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TRIGONOMETRIC BÉZIER-LIKE CURVES AND TRANSITION CURVES

Abstract. In this paper, planar cubic trigonometric Bézier curves with two shape parameters are considered. Appropriate conditions for these curves to be spiral are obtained and transition curves from the straight line to the straight line, from the straight line to the circle, from the circle to the circle with a C-shaped transition curve, from the circle to the circle with an S-shaped transition curve, from a circle to a circle where one of the circles lies inside of another circle are considered with the help of this spiral. Many numerical examples are also provided.

1. Introduction. Bézier curves are basic curves in computer-aided geometric design [5], providing useful features for designers. However, they have some disadvantages. For example, a local change on a Bézier curve causes the change in the entire curve. Moreover, Bézier curves are insufficient to design closed curves. For example, a circle is not described by a Bézier curve. To overcome these problems, rational Bézier curves have been defined. However, it is quite difficult to design a shape with these curves.

Therefore, many curves have been built to handle these situations with the use of other curves instead of Bézier curves (see [2], [3], [4], [20], [21], [22], [23] and references therein). In particular, planar cubic trigonometric Bézier curves with two shape parameters have been introduced by Han, Ma and Huang [12]. These curves eliminate the two deficiencies mentioned above. Moreover, these new curves approach the control polygon better than Bézier curves.

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In highway design, clothoids are used as optimal curves. However, it is very difficult to examine and control such curves. In highway design, there are five cases for transition curves, namely, straight line to circle, circle to circle with a C-shaped transition curve, circle to circle with an S-shaped transition curve, straight line to straight line, and circle to circle with one circle lying inside the other [1]. In [17], Walton and Meek considered Bézier curves and found appropriate conditions for Bézier curves to be spiral curves which are useful for highway designing, railway designing or satellite trajectories; they also investigated transition curves constructed by these spiral curves. Later they handled a more general case in [18]. These studies were continued by various authors [6], [7], [9], [10], [11], [15], [16], [19]. In [8], transition curves were examined with shape parameters.

In this article, we consider planar cubic trigonometric Bézier curves with two shape parameters given in [12]. We show that these curves are spiral curves under certain reasonable conditions. Then we construct transition curves from a straight line to a straight line, a straight line to a circle, a circle to a circle with a C-shaped transition curve, a circle to a circle with an S-shaped transition curve and a circle to a circle when one of the circles lies inside the other, with the help of those spirals. Since these curves approach the control polygon better than Bézier curves, the resulting transition curves are more controllable for designers. We also provide examples for every condition.

2. Preliminaries. For $t \in [0,1]$, the following four trigonometric polynomials are termed cubic trigonometric basis functions with two shape parameters λ and μ, where $\lambda, \mu \in [-2,1]$ [12]. Basis functions of planar cubic trigonometric Bézier curves with two shape parameters are defined by

$$\begin{aligned}
C_0(t) &= \left(1-\sin\tfrac{\pi t}{2}\right)^2\left(1-\lambda\sin\tfrac{\pi t}{2}\right),\\
C_1(t) &= \sin\tfrac{\pi t}{2}\left(1-\sin\tfrac{\pi t}{2}\right)\left(2+\lambda-\lambda\sin\tfrac{\pi t}{2}\right),\\
C_2(t) &= \cos\tfrac{\pi t}{2}\left(1-\cos\tfrac{\pi t}{2}\right)\left(2+\mu-\mu\cos\tfrac{\pi t}{2}\right),\\
C_3(t) &= \left(1-\cos\tfrac{\pi t}{2}\right)^2\left(1-\mu\cos\tfrac{\pi t}{2}\right).
\end{aligned}\tag{1}$$

The basis functions (1) have certain properties similar to those of Bernstein polynomials:

Theorem 1 ([12]). *The basis functions have the following properties:*

(a) Nonnegativity: $C_i(t) \geq 0$, $i = 0,1,2,3$.
(b) Partition of unity: $\sum_{i=0}^{3} C_i(t) = 1$.
(c) Monotonicity: *For a given parameter t, $C_0(t)$ and $C_3(t)$ are decreasing in λ and μ, respectively; similarly $C_1(t)$ and $C_2(t)$ are increasing in λ and μ, respectively. See Figure* 1.

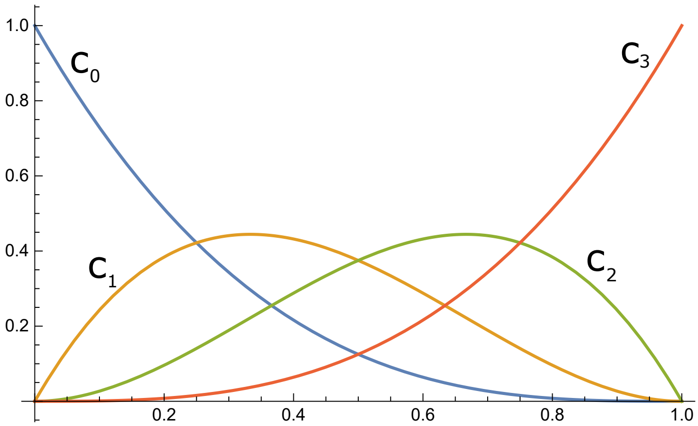

Fig. 1. Basis functions of trigonometric Bézier-like curves

(d) Symmetry: $C_0(t) = C_3(1-t)$ *and* $C_1(t) = C_2(1-t)$.

Proof. (a) For $t \in [0,1]$ and $\lambda, \mu \in [-2,1]$, we have

$$\begin{aligned}
&1 - \sin\tfrac{\pi t}{2} \geq 0,\\
&1 - \cos\tfrac{\pi t}{2} \geq 0,\\
&\sin\tfrac{\pi t}{2} \geq 0,\\
&1 - \lambda\sin\tfrac{\pi t}{2} \geq 0,\\
&2 + \lambda - \lambda\sin\tfrac{\pi t}{2} \geq 0,\\
&2 + \mu - \mu\cos\tfrac{\pi t}{2} \geq 0.
\end{aligned} \tag{2}$$

Therefore, $C_i(t) \geq 0$, $i = 0,1,2,3$.

(b) Using (1), we easily obtain

$$\sum_{i=0}^{3} C_i(t) = 1. \tag{3}$$

(c) We can obtain this case obviously. See Figure 1.

(d) This is obvious.

For total positivity of this basis, see also [13] and [14]. ■

Given control points P_i $(i = 0,1,2,3)$ in $\mathbb{R}^2$, planar cubic trigonometric Bézier curves with two shape parameters determined by these points are defined as (see [12])

$$\mathbf{f}(t) = \sum_{i=0}^{3} C_i(t)\mathbf{P}_i, \qquad t \in [0,1]. \tag{4}$$

Let $\mathbf{f}'(t)$ be the tangent vector of the curve (4). If $\mathbf{f}'(t) \neq 0$, the curvature $k(t)$ is defined as (see [5])

$$k(t) = \frac{\|\mathbf{f}'(t) \times \mathbf{f}''(t)\|}{\|\mathbf{f}'(t)\|^3}, \tag{5}$$

where $\times$ denotes cross-product and $\|\cdot\|$ is the Euclidean norm. The signed radius is the reciprocal of (5) and positive angles are measured anti-clockwise. We note, for G^2 continuity, the two curves must have the same curvature at the contact point.

3. Planar cubic trigonometric Bézier spiral curves with two shape parameters. In this section, we are going to obtain a planar cubic trigonometric Bézier spiral curve with two shape parameters under some conditions.

THEOREM 2. *Given a starting point P_0, starting unit tangent vector and unit normal vector T_0 and N_0, respectively, and an ending curvature value c, let control points of a planar cubic trigonometric Bézier curve with two shape parameters be*

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{P}_0 + a\mathbf{T}_0, \\ \mathbf{P}_2 &= \mathbf{P}_1 + b\mathbf{T}_0, \\ \mathbf{P}_3 &= \mathbf{P}_2 + d\cos(\phi)\mathbf{T}_0 + d\sin(\phi)\mathbf{N}_0 \end{aligned} \tag{6}$$

where ϕ is the anti-clockwise angle from $\mathbf{P}_3 - \mathbf{P}_2$ to $\mathbf{P}_2 - \mathbf{P}_1$, $a = \|\mathbf{P}_1 - \mathbf{P}_0\|$, $b = \|\mathbf{P}_2 - \mathbf{P}_1\|$ and $d = \|\mathbf{P}_3 - \mathbf{P}_2\|$ (see Figure 2). If

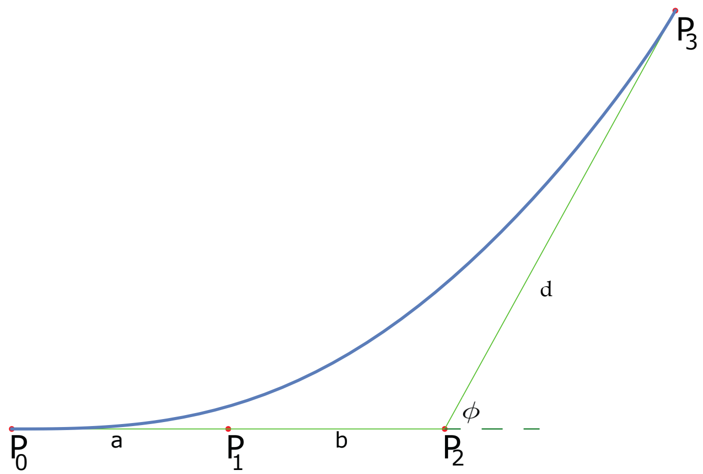

Fig. 2. Planar cubic trigonometric Bézier spiral curve with two shape parameters and its control points

$$a = b = \frac{2(1+2\mu)^2 \sin\phi\,(\cos\phi + \sin\phi)^2}{c(2+\mu)^2}, \tag{7}$$

$$d = \frac{2(1+2\mu)\sin\phi\,(\cos\phi + \sin\phi)}{c(2+\mu)^2}, \tag{8}$$

$\lambda = \mu = 1$ and $0 < \phi < \pi/2$, then this planar cubic trigonometric Bézier curve with two shape parameters is a spiral curve.

