \subset E$ with the same property.*

Proof. By Proposition 4, we may assume that $\|x_n\|_E/\|x_n\|_{L^1} \to \infty$ as $n \to \infty$. Then, by the Kadec–Pełczyński alternative, there is a subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ such that for some disjoint sequence $\{z_j\} \subseteq E$ we have

$$\|x_{n_j} - z_j\|_E \to 0 \quad \text{as } j \to \infty.$$

Since $\{x_{n_j}\}$ is equivalent in E to the unit vector ℓ^2-basis, applying now the principle of small perturbations (see e.g. [1, Theorem 1.3.9]) we can assume that $\{z_j\}_{j=1}^{\infty}$ is equivalent in E to the ℓ^2-basis as well. ∎

It is clear that for every r.i. space E on $[0,1]$ Rosenthal's space $\mathcal{U}_E$ (as a subspace of Z_E) contains a subspace isomorphic to ℓ^2. Hence, if $\mathcal{U}_E \overset{\subseteq}{\sim} E$, the space E must share the above property. So, if a r.i. space E does not contain a subspace isomorphic to ℓ^2, then $\mathcal{U}_E$ cannot be isomorphically embedded into E, which implies that $E \not\approx Z_E$. So, if E is a separable r.i. space such that $E \not\supset G$ and it does not contain disjoint sequences equivalent to the unit vector basis of ℓ^2, then $\mathcal{U}_E \overset{\not\subseteq}{\sim} E$ (see Corollary 1). In particular, if $p > 2$, the separable part $(\operatorname{Exp} L^p)_0$ of the exponential Orlicz space $\operatorname{Exp} L^p$ has the latter properties since each disjoint sequence of this space contains a subsequence equivalent to the unit vector basis of c_0 (see, e.g., [28]). As a result, we obtain the simplest examples of r.i. spaces E such that $E \not\approx Z_E$.

Further, it is known that if a r.i. space E is not equal to $L^\infty(0,1)$ up to an equivalent renorming, Rosenthal's space $\mathcal{U}_E$ contains a *complemented* subspace of Z_E isomorphic to ℓ^2 [16, Lemma 8.7 and subsequent Remark]. Therefore, if we know that $E \approx Z_E$, then E must contain a complemented subspace, which is isomorphic to ℓ^2 as well. According to [16, Proposition 8.17], there are some Orlicz spaces, "close" to L^1, that fail to contain such subspaces and hence are not isomorphic to Z_E (in fact, they are not isomorphic to any r.i. space on $(0,\infty)$; see [16, Corollary 8.15]). The simplest such example is the Orlicz space L_{F_α}, where $F_\alpha(u)$ is an Orlicz function equivalent to $u \log^\alpha u$ for large $u > 0$, where $0 < \alpha < 1/2$ (see also the discussion in the concluding part of Section 1).

Here, we prove results showing that the existence of complemented subspaces isomorphic to ℓ^2 does not guarantee that $\mathcal{U}_E$ is isomorphically embedded into E and, a fortiori, that $E \approx Z_E$. Specifically, we will provide examples of Lorentz spaces containing plenty of complemented subspaces isomorphic to ℓ^2, but without subspaces isomorphic to the corresponding Rosenthal spaces.

First, we introduce a lattice version of a notion from [27, p. 293]. We say that a Banach lattice E has the *disjoint Q_2-property* (briefly, $E \in \mathcal{D}Q_2$) whenever there is a constant $C_E > 0$ (depending only on E) such that given a disjoint sequence $\{h_n\}$ in E with $\|h_n\|_E = 1$ which is equivalent to the unit vector ℓ^2-basis, there exists a subsequence $\{h_{n_i}\} \subseteq \{h_n\}$ that is C_E-equivalent to the unit vector ℓ^2-basis.

