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Norm form equations with solutions taking values in a multi-recurrence

by

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1. Introduction. Let K be an algebraic number field of degree d and let $\alpha_1, \dots, \alpha_n$ be linearly independent elements of K over $\mathbb{Q}$. Thus we have $n \leq d$. We denote by $N_{K/\mathbb{Q}}$ the field norm of K, and for an integer m we consider the norm form equation given by

$$N_{K/\mathbb{Q}}(x_1\alpha_1 + \dots + x_n\alpha_n) = m. \tag{1.1}$$

It was proven by Schmidt (cf. [S71]) that if the $\mathbb{Z}$-module generated by $\alpha_1, \dots, \alpha_n$ contains a submodule which is a full module in a subfield of $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ different from the imaginary quadratic fields and $\mathbb{Q}$, then this equation has infinitely many solutions $(x_1, \dots, x_n) \in \mathbb{Z}^n$ for some m.

Observe that by multiplying with a common denominator we may assume that $\alpha_1, \dots, \alpha_n$ are algebraic integers; we will assume this from now on. In the present paper we are interested in those x_i which can be in the value set of a multi-recurrence. Since we are interested in proving finiteness results, we will always assume that (1.1) has infinitely many solutions. This immediately implies that there exists an index i such that we have infinitely many solutions of (1.1) with different x_i.

We remark that there are many results in the special case that the norm form equation is a Pell equation and/or the multi-recurrence is a linear recurrence sequence (see e.g. [BHP10] and the papers cited therein). A good overview on effective results can be found in [ST86]. Laurent proved in [L87] necessary and sufficient conditions under which two linear recurrence sequences have infinitely many common values. Since the coordinates of the solution vectors of a norm form equation can be described by finitely many multi-recurrences, the theorems proven in the present paper are a step for-

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ward to a generalization of Laurent's result to multi-recurrences. We thank the referee for pointing out this alternative point of view.

2. Notation and results. Let s be an arbitrary integer and G a *multi-recurrence*, that is, a mapping $\mathbb{Z}^s \to \mathbb{C}$ which is given by a function of polynomial-exponential type

$$G(k_1, \ldots, k_s) = \sum_{j=1}^{q} P_j(k_1, \ldots, k_s)\alpha_{j1}^{k_1} \cdots \alpha_{js}^{k_s} \tag{2.1}$$

with $P_j(X_1, \ldots, X_s) \in \mathbb{C}[X_1, \ldots, X_s]$ and non-zero $\alpha_{j1}, \ldots, \alpha_{js} \in \mathbb{C}$ for $j = 1, \ldots, q$. As is well-known, for such multi-sequences the value $G(k_1, \ldots, k_s)$ can be described by certain linear combinations of values of G with shifted entries and hence they are the natural extension of sequences which satisfy a linear recurrence relation. The multi-recurrence G is called *simple* if $\deg P_j = 0$ for all $j = 1, \ldots, q$; we put $P_j(X_1, \ldots, X_s) = p_j$ in this case. Moreover, we say that G *is defined over a number field* K if the α_{ji} and the coefficients of the P_j are elements of K for all j and i.

Clearly, G can have infinitely many zeros $(k_1, \ldots, k_s) \in \mathbb{Z}^s$. We mention that the structure of solutions $(k_1, \ldots, k_s)$ of $G(k_1, \ldots, k_s) = 0$, in the case that there are infinitely many of them, is not known (in contrast to the case $s = 1$ of linear recurring sequences where the Skolem–Mahler–Lech theorem says that all solutions lie in a finite union of arithmetic progressions). For more details on this see e.g. [S03].

It is also well-known and not too hard to prove (see e.g. [BHP10, proof of Theorem 2.1]) that each component of the solutions of (1.1) is contained in the union of finitely many multi-recurrences. In fact we know for each $\ell \in \{1, \ldots, n\}$ that

$$x_\ell = H(h_1, \ldots, h_r)$$

for a multi-recurrence H from a finite set of multi-recurrences all having the form

$$H(h_1, \ldots, h_r) := \sum_{i=1}^{n} \tau_i \sigma_i(\varepsilon_1)^{h_1} \cdots \sigma_i(\varepsilon_r)^{h_r}$$

where the τ_i are constants, $\varepsilon_1, \ldots, \varepsilon_r$ is a system of fundamental units in the ring of integers and $\sigma_1, \ldots, \sigma_n$ are embeddings of K in $\mathbb{C}$ such that the matrix $(\sigma_i(\alpha_j))$ has non-zero determinant. For the sake of completeness we give a short sketch of this proof. Let $\sigma_1, \ldots, \sigma_d$ be the elements of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$. For $i = 1, \ldots, d$ we get

$$x_1\sigma_i(\alpha_1) + \cdots + x_n\sigma_i(\alpha_n) = \sigma_i(\varepsilon)\sigma_i(\mu)$$

where ε is a unit and μ an element of norm m which can be chosen from a finite set by [EG97, Lemma 4]. Let us therefore consider a fixed value of μ.

