STUDIA MATHEMATICA

Online First version

Characterisations of pseudo-amenability

by

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Abstract. We use a Følner type condition to define pseudo-amenable groups. We then prove new characterisations of pseudo-amenable groups that are functional versions of that condition. This allows us to show new results about these groups, which closely mimic well-known results about amenable groups. For instance, we show that pseudo-amenability is preserved under closed subgroups and homomorphisms.

1. Introduction. The study of amenable groups has become increasingly important over the past few decades, with applications in a wide range of mathematical areas, including ergodic theory, statistics, and even cellular automata [1]. Amenable groups are usually defined by the existence of a left-invariant mean, however such groups have many different characterisations, among which are the *Følner conditions*. These conditions are often combinatorial in nature; the most well-known one is the following:

(F) For every $\varepsilon > 0$, and every compact set K, there is a compact C such that for every $s \in K$,

$$\lambda(sC \triangle C) < \varepsilon\lambda(C). \tag{1.1}$$

Here λ is a left Haar measure on G, and $\triangle$ denotes symmetric difference of sets. This condition was proved to be equivalent to amenability by Følner [6] in the discrete case and Namioka [10] in general for any locally compact group.

There are several variations of this condition that will be of interest to us. The first of these is the Reiter condition, which can be seen as a 'functional version' of Følner's condition. This condition is defined as follows:

2020 *Mathematics Subject Classification*: Primary 43A07; Secondary 43A15, 43A05.

Key words and phrases: amenable, Følner condition, Lebesgue space, locally compact group, pseudo-amenable.

Received 4 July 2020; revised 20 February 2021.

Published online 30 August 2021.

DOI: 10.4064/sm200704-20-1

(R) For every $\varepsilon > 0$ and any compact set C, there is some positive $f \in L^1(G)$ with unit norm such that

$$\|s * f - f\|_1 < \varepsilon \quad \text{for every } s \in C.$$

Here we define $s * f$ to be the left translation of f by s, that is, $(s * f)(x) := f(s^{-1}x)$. The main idea behind this paper is similar to the process above; we will construct 'functional versions' of known combinatorial conditions in order to obtain new results. We note here that there is nothing special about the space $L^1(G)$ in this condition: indeed, we may replace it with any $L^p(G)$ space for $1 \leq p < \infty$ (the case $p = \infty$ holds trivially in every group) [7, Section 3.2].

Another condition bearing similarity to (F) was introduced and shown to be equivalent to amenability by Følner [6, Section 2]. In this case, however, instead of a single translation, we allow finitely many:

(S) For every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for every finite set $E \subseteq G$ with $|E| \geq N$, there is a compact set $C \subseteq G$ such that

$$\frac{1}{|E|} \sum_{s \in E} \lambda(sC \setminus C) < \varepsilon \lambda(C). \tag{1.2}$$

The first result of this paper will be to introduce a new condition which one may see as a 'functional variation' of (S). We shall show this new condition is equivalent to (S), and hence amenability.

(FS_p) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every finite subset $F \subseteq G$ with $|F| \geq N$, there is a positive $f \in L^p(G)$ such that

$$\Big\|\sum_{s \in E} (s * f - f)\Big\|_p < \varepsilon |E| \, \|f\|_p \quad \text{for all } E \subseteq F \text{ with } |E| \geq N. \tag{1.3}$$

We note that this is formally weaker than (R).

This new condition is similar to (S), though there is one important difference: we require the inequality to hold for subsets of F rather than F itself.

THEOREM 1.1. *Let G be a locally compact group. Then G is amenable if and only if it satisfies condition* (FS_p) *for any $1 \leq p < \infty$.*

Proof. If G is amenable, then G satisfies (S). We can then simply take $f = \chi_C$ and directly obtain (FS_p).

