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EXTENDED CONVERGENCE ANALYSIS OF THE NEWTON–POTRA METHOD UNDER WEAK CONDITIONS

Abstract. We study a nonlinear equation with a nondifferentiable part. The semi-local convergence of the Newton–Potra method is proved under weaker (than in earlier research) conditions on derivatives and divided differences of the first order. Weaker semi-local convergence criteria and tighter error estimations are obtained. Hence, the applicability of this method is extended too. These advantages are obtained under the same computational effort.

1. Introduction. Consider the equation

$$H(x) \equiv F(x) + Q(x) = 0, \tag{1.1}$$

where F and Q are nonlinear operators, defined on a subset D of a Banach space E_1 with values in a Banach space E_2. We suppose that F is a Fréchet-differentiable operator, and Q is a continuous operator. In case of nondifferentiable H the classical Newton method is not applicable for solving equation (1.1).

The following Newton-type method was studied in [16]:

$$x_{n+1} = x_n - [F'(x_n)]^{-1} H(x_n), \quad n \geq 0.$$

This iterative process shows quite slow convergence rate in practice. Hence, divided difference methods could be used for solving (1.1), which do not

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require analytic information about the derivative. The most popular such method is the secant-type method [7, 10]

$$x_{n+1} = x_n - [H(x_n; x_{n-1})]^{-1} H(x_n), \qquad n \geq 0,$$

where $H(x_n; x_{n-1})$ is a first-order divided difference of the operator H. Another approach to solving equation (1.1) is based on combination of different methods for solving the equation $H(x) = 0$. Cãtinas [6] improved the convergence rate by studying the method

$$x_{n+1} = x_n - L_n^{-1} H(x_n),$$

where $L_n = Q(x_n; x_{n-1}) + F'(x_n)$. For this type of methods see [1, 2, 11, 12, 13, 15].

We are going to improve the results provided for the Newton–Potra method in [14]. The method is built on the Newton and Potra methods [8, 9]:

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1} H(x_n), \qquad n \geq 0, \\ A_n &= F'(x_n) + Q(x_n; x_{n-1}) + Q(x_{n-2}; x_n) - Q(x_{n-2}; x_{n-1}). \end{aligned} \tag{1.2}$$

In [12] the local and semi-local convergence of (1.2) was studied under classical Lipschitz conditions. It is known that the convergence order of the combined iterative process is the same as for the basic Potra method.

In this work we study the semi-local convergence of the Newton–Potra method under weak ω and ε type conditions for first order derivatives of the operator F and divided difference of order one of the operator Q. The Newton-type, secant-type and some other combined methods were studied under ω-conditions in [10, 11]. Note that classical Lipschitz and Hölder conditions are special cases of ω-conditions, which do not require the differentiability of the operator Q.

2. Semi-local convergence. Let $U(\overline{x}, \sigma) = \{x \in E_1 : \|x - \overline{x}\| < \sigma\}$ and $\overline{U(\overline{x}, \sigma)} = \{x \in E_1 : \|x - \overline{x}\| \leq \sigma\}$ for $\sigma > 0$, and $A_0 = F'(x_0) + Q(x_0; x_{-1}) + Q(x_{-2}; x_0) - Q(x_{-2}; x_{-1})$, where $x_{-2}, x_{-1}, x_0 \in D$.

THEOREM 2.1. *Let $F : D \subseteq E_1 \to E_2$ be a Fréchet-differentiable operator, $Q : D \subseteq E_1 \to E_2$ be a continuous operator, and $Q(\cdot\,;\cdot)$ be a divided difference of order one for the operator Q, defined on the subset D. Suppose that the operator A_0 is invertible and for each $x, y, u, v \in D$ the following conditions are satisfied:*

$$\|A_0^{-1}(F'(x) - F'(x_0))\| \leq \omega_1^0(\|x - x_0\|), \tag{2.1}$$

$$\|A_0^{-1}(Q(x; y) - Q(u; v))\| \leq \omega_2^0(\|x - u\|, \|y - v\|). \tag{2.2}$$