Choose the order of the isomorphisms $\sigma_1, \ldots, \sigma_d$ in such a way that the matrix

$$M = \begin{pmatrix} \sigma_1(\alpha_1) & \cdots & \sigma_1(\alpha_n) \\ \vdots & \ddots & \vdots \\ \sigma_n(\alpha_1) & \cdots & \sigma_n(\alpha_n) \end{pmatrix}$$

has non-zero determinant. This implies

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = M^{-1} \begin{pmatrix} \sigma_1(\varepsilon)\sigma_1(\mu) \\ \vdots \\ \sigma_n(\varepsilon)\sigma_n(\mu) \end{pmatrix}$$

and we get

$$x_\ell = \sum_{i=1}^{n} m_{\ell i}\sigma_i(\varepsilon)\sigma_i(\mu).$$

Applying Dirichlet's unit theorem gives the above statement.

We call the set $\Lambda(A, b) := \{(k_1, \ldots, k_s)A + b : k_1, \ldots, k_s \in \mathbb{Z}\}$ with A an $(s \times r)$-matrix with entries in $\mathbb{Z}$ and b a row vector with r entries from $\mathbb{Z}$ a *shifted sublattice* of $\mathbb{Z}^r$. We say that $(h_1, \ldots, h_r)$ *runs through a shifted sublattice* of $\mathbb{Z}^r$ if $(h_1, \ldots, h_r) \in \Lambda(A, b)$ for some $A \in \mathbb{Z}^{s\times r}$ and $b \in \mathbb{Z}^r$.

Let L be a finite extension of K. Then we can lift equation (1.1) to a norm form equation in L using the tower formula for the field norm

$$\begin{aligned} N_{L/\mathbb{Q}}(x_1\alpha_1 + \cdots + x_n\alpha_n) &= N_{K/\mathbb{Q}}(N_{L/K}(x_1\alpha_1 + \cdots + x_n\alpha_n)) \\ &= N_{K/\mathbb{Q}}((x_1\alpha_1 + \cdots + x_n\alpha_n)^{[L:K]}) = m^{[L:K]}. \end{aligned}$$

We call

$$N_{L/\mathbb{Q}}(x_1\alpha_1 + \cdots + x_n\alpha_n) = m^{[L:K]} \tag{2.2}$$

the *lifted norm form equation*. It is clear that all solutions of (1.1) are solutions of (2.2) as well, and conversely, all solutions of (2.2) correspond to solutions of $N_{K/\mathbb{Q}}(x_1\alpha_1 + \cdots + x_n\alpha_n) = \pm m$.

Our interest applies to solutions $(x_1, \ldots, x_n)$ of a norm form equation with the property that $x_\ell = G(k_1, \ldots, k_s)$ for some ℓ and a given multi-recurrence G. It is easy to see that this problem may have infinitely many solutions, e.g. this is the case if

$$G(k_1, \ldots, k_s) = H(h_1, \ldots, h_r)|_{(h_1,\ldots,h_r)=(k_1,\ldots,k_s)A+b} + G_0(k_1, \ldots, k_s) \tag{2.3}$$

for all $(k_1, \ldots, k_s)$ within an arithmetic progression of s-dimensional vectors (i.e. the cartesian product of s arithmetic progressions of integers), where H is a multi-recurrence coming from the solutions of the lifted norm form equation, evaluated at points that run through a shifted sublattice, and G_0 is a multi-recurrence that has infinitely many zeros along the intersection of that sublattice and the arithmetic progression. We call this case an *unavoidable exception.* If G has the form (2.3) of an unavoidable exception with the additional property that $G_0 = 0$ and that H comes directly from the solutions of (1.1), i.e. trivially lifted to itself ($L = K$), then we call it a *reduced unavoidable exception.*

We have the following first theorem that describes the situation in the general case:

THEOREM 2.1. *Let G be a simple multi-recurrence that is defined over K by* (2.1) *where all the α_{ji} are algebraic integers. Then for any $\ell \in \{1, \dots, n\}$ there are at most finitely many different values of x_ℓ such that $(x_1, \dots, x_n) \in \mathbb{Z}^n$ is a solution of* (1.1) *and*

$$x_\ell = G(k_1, \dots, k_s)$$

for suitable $(k_1, \dots, k_s) \in \mathbb{N}^s$ unless G has the form of an unavoidable exception.

In the special case $s = 1$ of a linear recurrence sequence we can prove the following stronger result:

THEOREM 2.2. *Let G be a simple linear recurrence sequence that is defined over K by* (2.1) *with $s = 1$. Then for any $\ell \in \{1, \dots, n\}$ there are at most finitely many different values of x_ℓ such that $(x_1, \dots, x_n) \in \mathbb{Z}^n$ is a solution of* (1.1) *and*

$$x_\ell = G(k_1)$$

for suitable $k_1 \in \mathbb{N}$ unless G has the form of a reduced unavoidable exception.