For the other direction, let $\varepsilon > 0$. Take any finite subset F with $|F| \geq N$. We can then find some positive f_F with unit norm that satisfies (FS_p). We can reformulate this in terms of the dual space:

$$\left|\sum_{s \in E} \frac{\langle \varphi, s * f_F - f_F \rangle}{|E|}\right| < \varepsilon \quad \text{for all } \varphi \in L^p(G)^* \text{ with } \|\varphi\| \leq 1. \tag{1.4}$$

In essence, (1.4) states that the 'average value of $\langle \varphi, s * f_F - f_F \rangle$' for any sufficiently large subset of F is small. So let us consider the exceptional set defined by

$$A_{\varphi,F} = \{s \in F : |\langle \varphi, s * f_F - f_F \rangle| \geq 2\varepsilon\}.$$

Now take any finite subset $A^0_{\varphi,F} \subseteq A_{\varphi,F}$ whose values are solely in the positive quadrant (i.e. with positive real and imaginary components). Then we would have

$$\Bigg| \sum_{s \in A^0_{\varphi,F}} \frac{\langle \varphi, s * f_F - f_F \rangle}{|A^0_{\varphi,F}|} \Bigg| \geq \varepsilon.$$

The only way that this is consistent with (1.4) is if $|A^0_{\varphi,F}| < N$. Following this logic for each of the four quadrants, we obtain $|A_{\varphi,F}| < 4N$. Note that this bound is independent of the choice of F.

Now, choose any finite set F with $|F| \geq 8N$, and define $F' = F \cup F^{-1}F$. Let $s \in F$. Take any $\varphi \in L^p(G)^*$ with $\|\varphi\| \leq 1$ and choose $t \in F$ so that

$$t \notin A_{\varphi,F'} \quad \text{and} \quad s^{-1}t \notin A_{s^{-1}*\varphi,F'}.$$

Such a t exists by the bound on the A sets. Using the definition of the A sets, we then have

$$\begin{aligned} |\langle \varphi, s * f_F - f_F \rangle| &\leq |\langle \varphi, s * f_F - t * f_F \rangle| + |\langle \varphi, t * f_F - f_F \rangle| \\ &\leq |\langle \varphi, s * f_F - t * f_F \rangle| + 2\varepsilon \\ &= |\langle s^{-1} * \varphi, f_F - (s^{-1}t) * f_F \rangle| + 2\varepsilon \leq 4\varepsilon. \end{aligned}$$

Hence for any $\varepsilon > 0$ and finite set F, there is a positive function $g \in L^p(G)$ with unit norm such that $\|s * g - g\|_p < \varepsilon$. In fact, we can assume without loss of generality that $p = 1$. This is enough to conclude that G is amenable (see Paterson [12, Proposition (0.8)]). ∎

There is also another condition that was introduced and shown to be equivalent to amenability by Pham [13, Theorem 5.1].

(WF) For every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for every finite $F \subseteq G$ with $|F| \geq N$, there is a compact C such that

$$\lambda(EC) < \varepsilon |E| \lambda(C) \quad \text{for all } E \subseteq F \text{ with } |E| \geq N. \tag{1.5}$$

This condition has the same 'quantify over subsets' flavour as (FS_p) has. This will be important to us later when we introduce the notion of pseudo-amenable groups. For now, we start with a useful lemma which allows us to evaluate the join of translations of a simple function. Recall that the *join* of two functions f and g, denoted $f \vee g$, is simply their pointwise supremum.

