STUDIA MATHEMATICA
Online First version

On the entropy and index of the winding endomorphisms of p-adic ring C^*-algebras

by

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Abstract. For $p \geq 2$, the p-adic ring C^*-algebra $\mathcal{Q}_p$ is the universal C^*-algebra generated by a unitary U and an isometry S_p such that $S_pU = U^pS_p$ and $\sum_{l=0}^{p-1} U^lS_pS_p^*U^{-l} = 1$. For any k coprime to p we define an endomorphism $\chi_k \in \mathrm{End}(\mathcal{Q}_p)$ by setting $\chi_k(U) := U^k$ and $\chi_k(S_p) := S_p$. We then compute the entropy of χ_k, which turns out to be $\log|k|$. Finally, for selected values of k we also compute the Watatani index of χ_k showing that the entropy is the natural logarithm of the index.

1. Introduction. First introduced by Adler, Konheim, and McAndrew [1], the topological entropy of a continuous map on a compact Hausdorff space soon proved to be a useful numerical invariant (under topological conjugacy) to tackle, for instance, dynamics that may be out of reach of the celebrated Halmos–von Neumann theorem, which only concerns those with topological discrete spectrum. Two results worth mentioning are that the entropy of any homeomorphism of the circle is null and the entropy of a differentiable map on a Riemannian manifold is finite, the latter known as Kushnirenko's theorem. A few years later, Dinaburg and Bowen gave a novel yet equivalent definition for maps on metric spaces, which is particularly suited to establishing connections with Kolmogorov's measure-theoretic entropy.

It was not until the mid-1990s, though, that Voiculescu [24] extended the original definition to endomorphisms, or more generally to completely positive maps, of nuclear C^*-algebras, thought of as the natural non-commutative counterpart of compact Hausdorff spaces. However, the computations involved to find the exact value of entropy are often rather demanding, so much so that not as many examples of endomorphisms as one would expect

2020 *Mathematics Subject Classification*: Primary 46L55, 46L40, 28D20; Secondary 37A25.
Key words and phrases: non-commutative dynamics, non-commutative topological entropy, Watatani index, p-adic ring C^*-algebras, Bunce–Deddens algebras, odometer, Cuntz algebras.
Received 25 November 2020; revised 1 February 2021.
Published online 18 October 2021.

DOI: 10.4064/sm201125-9-2

are known whose entropy has been computed. Of course, part of the difficulty also depends on the choice of the C^*-algebra.

Now the Cuntz algebras $\mathcal{O}_p$, $p \geq 2$, are a natural family of C^*-algebras to consider, not least because of their many connections with several research fields such as algebraic quantum field theory, index theory, and wavelets. The first example to be discussed was the so-called canonical shift of $\mathcal{O}_p$. In [15] Choda showed that its entropy is $\log p$, which is quite remarkable as this value is nothing but the entropy of a Bernoulli shift on the alphabet $\{1, \ldots, p\}$, and the restriction of the canonical shift to the diagonal subalgebra $\mathcal{D}_p \subset \mathcal{O}_p$ is just the Bernoulli shift. Soon after, this result was obtained by Boca and Goldstein [13] for shift-type endomorphisms on arbitrary Cuntz–Krieger algebras by using a different technique, and more recently by Skalski and Zacharias [23] for higher rank graph C^*-algebras. In [22] the last-mentioned authors provided an upper bound on the entropy of a general class of endomorphisms of $\mathcal{O}_p$ that leave the UHF subalgebra $\mathcal{F}_p$ invariant and satisfy a "finite-range" condition. Furthermore, they found the exact value of the entropy for all such endomorphisms of $\mathcal{O}_2$ associated with permutations of rank 2.

In this paper we aim to show that a suitable adaptation of the techniques employed in the aforementioned paper can be exploited to compute the entropy of a countable class of endomorphisms acting on the so-called p-adic ring C^*-algebras $\mathcal{Q}_p$. These and their generalizations have been the focus of much recent research [2–8, 10] and are here considered because they contain the Cuntz algebras in a natural way. Indeed, as we will see in the next section, each $\mathcal{O}_p$ is contained in $\mathcal{Q}_p$. Moreover, the commutative C^*-algebra of continuous functions on the one-dimensional torus $\mathbb{T}$ appears as a maximal abelian subalgebra of each $\mathcal{Q}_p$. Now the endomorphisms dealt with in our paper preserve this MASA, on which they simply act as $\mathbb{T} \ni z \mapsto z^k \in \mathbb{T}$, for some integer k. For this reason we will refer to them as *winding endomorphisms*. Quite interestingly, their entropy is completely determined by k. More precisely, the main result of the present paper is that their non-commutative entropy is $\log |k|$, which is exactly the classical entropy of the continuous map $\Phi_k(z) = z^k$, $z \in \mathbb{T}$, on the circle. This is much in the same spirit as Choda's result on the entropy of the canonical shift we recalled above.