3. Preliminaries. Let F be an algebraically closed field of characteristic 0. Denote by F^* the multiplicative group of non-zero elements and let $(F^*)^n$ be the direct product consisting of n-tuples $(y_1, \dots, y_n)$ with $y_\ell \in F^*$ for $\ell = 1, \dots, n$ equipped with componentwise multiplication. Let Γ be a subgroup of $(F^*)^n$ and suppose that $(a_1, \dots, a_n) \in (F^*)^n$. We will consider the generalized unit equation

$$a_1 y_1 + \dots + a_n y_n = 1 \tag{3.1}$$

in $(y_1, \dots, y_n) \in \Gamma$. A solution $(y_1, \dots, y_n)$ is called *non-degenerate* if no subsum of the left hand side of (3.1) vanishes, which means that $\sum_{i \in I} a_i y_i \neq 0$ for any non-empty subset I of $\{1, \dots, n\}$. The following lemma proved by Evertse, Schlickewei and Schmidt as Theorem 1.1 in [ESS02] will be used in our proofs:

LEMMA 3.1. *Suppose that Γ has finite rank r. Then the number of non-degenerate solutions $(y_1, \dots, y_n) \in \Gamma$ of equation* (3.1) *is bounded by*

$$\exp((6n)^{3n}(r + 1)).$$

In particular this implies that there are only finitely many non-degenerate solutions of the generalized unit equation.

4. Proofs. Now we are going to prove our two theorems. We start with the general case, which will also be the base for the special one.

Proof of Theorem 2.1. Assume that for some $\ell \in \{1, \dots, n\}$ there are infinitely many different values of x_ℓ such that $(x_1, \dots, x_n) \in \mathbb{Z}^n$ is a solution of (1.1) and

$$x_\ell = G(k_1, \dots, k_s)$$

for suitable $(k_1, \dots, k_s) \in \mathbb{N}^s$. Then we can choose an infinite sequence of such values for x_ℓ that are all non-zero and pairwise distinct. Each value corresponds to another vector $(k_1, \dots, k_s)$ satisfying $x_\ell = G(k_1, \dots, k_s)$. This vector is not necessarily uniquely determined, but we will fix one possible vector for each value of x_ℓ now.

Thus we get a sequence of vectors $(k_1, \dots, k_s)$. If the first component k_1 takes a fixed value for infinitely many elements of our sequence, then we pass to a subsequence where k_1 is constant. Otherwise we pass to a subsequence where k_1 is strictly increasing and non-zero. We perform the same procedure with the other components. After reindexing we can assume that $k_1, \dots, k_t$ are strictly increasing and that $k_{t+1}, \dots, k_s$ are constant.

The next step is to analyze whether there are linear dependences between the non-constant components $k_1, \dots, k_t$ of the vectors $(k_1, \dots, k_s)$. If

$$a_0^{(t)} + a_1^{(t)} k_1 + \dots + a_t^{(t)} k_t = 0$$

for infinitely many vectors $(k_1, \dots, k_s)$ and for constant rational integers $a_0^{(t)}, \dots, a_t^{(t)}$ which are not all zero, then we pass to an infinite subsequence where this equation holds. By reindexing we can assume $a_t^{(t)} \neq 0$ and get

$$k_t = \frac{1}{a_t^{(t)}} (-a_0^{(t)} - a_1^{(t)} k_1 - \dots - a_{t-1}^{(t)} k_{t-1}).$$

If in addition for infinitely many vectors $(k_1, \dots, k_s)$ and constant rational integers $a_0^{(t-1)}, \dots, a_{t-1}^{(t-1)}$ which are not all zero the equation

$$a_0^{(t-1)} + a_1^{(t-1)} k_1 + \dots + a_{t-1}^{(t-1)} k_{t-1} = 0$$

holds, then we perform the analogous procedure.

Thus we can assume that there is an index $l \leq t$ such that both

$$a_0^{(l)} + a_1^{(l)} k_1 + \dots + a_l^{(l)} k_l = 0$$

for infinitely many vectors $(k_1, \dots, k_s)$ and for constant rational integers

$a_0^{(l)}, \dots, a_l^{(l)}$ implies $a_0^{(l)} = \cdots = a_l^{(l)} = 0$, and

$$\begin{aligned} k_{l+1} &= \frac{1}{d_{l+1}}(\widetilde{a}_0^{(l+1)} + \widetilde{a}_1^{(l+1)} k_1 + \cdots + \widetilde{a}_l^{(l+1)} k_l), \\ &\vdots \\ k_t &= \frac{1}{d_t}(\widetilde{a}_0^{(t)} + \widetilde{a}_1^{(t)} k_1 + \cdots + \widetilde{a}_l^{(t)} k_l) \end{aligned} \tag{4.1}$$

with rational integers $d_j > 0$ and $\widetilde{a}_i^{(j)}$ for our infinite sequence. Therefore we define the finite extension L of K as

$$L = K(\{\sqrt[d_j]{\alpha_{\kappa\nu}} : j = l+1, \dots, t;\ \kappa = 1, \dots, q;\ \nu = 1, \dots, s\}).$$