Let E be a Banach lattice with the $\mathcal{D}Q_2$-property (with constant C_E). Suppose that $\{x_n\}_{n=1}^\infty \subset E$ is a disjoint sequence which is equivalent to the unit ℓ^2-basis and *semi-normalized* (i.e., $C^{-1} \le \|x_n\|_E \le C$ for some $C > 0$ and all $n = 1, 2, \dots$). Then it is obvious that $\{x_n\}_{n=1}^\infty$ contains a subsequence, which is C'-equivalent to the unit vector ℓ^2-basis, where $C' := C_E \cdot C$.

THEOREM 3. *Let E be a separable r.i. space such that $E \in \mathcal{D}Q_2$. If $\mathcal{U}_E \subsetneqq_{\sim} E$, then $E \supseteq G$.*

Proof. Let $\{A_n\}_{n=1}^\infty$ be a sequence of disjoint subsets of $(0, \infty)$ which satisfy conditions (2). Then, for every $l \in \mathbb{N}$, there are pairwise disjoint sets $S_i^l \subseteq \mathbb{N}$, $i = 1, 2, \dots$, such that

$$\sum_{n \in S_i^l} m(A_n) = \frac{1}{l}.$$

Denote $B_i^l := \bigcup_{n \in S_i^l} A_n$, $i = 1, 2, \dots$. Consider the block basis $\{\chi_{B_i^l}\}_{i=1}^\infty$ of $\{\chi_{A_n}\}_{n=1}^\infty$. According to the definition of the norm in Z_E (see (3)), each set consisting of l distinct functions $\chi_{B_i^l}$ is isometrically equivalent in Z_E to the set $\{\chi_{((i-1)/l,\ i/l)}\}_{i=1}^l$ in E, i.e., for all distinct $i_1, \dots, i_l \in \mathbb{N}$ and $a_j \in \mathbb{R}$,

$$\Big\| \sum_{j=1}^{l} a_j \chi_{B_{i_j}^l} \Big\|_{Z_E} = \Big\| \sum_{i=1}^{l} a_i \chi_{((i-1)/l,\ i/l)} \Big\|_E \tag{29}$$

(cf. [27, Corollary 8]).

On the other hand, the sequence $\{\chi_{B_i^l}\}_{i=1}^\infty$ is C_l-equivalent in Z_E to the unit vector ℓ^2-basis. Indeed, for arbitrary $a_i \in \mathbb{R}$ there is a set $S_l' \subseteq \mathbb{N}$ with $\operatorname{card} S_l' = l$ such that, with constants depending of l, we have

$$\begin{aligned} \Big\| \sum_{i=1}^{\infty} a_i \chi_{B_i^l} \Big\|_{Z_E} &= \Big\| \sum_{i \in S_l'} a_i \chi_{B_i^l} \Big\|_E + \Big\| \sum_{i \notin S_l'} a_i \chi_{B_i^l} \Big\|_{L^2} \\ &\overset{2C_l}{\asymp} \Big\| \sum_{i \in S_l'} a_i \chi_{B_i^l} \Big\|_{L^2} + \Big\| \sum_{i \notin S_l'} a_i \chi_{B_i^l} \Big\|_{L^2} \\ &\overset{2C_l}{\asymp} \Big\| \sum_{i=1}^{\infty} a_i \chi_{((i-1)/l,\ i/l)} \Big\|_{L^2} = \frac{1}{\sqrt{l}} \|(a_i)\|_2. \end{aligned} \tag{30}$$

From the hypothesis there exists an isomorphism $T \colon \mathcal{U}_E \to E$. Then, if $y_i^l := T(\chi_{B_i^l})$, $i = 1, 2, \dots$, by (30) we have

$$\Big\| \sum_{i=1}^{\infty} a_i y_i^l \Big\|_E \overset{\|T\|}{\asymp} \Big\| \sum_{i=1}^{\infty} a_i \chi_{B_i^l} \Big\|_{Z_E} \asymp \frac{1}{\sqrt{l}} \|(a_i)\|_2, \tag{31}$$

with constants depending on l and $\|T\|$.