Here ω_1^0 is a nondecreasing positive function on $[0, R]$ with $\omega_1^0(tr) \leq h(t)\omega_1^0(r)$, $h : [0, 1] \to \mathbb{R}$, $t \in [0, 1]$, $r \in [0, R]$; and $\omega_2^0 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous *nondecreasing function in both arguments. Assume that for all* $x, y, u, v \in D_0$,

$$\|A_0^{-1}(F'(x) - F'(y))\| \le \omega_1(\|x - y\|) \tag{2.3}$$

and

$$\|A_0^{-1}(Q(x;y) - Q(u;v))\| \le \omega_2(\|x - u\|, \|y - v\|), \tag{2.4}$$

where ω_1 *and* ω_2 *are as* ω_1^0 *and* ω_2^0, *respectively;* $D_0 = D \cap U(x_0, r_0)$, *and* r_0 *is the smallest positive solution of the equation* $\psi_0(\alpha, \beta, u) = 1$ *with*

$$\psi_0(\alpha,\beta,u) = \omega_1^0(u) + \omega_2^0(u, u+\alpha) + \omega_2^0(u+\alpha+\beta, u) + \omega_2^0(u+\alpha+\beta, u+\alpha).$$

Moreover, suppose that $\eta > 0$ *and* $\alpha, \beta \ge 0$ *are constants satisfying*

$$\|A_0^{-1}(F(x_0) + Q(x_0))\| \le \eta, \qquad \|x_0 - x_{-1}\| \le \alpha, \qquad \|x_{-1} - x_{-2}\| \le \beta. \tag{2.5}$$

Suppose that the equation

$$u\left(1 - \frac{\gamma}{1 - \psi_0(\alpha,\beta,u)}\right) - \eta = 0, \tag{2.6}$$

where $\gamma = \Phi\omega_1(\eta) + \max\{\omega_2(\eta,\eta) + \omega_2(0,\eta), \omega_2^0(\eta,\alpha) + \omega_2^0(0,\alpha)\}$ *and* $\Phi h = \int_0^1 h(t)\,dt$, *has at least one positive solution greater than* η, α *and* β. *Denote the smallest such solution by* R. *Furthermore, suppose*

$$\psi_0(\alpha,\beta,R) < 1, \qquad \delta = \frac{\gamma}{1 - \psi_0(\alpha,\beta,R)} < 1$$

and $\overline{U(x_0,R)} \subset D$. *Then the sequence* $\{x_n\}_{n\ge 0}$ *generated by the iterative process* (1.2) *is well-defined, remains in* $U(x_0, R)$ *and converges to the unique solution* $x^* \in \overline{U(x_0,R)}$ *of equation* (1.1).

Proof. Using (1.2) and (2.5) for $n = 0$ we obtain

$$\|x_1 - x_0\| \le \|A_0^{-1}(F(x_0) + Q(x_0))\| \le \eta < R.$$

Hence, $x_1 \in U(x_0, R)$.

We get in turn, using conditions (2.1) and (2.2),

$$\begin{aligned}
\|I - A_0^{-1}A_1\| = \|A_0^{-1}(A_0 - A_1)\| &\le \|A_0^{-1}(F'(x_0) - F'(x_1))\| \\
&\quad + \|A_0^{-1}(Q(x_0;x_{-1}) + Q(x_{-2};x_0) - Q(x_{-2};x_{-1}) \\
&\quad - Q(x_1;x_0) - Q(x_{-1};x_1) + Q(x_{-1};x_0))\| \le \tilde{\psi}_1 \\
&= \omega_1^0(\|x_1 - x_0\|) + \omega_2^0(\|x_1 - x_0\|, \|x_{-1} - x_0\|) \\
&\quad + \omega_2^0(\|x_{-2} - x_{-1}\|, \|x_0 - x_1\|) \\
&\quad + \omega_2^0(\|x_{-1} - x_{-2}\|, \|x_0 - x_{-1}\|) \\
&\le \omega_1^0(\eta) + \omega_2^0(\eta,\alpha) + \omega_2^0(\beta,\eta) + \omega_2^0(\beta,\alpha) \\
&\le \omega_1^0(R) + \omega_2^0(R,\alpha) + \omega_2^0(\beta,R) + \omega_2^0(\beta,\alpha) \\
&\le \omega_1^0(R) + \omega_2^0(R, R+\alpha) + \omega_2^0(R+\alpha+\beta, R) \\
&\quad + \omega_2^0(R+\alpha+\beta, R+\alpha) = \psi_0(\alpha,\beta,R) < 1.
\end{aligned}$$