Lemma 1.2. *Let f be a positive simple function such that $f=\sum_{i=1}^{k}\alpha_i\chi_{A_i}$, where $\alpha_i > 0$ and $A_1 \supseteq \cdots \supseteq A_k$. Let E be a finite set. Then*

$$\bigvee_{s\in E} s * f = \sum_{i=1}^{k} \alpha_i \chi_{EA_i},$$

Proof. Fix some $x \in G$. We define k_x to be the largest integer such that $x \in EA_{k_x}$, or 0 if there is no such integer. Since $EA_1 \supseteq \cdots \supseteq EA_k$, it follows that

$$\sum_{i=1}^{k} \alpha_i \chi_{EA_i}(x) = \sum_{i=1}^{k_x} \alpha_i.$$

On the other hand, we can equivalently say that k_x is the largest integer such that there exists a $t \in E$ with $x \in tA_{k_x}$. In other words, $(s * f)(x)$ is maximal when $s = t$, and so

$$\Big(\bigvee_{s\in E} s * f\Big)(x) = (t * f)(x) = \sum_{i=1}^{k_x} \alpha_i = \sum_{i=1}^{k} \alpha_i \chi_{EA_i}(x). \ \blacksquare$$

Our next proposition is perhaps one of the most important as it provides an equivalence between the combinatorial and functional versions of a condition. Because of this connection, we shall use this result later. Its proof (presented below) has similarities to the proof of the equivalence of (F) and (R) given by Greenleaf [7, p. 66].

Proposition 1.3. *Let G be a locally compact group, and $1 \le p < \infty$. Take any $\varepsilon > 0$, and a finite set $E \subseteq G$. Then there is a positive function $f \in L^p(G)$ with*

$$\Big\|\bigvee_{s\in E} s * f\Big\|_p^p < \varepsilon |E|\, \|f\|_p^p \tag{1.6}$$

if and only if there is a compact set $C \subseteq G$ such that

$$\lambda(EC) < \varepsilon |E| \lambda(C). \tag{1.7}$$

Proof. For the non-trivial direction, we start with the case $p = 1$ and take a positive simple function f satisfying (1.6). We can assume without loss of generality that f is simple ([1]) and has unit norm. In particular we write

$$f = \sum_{i=1}^{k} \alpha_i \chi_{A_i}$$

([1]) We can make this assumption as the simple functions are dense in $L^1(G)$. More explicitly, if we have an (arbitrary) positive $g \in L^1(G)$ with unit norm such that $\|\bigvee s * g\|_1 < (\varepsilon - \delta)|E|$ for $\delta > 0$, then we can take any positive simple function f with $\|f - g\| < \delta/(1 + \varepsilon)$. This is sufficient to prove that $\|\bigvee s * f\|_1 < \varepsilon |E|\, \|f\|_1$.

where $\alpha_i > 0$ and $A_1 \supseteq \cdots \supseteq A_k$ are λ-finite Borel sets. Now, we use Lemma 1.2 to see that

$$\Big\|\bigvee_{s\in E} s * f\Big\|_1 = \Big\|\sum_{i=1}^{k} \alpha_i \chi_{EA_i}\Big\|_1 = \sum_{i=1}^{k} \alpha_i \lambda(EA_i)$$
$$= \sum_{i=1}^{k} (\alpha_i \lambda(A_i))\bigg(\frac{\lambda(EA_i)}{\lambda(A_i)}\bigg).$$

The summation on the last line is a convex sum, as $\sum_{i=1}^{k} \alpha_i \lambda(A_i) = \|f\|_1 = 1$. This means that there must be some $A \in \{A_1, \ldots, A_k\}$ such that

$$\Big\|\bigvee_{s\in E} s * f\Big\| \geq \frac{\lambda(EA)}{\lambda(A)}.$$

It follows that

$$\lambda(EA) \leq \Big\|\bigvee_{s\in E} s * f\Big\|\lambda(A) < \varepsilon|E|\lambda(A).$$

Using the (inner and outer) regularity of λ, one can then show that there is a compact set satisfying (1.7).

The case of arbitrary p follows directly from the $p = 1$ version of (1.6). If $g \in L^p(G)$ satisfies this equation, then $f = g^{1/p} \in L^1(G)$ will satisfy the same equation (with $p = 1$). ∎

Using (WF), this gives us another characterisation of amenability.