When $\|y_i^l\|_E \asymp \|y_i^l\|_{L^1}$, $i = 1, 2, \ldots$, for some $l \in \mathbb{N}$, all the conditions of Proposition 4 are satisfied, and so the desired result follows.

Assume, conversely, that for each $l \in \mathbb{N}$ we have

$$\liminf_{i\to\infty} \frac{\|y_i^l\|_{L^1}}{\|y_i^l\|_E} = 0.$$

Denoting $u_i^l := (1/\phi_E(1/l))y_i^l$, $i, l = 1, 2, \ldots$, where ϕ_E is the fundamental function of E, we get

$$\|T\|^{-1} \le \|u_i^l\|_E \le \|T\|, \quad i, l = 1, 2, \ldots, \tag{32}$$

and clearly for every $l = 1, 2, \ldots$,

$$\liminf_{i\to\infty} \frac{\|u_i^l\|_{L^1}}{\|u_i^l\|_E} = 0.$$

Then again, by the Kadec–Pełczyński alternative, for each $l = 1, 2, \ldots$ there is subsequence $\{u_{i_j}^l\} \subseteq \{u_i^l\}$, where the sequence $\{i_j\}$ depends on $l \in \mathbb{N}$, such that for some disjoint sequence $\{z_j^l\} \subseteq E$,

$$\|u_{i_j}^l - z_j^l\|_E \to 0 \quad \text{as } j \to \infty.$$

Applying the principle of small perturbations (see e.g. [1, Theorem 1.3.9]), we can assume that $\{z_j^l\}_{j=1}^\infty$ is 2-equivalent in E to $\{u_{i_j}^l\}_{j=1}^\infty$, and so, by (32),

$$(2\|T\|)^{-1} \le \|z_j^l\|_E \le 2\|T\|, \quad j, l = 1, 2, \ldots,$$

which means that for every $l = 1, 2, \ldots$ the sequence $\{z_j^l\}_{j=1}^\infty$ is semi-normalized with a constant independent of l. Moreover, taking into account (31), we see that $\{z_j^l\}_{j=1}^\infty$ is equivalent in E to the unit vector ℓ^2-basis (with constants depending on $l = 1, 2, \ldots$). Since $E \in \mathcal{D}Q_2$, for each $l \in \mathbb{N}$ the sequence $\{z_j^l\}_{j=1}^\infty$ contains a further subsequence $\{z_{j_k}^l\}_{k=1}^\infty$ that is C'-equivalent to the unit vector ℓ^2-basis (here, $\{j_k\}$ depends on $l \in \mathbb{N}$ but C' does not). Clearly, $\{u_{i_{j_k}}^l\}_{k=1}^\infty$ is then $2C'$-equivalent to that basis, i.e.,

$$\Big\|\sum_{k=1}^\infty a_k u_{i_{j_k}}^l\Big\|_E \overset{2C'}{\asymp} \|(a_k)\|_2. \tag{33}$$

Moreover, from (29) and the above notation it follows that

$$\Big\|\sum_{k=1}^l a_k u_{i_{j_k}}^l\Big\|_E \overset{\|T\|}{\asymp} \frac{1}{\phi_E(1/l)}\Big\|\sum_{k=1}^l a_k \chi_{B_{i_{j_k}}^l}\Big\|_{Z_E} = \frac{1}{\phi_E(1/l)}\Big\|\sum_{j=1}^l a_j \chi_{((j-1)/l,\, j/l)}\Big\|_E$$

for all $a_j \in \mathbb{R}$. Combining this with (33), we obtain

$$\Big\|\sum_{j=1}^l a_j \chi_{((j-1)/l,\, j/l)}\Big\|_E \asymp \phi_E(1/l)\Big(\sum_{j=1}^l a_j^2\Big)^{1/2}, \quad l \in \mathbb{N}, \tag{34}$$

with constants independent of $l \in \mathbb{N}$ and $a_j \in \mathbb{R}$.