Proof. Let $\mathbf{f}(t)$ be defined by (4). Since

$$\begin{aligned}\mathbf{f}'(t) = \tfrac{\pi}{8}\Big(&(\mathbf{P}_1 - \mathbf{P}_0)\big((8+7\lambda)\cos\big(\tfrac{\pi t}{2}\big) - 3\lambda\cos\big(\tfrac{3\pi t}{2}\big)\big)\\ &- (\mathbf{P}_3 + \mathbf{P}_2)(8+7\mu)\sin\big(\tfrac{\pi t}{2}\big)\\ &+ 4\big(\mathbf{P}_0(1+2\lambda) + 2(-\mathbf{P}_1(1+\lambda) + (\mathbf{P}_2+\mathbf{P}_3)(1+\mu))\big)\sin(\pi t)\\ &- 3(\mathbf{P}_2+\mathbf{P}_3)\mu\sin\big(\tfrac{3\pi t}{2}\big)\Big)\end{aligned} \tag{9}$$

and

$$\begin{aligned}\mathbf{f}''(t) = \tfrac{\pi^2}{16}\Big(&-(\mathbf{P}_2+\mathbf{P}_3)(8+7\mu)\cos\big(\tfrac{\pi t}{2}\big)\\ &+ 8\big(\mathbf{P}_0 + 2\mathbf{P}_0\lambda + 2(-\mathbf{P}_1(1+\lambda) + (\mathbf{P}_2+\mathbf{P}_3)(1+\mu))\big)\cos(\pi t)\\ &- 9(\mathbf{P}_2+\mathbf{P}_3)\mu\cos\big(\tfrac{3\pi t}{2}\big)\\ &+ 2(\mathbf{P}_1-\mathbf{P}_0)\big(-4+\lambda+9\lambda\cos(\pi t)\sin\big(\tfrac{\pi t}{2}\big)\big)\Big),\end{aligned} \tag{10}$$

the curvature of $\mathbf{f}(t)$ is obtained as

$$\begin{aligned}\kappa(t) = {}&\\ &\Big\{\Big(a\cos\big(\tfrac{\pi t}{2}\big)(4+5\lambda-3\lambda\cos(\pi t)) + \Big(-4\cos\big(\tfrac{\pi t}{2}\big)(a(1+2\lambda) - b\cos\theta)\\ &+ 4d\big(2+\mu-3\mu\cos\big(\tfrac{\pi t}{2}\big)\big)\cos(\theta+\phi)\sin^2\big(\tfrac{\pi t}{2}\big)\Big)\sin\big(\tfrac{\pi t}{2}\big)\Big)\\ &\times\Big(8b\cos(\pi t)\sin\theta - 4d\Big((\mu-4)\big(1+2\cos\big(\tfrac{\pi t}{2}\big)\big) + 9\mu\cos(\pi t)\Big)\sin^2\big(\tfrac{\pi t}{4}\big)\\ &\times\sin(\phi+\theta)\Big) - 8\Big(-4\cos(\pi t)(a(1+2\lambda) - b\cos\theta) - 2d\\ &\times\Big((\mu-4)\big(1+2\cos\big(\tfrac{\pi t}{2}\big)\big) + 9\mu\cos(\pi t)\Big)\cos(\phi+\theta)\sin^2\big(\tfrac{\pi t}{4}\big)\\ &+ a(\lambda-4+9\lambda\cos(\pi t))\sin\big(\tfrac{\pi t}{2}\big)\Big)\Big(\tfrac{1}{2}b\sin(\pi t)\sin\theta + 2d\cos\big(\tfrac{\pi t}{4}\big)\\ &\times\big(2+\mu-3\mu\cos\big(\tfrac{\pi t}{2}\big)\big)\sin^3\big(\tfrac{\pi t}{4}\big)\sin(\phi+\theta)\Big)\Big\}\\ &\times\Big\{\Big(a\cos\big(\tfrac{\pi t}{2}\big)(4+5\lambda-3\lambda\cos(\pi t)) + \Big(-4\cos\big(\tfrac{\pi t}{2}\big)(a(1+2\lambda) - b\cos\theta)\\ &+ 4d\big(2+\mu-3\mu\cos\big(\tfrac{\pi t}{2}\big)\big)\cos(\theta+\phi)\sin^2\big(\tfrac{\pi t}{2}\big)\Big)\sin\big(\tfrac{\pi t}{2}\big)\Big)^2 + 16\Big(\tfrac{b}{2}\sin(\pi t)\\ &\times\sin\theta + 2d\cos\big(\tfrac{\pi t}{4}\big)\big(2+\mu-3\mu\cos\big(\tfrac{\pi t}{2}\big)\big)\sin^3\big(\tfrac{\pi t}{4}\big)\sin(\phi+\theta)\Big)^2\Big\}^{-3/2},\end{aligned} \tag{11}$$

where θ is the anti-clockwise angle from $\mathbf{P}_2 - \mathbf{P}_1$ to $\mathbf{P}_1 - \mathbf{P}_0$ (see Figure 3).

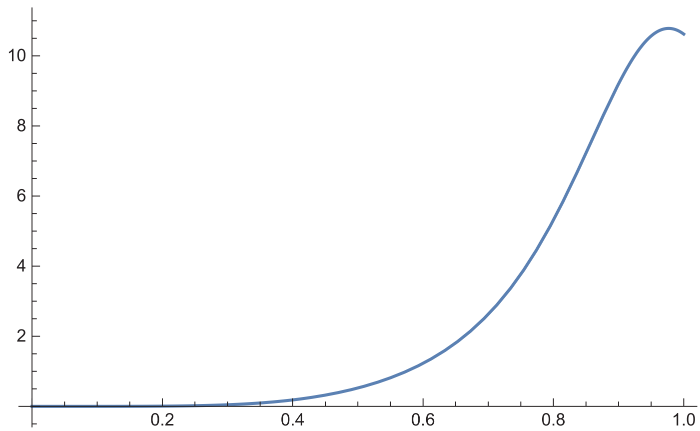

Fig. 3. Curvature function of a trigonometric Bézier curve

The curvature at the starting point is

$$\kappa(0) = \frac{2b\sin\theta}{(3+\lambda)^2 a^2}. \tag{12}$$

If $\kappa(0) = 0$, we obtain $\theta = 0$. Using $\theta = 0$ and $\kappa(1) = c$, we get

$$b = \frac{cd^2(2+\mu)^2}{2\sin\phi}. \tag{13}$$

From $\kappa(1) = c$, we easily deduce that $\kappa'(1) = 0$. Thus by using the equation

$$\begin{aligned}\kappa'(1) &= \frac{3c\pi(1+\cot\phi)(cd(2+\mu)^2 - 2(1+2\mu)\sin\phi(\cos\phi+\sin\phi))}{2(2+\mu)} \\ &= 0,\end{aligned} \tag{14}$$

we obtain

$$d = \frac{2(1+2\mu)\sin\phi(\cos\phi+\sin\phi)}{c(2+\mu)^2}. \tag{15}$$

If

$$a = b = \frac{2(1+2\mu)^2\sin\phi(\cos\phi+\sin\phi)^2}{c(2+\mu)^2}, \tag{16}$$

$$d = \frac{2(1+2\mu)\sin\phi(\cos\phi+\sin\phi)}{c(2+\mu)^2}, \tag{17}$$

$\lambda = \mu = 1$, $\theta = 0$ and $0 < \phi < \pi/2$ then the curvature function $\kappa'(t)$ does not change sign in $t \in [0,1]$ (Figure 4). Therefore $\mathbf{f}(t)$ is a spiral curve (Figure 5). ■

3.1. Cubic trigonometric Bézier-like transition curves. In this section, we will construct transition curves from a straight line to a circle,

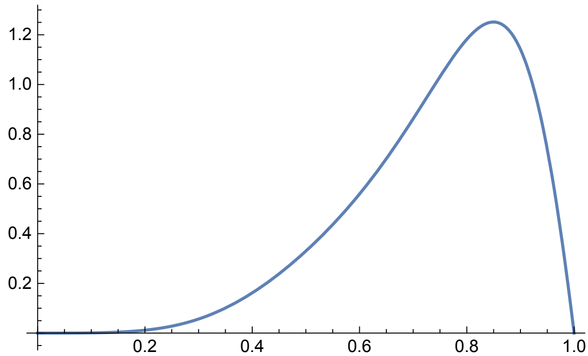

Fig. 4. The curvature function of a trigonometric spiral curve

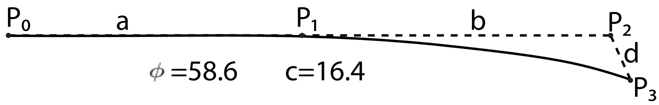

Fig. 5. A trigonometric spiral curve

from a circle to a circle with a C-shaped transition, from a circle to a circle with an S-shaped transition, from a straight line to a straight line, from a circle to a circle, one circle inside the other, by using the spiral curve obtained in Theorem 2.

3.1.1. *A transition spiral curve from a straight line to a circle.* In this subsection, we obtain a transition cubic trigonometric Bézier spiral curve with two shape parameters from a straight line to a circle and provide an example.

THEOREM 3. *Given a point* $\mathbf{P}$ *and a unit tangent vector* $\mathbf{T}$, *let* $\mathbf{d}$ *be a line through* $\mathbf{P}$ *parallel to* $\mathbf{T}$. *Let* M *be a circle centred at* $\mathbf{O}$ *with radius* $r > 0$, *where* $\mathbf{T} \times (\mathbf{O} - \mathbf{P}) > 0$ *and* $\mathbf{L} = \mathbf{O} - \mathbf{P}$. *Let* h *be the perpendicular distance from* $\mathbf{O}$ *to* $\mathbf{d}$. *If* $1 < h/r < 5\sqrt{2}/6$ *and* $\phi \in (0, \pi/4)$ *then there is a unique cubic trigonometric Bézier spiral as defined in Theorem 2 that joins* $\mathbf{d}$ *to* M *such that all points of contact are* G^2. *The angle between* $\mathbf{T}$ *and* $\mathbf{T}_1$ *of this spiral satisfies*

$$\frac{2\sin^2\phi\,(\cos\phi + \sin\phi)}{3} - \frac{h}{r} + \cos\phi = 0. \tag{18}$$

$\mathbf{T}_1$ *is the tangent vector at the endpoint of the spiral curve.*

Proof. At the starting point, the normal vector is

$$\mathbf{N} = \frac{\mathbf{L} - (\mathbf{L}.\mathbf{T})\mathbf{T}}{\|\mathbf{L} - (\mathbf{L}.\mathbf{T})\mathbf{T}\|}. \tag{19}$$

Here $(\mathbf{L}.\mathbf{T})$ denotes the dot product of two vectors. Since $\mathbf{f}(0)$ lies on the $\mathbf{d}$ and $\mathbf{f}(1)$ lies on M, it follows that

$$\mathbf{f}(0) = \mathbf{P} + \sigma \mathbf{T} \tag{20}$$

and

$$\mathbf{f}(1) = \mathbf{P} + (\mathbf{L}.\mathbf{T} + r\sin\phi)\mathbf{T} + (h - r\cos\phi)\mathbf{N}. \tag{21}$$

From Theorem 2, it follows that

$$\begin{aligned} \mathbf{f}(0) &= \mathbf{P}_0, \\ \mathbf{f}(1) &= \mathbf{P}_3 = \mathbf{P}_0 + 2a\mathbf{T} + d\mathbf{T}_1, \\ a = b &= \frac{2(1+2\mu)^2 \sin\phi\,(\cos\phi + \sin\phi)^2}{c(2+\mu)^2}, \\ d &= \frac{2(1+2\mu)\sin\phi\,(\cos\phi+\sin\phi)}{c(2+\mu)^2} \end{aligned} \tag{22}$$

and $\lambda = \mu = 1$. Inserting these equations in (20) and (21), we get

$$\begin{aligned} &\mathbf{f}(0) = \mathbf{P} + \sigma\mathbf{T} = \mathbf{P}_0, \\ &\mathbf{P} = \mathbf{P}_0 - \sigma\mathbf{T}, \\ &\mathbf{f}(1) = \mathbf{P} + (\mathbf{L}.\mathbf{T} + r\sin\phi)\mathbf{T} + (h - r\cos\phi)\mathbf{N} = \mathbf{P}_0 + 2a\mathbf{T} + d\mathbf{T}_1, \\ &\mathbf{P} = \mathbf{P}_0 - \sigma\mathbf{T} = \mathbf{P}_0 + 2a\mathbf{T} + d\mathbf{T}_1 - (\mathbf{L}.\mathbf{T} + r\sin\phi)\mathbf{T} - (h - r\cos\phi)\mathbf{N} \end{aligned}$$

and

$$\sigma\mathbf{T} + d\mathbf{T}_1 - (\mathbf{L}.\mathbf{T} + r\sin\phi - 2a)\mathbf{T} - (h - r\cos\phi)\mathbf{N} = 0. \tag{23}$$