THEOREM 1.4. *A locally compact group G is amenable if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for every finite set $F \subseteq G$ with $|F| \geq N$, there is a positive function $f \in L^p(G)$ such that*

$$\Big\|\bigvee_{s\in E} s * f\Big\|_p^p < \varepsilon|E|\,\|f\|_p^p \quad \text{for all } E \subseteq F \text{ with } |E| \geq N.$$

2. Pseudo-amenable groups. One may ask if quantifying over large subsets of F in (WF) (and (FS$_p$)) is necessary. Removing this constraint, we obtain a condition known as *pseudo-amenability*, which was introduced for discrete groups by Dales and Polyakov [3, Definition 5.5]. We restate this definition for general locally compact groups.

DEFINITION 2.1. Let G be a locally compact group with Haar measure λ. We say G is *pseudo-amenable* if it satisfies the following condition:

(PA) For every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that for every finite $F \subseteq G$ with $|F| \geq N$, there is some compact $C \subseteq G$ such that

$$\lambda(FC) < \varepsilon|F|\lambda(C). \tag{2.1}$$

By (WF), it is immediate that amenability implies pseudo-amenability, but the converse remains an open problem. Dales and Polyakov also showed that subgroups of discrete pseudo-amenable groups are themselves pseudo-amenable [3, Lemma 5.6]. We shall show this for the general case. This will be done with the aid of two new characterisations, obtained using the result from Proposition 1.3. These characterisations are (L) and (M) as given below.

(L) For every $\varepsilon \in (0,1)$, there is some $N \in \mathbb{N}$ such that for all $E \subseteq G$ with $|E| \geq N$, there is some positive $f \in L^1(G)$ such that

$$\Big\| \bigvee_{s \in E} s * f \Big\| < \varepsilon |E| \, \|f\|. \tag{2.2}$$

Here the norm is the usual L^1 norm $\|\cdot\|$.

(M) For every $\varepsilon \in (0,1)$, there is some $N \in \mathbb{N}$ such that for all $E \subseteq G$ with $|E| \geq N$, there is some positive $\mu \in M(G)$ such that

$$\Big\| \bigvee_{s \in E} s * \mu \Big\| < \varepsilon |E| \, \|\mu\|. \tag{2.3}$$

Here $M(G)$ denotes the set of all complex Borel regular measures on G, where the norm of a measure is its total variation. We also note that the join of two positive (or in general, real-valued) measures is defined as

$$[\mu \vee \nu](A) := \sup \{\mu(X) + \nu(A \setminus X) : X \subseteq A \text{ is Borel}\}$$

for all Borel sets A. If we consider $L^1(G) \subseteq M(G)$, then this definition coincides with the usual join of positive (or real-valued) functions.

We now present the central theorem of this paper.

THEOREM 2.2. *Let G be a locally compact group. Then the following are equivalent:*

- *G is pseudo-amenable.*
- *G satisfies* (L).
- *G satisfies* (M).

Proof. The equivalence of the first two conditions follows immediately from Proposition 1.3.

It is trivial that condition (L) implies (M) as we can embed $L^1(G)$ into $M(G)$. Note that the standard embedding $f \mapsto \mu_f$ where $\mu_f(E) = \int_E f(x)\,\mathrm{d}x$ is norm-preserving.

For the other direction, let $\varepsilon \in (0,1)$, and take any sufficiently large finite $E \subseteq G$. Take $\mu \in M(G)$ that satisfies (M). Now fix any positive $h \in L^1(G)$, and define $f \in L^1(G)$ by $f = \mu * h$, where

$$f(x) = (\mu * h)(x) = \int_G (y * h)(x)\,\mathrm{d}\mu(y)$$

and we have $\|f\|_1 = \|\mu\| \, \|h\|_1$. Now note that if we take arbitrary positive $\nu_1, \nu_2 \in M(G)$ and some positive $g \in L^1(G)$, we have

$$\begin{aligned}[\nu_1 * g \vee \nu_2 * g](x) &= \Big[\int_G (y * g)(x) \, \mathrm{d}\nu_1(y)\Big] \vee \Big[\int_G (y * g)(x) \, \mathrm{d}\nu_2(y)\Big] \\ &\le \int_G (y * g)(x) \, \mathrm{d}(\nu_1 \vee \nu_2)(y) = [(\nu_1 \vee \nu_2) * g](x).\end{aligned}$$