Now we mention that our sequence of values of x_ℓ corresponds to a sequence of vectors $(h_1, \dots, h_r)$ satisfying

$$x_\ell = H(h_1, \dots, h_r)$$

for a fixed multi-recurrence H coming from the solutions of the lifted norm form equation (2.2) if we pass to a subsequence once again. Moreover, the sequence of values of x_ℓ corresponds to a sequence of vectors $(h'_1, \dots, h'_{r'})$ satisfying

$$x_\ell = H'(h'_1, \dots, h'_{r'})$$

for a fixed multi-recurrence H' coming from the solutions of the norm form equation (1.1) if we pass to a subsequence once again.

Altogether we have the following correspondences which will be used tacitly in what follows:

$$x_\ell \leftrightarrow (k_1, \dots, k_s) \leftrightarrow (h_1, \dots, h_r) \leftrightarrow (h'_1, \dots, h'_{r'}).$$

When we say that something holds for infinitely many vectors, this means for infinitely many vectors in our sequence and we implicitly pass to a subsequence where this property is satisfied by all elements.

Let us take a closer look at the multi-recurrence G. Suppose that there is a constant c_{ij} for some distinct indices i, j such that for infinitely many vectors,

$$\alpha_{j1}^{k_1} \cdots \alpha_{js}^{k_s} = c_{ij} \alpha_{i1}^{k_1} \cdots \alpha_{is}^{k_s}. \tag{4.2}$$

Then we can construct a multi-recurrence

$$G_0^{(\mathrm{I})}(k_1, \dots, k_s) := \sum (p_j \alpha_{j1}^{k_1} \cdots \alpha_{js}^{k_s} - p_j c_{ij} \alpha_{i1}^{k_1} \cdots \alpha_{is}^{k_s})$$

which is zero for the vectors in our sequence such that

$$G_{\mathrm{red}}(k_1, \dots, k_s) := G(k_1, \dots, k_s) - G_0^{(\mathrm{I})}(k_1, \dots, k_s) = \sum_{j=1}^{\widetilde{q}} \widetilde{p}_j \alpha_{j1}^{k_1} \cdots \alpha_{js}^{k_s}$$

does not contain two summands which satisfy a relation of the shape (4.2). We perform the analogous procedure for the multi-recurrences H and H' to get

$$\begin{aligned} H_{\mathrm{red}}(h_1,\ldots,h_r) &:= H(h_1,\ldots,h_r) - H_0(h_1,\ldots,h_r) \\ &= \sum_{i=1}^{\widetilde{n}} \widetilde{\tau}_i \sigma_i(\varepsilon_1)^{h_1}\cdots\sigma_i(\varepsilon_r)^{h_r}, \\ H'_{\mathrm{red}}(h_1,\ldots,h_r) &:= H'(h_1,\ldots,h_r) - H'_0(h_1,\ldots,h_r) \\ &= \sum_{i=1}^{\widetilde{n}'} \widetilde{\tau}'_i \sigma'_i(\varepsilon'_1)^{h'_1}\cdots\sigma'_i(\varepsilon'_{r'})^{h'_{r'}}. \end{aligned}$$

For our sequence we have the equality

$$\begin{aligned} H_{\mathrm{red}}(h_1,\ldots,h_r) &= H(h_1,\ldots,h_r) - H_0(h_1,\ldots,h_r) \\ &= H(h_1,\ldots,h_r) = x_\ell = G(k_1,\ldots,k_s) \\ &= G(k_1,\ldots,k_s) - G_0^{(\mathrm{I})}(k_1,\ldots,k_s) = G_{\mathrm{red}}(k_1,\ldots,k_s). \end{aligned}$$

Putting in the sum representations yields

$$\sum_{i=1}^{\widetilde{n}} \widetilde{\tau}_i \sigma_i(\varepsilon_1)^{h_1}\cdots\sigma_i(\varepsilon_r)^{h_r} - \sum_{j=1}^{\widetilde{q}} \widetilde{p}_j \alpha_{j1}^{k_1}\cdots\alpha_{js}^{k_s} = 0, \tag{4.3}$$

which can be rewritten as

$$\sum_{i=1}^{\widetilde{n}} \widetilde{\tau}_i \sigma_i(\varepsilon_1)^{h_1}\cdots\sigma_i(\varepsilon_r)^{h_r} - \sum_{j=1}^{\widetilde{q}-1} \widetilde{p}_j \alpha_{j1}^{k_1}\cdots\alpha_{js}^{k_s} = \widetilde{p}_{\widetilde{q}} \alpha_{\widetilde{q}1}^{k_1}\cdots\alpha_{\widetilde{q}s}^{k_s}$$