Next, one can easily check that equivalence (34) implies that $\phi_E(t) \asymp t^{1/2}$, $0 < t \le 1$. Indeed, for every $l \in \mathbb{N}$ we have

$$\chi_{(0,1)} = \sum_{i=1}^{l} \chi_{(i-1)/l,i/l)},$$

whence, by (34),

$$1 = \|\chi_{(0,1)}\|_E \asymp \sqrt{l}\,\phi_E(1/l). \tag{35}$$

Therefore, $\phi_E(1/l) \asymp 1/\sqrt{l}$, $l \in \mathbb{N}$. Combining this with the quasi-concavity of ϕ_E, we find that $\phi_E(t) \asymp \sqrt{t}$, $0 < t \le 1$. As a consequence, from (34) it follows that

$$\Big\|\sum_{j=1}^{l} a_j \chi_{((j-1)/l,j/l)}\Big\|_E \asymp \frac{1}{\sqrt{l}}\Big(\sum_{j=1}^{l} a_j^2\Big)^{1/2} = \Big\|\sum_{j=1}^{l} a_j \chi_{((j-1)/l,j/l)}\Big\|_{L^2}, \quad l \in \mathbb{N},$$

with constants independent of $l \in \mathbb{N}$ and $a_j \in \mathbb{R}$. Clearly, this implies that $E \approx L^2$, and the desired result follows. ∎

THEOREM 4. *Let E be a separable r.i. space on $[0,1]$ such that both E and E' have the $\mathcal{D}Q_2$-property. If $E \approx Z_E$, then $G \subseteq E \subseteq G'$.*

Proof. It follows from Theorem 3 that we need only prove that $E \subseteq G'$.

Suppose that T is an isomorphism from Z_E onto E. Clearly, T^* is then an isomorphism from E^* onto $(Z_E)^*$. Since E is separable, we have $E^* = E'$ and, by Lemma 1, Z_E is a separable space with $(Z_E)^* = (Z_E)' = Z_{E'}$. Thus, $E' \approx Z_{E'}$, and hence, by Theorem 3, $E' \supseteq G$, which implies $E \subseteq E'' \subseteq G'$. ∎

Let $1 \le p \le \infty$. Recall that a Banach lattice E is said to be *p-disjointly homogeneous* (p-$\mathcal{DH}$) if every disjoint normalized sequence contains a subsequence equivalent to the unit vector ℓ^p-basis (c_0-basis if $p = \infty$). Moreover, E is called *uniformly p-$\mathcal{DH}$* if there is a constant B_E, which depends only on E, such that from every disjoint normalized sequence $\{x_n\}$ we can select a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ which is B_E-equivalent to the ℓ^p-basis (for a detailed account of these properties see the survey [14] and references therein).

From Lozanovsky's theorem (see e.g. [3, Theorem 4.71]) it follows that every p-$\mathcal{DH}$ Banach lattice for $1 < p < \infty$ is reflexive. Also, it is obvious that each uniformly 2-$\mathcal{DH}$ lattice has the $\mathcal{D}Q_2$-property.

THEOREM 5. *Let E be a uniformly 2-$\mathcal{DH}$ r.i. space on $[0,1]$. Suppose that at least one of the following conditions holds:*

(i) *Rosenthal's space* $\mathcal{U}_E$ *is isomorphically embedded into* E*;*
(ii) E *is isomorphic to a r.i. space on* $(0,\infty)$.

Then $E \supseteq G$.

Moreover, if additionally the Köthe dual E' *is uniformly* 2-$\mathcal{DH}$ *and* E' *satisfies* (i) *or* (ii), *then* $G \subseteq E \subseteq G'$.

Proof. Since E is a uniformly 2-$\mathcal{DH}$, condition (i) implies $E \supseteq G$ by Theorem 3.