More generally, $\bigvee(\nu_i * g) \le (\bigvee \nu_i) * g$ for finitely many joins. From this it follows that

$$\begin{aligned}\Big\|\bigvee_{s\in E} s * f\Big\|_1 &= \Big\|\bigvee_{s\in E} (s * \mu * h)\Big\|_1 \\ &\le \Big\|\Big(\bigvee_{s\in E} s * \mu\Big) * h\Big\|_1 = \Big\|\bigvee_{s\in E} s * \mu\Big\| \|h\|_1 \\ &< \varepsilon |E| \, \|\mu\| \|h\|_1 = \varepsilon |E| \, \|f\|_1.\end{aligned}$$

Hence f satisfies condition (L). ■

We shall now use this theorem to prove that pseudo-amenability is preserved under subgroups. Our approach is similar to that of Paterson [12, Section (1.11)], where the same statement is proved for amenable groups. However, the case of pseudo-amenable groups is more involved as we cannot simply take a left-invariant mean. We begin with the following definition.

Definition 2.3. Let G be a locally compact group, and H a closed subgroup. A *Bruhat function* for H is a function $\beta : G \to \mathbb{R}$ satisfying:

- For every compact $C \subseteq G$, there is a continuous function $\psi \ge 0$ on G such that $\psi|_{CH} = \beta|_{CH}$.
- For every $x \in G$ we have

$$\int_H \beta(xy) \, \mathrm{d}_H y = 1.$$

It is known (see Reiter [14, Chapter 8]) that for every locally compact group G and closed subgroup H, there exists a Bruhat function for H. The existence of such functions is precisely what we will need.

Theorem 2.4. *Let G be a pseudo-amenable locally compact group. Let H be a closed subgroup of G. Then H itself is pseudo-amenable.*

Proof. We let $\varepsilon > 0$ and take any sufficiently large finite set $E \subseteq H$. Choose some positive $f \in L^1(G)$ satisfying (L) for the choice of ε and E. Let $C_c(G)$ denote the set of continuous functions with compact support. We can, without loss of generality, assume that $f \in C_c(G)$ as these functions are dense in $L^1(G)$.

Now let β be a Bruhat function for H. For $y \in H$ we define

$$g(y) := \int_G f(x)\beta(x^{-1}y)\,\mathrm{d}_G x. \tag{2.4}$$

We will show $g \in L^1(H)$. Firstly, as $f \in C_c(G)$, we know that β is continuous on $\mathrm{supp}(f)H$. It is then clear that g is continuous. Furthermore, we have

$$\begin{aligned}\int_H g(y)\,\mathrm{d}_H y &= \int_H \int_G f(x)\beta(x^{-1}y)\,\mathrm{d}_G x\,\mathrm{d}_H y \\ &= \int_G f(x) \int_H \beta(x^{-1}y)\,\mathrm{d}_H y\,\mathrm{d}_G x = \int_G f(x)\,\mathrm{d}_G x\end{aligned}$$

so $\|g\|_{L^1(H)} = \|f\|_{L^1(G)}$. Now if we consider a translation of g by $s \in H$, it is not difficult to see that

$$(s * g)(y) = \int_G (s * f)(x)\beta(x^{-1}y)\,\mathrm{d}_G x.$$

We can show that $\|\bigvee_{s\in E} s * g\| \le \|\bigvee_{s\in E} s * f\|$. For simplicity we only handle the case of $|E| = 2$, but the argument can be generalised to any E. So take $E = \{s_1, s_2\}$. Then