Hence, by the Banach Lemma on invertible operators, $A_1^{-1}A_0$ exists and

$$\|A_1^{-1}A_0\| < \frac{1}{1-\psi_0(\alpha,\beta,R)}.$$

We can write

$$\begin{aligned}
&A_0^{-1}(F(x_1)+Q(x_1)) = A_0^{-1}(F(x_1)-F(x_0)-F(x_0)(x_1-x_0))\\
&+A_0^{-1}\big(Q(x_1)-Q(x_0)-\big(Q(x_0;x_{-1})+Q(x_{-2};x_0)-Q(x_{-2};x_{-1})\big)(x_1-x_0)\big)\\
&=\int_0^1 A_0^{-1}\big(F'(x_0+t(x_1-x_0))-F'(x_0)\big)\,dt\,(x_1-x_0)\\
&\qquad+A_0^{-1}\big(Q(x_1;x_0)-Q(x_0;x_{-1})-Q(x_{-2};x_0)+Q(x_{-2};x_{-1})\big)(x_1-x_0).
\end{aligned}$$

Using estimates (2.1) and (2.2) we obtain

$$\begin{aligned}
&\|x_2-x_1\| = \|A_1^{-1}(F(x_1)+Q(x_1))\| \le \|A_1^{-1}A_0\|\,\|A_0^{-1}(F(x_1)+Q(x_1))\|\\
&\qquad\le \delta_1\|x_1-x_0\| = \frac{\Phi\omega_1^0(\|x_0-x_1\|)}{1-\tilde\psi_1}\|x_1-x_0\|\\
&\quad+\frac{\omega_2^0(\|x_1-x_0\|,\|x_0-x_{-1}\|)+\omega_2^0(\|x_{-2}-x_{-2},\|x_{-1}-x_0\|)}{1-\tilde\psi_1}\|x_1-x_0\|\\
&\qquad\qquad\le \frac{\Psi\omega_1^0(\eta)+\omega_2^0(\eta,\alpha)+\omega_2^0(0,\alpha)}{1-\psi_0(\alpha,\beta,R)}\|x_1-x_0\| = \delta\|x_1-x_0\| < \eta.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_2-x_0\| &\le \|x_2-x_1\|+\|x_1-x_0\|\\
&\le (\delta+1)\|x_1-x_0\| \le (\delta+1)\eta = \frac{1-\delta^2}{1-\delta}\eta < \frac{1}{1-\delta}\eta = R
\end{aligned}$$

and so $x_2\in U(x_0,R)$.

For each $k=\overline{1,n-1}$ the following statements hold:

- $A_k^{-1}A_0$ exists and $\|A_k^{-1}A_0\| < \frac{1}{1-\psi_0(\alpha,\beta,R)}$;
- $\|x_{k+1}-x_k\| \le \delta\|x_k-x_{k-1}\| \le \delta^k\|x_1-x_0\| \le \eta$;
- $x_{k+1}\in U(x_0,R)$.