From (23), taking the dot product of this equation with $\mathbf{N}$ and using $\mathbf{T}.\mathbf{N} = 0$, $\mathbf{T}_1.\mathbf{N} = \sin\phi$, we obtain

$$\begin{aligned} \sigma\mathbf{T}.\mathbf{N} + d\mathbf{T}_1.\mathbf{N} - (\mathbf{L}.\mathbf{T} + r\sin\phi - 2a)\mathbf{T}.\mathbf{N} - (h - r\cos\theta)\mathbf{N}.\mathbf{N} &= 0, \\ d\sin\phi - (h - r\cos\phi) &= 0. \end{aligned}$$

Since $c = 1/r$ and the curve is a spiral, we obtain

$$r\frac{2\sin\phi\,(\cos\phi+\sin\phi)}{3}\sin\phi - (h - r\cos\phi) = 0.$$

Let

$$\begin{aligned} q(\phi) &= \frac{1}{r}\left(r\frac{2\sin\phi\,(\cos\phi+\sin\phi)}{3}\sin\phi - (h - r\cos\phi)\right) \\ &= \frac{2\sin\phi\,(\cos\phi+\sin\phi)}{3}\sin\phi - \frac{h}{r} + \cos\phi \\ &= \frac{2\sin^2\phi\,(\cos\phi+\sin\phi)}{3} - \frac{h}{r} + \cos\phi. \end{aligned} \tag{24}$$

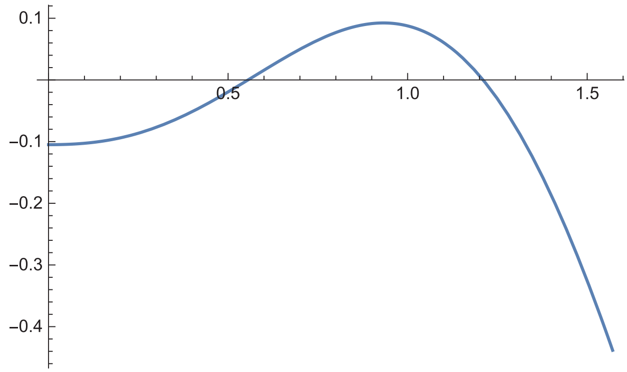

Fig. 6. The graph of $q(\phi)$

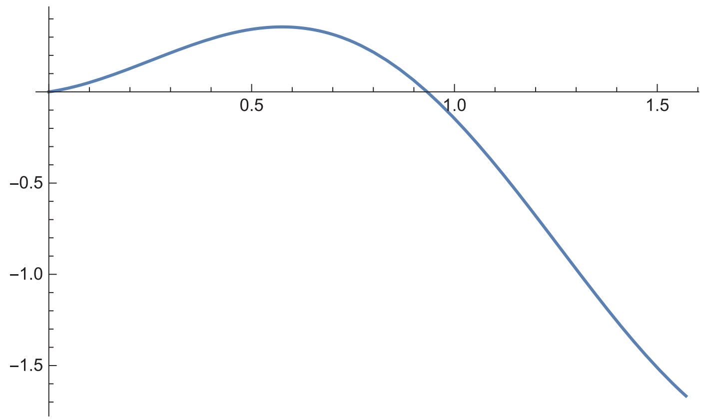

Fig. 7. The graph of $q'(\phi)$

If $\phi = 0$, then

$$q(0) = 1 - \frac{h}{r}.$$

By assumption (3) and $r < h$, we derive

$$q(0) = 1 - \frac{h}{r} < 0.$$

For $\phi = \pi/2$, it follows that

$$q(\pi/2) = \frac{2}{3} - \frac{h}{r}. \tag{25}$$

Since

$$1 < \frac{h}{r}, \tag{26}$$

we obtain

$$q(\pi/2) = \frac{2}{3} - \frac{h}{r} < 0. \tag{27}$$

But, for some values of ϕ, we get $q(\phi) > 0$ (Figure 6). For example, if $\phi = \pi/3$, then

$$q\left(\frac{\pi}{3}\right) = \frac{2\sin^2\frac{\pi}{3}\left(\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\right)}{3} - \frac{h}{r} + \cos\frac{\pi}{3} = \frac{\sqrt{3}+3}{4} - \frac{h}{r}. \tag{28}$$

Here,

$$q\left(\frac{\pi}{3}\right) > 0 \tag{29}$$

when

$$1 < \frac{h}{r} < \frac{\sqrt{3}+3}{4}. \tag{30}$$

From (24), we obtain

$$\begin{aligned} q'(\phi) &= \tfrac{1}{3}\sin\phi\,(-2\sin^2\phi + 4\cos^2\phi + 6\cos\phi\sin\phi - 3) \\ &= \tfrac{1}{3}\sin\phi\,(-5 + 6\cos^2\phi + 6\cos\phi\sin\phi) \\ &= \tfrac{1}{3}\sin\phi\,(3(\sin 2\phi + \cos 2\phi) - 2) \\ &= \tfrac{1}{3}\sin\phi\,(3(\sin 2\phi + \cos 2\phi - 1) + 1) \\ &= \tfrac{1}{3}\sin\phi\,(6(\cos\phi - \sin\phi)\sin\phi + 1) \end{aligned}$$

(see Figure 7). If $\cos\phi - \sin\phi > 0$, then $q'(\phi) > 0$. Therefore, we set $0 < \phi < \pi/4$. Since

$$q\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{6} - \frac{h}{r}, \tag{31}$$

if

$$1 < \frac{h}{r} < \frac{5\sqrt{2}}{6} \tag{32}$$

and $0 < \phi < \pi/4$, we conclude that

$$q(0) < 0, \qquad q(\phi) > 0, \qquad q'(\phi) > 0.$$

So, (18) has a unique solution for $0 < \phi < \pi/4$ (see also Figure 8). In that case, the formula for σ is

$$\sigma = (\mathbf{L}.\mathbf{T} + r\sin\phi - 2a) - d\cos\phi. \;\blacksquare \tag{33}$$

We now give a numerical example for this transition curve.

Example 1. We choose $\mathbf{P} = (0,0)$ as a point, $\mathbf{T} = (1,0)$ and $\mathbf{N} = (0,1)$ as a unit tangent vector and a unit normal vector at $\mathbf{P}$, respectively. Then $\mathbf{d} : y = 0$ is the straight line through $\mathbf{P}$ parallel to $\mathbf{T}$. For a circle M with radius 5.8 and centre $\mathbf{O} = (1,6)$, the perpendicular distance from $\mathbf{O}$ to $\mathbf{d}$ is $h = 6$. The sufficient condition of Theorem 3, $1 < h/r < 5\sqrt{2}/6$, is fulfilled. Also, $\lambda = \mu = 1$ and $\theta = 0$. We solve (18) for ϕ and obtain a unique solution for $0 < \phi < \pi/4$, $\phi = 0.328029$. We use Mathematica 10.4 to obtain the

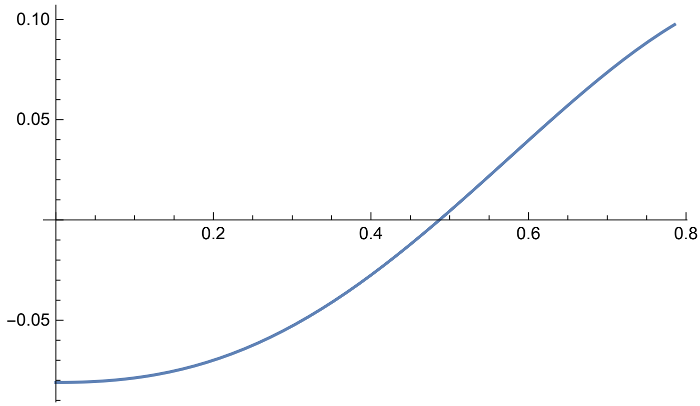

Fig. 8. The graph of $q(\phi)$ where $\phi \in [0, \pi/4]$ and $r = 0.74, h = 0.80$

solution and we also give the angles in radians. Using ϕ in (33), we obtain $\sigma = -10.6617$. Therefore, the starting point of the spiral is $\mathbf{P}_0 = (\sigma, 0)$. From Theorem 2, the other control points are $\mathbf{P}_1 = (-4.64475, 0)$, $\mathbf{P}_2 = (1.37223, 0)$, $\mathbf{P}_3 = (2.86863, 0.50926)$. We also obtain $a = b = 6.01698$ and $d = 1.58068$. Thus the equation of the spiral curve (see Figure 9) is obtained as

$$\begin{aligned} x(t) &= -7.79309 - 4.4892\cos\left(\tfrac{\pi t}{2}\right) + 3.11697\cos^2\left(\tfrac{\pi t}{2}\right) - 1.4964\cos^3\left(\tfrac{\pi t}{2}\right) \\ &\quad + 18.0509\sin\left(\tfrac{\pi t}{2}\right) - 13.4062\sin^2\left(\tfrac{\pi t}{2}\right) + 6.01698\sin^3\left(\tfrac{\pi t}{2}\right), \\ y(t) &= -0.50926\left(\cos\left(\tfrac{\pi t}{2}\right) - 1\right)^3. \end{aligned}$$

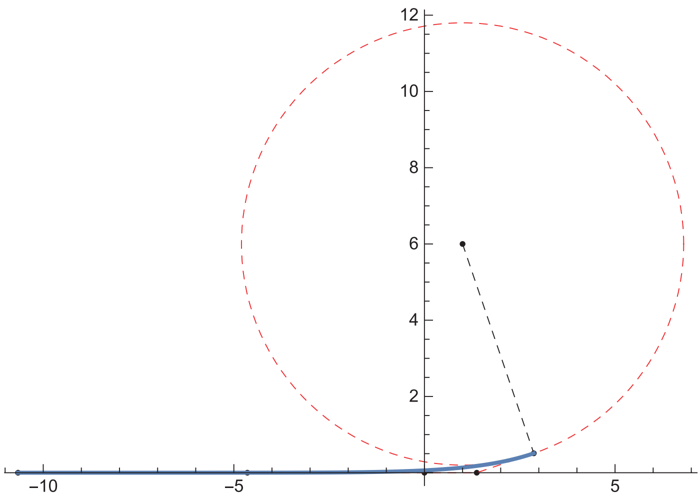

Fig. 9. A spiral transition curve from a straight line to a circle

3.1.2. *A C-shaped transition spiral curve from a circle to a circle.* In this subsection, a special transition spiral curve from a circle to a circle is considered. It is a C-shaped curve. A numerical example is also provided.

THEOREM 4. *Given two circles M_0 and M_1, centered at $\mathbf{O}_0$ and $\mathbf{O}_1$ with radii $r_0 < 0$ and $r_1 > 0$. Here, minus is used to define the direction. If one circle does not enclose the other, i.e.*

$$\big||r_1| - |r_0|\big| = |r_1 + r_0| < \|\mathbf{O}_1 - \mathbf{O}_0\| < \frac{23\sqrt{2}}{3}|r_1 - r_0|$$

and $\phi \in (0, \pi/4)$, then the circles M_0 and M_1 can be joined by a pair of planar trigonometric cubic Bézier-like spirals given in Theorem 2 forming a C-shaped transition spiral curve such that all points of contact are G^2.