and after dividing by the right hand side this is

$$\sum_{i=1}^{\widetilde{n}} \frac{\widetilde{\tau}_i}{\widetilde{p}_{\widetilde{q}}} \sigma_i(\varepsilon_1)^{h_1}\cdots\sigma_i(\varepsilon_r)^{h_r} \alpha_{\widetilde{q}1}^{-k_1}\cdots\alpha_{\widetilde{q}s}^{-k_s} - \sum_{j=1}^{\widetilde{q}-1} \frac{\widetilde{p}_j}{\widetilde{p}_{\widetilde{q}}} \left(\frac{\alpha_{j1}}{\alpha_{\widetilde{q}1}}\right)^{k_1}\cdots\left(\frac{\alpha_{js}}{\alpha_{\widetilde{q}s}}\right)^{k_s} = 1.$$

The previous line can be seen as a generalized unit equation in the $\widetilde{n}+\widetilde{q}-1$ unknowns

$$\sigma_i(\varepsilon_1)^{h_1}\cdots\sigma_i(\varepsilon_r)^{h_r} \alpha_{\widetilde{q}1}^{-k_1}\cdots\alpha_{\widetilde{q}s}^{-k_s}, \qquad \left(\frac{\alpha_{j1}}{\alpha_{\widetilde{q}1}}\right)^{k_1}\cdots\left(\frac{\alpha_{js}}{\alpha_{\widetilde{q}s}}\right)^{k_s}, \tag{4.4}$$

and by Lemma 3.1 either there are only finitely many solutions or we have a vanishing subsum. In the first case all expressions (4.4) are constant for infinitely many vectors. In the second case some expressions (4.4) are constant for infinitely many vectors and the remaining terms make up a vanishing subsum. We multiply the vanishing subsum by $\widetilde{p}_{\widetilde{q}} \alpha_{\widetilde{q}1}^{k_1}\cdots\alpha_{\widetilde{q}s}^{k_s}$ and then do the same as we have done with (4.3). Since in each step we have fewer summands in the equation of the form (4.3) than in the step before, this procedure ends after finitely many steps.

This gives us a set of equations, valid for infinitely many vectors, of the following three types (in all three types it is the case $i \neq j$):

$$\sigma_i(\varepsilon_1)^{h_1} \cdots \sigma_i(\varepsilon_r)^{h_r} = C_{ij}\alpha_{j1}^{k_1} \cdots \alpha_{js}^{k_s}, \tag{4.5}$$

$$\sigma_i(\varepsilon_1)^{h_1} \cdots \sigma_i(\varepsilon_r)^{h_r} = D_{ij}\sigma_j(\varepsilon_1)^{h_1} \cdots \sigma_j(\varepsilon_r)^{h_r}, \tag{4.6}$$

$$\alpha_{i1}^{k_1} \cdots \alpha_{is}^{k_s} = E_{ij}\alpha_{j1}^{k_1} \cdots \alpha_{js}^{k_s}. \tag{4.7}$$

Each expression $\sigma_i(\varepsilon_1)^{h_1} \cdots \sigma_i(\varepsilon_r)^{h_r}$ for $i = 1, \ldots, \widetilde{n}$ and each expression $\alpha_{j1}^{k_1} \cdots \alpha_{js}^{k_s}$ for $j = 1, \ldots, \widetilde{q}$ occurs at least once among those equations.

By our construction of the multi-recurrences G_{red} and H_{red} there cannot be an equation of type (4.6) or (4.7). Moreover, no expression $\sigma_i(\varepsilon_1)^{h_1} \cdots \sigma_i(\varepsilon_r)^{h_r}$ for $i = 1, \ldots, \widetilde{n}$ and no expression $\alpha_{j1}^{k_1} \cdots \alpha_{js}^{k_s}$ for $j = 1, \ldots, \widetilde{q}$ can occur more than once among the equations of type (4.5) since otherwise we could deduce an equation of type (4.6) or (4.7). Thus each expression $\sigma_i(\varepsilon_1)^{h_1} \cdots \sigma_i(\varepsilon_r)^{h_r}$ for $i = 1, \ldots, \widetilde{n}$ and each expression $\alpha_{j1}^{k_1} \cdots \alpha_{js}^{k_s}$ for $j = 1, \ldots, \widetilde{q}$ occurs exactly once among the equations of type (4.5). Therefore we have $\widetilde{n} = \widetilde{q}$ and after a suitable reindexing

$$\sigma_i(\varepsilon_1)^{h_1} \cdots \sigma_i(\varepsilon_r)^{h_r} = C_{ii}\alpha_{i1}^{k_1} \cdots \alpha_{is}^{k_s}$$

for $i = 1, \ldots, \widetilde{q}$. Since $k_{t+1}, \ldots, k_s$ are constant in our sequence, this can be rewritten as

$$\sigma_i(\varepsilon_1)^{h_1} \cdots \sigma_i(\varepsilon_r)^{h_r} = \widetilde{C}_i\alpha_{i1}^{k_1} \cdots \alpha_{it}^{k_t} \tag{4.8}$$

for $i = 1, \ldots, \widetilde{q}$.