Using also conditions (2.1) and (2.2), for $k=n$, we obtain

$$\begin{aligned}
&\|I-A_0^{-1}A_n\| = \|A_0^{-1}(A_0-A_n)\|\\
&\quad\le \|A_0^{-1}(F'(x_0-F'(x_n))\| + \|A_0^{-1}(Q(x_0;x_{-1})+Q(x_{-2};x_0)-Q(x_{-2};x_{-1})\\
&\qquad-Q(x_n;x_{n-1})-Q(x_{n-2};x_n)+Q(x_{n-2};x_{n-1}))\|\\
&\quad\le \tilde\psi_n = \omega_1^0(\|x_0-x_n\|)+\omega_2^0(\|x_0-x_n\|,\|x_{-1}-x_{n-1}\|)\\
&\qquad+\omega_2^0(\|x_{-2}-x_{n-2}\|,\|x_0-x_n\|)+\omega_2^0(\|x_{n-2}-x_{-2}\|,\|x_{n-1}-x_{-1}\|)\\
&\quad\le \omega_1^0(R)+\omega_2^0(R,R+\alpha)+\omega_2^0(R+\alpha+\beta,R)+\omega_2^0(R+\alpha+\beta,R+\alpha)\\
&\quad=\psi_0(\alpha,\beta,R)<1.
\end{aligned}$$

Then, by the Banach Lemma on invertible operators, $A_n^{-1}A_0$ exists and

$$\|A_n^{-1}A_0\| < \frac{1}{1-\psi_0(\alpha,\beta,R)}.$$

We can write

$$\begin{aligned}
&A_0^{-1}(F(x_n)+Q(x_n))\\
&= A_0^{-1}\big(F(x_n)-F(x_{n-1})-F'(x_{n-1})(x_n-x_{n-1})\big) + A_0^{-1}\big(Q(x_n)-Q(x_{n-1})\\
&\quad -(Q(x_{n-1};x_{n-2})-Q(x_{n-3};x_{n-1})+Q(x_{n-3};x_{n-2}))(x_n-x_{n-1})\big)\\
&= \int_0^1 A_0^{-1}\big(F'(x_{n-1}+t(x_n-x_{n-1}))-F'(x_{n-1})\big)\,dt\,(x_n-x_{n-1})\\
&\quad + A_0^{-1}\big(Q(x_n;x_{n-1})-Q(x_{n-1};x_{n-2})-Q(x_{n-3};x_{n-1})\\
&\quad + Q(x_{n-3};x_{n-2})\big)(x_n-x_{n-1}).
\end{aligned}$$

We get in turn, using conditions (2.3) and (2.4),

$$\begin{aligned}
&\|x_{n+1}-x_n\| = \|A_n^{-1}(F(x_n)+Q(x_n))\|\\
&\le \|A_n^{-1}A_0\|\|A_0^{-1}(F(x_n)+Q(x_n))\| \le \delta_n\|x_n-x_{n-1}\|\\
&= \frac{\Phi\omega_1(\|x_n-x_{n-1}\|)}{1-\tilde{\psi}_n}\|x_n-x_{n-1}\|\\
&\quad + \frac{\omega_2(\|x_n-x_{n-1}\|,\|x_{n-1}-x_{n-2}\|)+\omega_2(\|x_{n-3}-x_{n-3}\|,\|x_{n-2}-x_{n-1}\|)}{1-\tilde{\psi}_n}\\
&\quad \times \|x_n-x_{n-1}\|\\
&\le \frac{\Phi\omega_1(\eta)+\omega_2(\eta,\eta)+\omega_2(0,\eta)}{1-\psi_0(\alpha,\beta,R)}\|x_n-x_{n-1}\|\\
&= \delta\|x_n-x_{n-1}\| \le \delta^n\|x_1-x_0\| < \eta.
\end{aligned}$$

Let us prove that $x_{n+1}\in U(x_0,R)$:

$$\begin{aligned}
\|x_{n+1}-x_0\| &\le \|x_{n+1}-x_n\|+\|x_n+x_{n-1}\|+\cdots+\|x_1-x_0\|\\
&\le (\delta^n+\delta^{n-1}+\cdots+\delta+1)\|x_1-x_0\| = \frac{1-\delta^{n+1}}{1-\delta}\eta < \frac{1}{1-\delta}\eta = R
\end{aligned}$$

and so $x_{n+1}\in U(x_0,R)$.