Let now E be isomorphic to a r.i. space Y on $(0,\infty)$. Denote $x_{n,i} := \chi_{[(i-1)/n,i/n)}$, $n,i \in \mathbb{N}$, and assume first that, for every $n \in \mathbb{N}$, the sequence $\{x_{n,i}\}_{i=1}^{\infty}$ is equivalent in Y to the unit vector ℓ^2-basis. Then, if T is an isomorphism of Y onto E, each sequence $\{y_{n,i}\}_{i=1}^{\infty}$, $n \in \mathbb{N}$, where $y_{n,i} := T(x_{n,i})$, $n,i \in \mathbb{N}$, is equivalent in E to the unit vector ℓ^2-basis as well. When $\|y_{n,i}\|_E \asymp \|y_{n,i}\|_{L^1}$, $i = 1,2,\ldots$, for some $n \in \mathbb{N}$, the desired result follows, as above, by Proposition 4. Hence, it remains to consider the case when for each $n \in \mathbb{N}$ we have

$$\liminf_{i\to\infty} \frac{\|y_{n,i}\|_{L^1}}{\|y_{n,i}\|_E} = 0.$$

Then, denoting $u_{n,i} := (1/\phi_E(1/n))y_{n,i}$, $i,n = 1,2,\ldots$, and reasoning as in the proof of Theorem 4, we can find, for every $n \in \mathbb{N}$, a subsequence $\{u_{n,i_j}\}_{j=1}^{\infty}$ which is 2-equivalent in E to some disjoint semi-normalized (with a constant independent of n) sequence $\{z_{n,j}\}_{j=1}^{\infty}$. Thanks to the uniform 2-$\mathcal{DH}$ property of E, passing to a further subsequence if necessary, we can assume that there is a constant $D' > 0$ such that for every $n \in \mathbb{N}$ the sequence $\{u_{n,i_j}\}_{j=1}^{\infty}$ is D'-equivalent in Y to the unit vector ℓ^2-basis. On the other hand, for every $n \in \mathbb{N}$ the sequence $\{y_{n,i}\}_{i=1}^{\infty}$ (together with $\{x_{n,i}\}_{i=1}^{\infty}$ in Y) is B-symmetric (1) in E for some $B > 0$. Consequently, for every $n \in \mathbb{N}$ the sequence $\{u_{n,i}\}_{i=1}^{\infty}$ and hence $\{(1/\phi_E(1/n))x_{n,i}\}_{i=1}^{\infty}$ is D-equivalent in Y to the unit vector ℓ^2-basis for some $D > 0$, i.e.,

$$D^{-1}\phi_E(1/n)\|(a_i)\|_2 \le \Big\|\sum_{i=1}^{\infty} a_i x_{n,i}\Big\|_Y \le D\phi_E(1/n)\|(a_i)\|_2$$

for all $n \in \mathbb{N}$ and $(a_i) \in \ell^2$. Clearly, this implies that $Y = L^2(0,\infty)$ (see the concluding part of the proof of Theorem 4). Since $E \approx Y$ by assumption, we infer that $E = L^2[0,1]$ (with equivalence of norms), and so in this case everything is done.

Conversely, suppose that the sequence $\{x_{1,i}\}_{i=1}^{\infty}$ is not equivalent in Y to the unit vector ℓ^2-basis; then the same is true also for all sequences $\{x_{n,i}\}_{i=1}^{\infty}$,

(1) A sequence $\{v_k\}_{k=1}^{\infty}$ in a Banach space X is called *B-symmetric* if for every permutation $\pi : \mathbb{N} \to \mathbb{N}$ and all $a_k \in \mathbb{R}$, $k = 1,2,\ldots$, we have $B^{-1}\|\sum_{k=1}^{\infty} a_k v_k\|_X \le \|\sum_{k=1}^{\infty} a_k v_{\pi(k)}\|_X \le B\|\sum_{k=1}^{\infty} a_k v_k\|_X$.