Proof. Let a trigonometric cubic Bézier-like spiral that meets M_0 be $\mathbf{f}_0(t)$ and another one that meets M_1 be $\mathbf{f}_1(t)$. Two Bézier-like spirals have fourteen degrees of freedom. These are the starting and ending points, unit tangent vectors at the starting and ending points, the angle between these unit tangent vectors, the value of the curvature at the ending point and the λ and μ shape parameters for $\mathbf{f}_0(t)$, and similarly for $\mathbf{f}_1(t)$. Let two spirals be joined at the starting points to ensure G^2 at the meeting points of the trigonometric cubic Bézier-like spirals. At the starting points, they both have zero curvature. From the spiral condition, $\lambda = 1$ and $\mu = 1$. Let $-\mathbf{T}$ and $\mathbf{T}$ be the unit tangent vectors of $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$ at the starting points, respectively. $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$ meet circles so that contact points are G^2. There is one remaining degree. For this degree of freedom, according to Theorem 2, the angle from $-\mathbf{T}$ to $\mathbf{T}_0$ is ϕ_0 and the angle from $\mathbf{T}$ to $\mathbf{T}_1$ is ϕ_1 where $-\pi/4 < \phi_0 < 0$ and $0 < \phi_1 < \pi/4$. Here, $\mathbf{T}_0$ and $\mathbf{T}_1$ are the unit tangent vectors of $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$ at the endpoints of the curves, respectively. Set $\phi = -\phi_0 = \phi_1$; then $0 < \phi < \pi/4$ is the remaining degree of freedom (see Figure 10). Therefore, it follows from Theorem 2 that

$$\begin{aligned}\mathbf{f}_0(1) &= \mathbf{P}_0 + (a_0 + b_0)\mathbf{T} - d_0\mathbf{T}_0,\\ \mathbf{f}_1(1) &= \mathbf{P}_0 + (a_1 + b_1)\mathbf{T} + d_1\mathbf{T}_1.\end{aligned}$$

Here,

$$\begin{aligned}a_0 = b_0 &= \frac{2r_0(1+2\mu)^2 \sin\phi\,(\cos\phi + \sin\phi)^2}{(2+\mu)^2},\\ d_0 &= \frac{2r_0(1+2\mu)\sin\phi\,(\cos\phi + \sin\phi)}{(2+\mu)^2},\\ a_1 = b_1 &= \frac{2r_1(1+2\mu)^2 \sin\phi\,(\cos\phi + \sin\phi)^2}{(2+\mu)^2},\\ d_1 &= \frac{2r_1(1+2\mu)\sin\phi\,(\cos\phi + \sin\phi)}{(2+\mu)^2}.\end{aligned}$$

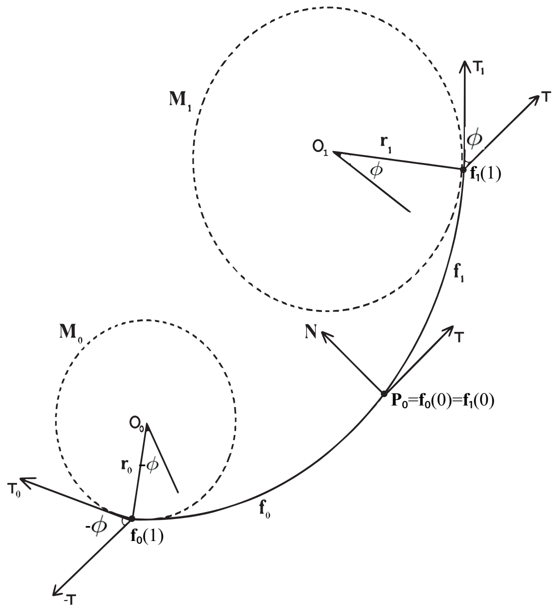

Fig. 10. A C-shaped transition spiral curve from a circle to a circle

So

$$\mathbf{f}_1(1) - \mathbf{f}_0(1) = \frac{(r_1 - r_0)4(1+2\mu)^2 \sin\phi\,(\cos\phi + \sin\phi)^2}{(2+\mu)^2}\mathbf{T} + \frac{2(1+2\mu)\sin\phi\,(\cos\phi+\sin\phi)}{(2+\mu)^2}(r_1\mathbf{T}_1 + r_0\mathbf{T}_0). \tag{34}$$

Here, we assume that the orientation of the unit tangent vector $\mathbf{T}$ is $\mathbf{T}.(\mathbf{O}_1 - \mathbf{O}_0) > 0$. Therefore, we select one of two solutions. One can choose $\mathbf{T}.(\mathbf{O}_1 - \mathbf{O}_0) < 0$ to obtain the other solution. It follows that

$$\{(\mathbf{f}_1(1) - \mathbf{O}_1) - (\mathbf{f}_0(1) - \mathbf{O}_0)\}.\mathbf{N} = -(r_1 + r_0)\cos\phi, \tag{35}$$

$$\{(\mathbf{f}_1(1) - \mathbf{O}_1) - (\mathbf{f}_0(1) - \mathbf{O}_0)\}.\mathbf{T} = (r_1 - r_0)\sin\phi, \tag{36}$$

where the angle from $\mathbf{T}$ to $\mathbf{T}_1$ is ϕ. Using (34) and

$$\begin{aligned} \mathbf{T}_1.\mathbf{T} &= -\mathbf{T}_0.\mathbf{T} = \cos\phi, \\ \mathbf{T}_1.\mathbf{N} &= \mathbf{T}_0.\mathbf{N} = \sin\phi, \end{aligned} \tag{37}$$

we obtain

$$\{\mathbf{f}_1(1) - \mathbf{f}_0(1)\}.\mathbf{N} = \frac{2(1+2\mu)\sin^2\phi\,(\cos\phi+\sin\phi)}{(2+\mu)^2}(r_1 + r_0) \tag{38}$$

and

$$\{\mathbf{f}_1(1)-\mathbf{f}_0(1)\}.\mathbf{T} = \frac{(r_1-r_0)4(1+2\mu)^2\sin\phi\,(\cos\phi+\sin\phi)^2}{(2+\mu)^2} + \frac{2(1+2\mu)\sin\phi\,(\cos\phi+\sin\phi)}{(2+\mu)^2}(r_1\cos\phi - r_0\cos\phi). \tag{39}$$

It also follows by using (35) and (36) that

$$\{(\mathbf{O}_1-\mathbf{O}_0)\}.\mathbf{N} = (r_1+r_0)g_1(\phi), \tag{40}$$

$$\{(\mathbf{O}_1-\mathbf{O}_0)\}.\mathbf{T} = (r_1-r_0)g_2(\phi). \tag{41}$$

Since

$$\{(\mathbf{O}_1-\mathbf{O}_0)\}.\mathbf{N} = \{(\mathbf{f}_1(1)-\mathbf{f}_0(1))\}.\mathbf{N} + (r_1+r_0)\cos\phi = (r_1+r_0)g_1(\phi),$$

from (38) we get

$$(r_1+r_0)g_1(\phi) = \frac{2(1+2\mu)\sin^2\phi\,(\cos\phi+\sin\phi)}{(2+\mu)^2}(r_1+r_0) + (r_1+r_0)\cos\phi.$$

Thus, we obtain

$$g_1(\phi) = \frac{2(1+2\mu)\sin^2\phi\,(\cos\phi+\sin\phi)}{(2+\mu)^2} + \cos\phi. \tag{42}$$

Similarly, since

$$\{(\mathbf{O}_1-\mathbf{O}_0)\}.\mathbf{T} = \{(\mathbf{f}_1(1)-\mathbf{f}_0(1))\}.\mathbf{T} - (r_1-r_0)\sin\phi = (r_1-r_0)g_2(\phi),$$

using (39) we get

$$g_2(\phi) = \frac{2(1+2\mu)\sin\phi\,(\cos\phi+\sin\phi)}{(2+\mu)^2}\big(2(1+2\mu)(\cos\phi+\sin\phi)+\cos\phi\big) - \sin\phi. \tag{43}$$

By using Theorem 2, (42) and (43), we have

$$g_1(\phi) = \frac{2\sin^2\phi\,(\cos\phi+\sin\phi)}{3} + \cos\phi, \tag{44}$$

$$g_2(\phi) = \frac{2\sin\phi\,(\cos\phi+\sin\phi)(7\cos\phi+6\sin\phi)}{3} - \sin\phi. \tag{45}$$

Hence, the pair of trigonometric cubic Bézier-like spirals is obtained by solving the equation

$$(r_1+r_0)^2\{g_1(\phi)\}^2 + (r_1-r_0)^2\{g_2(\phi)\}^2 = \|\mathbf{O}_1-\mathbf{O}_0\|^2. \tag{46}$$

We set

$$q(\phi) = (r_1+r_0)^2\{g_1(\phi)\}^2 + (r_1-r_0)^2\{g_2(\phi)\}^2 - \|\mathbf{O}_1-\mathbf{O}_0\|^2.$$

Then it follows that

$$g_1(0) = 1, \quad g_2(0) = 0.$$

From these equations, we have

$$q(0) = (r_1 + r_0)^2 - \|\mathbf{O}_1 - \mathbf{O}_0\|^2. \tag{47}$$

Since $|r_1 - r_0| < \|\mathbf{O}_1 - \mathbf{O}_1\|$, we have

$$|r_1 - r_0|^2 < \|\mathbf{O}_1 - \mathbf{O}_1\|^2, \quad q(0) < 0.$$

Also, since

$$g_1(\pi/4) = \frac{5\sqrt{2}}{6} > 0 \quad \text{and} \quad g_2(\pi/4) = \frac{23\sqrt{2}}{6} > 0,$$

we obtain

$$\begin{aligned} q(\pi/4) &= (r_1 + r_0)^2 \cdot \frac{50}{36} + (r_1 - r_0)^2 \cdot \frac{1058}{36} - \|\mathbf{O}_1 - \mathbf{O}_0\|^2 \\ &= (r_1 - r_0)^2 \left(\frac{(r_1 + r_0)^2}{(r_1 - r_0)^2} \cdot \frac{50}{36} + \frac{1058}{36} - \frac{\|\mathbf{O}_1 - \mathbf{O}_0\|^2}{(r_1 - r_0)^2} \right). \end{aligned}$$

By assumption, we have $\|\mathbf{O}_1 - \mathbf{O}_0\| < \frac{23\sqrt{2}}{6}|r_1 - r_0|$. Therefore, we derive

$$q(\pi/4) > 0.$$

We put

$$q'(\phi) = 2(r_1 + r_0)^2 g_1(\phi) g_1'(\phi) + 2(r_1 - r_0)^2 g_2(\phi) g_2'(\phi),$$

where

$$\begin{aligned} g_1'(\phi) &= \tfrac{1}{6}(3\cos\phi - 3\cos(3\phi) - 7\sin\phi + 3\sin(3\phi)), \\ g_2'(\phi) &= \tfrac{1}{6}(31\cos\phi - 33\cos(3\phi) - 13\sin\phi + 39\sin(3\phi)). \end{aligned}$$

Then

$$g_1'(\pi/4) = \frac{\sqrt{2}}{6} > 0 \quad \text{and} \quad g_2'(\pi/4) = \frac{15\sqrt{2}}{2} > 0.$$

Hence, we conclude that

$$q(0) < 0, \qquad q(\phi) > 0, \qquad q'(\phi) > 0.$$

Therefore (46) has a unique solution for $0 < \theta < \pi/4$ and $\big||r_1| - |r_0|\big| = |r_1 + r_0| < \|\mathbf{O}_1 - \mathbf{O}_0\| < \frac{23\sqrt{2}}{6}|r_1 - r_0|$. By solving (46) for ϕ, we determine $\mathbf{T}$, $\mathbf{N}$, $\mathbf{T}_0$ and $\mathbf{T}_1$. Then, applying Theorem 2, the pair of trigonometric cubic Bézier-like spirals are obtained. ∎

Example 2. Let M_0 be the circle with radius $r_0 = -0.5 < 0$ and centre $\mathbf{O}_0 = (-4, 1)$ and M_1 be the circle with radius $r_1 = 2 > 0$ and centre $\mathbf{O}_1 = (5, 3)$. Here,

$$\begin{gathered} \big||r_1| - |r_0|\big| = |r_1 + r_0| < \|\mathbf{O}_1 - \mathbf{O}_0\| < \frac{23\sqrt{2}}{6}|r_1 - r_0|, \\ 1.5 < 9.21954 < 13.5529. \end{gathered}$$