The same steps we have done for H_{red} in the last paragraphs can be done for H'_{red} as well. So we have $\widetilde{n}' = \widetilde{q}$ and

$$\sigma'_i(\varepsilon'_1)^{h'_1} \cdots \sigma'_i(\varepsilon'_{r'})^{h'_{r'}} = \widetilde{C}'_i\alpha_{i1}^{k_1} \cdots \alpha_{it}^{k_t}$$

for $i = 1, \ldots, \widetilde{q}$. Let $(\widehat{k}_1, \ldots, \widehat{k}_s)$ be the first (smallest) element in our sequence. Then we get from the last equation by division the following one:

$$\sigma'_i(\varepsilon'_1)^{h'_1 - \widehat{h}'_1} \cdots \sigma'_i(\varepsilon'_{r'})^{h'_{r'} - \widehat{h}'_{r'}} = \alpha_{i1}^{k_1 - \widehat{k}_1} \cdots \alpha_{it}^{k_t - \widehat{k}_t}. \tag{4.9}$$

Since the left hand side is a unit in the ring of integers (of K and thus also of L), the right hand side must be a unit, too. Moreover, the exponents $k_1 - \widehat{k}_1, \ldots, k_t - \widehat{k}_t$ are all positive rational integers by construction and the bases $\alpha_{i1}, \ldots, \alpha_{it}$ are algebraic integers by assumption. Thus, $\alpha_{i1}, \ldots, \alpha_{it}$ are units in the ring of integers for $i = 1, \ldots, \widetilde{q}$ (of K and thus also of L).

From here on we will work over L. We use the representations (4.1) to rewrite equation (4.8) as

$$\begin{aligned} \sigma_i(\varepsilon_1)^{h_1} \cdots \sigma_i(\varepsilon_r)^{h_r} &= \widetilde{C}_i\alpha_{i1}^{k_1} \cdots \alpha_{it}^{k_t} \\ &= \widetilde{C}_i\alpha_{i1}^{k_1} \cdots \alpha_{il}^{k_l} \prod_{u=l+1}^{t} \alpha_{iu}^{(\widetilde{a}_0^{(u)} + \widetilde{a}_1^{(u)}k_1 + \cdots + \widetilde{a}_l^{(u)}k_l)/d_u} \end{aligned}$$

$$= B_i\Big(\alpha_{i1}\prod_{u=l+1}^{t}\alpha_{iu}^{\widetilde{a}_1^{(u)}/d_u}\Big)^{k_1}\cdots\Big(\alpha_{il}\prod_{u=l+1}^{t}\alpha_{iu}^{\widetilde{a}_l^{(u)}/d_u}\Big)^{k_l}$$
$$= B_i\beta_{i1}^{k_1}\cdots\beta_{il}^{k_l}$$

with

$$\beta_{ij} := \alpha_{ij}\prod_{u=l+1}^{t}\alpha_{iu}^{\widetilde{a}_j^{(u)}/d_u}$$

for $j = 1, \dots, l$ and $i = 1, \dots, \widetilde{q}$. As the α_{ij} are units in the ring of integers, by our construction of the number field L the β_{ij} are units in the ring of integers of L.

Therefore

$$\sigma_i(\varepsilon_1)^{h_1}\cdots\sigma_i(\varepsilon_r)^{h_r} = B_i\beta_{i1}^{k_1}\cdots\beta_{il}^{k_l}$$

and by division

$$\sigma_i(\varepsilon_1)^{h_1-\widehat{h}_1}\cdots\sigma_i(\varepsilon_r)^{h_r-\widehat{h}_r} = \beta_{i1}^{k_1-\widehat{k}_1}\cdots\beta_{il}^{k_l-\widehat{k}_l} \tag{4.10}$$

for $i = 1, \dots, \widetilde{q}$. Since β_{ij} is a unit, also $\sigma_i^{-1}(\beta_{ij})$ is a unit and by the Dirichlet unit theorem we can write this as

$$\sigma_i^{-1}(\beta_{ij}) = \zeta^{(ij)}\varepsilon_1^{w_1^{(ij)}}\cdots\varepsilon_r^{w_r^{(ij)}}$$

for rational integers $w_1^{(ij)}, \dots, w_r^{(ij)}$ and a root of unity $\zeta^{(ij)}$. Applying σ_i yields

$$\beta_{ij} = \sigma_i(\zeta^{(ij)})\sigma_i(\varepsilon_1)^{w_1^{(ij)}}\cdots\sigma_i(\varepsilon_r)^{w_r^{(ij)}}.$$