Next, we prove that $\{x_n\}_{n\ge 0}$ is a Cauchy sequence:

$$\begin{aligned}
\|x_{n+p}-x_n\| &\le \|x_{n+p}-x_{n+p-1}\|+\|x_{n+p-1}-x_{n+p-2}\|+\cdots+\|x_{n+1}-x_n\|\\
&\le (\delta^{p-1}+\delta^{p-2}+\cdots+1)\|x_{n+1}-x_n\| = \frac{1-\delta^p}{1-\delta}\delta^n\eta < \frac{\delta^n}{1-\delta}\eta.
\end{aligned}$$

Hence, $\{x_n\}_{n\ge 0}$ is a Cauchy sequence, and converges to $x^*\in\overline{U(x_0,R)}$.

Let us prove that x^* is the unique solution of equation (1.1). Then, since

$$\|A_0^{-1}H(x_n)\| \le (\Phi\omega_1(\eta)+\omega_2(\eta,\eta)+\omega_2(0,\eta))\|x_n-x_{n-1}\|,$$

and $\|x_n-x_{n-1}\|\to 0$ as $n\to\infty$, we deduce that $H(x^*)=0$.

The uniqueness will be shown by contradiction. Suppose that there exists $x^{**} \in U(x_0, R)$, $x^{**} \neq x^*$, with $H(x^{**}) = 0$. Denote

$$T \equiv \int_0^1 F'(x^* + t(x^{**} - x^*))\,dt + Q(x^{**}; x^*).$$

Thus, the equality $T(x^{**} - x^*) = H(x^{**}) - H(x^*)$ is well-defined. If the operator T^{-1} is invertible, then $x^{**} = x^*$. We get

$$\begin{aligned}
\|I - A_0^{-1}T\| &= \|A_0^{-1}(A_0 - T)\| \\
&\le \Big\| A_0^{-1} \int_0^1 \big(F'(x_0) - F'(x_* + t(x^{**} - x^*))\big)\,dt \Big\| \\
&\quad + \|A_0^{-1}[Q(x^{**}; x^*) - Q(x_0; x_{-1}) - Q(x_{-2}; x_0) + Q(x_{-2}; x_{-1})]\| \\
&\le \int_0^1 \omega_1^0\big((1-t)\|x_0 - x^*\| + t\|x_0 - x^{**}\|\big)\,dt \\
&\quad + \omega_2^0(\|x^{**} - x_0\|, \|x^* - x_{-1}\|) + \omega_2^0(\|x_{-2} - x_{-2}\|, \|x_{-1} - x_0\|) \\
&\le \omega_1^0(R) + \omega_2^0(R, R + \alpha) + \omega_2^0(0, \alpha) < 1.
\end{aligned}$$

Hence, T^{-1} exists. ∎

Let D_0 be some domain defined similarly to Theorem 2.1.

THEOREM 2.2. *Let $F : D \subseteq E_1 \to E_2$ be a Fréchet-differentiable operator, $Q : D \subseteq E_1 \to E_2$ be a continuous operator, and $Q(\cdot\,; \cdot)$ be a divided difference of order one for the operator Q, defined on the subset D. Suppose that the operator A_0 is invertible and the following conditions are satisfied for all $x, y, u, v \in D$:*

$$\|A_0^{-1}(F'(x) - F'(x_0))\| \le \varepsilon_1^0,$$
$$\|A_0^{-1}(Q(x; y) - Q(u; v))\| \le \varepsilon_2^0,$$

and for $x, y, u, v \in D_0 \subseteq D$,

$$\|A_0^{-1}(F'(x) - F'(y))\| \le \varepsilon_1,$$
$$\|A_0^{-1}(Q(x; y) - Q(u; v))\| \le \varepsilon_2.$$