Let $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$ be trigonometric Bézier spiral curves with two shape parameters. For these spirals, $\lambda = 1$, $\mu = 1$ and also $\theta = 0$. Choose $\mathbf{P} = (0, 0)$. $\mathbf{P}$ is the starting point for both spirals. $\mathbf{T}$ is the unit tangent and $\mathbf{N}$ is the unit normal vector at $\mathbf{P}$ for $\mathbf{f}_0(t)$. Similarly, $-\mathbf{T}$ is the unit tangent and $\mathbf{N}$ is the unit normal vector at $\mathbf{P}$ for $\mathbf{f}_1(t)$. $\mathbf{T}_0$ and $\mathbf{T}_1$ are respectively unit tangent vectors at the ending points of $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$. The angle from $-\mathbf{T}$ to $\mathbf{T}_0$ is ϕ_0, and the angle from $\mathbf{T}$ to $\mathbf{T}_1$ is ϕ_1 and we set $\phi = -\phi_0 = \phi_1$; then $0 < \phi < \pi/4$. We solve (46) for ϕ, and we obtain $\phi = 0.523913$. From (37), (40) and (41), we get $\mathbf{T}$, $\mathbf{N}$, $\mathbf{T}_0$ and $\mathbf{T}_1$. Therefore, $\mathbf{T} = (0.99921, 0.0397456)$, $\mathbf{N} = (-0.0397456, 0.99921)$, $\mathbf{T}_0 = (-0.885068, 0.465462)$ and $\mathbf{T}_1 = (0.8453, 0.534291)$. Applying Theorem 2, the control points of $\mathbf{f}_0(t)$ are

$$\mathbf{P} = (0, 0), \qquad \mathbf{P}_1 = (-0.93294, -0.0371096),$$
$$\mathbf{P}_2 = (-1.86588, -0.0742191), \qquad \mathbf{P}_3 = (-2.69225, 0.360373).$$

Similarly, the control points of $\mathbf{f}_1(t)$ are

$$\mathbf{P} = (0, 0), \qquad \mathbf{Q}_1 = (3.7317, 0.1484),$$
$$\mathbf{Q}_2 = (7.4635, 0.2968), \qquad \mathbf{Q}_3 = (10.6205, 2.2923).$$

Also, we obtain $a_0 = b_0 = -0.933678$, $d_0 = -0.227814$, $a_1 = b_1 = 3.73471$ and $d_1 = 0.911256$ for $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$ respectively. Consequently, the expressions

$$\mathbf{f}_0(t) = (\mathbf{x}_0(t), \mathbf{y}_0(t)) \quad \text{and} \quad \mathbf{f}_1(t) = (\mathbf{x}_1(t), \mathbf{y}_1(t))$$

can be written as

$$\begin{aligned}
\mathbf{x}_0(t) &= -2.69225 + 2.4791\cos\left(\tfrac{\pi t}{2}\right) - 0.613224\cos^2\left(\tfrac{\pi t}{2}\right) + 0.826368\cos^3\left(\tfrac{\pi t}{2}\right) \\
&\quad - 2.79882\sin\left(\tfrac{\pi t}{2}\right) + 3.73176\sin^2\left(\tfrac{\pi t}{2}\right) - 0.93294\sin^3\left(\tfrac{\pi t}{2}\right), \\
\mathbf{y}_0(t) &= 1.12359 - 1.62972\cos\left(\tfrac{\pi t}{2}\right) + 0.614778\cos(\pi t) - 0.108648\cos\left(\tfrac{3\pi t}{2}\right) \\
&\quad - 0.139161\sin\left(\tfrac{\pi t}{2}\right) + 0.00927739\sin\left(\tfrac{3\pi t}{2}\right), \\
\mathbf{x}_1(t) &= 10.6205 - 9.47086\cos\left(\tfrac{\pi t}{2}\right) + 2.00734\cos^2\left(\tfrac{\pi t}{2}\right) - 3.15695\cos^3\left(\tfrac{\pi t}{2}\right) \\
&\quad + 11.1953\sin\left(\tfrac{\pi t}{2}\right) - 14.927\sin^2\left(\tfrac{\pi t}{2}\right) + 3.73176\sin^3\left(\tfrac{\pi t}{2}\right),
\end{aligned}$$

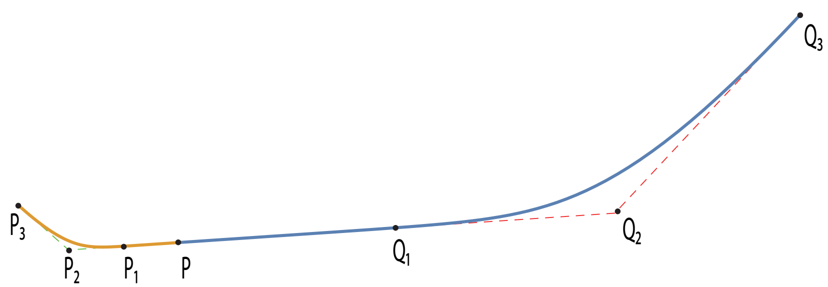

Fig. 11. Trigonometric Bézier-like C-shaped transition spirals from a circle to a circle

$$\mathbf{y}_1(t) = 4.84012 - 7.48284\cos\left(\tfrac{\pi t}{2}\right) + 3.14157\cos(\pi t) - 0.498856\cos\left(\tfrac{3\pi t}{2}\right) + 0.556644\sin\left(\tfrac{\pi t}{2}\right) - 0.0371096\sin\left(\tfrac{3\pi t}{2}\right)$$

(see also Figure 11).

3.1.3. *An S-shaped transition spiral curve from a circle to a circle.* In this section, the second part of a transition spiral curve from a circle to a circle is considered. It is an S-shaped curve. A numerical example of this case is also given here.

THEOREM 5. *Given two circles M_0 and M_1, centered at $\mathbf{O}_0$ and $\mathbf{O}_1$ with radii $r_0 > 0$ and $r_1 > 0$ respectively.* (*Here, we obtain two cases, the second case is $r_0, r_1 < 0$. Minus is used to define the direction.*)

If $|r_1 + r_0| < \|\mathbf{O}_1 - \mathbf{O}_0\| < \frac{\sqrt{277}}{3}|r_1 + r_0|$, then these two circles can be joined by a pair of planar trigonometric cubic Bézier-like spirals given in Theorem 2 forming S-shaped curves such that all points of contact are G^2 as shown in Figure 12.

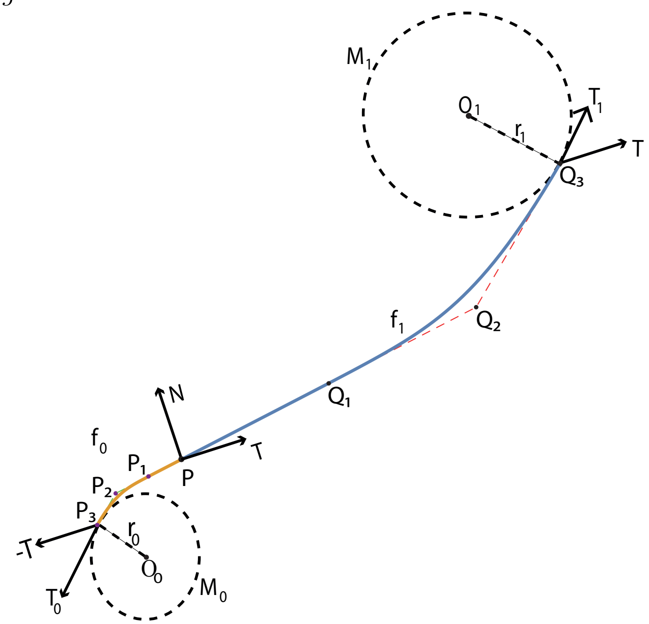

Fig. 12. S-shaped transition curves from a circle to another circle

Proof. Let a trigonometric cubic Bézier-like spiral that meets M_0 be $\mathbf{f}_0(t)$ and the other that meets M_1 be $\mathbf{f}_1(t)$. The following equation is obtained similar to Theorem 4:

$$\mathbf{f}_1(1) - \mathbf{f}_0(1) = \frac{(r_1 + r_0)4(1+2\mu)^2 \sin\phi\,(\cos\phi + \sin\phi)^2}{(2+\mu)^2}\mathbf{T} + \frac{2(1+2\mu)\sin\phi\,(\cos\phi+\sin\phi)}{(2+\mu)^2}(r_1\mathbf{T}_1 - r_0\mathbf{T}_0). \tag{48}$$

We choose the direction of $\mathbf{T}$ as $\mathbf{T}(\mathbf{O}_1 - \mathbf{O}_0) > 0$. The other direction of $\mathbf{T}$ is $\mathbf{T}(\mathbf{O}_1 - \mathbf{O}_0) < 0$. The other direction of $\mathbf{T}$ provides that $\mathbf{f}_0(t)$ is replaced with $\mathbf{f}_1(t)$. Since ϕ is an angle from $\mathbf{T}$ to $\mathbf{T}_1$, using the following equations:

$$\begin{aligned} \mathbf{T}_1.\mathbf{T} &= -\mathbf{T}_0.\mathbf{T} = \cos\phi, \\ \mathbf{T}_1.\mathbf{N} &= -\mathbf{T}_0.\mathbf{N} = \sin\phi \end{aligned} \tag{49}$$

in (48), we obtain

$$\{(\mathbf{f}_1(1) - \mathbf{O}_1) - (\mathbf{f}_0(1) - \mathbf{O}_0)\}.\mathbf{N} = -(r_1 + r_0)\cos\phi, \tag{50}$$

$$\{(\mathbf{f}_1(1) - \mathbf{O}_1) - (\mathbf{f}_0(1) - \mathbf{O}_0)\}.\mathbf{T} = (r_1 + r_0)\sin\phi. \tag{51}$$

Similar to Theorem 4, by solving

$$(r_1+r_0)^2\{g_1(\phi)\}^2 + (r_1+r_0)^2\{g_2(\phi)\}^2 = \|\mathbf{O}_1 - \mathbf{O}_0\|^2, \quad 0 < \phi < \pi/2, \tag{52}$$

trigonometric Bézier-like spirals are obtained. Using (48) and (49), we obtain

$$\{\mathbf{f}_1(1) - \mathbf{f}_0(1)\}.\mathbf{N} = \frac{2(1+2\mu)\sin^2\phi\,(\cos\phi + \sin\phi)}{(2+\mu)^2}(r_1 + r_0) \tag{53}$$

and

$$\begin{aligned} \{\mathbf{f}_1(1) - \mathbf{f}_0(1)\}.\mathbf{T} = &\frac{(r_1 + r_0)4(1+2\mu)^2\sin\phi\,(\cos\phi + \sin\phi)^2}{(2+\mu)^2} \\ &+ \frac{2(1+2\mu)\sin\phi\,(\cos\phi + \sin\phi)}{(2+\mu)^2}\cos\phi(r_1 + r_0). \end{aligned} \tag{54}$$

Using these equations, (50) and (51) are rewritten as

$$\{(\mathbf{O}_1 - \mathbf{O}_0)\}.\mathbf{N} = (r_1 + r_0)g_1(\phi), \tag{55}$$

$$\{(\mathbf{O}_1 - \mathbf{O}_0)\}.\mathbf{T} = (r_1 + r_0)g_2(\phi). \tag{56}$$