Now we put this into equation (4.10) and apply σ_i^{-1} to get

$$\varepsilon_1^{h_1-\widehat{h}_1}\cdots\varepsilon_r^{h_r-\widehat{h}_r} = (\zeta^{(i1)}\varepsilon_1^{w_1^{(i1)}}\cdots\varepsilon_r^{w_r^{(i1)}})^{k_1-\widehat{k}_1}\cdots(\zeta^{(il)}\varepsilon_1^{w_1^{(il)}}\cdots\varepsilon_r^{w_r^{(il)}})^{k_l-\widehat{k}_l}$$
$$= \zeta^{(i1)^{k_1-\widehat{k}_1}}\cdots\zeta^{(il)^{k_l-\widehat{k}_l}}\prod_{v=1}^{r}\varepsilon_v^{(k_1-\widehat{k}_1)w_v^{(i1)}+\cdots+(k_l-\widehat{k}_l)w_v^{(il)}}.$$

Since the representation of any unit in the Dirichlet unit theorem is unique, we get

$$1 = \zeta^{(i1)^{k_1-\widehat{k}_1}}\cdots\zeta^{(il)^{k_l-\widehat{k}_l}},$$
$$h_1-\widehat{h}_1 = (k_1-\widehat{k}_1)w_1^{(i1)}+\cdots+(k_l-\widehat{k}_l)w_1^{(il)},$$
$$\vdots$$
$$h_r-\widehat{h}_r = (k_1-\widehat{k}_1)w_r^{(i1)}+\cdots+(k_l-\widehat{k}_l)w_r^{(il)}.$$

We rewrite this in matrix notation, which results in

$$\begin{pmatrix} h_1-\widehat{h}_1 \\ h_2-\widehat{h}_2 \\ \vdots \\ h_r-\widehat{h}_r \end{pmatrix} = \begin{pmatrix} w_1^{(i1)} & w_1^{(i2)} & \cdots & w_1^{(il)} \\ w_2^{(i1)} & w_2^{(i2)} & \cdots & w_2^{(il)} \\ \vdots & \vdots & \ddots & \vdots \\ w_r^{(i1)} & w_r^{(i2)} & \cdots & w_r^{(il)} \end{pmatrix} \begin{pmatrix} k_1-\widehat{k}_1 \\ k_2-\widehat{k}_2 \\ \vdots \\ k_l-\widehat{k}_l \end{pmatrix}$$

$$= \begin{pmatrix} w_1^{(i1)} & w_1^{(i2)} & \cdots & w_1^{(il)} & 0 & \cdots & 0 \\ w_2^{(i1)} & w_2^{(i2)} & \cdots & w_2^{(il)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ w_r^{(i1)} & w_r^{(i2)} & \cdots & w_r^{(il)} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} k_1-\widehat{k}_1 \\ k_2-\widehat{k}_2 \\ \vdots \\ k_s-\widehat{k}_s \end{pmatrix}.$$

Now we transpose the equation and get

$$(h_1-\widehat{h}_1, h_2-\widehat{h}_2, \ldots, h_r-\widehat{h}_r) = (k_1-\widehat{k}_1, k_2-\widehat{k}_2, \ldots, k_s-\widehat{k}_s)\cdot A^{(i)} \tag{4.11}$$

with

$$A^{(i)} = \begin{pmatrix} w_1^{(i1)} & w_2^{(i1)} & \cdots & w_r^{(i1)} \\ w_1^{(i2)} & w_2^{(i2)} & \cdots & w_r^{(i2)} \\ \vdots & \vdots & \ddots & \vdots \\ w_1^{(il)} & w_2^{(il)} & \cdots & w_r^{(il)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{Z}^{s\times r}$$

for $i = 1, \ldots, \widetilde{q}$. Since the $\widehat{\ }$-vector is a fixed element of our sequence (namely the first one), we can define the vector $b^{(i)} \in \mathbb{Z}^r$ to be the solution of

$$(\widehat{h}_1, \widehat{h}_2, \ldots, \widehat{h}_r) = (\widehat{k}_1, \widehat{k}_2, \ldots, \widehat{k}_s)\cdot A^{(i)} + b^{(i)} \quad \text{for } i = 1, \ldots, \widetilde{q}. \tag{4.12}$$

Adding equations (4.11) and (4.12) yields

$$(h_1, h_2, \ldots, h_r) = (k_1, k_2, \ldots, k_s)\cdot A^{(i)} + b^{(i)} \quad \text{for } i = 1, \ldots, \widetilde{q}. \tag{4.13}$$

In the next step we consider two instances of equation (4.13) for different indices i_1 and i_2. Subtracting one of them from the other gives

$$0 = (k_1, k_2, \ldots, k_s)\cdot (A^{(i_1)} - A^{(i_2)}) + b^{(i_1)} - b^{(i_2)}.$$

Since we have excluded any further linear dependences of $k_1, \ldots, k_l$ in the paragraph containing equation (4.1), we must have $A^{(i_1)} = A^{(i_2)}$ and $b^{(i_1)} = b^{(i_2)}$. Therefore we can omit the superscript and write

$$A := A^{(1)} = A^{(2)} = \cdots = A^{(\widetilde{q})} \quad \text{as well as} \quad b := b^{(1)} = b^{(2)} = \cdots = b^{(\widetilde{q})}$$

in what follows.