Suppose that

$$\|A_0^{-1}(F(x_0) + Q(x_0))\| \le \eta, \qquad 0 < \lambda = \frac{\varepsilon_1 + 2\varepsilon_2}{1 - (\varepsilon_1^0 + 3\varepsilon_2^0)} < 1, \qquad \frac{\eta}{1 - \lambda} < R$$

and $\overline{U(x_0, R)} \subset D$. Then the sequence $\{x_n\}_{n \ge 0}$ generated by the iterative process (1.2) *is well-defined, remains in $U(x_0, R)$ and converges to the unique solution $x^* \in \overline{U(x_0, R)}$ of equation* (1.1). *Moreover, for each $n \ge 0$,*

$$\|x_n - x^*\| \le \frac{\lambda^n}{1 - \lambda}\eta. \tag{2.7}$$

Proof. We can prove the convergence of method (1.2) by induction, in an analogous way to Theorem 2.1. Let us prove estimate (2.7). It is known that for $n, p \in \mathbb{N}$,

$$\|x_{n+p} - x_n\| \le \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \cdots + \|x_{n+1} - x_n\| \le (\lambda^{p-1} + \lambda^{p-2} + \cdots + 1)\|x_{n+1} - x_n\| = \frac{1-\lambda^p}{1-\lambda}\lambda^n\eta < \frac{\lambda^n}{1-\lambda}\eta.$$

Hence, we obtain (2.7) for $p \to \infty$. ∎

REMARK 2.3. The corresponding conditions in [14] are, for $x, y, u, v \in D$,

$$\|A_0^{-1}(F'(x) - F'(y))\| \le \omega_1^1(\|x - y\|),$$
$$\|A_0^{-1}(Q(x;y) - Q(u;v))\| \le \omega_2^1(\|x - u\|, \|y - v\|),$$
$$\gamma^1 = \varPhi\omega_1^1(\eta) + \max\{\omega_2^1(\eta,\eta) + \omega_2^1(0,\eta), \omega_2^1(\eta,\alpha) + \omega_2^1(0,\alpha)\},$$
$$\psi^1(\alpha,\beta,u) = \omega_1^1(u) + \omega_2^1(u, u+\alpha) + \omega_2^1(u+\alpha+\beta, u) + \omega_2^1(u+\alpha+\beta, u+\alpha),$$
$$u\left(1 - \frac{\gamma^1}{\psi^1(\alpha,\beta,u)}\right) - \eta = 0,$$
$$\psi^1(\alpha,\beta,\eta^1) < 1,$$
$$\delta^1 = \frac{\gamma^1}{1 - \psi^1(\alpha,\beta,R^1)} < 1,$$
$$\overline{U(x_0, R^1)} \subseteq D.$$

But $D_0 \subseteq D$, so we get

$$\omega_1^0(t) \le \omega_1^1(t), \quad \omega_1(t) \le \omega_1^1(t), \quad \omega_2^0(t) \le \omega_2^1(t), \quad \omega_2(t) \le \omega_2^1(t),$$
$$\gamma \le \gamma^1, \quad \psi_0(t) \le \psi^1(t), \quad \delta_n \le \delta_n^1,$$

and

$$\delta \le \delta^1 \quad \text{if } R \le R^1,$$
$$\delta \ge \delta^1 \quad \text{if } R \ge R^1.$$

Examples where the preceding inequalities are strict can be found in [3, 4, 5, 6].

Hence, we obtain the following improvements:

- weaker sufficient semi-local convergence criteria;
- at least as tight estimations on $\|x_n - x_{n-1}\|$;
- at least as precise information on location of solution.

The improvements are obtained with the same information, since the new ω functions are special cases of the old ones in [14]. A further improvement can be obtained if we replace D_0 by $D_1 = D \cap U(x_1, r_0 - \mu)$, where $\mu = \max\{\alpha, \beta, \eta\}$ since $D_1 \subseteq D_0$ provided that $r_0 > \mu$. Then the resulting ω functions will be at least as tight as the ones in Theorem 2.1. Hence, we further extended the applicability of method (1.2). This technique is also applicable to other methods.