$n \in \mathbb{N}$. As mentioned above, for every $n \in \mathbb{N}$ the sequence $\{y_{n,i}\}_{i=1}^{\infty}$ is B-symmetric in E for some $B > 0$. Moreover, since $\{x_{n,i}\}_{i=1}^{\infty}$, $n \in \mathbb{N}$, spans a 1-complemented subspace in Y (see e.g. [21, Ch. II, §3.2]), we can assume that, for every $n \in \mathbb{N}$, the span $[y_{n,i}, i \in \mathbb{N}]$ is a B-complemented subspace in E. Then, according to [16, Lemma 8.10], there is a constant $A' > 0$ such that for every $n \in \mathbb{N}$ the sequence $\{y_{n,i}\}_{i=1}^{\infty}$ is A'-equivalent in E to a disjoint sequence in E. Since E is uniformly 2-$\mathcal{DH}$ and $\{x_{n,i}\}_{i=1}^{\infty}$ is a B-symmetric sequence in E, we conclude that there is a constant $A > 0$ such that for every $n \in \mathbb{N}$ the sequence $\{(1/\phi_E(1/n))x_{n,i}\}_{i=1}^{\infty}$ is A-equivalent in Y to the unit vector ℓ^2-basis. As above, this implies that $Y = L^2(0,\infty)$ and hence $E = L^2[0,1]$ (with equivalence of norms), which completes the proof. ∎

It is well known that every Lorentz space $\Lambda_2(\varphi)$ has the uniform 2-$\mathcal{DH}$ property (see e.g. [13, Theorem 5.1]). Therefore, since the embedding $\Lambda_2(\varphi) \supseteq G$ is equivalent to the condition $\sum_{k=1}^{\infty}\varphi(e^{-k}) < \infty$ (see e.g. [6, Lemma 3]), we get the following consequence of Theorem 5.

Corollary 2. *Let φ be an increasing concave function on $[0,1]$ with $\varphi(0) = 0$. Suppose that at least one of the following conditions holds:*

(i) *Rosenthal's space $\mathcal{U}_{\Lambda_2(\varphi)}$ is isomorphically embedded into $\Lambda_2(\varphi)$;*
(ii) *$\Lambda_2(\varphi)$ is isomorphic to a r.i. space on $(0,\infty)$.*

Then $\sum_{k=1}^{\infty}\varphi(e^{-k}) < \infty$.

In particular, we get the following new examples of r.i. spaces on $[0,1]$ that are not equivalent to any r.i. spaces on $(0,\infty)$.

Corollary 3. *Let $0 < \alpha \le 1$. Then the Lorentz space $\Lambda_2(\log^{-\alpha}(e/u))$ has the following properties:*

(a) *any disjoint sequence in $\Lambda_2(\log^{-\alpha}(e/u))$ contains a subsequence that is 2-equivalent to the unit vector basis of ℓ^2 and spans a 2-complemented subspace in $\Lambda_2(\log^{-\alpha}(e/u))$;*
(b) *Rosenthal's space $\mathcal{U}_{\Lambda_2(\log^{-\alpha}(e/u))}$ fails to be isomorphically embedded into $\Lambda_2(\log^{-\alpha}(e/u))$ and $\Lambda_2(\log^{-\alpha}(e/u))$ is not isomorphic to any r.i. space on $(0,\infty)$.*

5. Existence of an isomorphic embedding $T\colon \mathcal{U}_E \to E$ when the functions $T(\chi_{A_n})$, $n = 1,2,\ldots$, are independent. In this final section, we treat the special case when there is an isomorphic embedding $T\colon \mathcal{U}_E \to E$ such that the functions $T(\chi_{A_n})$, $n = 1,2,\ldots$, are independent and symmetrically distributed.

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of disjoint measurable subsets of $(0,\infty)$ which satisfy conditions (2). As at the beginning of the proof of Theorem 3, for every $l \in \mathbb{N}$ we find pairwise disjoint sets $S_i^l \subseteq \mathbb{N}$, $i = 1, 2, \ldots$, such that $\sum_{n\in S_i^l} m(A_n) = 1/l$ and denote $B_i^l := \bigcup_{n\in S_i^l} A_n$, $i = 1, 2, \ldots$.