Finally, in the last section we attack the problem of computing the Watatani index of winding endomorphisms so as to spot possible relations with entropy, very much in line with what was done in [16], where the quadratic permutation endomorphisms of the Cuntz algebra $\mathcal{O}_2$ were studied. The technique we employ can be applied only to values of k of the form $\pm(p-1)^i$, $i \in \mathbb{N}$, and the Watatani index of the corresponding endomorphism turns out to be exactly $|k| = (p-1)^i$. Nevertheless the index of the restriction of a winding endomorphism to a remarkable subalgebra of $\mathcal{Q}_p$, the so-called gauge invariant subalgebra $\mathcal{Q}_p^{\mathbb{T}}$, which is isomorphic to the Bunce–Deddens algebra of type p^∞, can be computed for all values of k, and again is given by $|k|$. In particular, in all cases where the index can be computed, the entropy is the natural logarithm of the index.

2. Preliminaries and notation. Let $p \geq 2$ be a natural number. The *p-adic ring C^*-algebra* $\mathcal{Q}_p$ is the universal C^*-algebra generated by a unitary U and an isometry S_p such that

$$U^p S_p = S_p U \quad \text{and} \quad \sum_{l=0}^{p-1} U^l S_p S_p^* U^{-l} = 1$$

(see also [19] for $\mathcal{Q}_2$, and [8] for the general case). Note that $US_p^* = S_p^* U^p$, $U^* S_p^* = S_p^* U^{-p}$, and $\sum_{j=0}^{p^k} U^j S_p^k (S_p^*)^k U^{-j} = 1$ for all $k \in \mathbb{N}$. Furthermore, $(S_p^*)^m U^{-i} U^j S_p^m = \delta_{i,j}$ for $0 \leq i,j \leq p^m - 1$. All of these equalities can also be checked by means of the so-called *canonical representation* $\pi : \mathcal{Q}_p \to \mathcal{B}(\ell^2(\mathbb{Z}))$ defined by $\pi(S_p)e_k := e_{pk}$ and $\pi(U)e_k := e_{k+1}$ for all $k \in \mathbb{Z}$, where $\{e_k : k \in \mathbb{Z}\}$ is the canonical basis of $\ell^2(\mathbb{Z})$, that is, $e_k(l) := \delta_{k,l}$ for any $k, l \in \mathbb{Z}$. The canonical representation of a p-adic ring C^*-algebra is irreducible: see [8, Proposition 2.3], where the result is proved for a broad class of C^*-algebras, including all p-adic C^*-algebras, for which a canonical representation is always defined. As is known, the Cuntz algebra $\mathcal{O}_p$ is the universal C^*-algebra generated by p isometries T_j, $j = 0, 1, \ldots, p-1$, such that $\sum_{j=0}^{p-1} T_j T_j^* = 1$ [17]. We recall that $\mathcal{O}_p$ injects into $\mathcal{Q}_p$ through the *-homomorphism that sends T_j to $U^j S_p$ for $j = 0, \ldots, p-1$. Henceforth, we will always think of $\mathcal{O}_p$ as a subalgebra of $\mathcal{Q}_p$.