$$\begin{aligned}\|s_1 * g \vee s_2 * g\| &= \int_H (s_1 * g \vee s_2 * g)(y)\,\mathrm{d}_H y \\ &= \int_H \Big[\int_G (s_1 * f)(x)\beta(x^{-1}y)\,\mathrm{d}_G x\Big] \vee \Big[\int_G (s_2 * f)(x)\beta(x^{-1}y)\,\mathrm{d}_G x\Big]\,\mathrm{d}_H y \\ &\le \int_H \int_G [(s_1 * f)(x)\beta(x^{-1}y)] \vee [(s_1 * g)(x)\beta(x^{-1}y)]\,\mathrm{d}_G x\,\mathrm{d}_H y \\ &= \int_G (s_1 * f \vee s_2 * f)(x) \int_H \beta(x^{-1}y)\,\mathrm{d}_H y\,\mathrm{d}_G x \\ &= \int_G (s_1 * f \vee s_2 * f)(x)\,\mathrm{d}_G x = \|s_1 * f \vee s_2 * f\|.\end{aligned}$$

The inequality holds because for any positive functions f_1, f_2 we have

$$\max\Big\{\int f_1\,\mathrm{d}x, \int f_2\,\mathrm{d}x\Big\} \le \int f_1 \vee f_2\,\mathrm{d}x.$$

Thus g satisfies (L), since

$$\Big\|\bigvee_{s\in E} s * g\Big\| \le \Big\|\bigvee_{s\in E} s * f\Big\| < \varepsilon|E|\,\|f\| = \varepsilon|E|\,\|g\|.$$

So H is pseudo-amenable. ∎

We know that amenable groups are closed under a directed union. We can equivalently state this by saying that if the closure of any finitely generated subgroup of G is amenable, then the group G itself is amenable. We have a similar result for pseudo-amenable groups, though with a slight modification.

PROPOSITION 2.5. *Let G be a locally compact group. Suppose that every separable subgroup of G is pseudo-amenable. Then G is pseudo-amenable.*

Proof. Suppose that G is not pseudo-amenable. Then by definition, there is some $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there is some finite $F_N \subseteq G$ with $|F_N| \geq N$ such that for every compact set C, we have

$$\lambda(F_N C) \geq \varepsilon |F_N| \lambda(C). \tag{2.5}$$

Let H be the closure of the subgroup generated by $\bigcup_{N\in\mathbb{N}} F_N$. Now, $F_N \subseteq H$, and furthermore, for any $C \subseteq H$, C is compact in G if and only if C is compact in H. From this statement and (2.5), it follows by definition that H is not pseudo-amenable. ∎

Unfortunately this argument does not work for finitely generated subgroups.

Our next theorem will show that the image of a pseudo-amenable group under a homomorphism is also pseudo-amenable. We shall use condition (M) and the push-forward measure to show this.

THEOREM 2.6. *Let G and H be locally compact groups, and $\pi : G \to H$ a surjective homomorphism. If G is pseudo-amenable, then so is H.*

Proof. First take any $\mu \in M(G)$. Let $\pi_*(\mu)$ be the push-forward measure, defined by $\pi_*(\mu)(A) := \mu(\pi^{-1}(A))$. Now, the mapping $\mu \mapsto \pi_*(\mu)$ is useful as if we take a positive $\mu \in M(G)$ that satisfies (M), then the corresponding $\pi_*(\mu) \in M(H)$ will also satisfy (M) for the same values of ε and N.

To see this, one can show that this mapping preserves norms, joins, and translations in the sense that

$$\|\pi_*(\mu)\| = \|\mu\|, \quad \pi_*(\mu) \vee \pi_*(\nu) \leq \pi_*(\mu \vee \nu), \quad \pi(s) * \pi_*(\mu) = \pi_*(s * \mu)$$

for any positive $\mu, \nu \in M(G)$ and $s \in G$.