Next, suppose that E is a r.i. space such that $\mathcal{U}_E$ is isomorphically embedded into E, $T\colon \mathcal{U}_E \to E$ is an isomorphism, and $y_i^l := T(\chi_{B_i^l})$, $i, l \in \mathbb{N}$. In contrast to the preceding section, we assume that the sequences $\{y_i^l\}_{i=1}^\infty$, $l \in \mathbb{N}$, do not contain "almost" disjoint subsequences, which means (see the proof of Theorem 3) that $\|y_i^l\|_E \asymp \|y_i^l\|_{L^1}$, $i = 1, 2, \ldots$, for each $l \in \mathbb{N}$. Then it is easy to check (see also [18]) that for every $l \in \mathbb{N}$ there exists a constant $\varepsilon_l > 0$ such that

$$m(\{t : |y_i^l(t)| > \varepsilon_l \|y_i^l\|_E\}) \geq \varepsilon_l.$$

However, we will need the following stronger condition: there are $\alpha, \beta, \gamma > 0$, an infinite sequence $\{l_k\}_{k=1}^\infty \subset \mathbb{N}$, and a sequence of sets $F_k \subset \mathbb{N}$, $k = 1, 2, \ldots$, such that $\gamma l_k \leq \operatorname{card} F_k \leq l_k$ and for each $i \in F_k$,

$$m(\{t : |y_i^{l_k}(t)| > \alpha\}) \geq \beta/l_k. \tag{36}$$

Furthermore, consider the family $\{B_i^{l_k} : i \in F_k, k \in \mathbb{N}\}$. One can readily check that the definition of the sets B_i^l, $i, l \in \mathbb{N}$, and the conditions imposed on the sets F_k, $k \in \mathbb{N}$, ensure that the latter family satisfies requirements (2). Since Rosenthal's space $\mathcal{U}_E$ does not depend (up to isomorphism) on the particular choice of a sequence of sets which satisfy (2) [16, Lemma 8.7], without loss of generality we can replace the initial sequence $\{A_n\}_{n=1}^\infty$ with $\{B_i^{l_k} : i \in F_k,\ k \in \mathbb{N}\}$.

Theorem 6. *Let E be a r.i. space on $[0, 1]$ such that there exists an isomorphic embedding $T\colon \mathcal{U}_E \to E$. Suppose that there is a sequence $\{l_k\}_{k=1}^\infty \subset \mathbb{N}$ such that the functions $y_i^{l_k} := T(\chi_{B_i^{l_k}})$, $k, i \in \mathbb{N}$, are independent and symmetrically distributed r.v.'s which satisfy the above conditions* (36). *Then the Kruglov operator K is bounded from E into E''. Moreover, there is a constant $C > 0$ such that*

$$\phi_E\biggl(\Bigl(\frac{\beta}{2l_k}\Bigr)^{\gamma l_k}\biggr) \leq \frac{C}{l_k}, \qquad k = 1, 2, \ldots, \tag{37}$$

where ϕ_E is the fundamental function of E.

Proof. First, for each $k = 1, 2, \ldots$, we compare the finite sequences $\{y_i^{l_k}\}_{i\in F_k}$ and $\{u_i^{l_k}\}_{i\in F_k}$, where, as above, u_i^l are independent and symmetrically distributed r.v.'s equimeasurable with the characteristic functions $\chi_{B_i^{l_k}}$, $k, i = 1, 2, \ldots$. From (36) it follows that for all $\tau > 0$,

$$m(\{t : |y_i^{l_k}(t)| > \tau\}) \geq \beta m(\{t : \alpha|u_i^{l_k}(t)| > \tau\}), \qquad i \in F_k,\ k = 1, 2, \ldots.$$

Hence, applying the result of Kwapień–Rychlik, [29, Ch. V, Theorem 4.4], for all $\tau > 0$ and $a_i^k \in \mathbb{R}$ we get

$$m\Big(\Big\{t : \Big|\sum_{k=1}^{\infty}\sum_{i\in F_k} a_i^k u_i^{l_k}(t)\Big| > \tau\Big\}\Big) \le \frac{2}{\beta} m\Big(\Big\{t : \Big|\sum_{k=1}^{\infty}\sum_{i\in F_k} a_i^k y_i^{l_k}(t)\Big| > \beta\alpha\tau\Big\}\Big).$$