The p-adic ring C^*-algebra is acted upon by $\mathbb{T}$ in a natural way through the so-called gauge automorphisms $\{\alpha_z : z \in \mathbb{T}\}$. These are defined as $\alpha_z(S_p) := zS_p$ and $\alpha_z(U) := U$. We denote by $\mathcal{Q}_p^{\mathbb{T}} \subset \mathcal{Q}_p$ the subalgebra fixed by the gauge action of $\mathbb{T}$, i.e. $\mathcal{Q}_p^{\mathbb{T}} := \{x \in \mathcal{Q}_p : \alpha_z(x) = x$, for any $z \in \mathbb{T}\}$. By definition, it is easy to check that $\mathcal{Q}_p$ is the norm closure of the linear span of monomials of the type $U^i S_p^h (S_p^*)^h U^j$, $h \in \mathbb{N}$ and $i, j \in \mathbb{Z}$. Moreover, $\mathcal{Q}_p^{\mathbb{T}}$ is known to be isomorphic to the Bunce–Deddens algebra of type p^∞ [11, Remark 2.8]. Another notable subalgebra of $\mathcal{Q}_p$, which will play a key role in Section 4, is the so-called diagonal subalgebra, $\mathcal{D}_p$, which is the abelian C^*-algebra generated by all projections of the form $U^i S_p^m (S_p^*)^m U^{-i}$. It turns out that $\mathcal{D}_p$ is linearly generated by the above projections. Furthermore, $\mathcal{D}_p$ is known to be maximal abelian [8]. The Gelfand spectrum of $\mathcal{D}_p$ can be seen to be homeomorphic to the Cantor set K, and the adjoint action of U restricts to $\mathcal{D}_p$ as the p-adic odometer, which we denote by T. The endomorphisms of $\mathcal{Q}_p$ we will focus on are those that fix S_p while mapping U to a power of it, say U^k. Set $\tilde{U} := U^k$ and $\tilde{S}_p = S_p$. For such an endomorphism to exist, by universality it is necessary and sufficient that $\tilde{U}$ and $\tilde{S}_p$ continue to satisfy the defining relations. Now the relation $\tilde{U}^p\tilde{S}_p = \tilde{S}_p\tilde{U}$ does not yield any restriction on k since it is trivially satisfied. Because U^p commutes with $S_pS_p^*$, the relation $\sum_{l=0}^{p-1}\tilde{U}^l\tilde{S}_p\tilde{S}_p^*\tilde{U}^{-l} = 1$ does entail a restriction on the possible values of k, for we must have $\{[0],[k],[2k],\ldots,[(p-1)k]\} = \mathbb{Z}_p$, where $[l]$ denotes the congruence class of l modulo p. This condition is fulfilled if and only if k and p are coprime, written $(k,p)=1$. This is a consequence of a simple result, which we single out for the reader's convenience.

PROPOSITION 2.1. *Let $p>1$ be a fixed integer. Then the group homomorphism Ψ_k defined on $(\mathbb{Z}_p,+)$ by $\Psi_k([n]) := [kn]$, for any $[n]\in\mathbb{Z}_p$, is surjective if and only if $(k,p)=1$.*

Thus, for any k coprime to p we can introduce the *winding endomorphism* $\chi_k : \mathcal{Q}_p \to \mathcal{Q}_p$ given by $\chi_k(U) := U^k$, $\chi_k(S_p) := S_p$. Except when $k=\pm 1$, these are all proper endomorphisms [2, Proposition 6.1]. When $p=2$, these endomorphisms were originally introduced in [2, Section 6] for $\mathcal{Q}_2$. Note that $\chi_{k_1}\circ\chi_{k_2} = \chi_{k_1k_2}$ for any integers k_1,k_2 coprime to p.

We now recall Voiculescu's definition of topological entropy [24, Section 4]. Since the C^*-algebras dealt with in this paper are all unital and nuclear, we limit ourselves to recalling the definition of this class, although a more general definition can be given for arbitrary exact C^*-algebras [14].

Given a *nuclear* C^*-algebra $\mathcal{A}$ and an endomorphism $\alpha:\mathcal{A}\to\mathcal{A}$, we denote by $\mathrm{CPA}(\mathcal{A})$ the set of triples $(\phi,\psi,\mathcal{B})$, where $\mathcal{B}$ is a finite-dimensional C^*-algebra, and $\phi:\mathcal{A}\to\mathcal{B}$ and $\psi:\mathcal{B}\to\mathcal{A}$ are unital *completely positive* maps (u.c.p. for short). For any $\epsilon>0$ and any finite subset $\omega\subset\mathcal{A}$ (for brevity we write $\omega\in\mathcal{P}f(\mathcal{A})$), we denote by $\mathrm{CPA}(\mathcal{A},\omega,\epsilon)$ the set of triples $(\phi,\psi,\mathcal{B})\in\mathrm{CPA}(\mathcal{A})$ such that $\|(\psi\circ\phi)(a)-a\|<\epsilon$ for all $a\in\omega$. As is known, the nuclearity of $\mathcal{A}$ is equivalent to the existence of a triple $(\phi,\psi,\mathcal{B})\in\mathrm{CPA}(\mathcal{A},\omega,\epsilon)$ for any $\omega\in\mathcal{P}f(\mathcal{A})$ and $\epsilon>0$. For a thorough account of completely positive maps and nuclear (also known as amenable) C^*-algebras, we refer to [20].