3. Numerical experiments. Let $E_1 = E_2 = \mathbb{R}^2$, $D = (1,2) \times (1,2)$ and define $F, Q : D \to \mathbb{R}^2$ by

$$F(x) = \begin{pmatrix} x_1^3 - x_2 - 1 \\ x_1 + x_2^3 - 5 \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} \frac{1}{9}|x_1 - 1| \\ \frac{1}{9}|x_2| \end{pmatrix}.$$

We can write

$$F'(x) = \begin{pmatrix} 3x_1^2 & -1 \\ 1 & 3x_2^2 \end{pmatrix}, \qquad Q(x,y) = \begin{pmatrix} \frac{|x_1-1|-|y_1-1|}{9(x_1-y_1)} & 0 \\ 0 & \frac{|x_2|-|y_2|}{9(x_2-y_2)} \end{pmatrix}.$$

Then

$$w_1^0(\|x-\xi\|) = 3\|A_0^{-1}\| \max_{x\in D} \{|x_1+\xi_1|, |x_2+\xi_2|\}\|x-\xi\|, \qquad \xi = x_0,$$
$$w_1(\|x-y\|) = 3\|A_0^{-1}\| \max_{x,y\in D_0} \{|x_1+y_1|, |x_2+y_2|\}\|x-y\|,$$
$$w_2^0(\|x-u\|, \|y-v\|) = w_2(\|x-u\|, \|y-v\|) = 2\|A_0^{-1}\|/9.$$

Let us choose $x_0 = (1.4800, 1.6000)$, $x_{-1} = (1.4801, 1.6001)$ and $x_{-2} = (1.4802, 1.6002)$. The solution $x^* \approx (1.3529, 1.5152)$ was obtained by method (1.2) after five iterations.

Now, we verify Theorem 2.1. In view of the choice of initial approximations, we obtain $\alpha = \beta = 10^{-4}$, $\eta \approx 0.1163$. The equation $\psi_0(\alpha, \beta, u) = 1$ has the only solution $r_0 \approx 0.4972$. Therefore, $D_0 \approx (1, 1.9772) \times (1.1028, 2)$ and $\gamma \approx 0.1892$. Equation (2.6) has two solutions greater than η, α and β: $u_1 \approx 0.1724$ and $u_2 \approx 0.3353$. Hence, $R \approx 0.1724$, $\psi(\alpha, \beta, R) \approx 0.4189 < 1$ and $\delta \approx 0.3256 < 1$. According to the corresponding theorem in [14], we get $\gamma^1 \approx 0.1892$, $R^1 \approx 0.1808$, $\psi^1(\alpha, \beta, R) \approx 0.4699 < 1$ and $\delta^1 \approx 0.3569 < 1$.

Consequently, Theorem 2.1 is applicable and the unique solution x^* is in $\overline{U(x_0, R)} \subset D$. Moreover, the new contraction factor δ is less than the corresponding one in [14].

Now, we verify Theorem 2.2. Define $D = (1.26, 1.48) \times (1.47, 1.65)$ and choose $x_0 = (1.3700, 1.5300)$, $x_{-1} = (1.3701, 1.5301)$, $x_{-2} = (1.3702, 1.5302)$. Then we have

$$\varepsilon_1^0 = 3\|A_0^{-1}\| \max_{x\in D} \{|x_1^2 - \xi_1^2|, |x_2^2 - \xi_2^2|\}, \qquad \xi = x_0,$$
$$\varepsilon_1 = 3\|A_0^{-1}\| \max_{x,y\in D_0} \{|x_1^2 - y_1^2|, |x_2^2 - y_2^2|\}, \qquad \varepsilon_2^0 = \varepsilon_2 = 2\|A_0^{-1}\|/9.$$

From the corresponding relations we deduce that $\eta \approx 0.0169$, $r_0 \approx 0.4709$, $\lambda \approx 0.6730 < 1$, $R > 0.0517$, $\lambda^1 \approx 0.8394 < 1$, and $R^1 > 0.1054$.

So, $\lambda < \lambda^1$, all conditions of Theorem 2.2 are satisfied and there exists $\tilde{R} > R$ such that $x^* \in \overline{U(x_0, \tilde{R})} \subset D$. However, a condition of the corresponding theorem in [14], $\overline{U(x_0, R^1)} \subset D$, is not fulfilled.

4. Conclusions. In this work, the combined Newton–Potra method for solving nonlinear operator equations was studied under weak ω and ε conditions. Basically, we have extended the applicability of the Newton–Potra method. Therefore, the method under study is an effective alternative for solving nonlinear equations with nondifferentiable operators.

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