So, by [21, Ch. II, §4.3, Corollary 2],

$$\Big\|\sum_{k=1}^{\infty}\sum_{i\in F_k} a_i^k u_i^{l_k}\Big\|_E \le \frac{2}{\beta^2\alpha}\Big\|\sum_{k=1}^{\infty}\sum_{i\in F_k} a_i^k y_i^{l_k}\Big\|_E.$$

On the other hand, since T is an isomorphism, we have

$$\Big\|\sum_{k=1}^{\infty}\sum_{i\in F_k} a_i^k y_i^{l_k}\Big\|_E \overset{\|T\|}{\asymp} \Big\|\sum_{k=1}^{\infty}\sum_{i\in F_k} a_i^k \chi_{B_i^{l_k}}\Big\|_{Z_E}. \tag{38}$$

Combining the last inequalities, we infer that

$$\Big\|\sum_{k=1}^{\infty}\sum_{i\in F_k} a_i^k u_i^{l_k}\Big\|_E \le \frac{2\|T\|}{\beta^2\alpha}\Big\|\sum_{k=1}^{\infty}\sum_{i\in F_k} a_i^k \chi_{B_i^{l_k}}\Big\|_{Z_E}.$$

Applying now Theorem 1 (to the family $\{B_i^{l_k} : i \in F_k,\ k \in \mathbb{N}\}$), we complete the proof of the first assertion.

Further, since $\operatorname{card} F_k \le l_k$ and $m(B_i^{l_k}) = 1/l_k$, from (38) it follows that

$$\Big\|\sum_{i\in F_k} y_i^{l_k}\Big\|_E \le C'\Big\|\sum_{i\in F_k} \chi_{B_i^{l_k}}\Big\|_E \le C', \qquad k = 1, 2, \ldots.$$

Moreover, taking into account the fact that $y_i^{l_k}$, $i \in F_k$, are independent symmetrically distributed r.v.'s, the inequality $\operatorname{card} F_k \ge \gamma l_k$ and (36), we get

$$\begin{aligned}\Big\|\sum_{i\in F_k} y_i^{l_k}\Big\|_E &\ge \alpha\gamma l_k \cdot \|\chi_{\bigcap_{i\in F_k}\{y_i^{l_k}\ge\alpha\}}\|_E \\ &= \alpha\gamma l_k \cdot \phi_E\Big(\prod_{i\in F_k} m(\{y_i^{l_k} \ge \alpha\})\Big) \\ &\ge \alpha\gamma l_k \cdot \phi_E\Big(\Big(\frac{\beta}{2l_k}\Big)^{\gamma l_k}\Big).\end{aligned}$$

Combining these inequalities, we obtain (37). ∎

COROLLARY 4. *Let E be the exponential Orlicz space* $\operatorname{Exp} L^p$, $p > 0$. *There exists an isomorphic embedding $T : \mathcal{U}_E \to E$ which satisfies the conditions of Theorem* 6 *if and only if* $0 < p \le 1$.

Proof. One can easily check that, for $E = \operatorname{Exp} L^p$, we have $\phi_E(t) \asymp \log^{-1/p}(e/t)$, $0 < t \le 1$. Therefore, a direct calculation shows that (37) is fulfilled in this case if and only if $0 < p \le 1$. Moreover, if $0 < p \le 1$, the space $\operatorname{Exp} L^p$ has the Kruglov property (see [12, beginning of §2.4] and [10, 4.3.1]), which implies that there exists an isomorphic embedding $T\colon \mathcal{U}_E \to E$ which satisfies the conditions of Theorem 6 (indeed, we take u_i^l for y_i^l, an arbitrary sequence $\{l_k\}_{k=1}^{\infty}$ of positive integers and any set of cardinality l_k for F_k, $k = 1, 2, \dots$). Thus, the desired result follows. ∎

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