With this, we can take any $\varepsilon > 0$, and take any sufficiently large finite set $E \subseteq H$. Since π is surjective, we can find a finite set $F \subseteq G$ such that $\pi(F) = E$ and $|F| = |E|$. Now choose $\mu \in M(G)$ satisfying condition (M) for F. Then

$$\Big\|\bigvee_{t\in E} t * \pi_*(\mu)\Big\| = \Big\|\bigvee_{s\in F} \pi_*(s * \mu)\Big\| \leq \Big\|\pi_*\Big(\bigvee_{s\in F} s * \mu\Big)\Big\| = \Big\|\bigvee_{s\in F} s * \mu\ \Big\|$$
$$< \varepsilon |F|\, \|\mu\| = \varepsilon |E|\, \|\pi_*(\mu)\|.$$

Hence $\pi_*(\mu)$ satisfies (M), and so H is pseudo-amenable. ∎

This theorem also has the following obvious corollary involving quotient groups.

COROLLARY 2.7. *Let G be a pseudo-amenable group, and H a closed normal subgroup. Then the quotient group G/H is pseudo-amenable.*

We now wish to see if we can ‘reverse the process’ given in this corollary. In particular we want to show that a given group is pseudo-amenable using the properties of a (closed normal) subgroup and the corresponding quotient group. Before we attempt this, we need some method of relating the Haar measures of G, H and G/H. Details of this are covered by Folland [5, pp. 56–57]. Most important is the result stated in [5, Theorem 2.49], which tells us that if we have a closed normal subgroup H, then we can normalise the Haar measures in each of G, H, and G/H in such a way that for any function $f \in C_c(G)$ we have

$$\int_G f(x)\,\mathrm{d}x = \int_{G/H}\int_H f(xh)\,\mathrm{d}h\,\mathrm{d}(xH).$$

We will need an equivalent version of this statement but with characteristic functions:

Lemma 2.8. *Let G be a locally compact group, and H a closed normal subgroup. For any compact set K, we have*

$$\lambda(K) = \int_{G/H}\int_H \chi_K(xh)\,\mathrm{d}h\,\mathrm{d}(xH).$$

This follows from the aforementioned Theorem 2.49 in Folland [5] and Proposition 7.4.4 in Cohn [2].

We shall also need a characterisation of amenability that was introduced by Emerson and Greenleaf [4]. One may view this as a strengthening of Følner’s original condition. This is as follows:

(SF) For every $\varepsilon > 0$, and every compact set K, there is a compact set C such that

$$\lambda(KC \setminus C) < \varepsilon\lambda(C). \tag{2.6}$$

In the ideal case, we would only need a closed normal subgroup H and the corresponding quotient group G/H to be pseudo-amenable in order to induce pseudo-amenability in G. Unfortunately, whether this is sufficient is unknown. However, we can strengthen the condition on H by requiring it to be amenable, which allows us to prove the following theorem.

Theorem 2.9. *Let G be a locally compact group, and H a closed normal subgroup. If H is amenable and G/H is pseudo-amenable, then G is pseudo-amenable.*

Proof. Firstly we let $\pi : G \to G/H$ denote the canonical mapping where $\pi(x) = xH$. We denote the Haar measures on G, H and G/H by λ_G, λ_H and $\lambda_{G/H}$ respectively.

Let $\varepsilon > 0$, and choose N as required for the pseudo-amenability of G/H. Take any finite set $F \subseteq G$ with $|F| \geq N$. Now it is possible that $|\pi(F)| < N$, so we cannot directly apply pseudo-amenability here. However, in this case we may simply take any finite set X such that $\pi(F) \subseteq X$ and $|X| = |F| \geq N$. This implies that there is some compact set $C' \subseteq G/H$ such that

$$\lambda_{G/H}(\pi(F)C') \leq \lambda_{G/H}(XC') < \frac{\varepsilon}{2}|F|\lambda_{G/H}(C').$$

We note that the right side of this equation features $|F|$ instead of $|\pi(F)|$.