Therefore, $g_1(\phi)$ and $g_2(\phi)$ are obtained as

$$\begin{aligned} g_1(\phi) &= \frac{2(1+2\mu)\sin^2\phi\,(\cos\phi + \sin\phi)}{(2+\mu)^2} + \cos\phi, \\ g_2(\phi) &= \frac{2(1+2\mu)\sin\phi\,(\cos\phi + \sin\phi)}{(2+\mu)^2}\big(2(1+2\mu)(\cos\phi + \sin\phi) + \cos\phi\big) \\ &\quad - \sin\phi. \end{aligned}$$

From Theorem 2, we obtain

$$\begin{aligned} g_1(\phi) &= \frac{2\sin^2\phi\,(\cos\phi + \sin\phi)}{3} + \cos\phi, \\ g_2(\phi) &= \frac{2\sin\phi\,(\cos\phi + \sin\phi)}{3}(6(\cos\phi + \sin\phi) + \cos\phi) - \sin\phi. \end{aligned}$$

Using (55) and (56), we get

$$(r_1 + r_0)^2\{g_1(\phi)\}^2 + (r_1 + r_0)^2\{g_2(\phi)\}^2 = \|\mathbf{O}_1 - \mathbf{O}_0\|^2.$$

Therefore, we arrive at

$$q(\phi) = (r_1 + r_0)^2[\{g_1(\phi)\}^2 + \{g_2(\phi)\}^2] - \|\mathbf{O}_1 - \mathbf{O}_0\|^2.$$

Since

$$g_1(0) = 1 \quad \text{and} \quad g_2(0) = 0,$$

we derive

$$q(0) = (r_1 + r_0)^2 - \|\mathbf{O}_1 - \mathbf{O}_0\|^2.$$

Hence we have

$$|r_1 + r_0| < \|\mathbf{O}_1 - \mathbf{O}_0\|, \qquad |r_1 + r_0|^2 < \|\mathbf{O}_1 - \mathbf{O}_0\|^2, \qquad q(0) < 0.$$

Also, since

$$g_1(\pi/4) = \frac{5\sqrt{2}}{6} \quad \text{and} \quad g_2(\pi/4) = \frac{23\sqrt{2}}{6},$$

we obtain

$$q(\pi/4) = (r_1 + r_0)^2 \frac{277}{9} - \|\mathbf{O}_1 - \mathbf{O}_0\|^2.$$

By assumption, since $|r_1 + r_0| < \|\mathbf{O}_1 - \mathbf{O}_0\| < \frac{\sqrt{277}}{3}|r_1 + r_0|$, we get

$$q(\pi/4) > 0.$$

By direct computation, we have

$$q'(\phi) = 2(r_1 + r_0)^2\big(g_1(\phi)g_1'(\phi) + g_2(\phi)g_2\prime(\phi)\big).$$

Since

$$g_1'(\pi/4) = \frac{\sqrt{2}}{6} > 0 \quad \text{and} \quad g_2'(\pi/4) = \frac{25\sqrt{2}}{6} > 0,$$

we derive

$$q(\pi/4) > 0.$$

As a result,

$$q(0) < 0, \qquad q(\phi) > 0, \quad q'(\phi) > 0.$$

Consequently, (52) has a unique solution when $0 < \phi < \pi/4$ and $|r_1 + r_0| < \|\mathbf{O}_1 - \mathbf{O}_0\| < \frac{\sqrt{277}}{3}|r_1 + r_0|$. Similar to the previous theorem, by solving (52) for ϕ, we determine $\mathbf{T}$, $\mathbf{N}$, $\mathbf{T}_0$ and $\mathbf{T}_1$. Applying Theorem 2, a pair of trigonometric cubic Bézier-like spirals are obtained. ∎

EXAMPLE 3. Let M_0 be the circle with radius $r_0 = 0.2 > 0$ and centre $\mathbf{O}_0 = (-1.5, -0.6)$ and M_1 be the circle with radius $r_1 = 0.9 > 0$ and centre $\mathbf{O}_1 = (1.8, 2.5)$. Here,

$$|r_1 + r_0| = 1.1 < \|\mathbf{O}_1 - \mathbf{O}_0\| \approx 4.52769 < \frac{\sqrt{277}}{3}|r_1 + r_0| \approx 6.10255.$$

Let $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$ be trigonometric Bézier spiral curves with two shape parameteres. For these spirals, $\lambda = 1$, $\mu = 1$ and also $\theta = 0$. Choose $\mathbf{P} = (0, 0)$. $\mathbf{P}$ is the starting point for both spirals. Therefore, $\mathbf{T}$ is the unit tangent and $\mathbf{N}$ is the unit normal vector at $\mathbf{P}$ for $\mathbf{f}_0(t)$. Similarly, $-\mathbf{T}$ is the unit tangent and $-\mathbf{N}$ is the unit normal vector at $\mathbf{P}$ for $\mathbf{f}_1(t)$. $\mathbf{T}_0$ and $\mathbf{T}_1$ are respectively the unit tangent vectors at the ending points of $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$. The angle from $-\mathbf{T}$ to $\mathbf{T}_0$ is ϕ_0, and the angle from $\mathbf{T}$ to $\mathbf{T}_1$ is ϕ_1 and we set $\phi = -\phi_0 = \phi_1$; then $0 < \phi < \pi/4$. We solve (52) for ϕ, and we obtain $\phi = 0.565604$. From (49), (55) and (56), we get

$$\mathbf{T} = (0.886314, 0.463084), \qquad \mathbf{N} = (-0.463084, 0.886314),$$
$$\mathbf{T}_0 = (-0.500105, -0.865965), \quad \mathbf{T}_1 = (0.500105, 0.865965).$$

Applying Theorem 2, the control points of $\mathbf{f}_0(t)$ are

$$\mathbf{P} = (0, 0), \qquad \mathbf{P}_1 = (-0.361935, -0.189105),$$
$$\mathbf{P}_2 = (-0.72387, -0.3782), \quad \mathbf{P}_3 = (-0.92809, -0.73183)$$

and the control points of $\mathbf{f}_1(t)$ are

$$\mathbf{P} = (0, 0), \qquad \mathbf{Q}_1 = (1.62871, 0.850973),$$
$$\mathbf{Q}_2 = (3.25742, 1.70195), \quad \mathbf{Q}_3 = (4.17642, 3.29326).$$

Also, for $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$, we obtain $a_0 = b_0 = 0.40836$, $d_0 = 0.098624$, $a_1 = b_1 = 1.83762$ and $d_1 = 0.443808$. Consequently, we write $\mathbf{f}_0(t) = (\mathbf{x}_0(t), \mathbf{y}_0(t))$ as

$$\begin{aligned}\mathbf{x}_0(t) = &-0.928093 + 0.612669 \cos\left(\tfrac{\pi t}{2}\right) + 0.111202 \cos^2\left(\tfrac{\pi t}{2}\right) \\ &+ 0.204223 \cos^3\left(\tfrac{\pi t}{2}\right) - 1.08581 \sin\left(\tfrac{\pi t}{2}\right) + 1.44774 \sin^2\left(\tfrac{\pi t}{2}\right) \\ &- 0.361935 \sin^3\left(\tfrac{\pi t}{2}\right),\end{aligned}$$

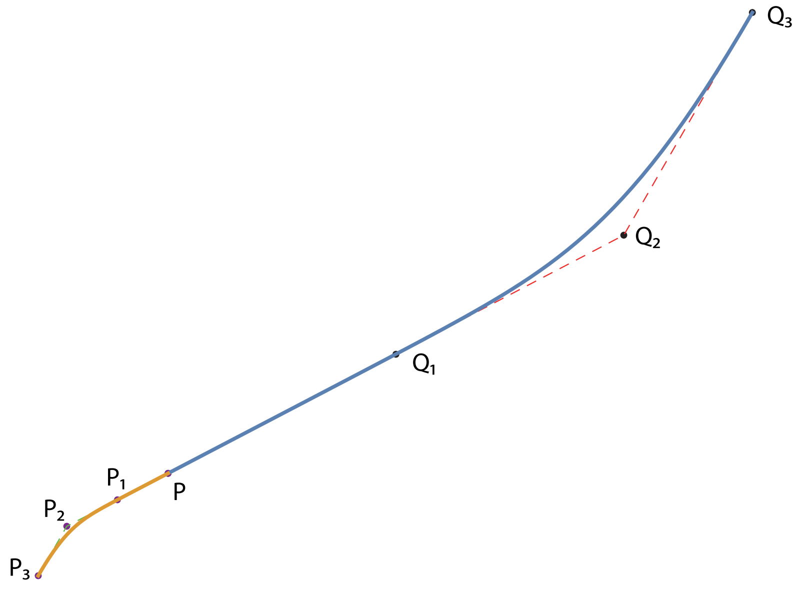

Fig. 13. A trigonometric Bézier-like S-shaped transition spiral from a circle to a circle

$$\mathbf{y}_0(t) = -0.694958 + 1.3261\cos\left(\tfrac{\pi t}{2}\right) - 0.719543\cos(\pi t) + 0.0884064\cos\left(\tfrac{3\pi t}{2}\right) - 0.709145\sin\left(\tfrac{\pi t}{2}\right) + 0.0472763\sin\left(\tfrac{3\pi t}{2}\right),$$

and $\mathbf{f}_1(t) = (\mathbf{x}_1(t), \mathbf{y}_1(t))$ as

$$\mathbf{x}_1(t) = 4.17642 - 2.75701\cos\left(\tfrac{\pi t}{2}\right) - 0.500409\cos^2\left(\tfrac{\pi t}{2}\right) - 0.919003\cos^3\left(\tfrac{\pi t}{2}\right) + 4.88613\sin\left(\tfrac{\pi t}{2}\right) - 6.51484\sin^2\left(\tfrac{\pi t}{2}\right) + 1.62871\sin^3\left(\tfrac{\pi t}{2}\right),$$

$$\mathbf{y}_1(t) = 3.12731 - 5.96743\cos\left(\tfrac{\pi t}{2}\right) + 3.23795\cos\pi - 0.397829\cos\left(\tfrac{3\pi t}{2}\right) + 3.19115\sin\left(\tfrac{\pi t}{2}\right) - 0.212743\sin\left(\tfrac{3\pi t}{2}\right).$$

See Figure 13.

3.1.4. *Transition spiral curves from a straight line to a straight line.* In this subsection, we are going to examine the possibility of a transition spiral curve from a straight line to a straight line. Indeed, we show that it is possible to construct such a spiral and provide a numerical example.