We will now evaluate the multi-recurrence H_{red} at the shifted sublattice given by A and b. For each summand we get the identity

$$\begin{aligned}\widetilde{\tau}_i\sigma_i(\varepsilon_1)^{h_1}\cdots\sigma_i(\varepsilon_r)^{h_r}|_{(h_1,\dots,h_r)=(k_1,\dots,k_s)A+b}&\\ &= \widetilde{q}_i\prod_{v=1}^{r}\sigma_i(\varepsilon_v)^{w_v^{(i1)}k_1+\cdots+w_v^{(il)}k_l}\\ &= \widetilde{q}_i(\sigma_i(\zeta^{(i1)^{k_1}}\cdots\zeta^{(il)^{k_l}}))^{-1}\prod_{u=1}^{l}(\sigma_i(\zeta^{(iu)})\sigma_i(\varepsilon_1)^{w_1^{(iu)}}\cdots\sigma_i(\varepsilon_r)^{w_r^{(iu)}})^{k_u}\\ &= \widetilde{q}_i(\sigma_i(\zeta^{(i1)^{k_1}}\cdots\zeta^{(il)^{k_l}}))^{-1}\beta_{i1}^{k_1}\cdots\beta_{il}^{k_l} = q_i\beta_{i1}^{k_1}\cdots\beta_{il}^{k_l},\end{aligned}$$

where the last equality holds (only) for $(k_1,\dots,k_s)$ within an arithmetic progression of s-dimensional vectors. This arithmetic progression can be chosen in such a way that it contains infinitely many of our vectors and that it is the same for all summands, i.e. for $i = 1,\dots,\widetilde{n}$. Thus, along an arithmetic progression we have the identity

$$H_{\mathrm{red}}(h_1,\dots,h_r)|_{(h_1,\dots,h_r)=(k_1,\dots,k_s)A+b} = \sum_{i=1}^{\widetilde{n}} q_i\beta_{i1}^{k_1}\cdots\beta_{il}^{k_l} =: G^*(k_1,\dots,k_s).$$

Since we have seen above that $H_{\mathrm{red}} = G_{\mathrm{red}}$ for the vectors in our sequence, $G_0^{(\mathrm{II})} := G_{\mathrm{red}} - G^*$ is zero for the vectors in our sequence.

Below, we will use the shortcut $|_\sharp$ for $|_{(h_1,\dots,h_r)=(k_1,\dots,k_s)A+b}$ to make the chain of equalities more readable. Moreover, we define

$$G_0 := G_0^{(\mathrm{I})} + G_0^{(\mathrm{II})} - H_0|_\sharp.$$

We emphasize that G_0 is zero for the vectors in our sequence. Putting all things together we get

$$\begin{aligned}G &= G_{\mathrm{red}} + G_0^{(\mathrm{I})} = G^* + G_0^{(\mathrm{II})} + G_0^{(\mathrm{I})}\\ &= H_{\mathrm{red}}|_\sharp + G_0^{(\mathrm{II})} + G_0^{(\mathrm{I})} = H|_\sharp - H_0|_\sharp + G_0^{(\mathrm{II})} + G_0^{(\mathrm{I})} = H|_\sharp + G_0\end{aligned}$$

as an identity along an arithmetic progression. Thus G has the form of an unavoidable exception. ■

It remains to prove Theorem 2.2. Since the procedure is the same as in the proof of Theorem 2.1, we will only describe the differences.

Proof of Theorem 2.2. We have $s = 1$. For some $\ell \in \{1,\dots,n\}$ assume that there are infinitely many values of x_ℓ such that $(x_1,\dots,x_n) \in \mathbb{Z}^n$ is a solution of (1.1) and

$$x_\ell = G(k_1)$$

for suitable $k_1 \in \mathbb{N}$. Then there are obviously no equations of the form $a^{(1)}k_1 = b^{(1)}$ valid for infinitely many k_1 unless $a^{(1)} = 0$. Therefore in the construction in the previous proof we get $L = K$.

In the new equation (4.9) we have on the right hand side the expression

$$\alpha_{i1}^{k_1-\widehat{k}_1}.$$

Thus we can deduce that α_{i1} must be an algebraic integer. It is not necessary to assume this.

In the same way as in the proof of Theorem 2.1 we get

$$G = H|_\sharp + G_0$$

along an arithmetic progression. Since $L = K$, the recurrence H comes directly from the solutions of (1.1). Since G_0 has infinitely many zeros, by the Skolem–Mahler–Lech theorem we have $G_0 = 0$ if we go to a new arithmetic progression. Thus G has the form of a reduced unavoidable exception. ■

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