Now, for any compact set C' in G/H, we can find a compact set C in G such that $\pi(C) = C'$ (the proof was given by Folland [5, Lemma 2.46]). This means we can find a compact set $C \subseteq G$ such that

$$\lambda_{G/H}(\pi(FC)) < \frac{\varepsilon}{2}|F|\lambda_{G/H}(\pi(C)). \tag{2.7}$$

Next we define

$$K = (FC)^{-1}FC \cap H.$$

It is clear from this definition that $K \subseteq H$ and furthermore K is compact. We can use the amenability of H via (2.6) to find some compact set $L \subseteq H$ such that

$$\lambda_H(KL) < 2\lambda_H(L). \tag{2.8}$$

Now, for any $x \in G$ and $h \in H$ where $xh \in FCL$, we clearly have $xH \in \pi(FC)$. Using Lemma 2.8 gives

$$\begin{aligned}
\lambda_G(FCL) &= \int_{G/H}\int_H \chi_{FCL}(xh)\,\mathrm{d}h\,\mathrm{d}(xH) = \int_{\pi(FC)}\int_H \chi_{FCL}(xh)\,\mathrm{d}h\,\mathrm{d}(xH) \\
&= \int_{\pi(FC)} \lambda_H(x^{-1}FCL \cap H)\,\mathrm{d}(xH) \\
&= \int_{\pi(FC)} \lambda_H((x^{-1}FC \cap H)L)\,\mathrm{d}(xH) \\
&\leq \int_{\pi(FC)} \lambda_H(KL)\,\mathrm{d}(xH) = \lambda_{G/H}(\pi(FC))\lambda_H(KL).
\end{aligned}$$

Next, for any $x \in C$ and $h \in H$, we have $\chi_L(h) \leq \chi_{CL}(xh)$. From this we obtain

$$\begin{aligned}
\lambda_G(CL) &= \int_{G/H}\int_H \chi_{CL}(xh)\,\mathrm{d}h\,\mathrm{d}(xH) = \int_{\pi(C)}\int_H \chi_{CL}(xh)\,\mathrm{d}h\,\mathrm{d}(xH) \\
&\geq \int_{\pi(C)}\int_H \chi_L(h)\,\mathrm{d}h\,\mathrm{d}(xH) = \lambda_{G/H}(\pi(C))\lambda_H(L).
\end{aligned}$$

We can combine these results with (2.7) and (2.8) to get

$$\lambda_G(FCL) \leq \lambda_{G/H}(\pi(FC))\lambda_H(KL) < \varepsilon|F|\lambda_{G/H}(\pi(C))\lambda_H(L) \leq \varepsilon|F|\lambda_G(CL),$$

which is precisely what we need to show the pseudo-amenability of G. ∎

Corollary 2.10. *If G is an amenable group and H a pseudo-amenable group, then $G \times H$ is pseudo-amenable.*

All of these results suggest a similarity between pseudo-amenable and amenable groups. It is still an open problem whether these classes are the same. If we were however to show that pseudo-amenability and amenability were two distinct classes of groups by way of an example, then such an example would necessarily be a non-amenable group that does not contain F_2 (it is known that F_2 is not pseudo-amenable; see [3, Theorem 5.9] and [13, Proposition 5.9]). The existence of such groups was a long-standing problem, commonly referred to as the *Von Neumann Conjecture*, and was resolved by Ol'shanskiĭ [11] in 1980 by constructing an explicit (though very complex) example of such a group. Recently, simpler counterexamples have been found by Monod [9] and by Lodha and Moore [8], so these may prove to be easier to analyse. It is not known if any of these groups are pseudo-amenable.

Acknowledgements. I would like to thank Hung Pham for his assistance and for many engaging discussions that have contributed to this paper.

Most of this work grew out of my Masters Thesis at Victoria University of Wellington, which was supported by the Victoria Masters by Thesis Scholarship.

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