THEOREM 6. *Assume that* $\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}$ *are given three points and* $\mathbf{T}_0$, $\mathbf{T}_1$ *are given as*

$$\mathbf{T}_0 = \frac{\mathbf{F} - \mathbf{F}_0}{\|\mathbf{F} - \mathbf{F}_0\|} \quad \text{and} \quad \mathbf{T}_1 = \frac{\mathbf{F} - \mathbf{F}_1}{\|\mathbf{F} - \mathbf{F}_1\|}.$$

Let $\alpha < \pi$ *be an angle at* $\mathbf{F}$ *formed by* $\mathbf{F}_0$, $\mathbf{F}$ *and* $\mathbf{F}_1$, *and* $c > 0$. *The pair of trigonometric Bézier-like spirals*

$$\mathbf{f}_0(t) = C_0(t)\mathbf{P}_0 + C_1(t)\mathbf{P}_1 + C_2(t)\mathbf{P}_2 + C_3(t)\mathbf{P}_3 \tag{57}$$

and

$$\mathbf{f}_1(t) = C_0(t)\mathbf{B}_0 + C_1(t)\mathbf{B}_1 + C_2(t)\mathbf{B}_2 + C_3(t)\mathbf{B}_3 \tag{58}$$

with the control points of $\mathbf{f}_0(t)$ *and* $\mathbf{f}_1(t)$ *given by*

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{P}_0 + a\mathbf{T}_0, \\ \mathbf{P}_2 &= \mathbf{P}_1 + a\mathbf{T}_0 = \mathbf{P}_0 + 2a\mathbf{T}_0, \\ \mathbf{P}_3 &= \mathbf{P}_2 + d\mathbf{T} = \mathbf{P}_0 + 2a\mathbf{T}_0 + d\mathbf{T} \end{aligned} \tag{59}$$

and

$$\begin{aligned} \mathbf{B}_1 &= \mathbf{B}_0 + a\mathbf{T}_1, \\ \mathbf{B}_2 &= \mathbf{B}_1 + a\mathbf{T}_1 = \mathbf{B}_0 + 2a\mathbf{T}_1, \\ \mathbf{B}_3 &= \mathbf{B}_2 - d\mathbf{T} = \mathbf{B}_0 + 2a\mathbf{T}_1 - d\mathbf{T}, \end{aligned} \tag{60}$$

respectively, where

$$\phi = \tfrac{1}{2}(\pi - \alpha),$$

$$\mathbf{T} = \frac{\mathbf{T}_1 - \mathbf{T}_0}{\|\mathbf{T}_1 - \mathbf{T}_0\|},$$

$$\mathbf{P}_0 = \mathbf{F} - \sigma \mathbf{T}_0,$$
$$\mathbf{B}_0 = \mathbf{F} - \sigma \mathbf{T}_1$$

and

$$\sigma = \frac{2a\cos\phi + d}{\cos\phi} = 2a + \frac{d}{\cos\phi},$$
$$\sigma = \frac{2\sin\phi\,(\cos\phi + \sin\phi)}{3c}(6(\cos\phi + \sin\phi) + \sec\phi), \tag{61}$$

joins the two directed lines emanating from $\mathbf{F}_0$ *and* $\mathbf{F}_1$*, meeting at* $\mathbf{P}_3 = \mathbf{B}_3$ *such that all points of contact are* G^2 *as shown in Figure* 14*; the absolute value of the curvature of the two spirals at their joint point is* c*.*

Proof. Trigonometric Bézier-like spirals have fourteen degrees of freedom. Here λ and μ are equal to 1 on both spirals. Let $\mathbf{T}$ be the unit tangent vector at $t = 1$ and c be the curvature of $\mathbf{f}_0(t)$ at $t = 1$. Because of the requirement of G^2, we set

$$\mathbf{f}_0(1) = \mathbf{f}_1(1). \tag{62}$$

At $t = 1$, the unit tangent vector of $\mathbf{f}_1(t)$ is $-\mathbf{T}$ and the curvature of the spiral is $-c$. Therefore, there are six remaining degrees of freedom. $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$ are each free to move along a straight line, i.e.

$$\mathbf{P}_0 = \mathbf{F} - \sigma_0 \mathbf{T}_0, \tag{63}$$
$$\mathbf{B}_0 = \mathbf{F} - \sigma_1 \mathbf{T}_1. \tag{64}$$

Both spirals have zero curvature at the starting points. Also, the unit tangent vectors of $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$ are respectively $\mathbf{T}_0$ and $\mathbf{T}_1$. According to Theorem 2, the angle from $\mathbf{T}_0$ to $\mathbf{T}$ is ϕ_0, and the angle from $\mathbf{T}_1$ to $-\mathbf{T}$ is ϕ_1. Here, $0 < \phi_0 < \pi/2$ and $-\pi/2 < \phi_1 < 0$. Since $\phi = \frac{1}{2}(\pi - \alpha)$ and

$$\mathbf{T} = \frac{\mathbf{T}_1 - \mathbf{T}_0}{\|\mathbf{T}_1 - \mathbf{T}_0\|},$$

we set $\phi = \phi_0 = -\phi_1$. The last degree of freedom is c and we use it as a shape parameter by choosing a value for it. Since $\mathbf{P}_0 = \mathbf{F} - \sigma_0\mathbf{T}_0$ and $\mathbf{B}_0 = \mathbf{F} - \sigma_1\mathbf{T}_1$, we obtain

$$\mathbf{P}_0 - \mathbf{B}_0 = \sigma_1\mathbf{T}_1 - \sigma_0\mathbf{T}_0. \tag{65}$$

Thus (62) can be written as

$$\mathbf{f}_0(1) = \mathbf{f}_1(1), \quad \mathbf{P}_0 - \mathbf{B}_0 = 2a(\mathbf{T}_1 - \mathbf{T}_0) - 2d\mathbf{T}. \tag{66}$$

(65) is equal to (66). Taking the dot product of these equations with $\mathbf{N}$ and using $\mathbf{T}_0.\mathbf{N} = \mathbf{T}_1.\mathbf{N} = -\sin\phi$, we obtain

$$\sigma_0 = \sigma_1 = \sigma.$$

In a similar way, taking the dot product of these equations with $\mathbf{T}$ and using $\mathbf{T}_0.\mathbf{T} = -\mathbf{T}_1.\mathbf{T} = \cos\phi$, we get

$$\sigma = \frac{2a\cos\phi + d}{\cos\phi} = 2a + \frac{d}{\cos\phi}.$$

Therefore, we obtain

$$\begin{aligned}\sigma &= 2\left(\frac{2(1+2\mu)^2\sin\phi\,(\cos\phi+\sin\phi)^2}{c(2+\mu)^2}\right) + \frac{2(1+2\mu)\sin\phi\,(\cos\phi+\sin\phi)}{c(2+\mu)^2\cos\phi} \\ &= \frac{2(1+2\mu)\sin\phi\,(\cos\phi+\sin\phi)}{c(2+\mu)^2}\big(2(1+2\mu)(\cos\phi+\sin\phi)+\sec\phi\big)\end{aligned}$$

where

$$a = b = \frac{2(1+2\mu)^2\sin\phi\,(\cos\phi+\sin\phi)^2}{c(2+\mu)^2}, \tag{67}$$

$$d = \frac{2(1+2\mu)\sin\phi\,(\cos\phi+\sin\phi)}{c(2+\mu)^2}. \tag{68}$$

Since $\mu = 1$, it follows that

$$\sigma = \frac{2\sin\phi\,(\cos\phi+\sin\phi)}{3c}(6(\cos\phi+\sin\phi)+\sec\phi), \tag{69}$$

$$c = \frac{2\sin\phi\,(\cos\phi+\sin\phi)}{3\sigma}(6(\cos\phi+\sin\phi)+\sec\phi), \tag{70}$$

where $\sigma = \min(\|\mathbf{F}-\mathbf{F}_0\|, \|\mathbf{F}-\mathbf{F}_1\|)$. ∎

EXAMPLE 4. Let $\mathbf{F}_0 = (0,0)$, $\mathbf{F}_1 = (2,5)$ and $\mathbf{F} = (4,3)$. Then

$$\begin{aligned}\mathbf{T}_0 &= \frac{\mathbf{F}-\mathbf{F}_0}{\|\mathbf{F}-\mathbf{F}_0\|} = \left(\frac{4}{5}, \frac{3}{5}\right), \\ \mathbf{T}_1 &= \frac{\mathbf{F}-\mathbf{F}_1}{\|\mathbf{F}-\mathbf{F}_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right).\end{aligned}$$

$\alpha < \pi$ is the angle at $\mathbf{F}$ formed by $\mathbf{F}_0$, $\mathbf{F}$ and $\mathbf{F}_1$. Taking the dot product of $\mathbf{T}_0$ and $\mathbf{T}_1$, we obtain $\cos\alpha = 0.141421$ and $\alpha = 1.4289$, $\alpha \in (0, \pi/2)$. Therefore, $\phi = \frac{1}{2}(\pi - \alpha) = 0.856347$. Here, $\|\mathbf{F}-\mathbf{F}_0\| = 2\sqrt{2}$ and $\|\mathbf{F}-\mathbf{F}_1\| = 5$. Thus, $\sigma = \min(\|\mathbf{F}-\mathbf{F}_0\|, \|\mathbf{F}-\mathbf{F}_1\|) = 0.282843$. Using (70), we get $c = 3.74842$. By assumption, we have

$$\begin{aligned}\mathbf{T} &= (0.070889, 0.997484), \\ \mathbf{P}_0 &= \mathbf{F} - \sigma\mathbf{T}_0 = (1.73726, 1.30294), \\ \mathbf{B}_0 &= \mathbf{F} - \sigma\mathbf{T}_1 = (2, 5).\end{aligned}$$

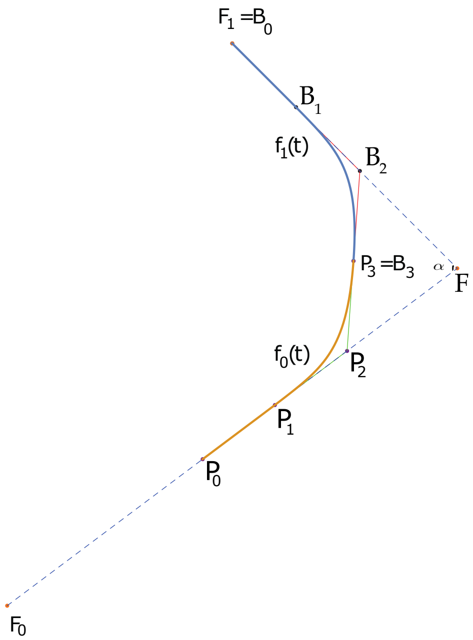

Fig. 14. A trigonometric Bézier transition spiral from a straight line to a straight line

And also, from (59) and (60), we obtain

$$\mathbf{P}_1 = (2.37894, 1.78421), \quad \mathbf{B}_1 = (2.56718, 4.43282),$$
$$\mathbf{P}_2 = (3.02063, 2.26547), \quad \mathbf{B}_2 = (3.13435, 3.86565),$$
$$\mathbf{P}_3 = (3.07749, 3.06556), \quad \mathbf{B}_3 = (3.07749, 3.06556).$$

Finally, from (57) and (58), we compute $\mathbf{f}_0(t)$ and $\mathbf{f}_1(t)$. These trigonometric Bézier-like spirals are joined at the point of $\mathbf{P}_3 = \mathbf{B}_3$ in such a way that the contact point is G^2 (Figure 14).

3.1.5. *Circle to circle, one circle inside the other.* In this section, a transition curve which is a single curve from a circle to a circle (one circle inside the other one) is introduced. In this case, there are eight degrees of freedom. The starting point of the spiral cannot be used as a point of contact, because the curvature at the starting point is zero. An additional degree of freedom is the parameter value of the unknown point of contact. The radii of the circles have the same signs. We assume that they are both positive. The opposite case can be obtained in a similar way.

Theorem 7. *Given two circles M_0 and M_1, respectively centered at $\mathbf{O}_0$ and $\mathbf{O}_1$ with radii $r_0 > 0$ and $r_1 > 0$. Suppose that M_1 is completely contained inside M_0. Then these circles are joined by a single trigonometric cubic Bézier-like spiral such that both points of contact are G^2 (Figure 15).*

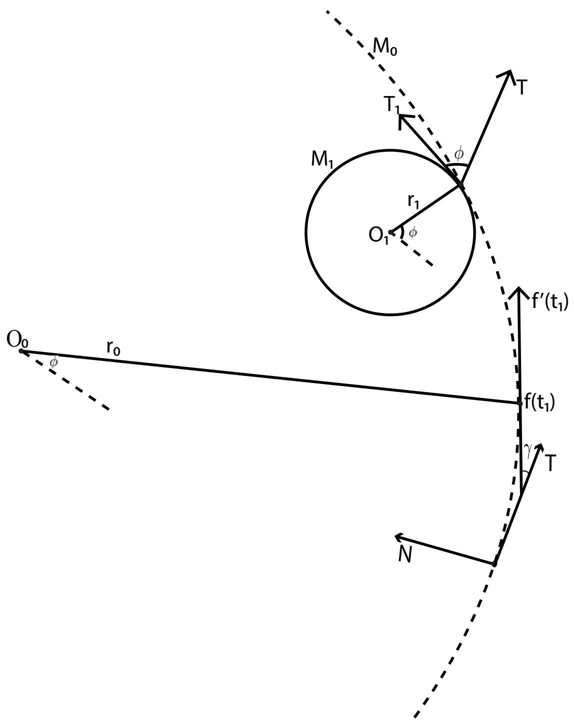

Fig. 15. Circle to circle with a single trigonometric Bézier-like spiral transition curve

Proof. A trigonometric Bézier-like spiral defined in Theorem 2 is joined to M_0 at $t = t_1$ and M_1 at $t = 1$. Let the angle from $\mathbf{T}$ to the unit tangent vector at $t = t_1$ be γ. From (11), it follows that

$$\kappa(t_1) = \frac{r_1}{r_0}. \tag{71}$$

If ϕ is known, then (71) can be solved only for $t_1 \in (0, 1)$ by using Theorem 2. By taking the dot product of $\mathbf{f}'(t)$ at $t = t_1$ with the unit normal vector $\mathbf{N}$ and the unit tangent vector $\mathbf{T}$ of $\mathbf{f}(t)$ at $t = 0$, we obtain

$$\begin{aligned}
(72)\quad & \mathbf{f}'(t_1).\mathbf{T} \\
&= \frac{\pi(1+2\mu)}{4c(2+\mu)^2}\Big(\big(4+5\mu-8(1+\mu)\cos\big(\tfrac{\pi t_1}{2}\big)+3\mu\cos(\pi t_1)\big)\sin\big(\tfrac{\pi t_1}{2}\big) \\
&\quad \times ((3+4\mu)\cos\phi+2(1+2\mu)\sin\phi)(-1+\cos(2\phi)-\sin(2\phi)) \\
&\quad - 4(1+2\mu)\big(4+5\mu-8(1+\mu)\cos\big(\tfrac{\pi t_1}{2}\big)+3\mu\cos(\pi t_1)\big)\sin\big(\tfrac{\pi t_1}{2}\big) \\
&\quad \times \sin\phi\,(1+\sin(2\phi)) \\
&\quad + 2(1+2\mu)\big(-4(1+\lambda)+(2+\lambda)\csc\big(\tfrac{\pi t_1}{2}\big)+3\lambda\sin\big(\tfrac{\pi t_1}{2}\big)\sin(\pi t_1) \\
&\quad \times \sin\phi\,(1+\sin(2\phi))\big)\Big),
\end{aligned}$$

$$(73)\quad \mathbf{f}'(t_1).\mathbf{N} = \frac{-\pi(1+2\mu)\big(4+5\mu-8(1+\mu)\cos\big(\frac{\pi t_1}{2}\big)+3\mu\cos(\pi t_1)\sin\big(\frac{\pi t_1}{2}\big)\sin^2\phi\,(\cos\phi+\sin\phi)\big)}{2c(2+\mu)^2}.$$

Since

$$\begin{aligned}
\mathbf{f}'(t_1) &= \cos\gamma\,\mathbf{T}+\sin\gamma\,\mathbf{N},\\
\mathbf{f}'(t_1).\mathbf{T} &= \cos\gamma\,\mathbf{T}.\mathbf{T}+\sin\gamma\,\mathbf{N}.\mathbf{T}=\cos\gamma,\\
\mathbf{f}'(t_1).\mathbf{N} &= \cos\gamma\,\mathbf{T}.\mathbf{N}+\sin\gamma\,\mathbf{N}.\mathbf{N}=\sin\gamma,
\end{aligned}$$

it follows from (72) that

$$\begin{aligned}
(74)\quad \mathbf{f}'(t_1).\mathbf{T} &= \cos\gamma\\
&= \frac{\pi(1+2\mu)}{4c(2+\mu)^2}\Big(\big(4+5\mu-8(1+\mu)\cos\big(\tfrac{\pi t_1}{2}\big)+3\mu\cos(\pi t_1)\big)\sin\big(\tfrac{\pi t_1}{2}\big)\\
&\quad\times((3+4\mu)\cos\phi+2(1+2\mu)\sin\phi)(-1+\cos(2\phi)-\sin(2\phi))\\
&\quad-4(1+2\mu)\big(4+5\mu-8(1+\mu)\cos\big(\tfrac{\pi t_1}{2}\big)+3\mu\cos(\pi t_1)\big)\sin\big(\tfrac{\pi t_1}{2}\big)\\
&\quad\times\sin\phi\,(1+\sin(2\phi))\\
&\quad+2(1+2\mu)\big(-4(1+\lambda)+(2+\lambda)\csc\big(\tfrac{\pi t_1}{2}\big)+3\lambda\sin\big(\tfrac{\pi t_1}{2}\big)\sin(\pi t_1)\\
&\quad\times\sin\phi\,(1+\sin(2\phi))\big)\Big),
\end{aligned}$$

and from (73) that

$$(75)\quad \mathbf{f}'(t_1).\mathbf{N} = \sin\gamma = \frac{-\pi(1+2\mu)\big(4+5\mu-8(1+\mu)\cos\big(\frac{\pi t_1}{2}\big)+3\mu\cos(\pi t_1)\sin\big(\frac{\pi t_1}{2}\big)\sin^2\phi\,(\cos\phi+\sin\phi)\big)}{2c(2+\mu)^2}.$$

Hence, we have

$$\frac{\mathbf{f}'(t_1)\mathbf{N}}{\mathbf{f}'(t_1)\mathbf{T}} = \frac{\sin\gamma}{\cos\gamma} = \tan\gamma,$$

and $\lambda=\mu=1$ implies that

$$\begin{aligned}
(76)\quad \tan\gamma &= -\Big(2\big(9-16\cos\big(\tfrac{\pi t_1}{2}\big)+3\cos(\pi t_1)\big)\sin\big(\tfrac{\pi t_1}{2}\big)\sin^2\phi\,(\cos\phi+\sin\phi)\Big)\\
&\quad\times\Big(\big(9-16\cos\big(\tfrac{\pi t_1}{2}\big)+3\cos(\pi t_1)\big)\sin\big(\tfrac{\pi t_1}{2}\big)(7\cos\phi+6\sin\phi)\\
&\quad\times(\cos(2\phi)-1-\sin(2\phi))\\
&\quad+12\big(16\cos\big(\tfrac{\pi t_1}{2}\big)-3(3+\cos(\pi t_1))\big)\sin\big(\tfrac{\pi t_1}{2}\big)\sin\phi\,(1+\sin(2\phi))\\
&\quad+6\big(-8+3\csc\big(\tfrac{\pi t_1}{2}\big)+3\sin\big(\tfrac{\pi t_1}{2}\big)\big)\sin(\pi t_1)\sin\phi\,(1+\sin(2\phi))\Big)^{-1}.
\end{aligned}$$

Then Theorem 2 and $a=b$ give us

$$\mathbf{f}(1) = \mathbf{P}_0 + 2a\mathbf{T} + d\mathbf{T}_1$$

and

$$\mathbf{f}(t_1) = \mathbf{P}_0 + a\big(C_1(t_1) + 2C_2(t_1) + 2C_3(t_1)\big)\mathbf{T} + C_3(t_1)d\mathbf{T}_1.$$

It follows from (1) that

$$\mathbf{f}(t_1) = \mathbf{P}_0 + a\big(\cos(\pi t_1) - 1 + \tfrac{15}{4}\sin\big(\tfrac{\pi t_1}{2}\big) - \tfrac{1}{4}\sin\big(\tfrac{3\pi t_1}{2}\big)\big)\mathbf{T} + d\big(1 - \cos\big(\tfrac{\pi t_1}{2}\big)\big)^3\mathbf{T}_1. \tag{77}$$

Using these equations, we obtain

$$\mathbf{f}(1) - \mathbf{f}(t_1) = a\big(3 - \cos(\pi t_1) - \tfrac{15}{4}\sin\big(\tfrac{\pi t_1}{2}\big) + \tfrac{1}{4}\sin\big(\tfrac{3\pi t_1}{2}\big)\big)\mathbf{T} + d\big(1 - \big(1 - \cos\big(\tfrac{\pi t_1}{2}\big)\big)^3\big)\mathbf{T}_1. \tag{78}$$

Also, from

$$\{(\mathbf{f}(1) - \mathbf{O}_1) - (\mathbf{f}(t_1) - \mathbf{O}_0)\}.\mathbf{N} = -(r_1\cos\phi - r_0\cos\gamma), \tag{79}$$

$$\{(\mathbf{f}(1) - \mathbf{O}_1) - (\mathbf{f}(t_1) - \mathbf{O}_0)\}.\mathbf{T} = r_1\sin\phi - r_0\sin\gamma \tag{80}$$

and by taking the dot product of (78) with **T** and **N** and using (79) and (80), it follows that

$$\{\mathbf{O}_1 - \mathbf{O}_0\}.\mathbf{N} = \big\{d\big(1 - \big(1 - \cos\big(\tfrac{\pi t_1}{2}\big)\big)^3\big)\big\}\sin\phi + r_1\cos\phi - r_0\cos\gamma \tag{81}$$

and

$$\{\mathbf{O}_1 - \mathbf{O}_0\}.\mathbf{T} = \big\{a\big(3 - \cos(\pi t_1) - \tfrac{15}{4}\sin\big(\tfrac{\pi t_1}{2}\big) + \tfrac{1}{4}\sin\big(\tfrac{3\pi t_1}{2}\big)\big)\big\} + \big\{d\big(1 - \big(1 - \cos\big(\tfrac{\pi t_1}{2}\big)\big)^3\big)\big\}\cos\phi - r_1\sin\phi + r_0\sin\gamma. \tag{82}$$

Therefore, we obtain

$$q(\phi) = \{g_1(\phi)\}^2 + \{g_2(\phi)\}^2 - \|\mathbf{O}_1 - \mathbf{O}_0\|^2 = 0 \tag{83}$$

where

$$g_1(\phi) = \big\{\mathbf{O}_1 - \mathbf{O}_0\big\}.\mathbf{N} = \big\{d\big(1 - \big(1 - \cos\big(\tfrac{\pi t_1}{2}\big)\big)^3\big)\big\}\sin\phi + r_1\cos\phi - r_0\cos\gamma \tag{84}$$

and

$$g_2(\phi) = \{\mathbf{O}_1 - \mathbf{O}_0\}.\mathbf{T} = \big\{a\big(3 - \cos(\pi t_1) - \tfrac{15}{4}\sin\big(\tfrac{\pi t_1}{2}\big) + \tfrac{1}{4}\sin\big(\tfrac{3\pi t_1}{2}\big)\big)\big\} + \big\{d\big(1 - \big(1 - \cos\big(\tfrac{\pi t_1}{2}\big)\big)^3\big)\big\}\cos\phi - r_1\sin\phi + r_0\sin\gamma. \tag{85}$$

By solving (83), we obtain a trigonometric cubic Bézier-like spiral. (71), (76) and (83) are nonlinear equations in t_1, ϕ and γ. If a solution exists, it can be found by solving (83) as a nonlinear equation in a single unknown. Because any values of ϕ and t_1 can be obtained from (71) and γ can be obtained from (76), after obtaining these values, $q(\phi)$ is evaluated. ∎

4. Concluding remarks. The cubic trigonometric Bézier curves with two shape parameters are closer to the control polygon, making the design easier for the designer. This is valid for highway design, railroad design or satellite path design where spiral curves will be used.

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