The *completely positive ϵ-rank* of an endomorphism α is then defined by

$$\mathrm{rcp}(\omega,\epsilon) := \inf\{\mathrm{rank}(\mathcal{B}) \mid (\phi,\psi,\mathcal{B})\in\mathrm{CPA}(\mathcal{A},\omega,\epsilon)\}$$

where $\mathrm{rank}(\mathcal{B})$ denotes the dimension of a maximal abelian subalgebra of $\mathcal{B}$. We set

$$\mathrm{ht}(\alpha,\omega;\epsilon) := \limsup_{n\to\infty}\frac{\log\mathrm{rcp}(\omega\cup\alpha(\omega)\cup\cdots\cup\alpha^{n-1}(\omega);\epsilon)}{n},$$

$$\mathrm{ht}(\alpha,\epsilon) := \sup_{\epsilon>0}\mathrm{ht}(\alpha,\omega;\epsilon),$$

and the *topological entropy* of α is finally defined as

$$\mathrm{ht}(\alpha) := \sup_{\omega \in \mathcal{P}f(\mathcal{A})} \mathrm{ht}(\alpha, \omega).$$

One way to obtain a lower bound for the topological entropy is to consider a commutative C^*-algebra $\mathcal{C}$ of $\mathcal{A}$ that is invariant under α. Then

$$\mathrm{ht}(\alpha) \geq \mathrm{ht}(\alpha\restriction_{\mathcal{C}}) = \mathrm{h_{top}}(T)$$

where T is the map induced by $\alpha\restriction_{\mathcal{C}}$ at the level of the spectrum of $\mathcal{C}$ [24]. Sometimes a lower bound thus obtained is just the exact value of the entropy. However, in [21] examples are given of automorphisms on non-commutative C^*-algebras whose entropy is in fact greater than the supremum of the lower bounds provided by considering the restriction to all classical subsystems. Another fundamental tool is the so-called *Kolmogorov–Sinai property* which says that if $(\omega_i)_{i\in I}$ is a family of finite subsets of $\mathcal{A}$ such that the linear span of $\bigcup_{i\in I,\, n\in\mathbb{N}} \alpha^n(\omega_i)$ is dense in $\mathcal{A}$, then

$$\mathrm{ht}(\alpha) = \sup_{\epsilon>0,\, i\in I} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{rcp}(\alpha^n(\omega_i), \epsilon).$$

3. Main result

THEOREM 3.1. *For any k coprime to p, the entropy of the winding endomorphism χ_k is*

$$\mathrm{ht}(\chi_k) = \log |k|.$$

The proof requires some technical preliminary results, which are given below. First, we introduce a countable family of finite sets whose linear span coincides with the whole p-adic ring C^*-algebra.

For any $l, m, n \in \mathbb{N}$, the set $\mathcal{A}_{l,m,n}$ is by definition the set of all monomials of the form $U^i S_p^m (S_p^*)^n U^j$, where $|i|, |j| \leq l$, and one of the following three conditions holds:

- $p^m > p^n > |j|$.
- $|i| < p^m < p^n$.
- $p^m = p^n > |j|$.

Finally, $\mathcal{B}_{l,m,n}$ is the vector space generated by $\mathcal{A}_{l,m,n}$.

REMARK 3.1. The set $\mathcal{A}_{l,m,n}$ is mapped to $\mathcal{A}_{lk,m,n}$ by the winding endomorphism $\chi_k \in \mathrm{Aut}(\mathcal{Q}_p)$. Indeed, $\chi_k(U^i S_p^m (S_p^*)^n U^j) = U^{ik} S_p^m (S_p^*)^n U^{kj}$. This means that the vector space $\mathcal{B}_{l,m,n}$ is also mapped to $\mathcal{B}_{lk,m,n}$ by χ_k.

LEMMA 3.1. *The set $\bigcup_{l,m,n=0}^{\infty} \mathcal{A}_{l,m,n}$ linearly generates a dense subspace of $\mathcal{Q}_p$.*

Proof. The monomials $\{U^i S_p^m (S_p^*)^n U^j \mid i, j \in \mathbb{Z},\, m, n \in \mathbb{N}\}$ generate a dense subspace of $\mathcal{Q}_p$ (see [8, Section 2] and the references therein). The fact that we only need to consider the three aforementioned cases is explained below.

If $p^m \geq p^n \leq |j|$, then $j = p^n a + b$ (with $|b| < p^n$) and

$$\begin{aligned} U^i S_p^m (S_p^*)^n U^j &= U^i S_p^m (S_p^*)^n U^{p^n a + b} = U^i S_p^m U^a (S_p^*)^n U^b \\ &= U^{i + p^m a} S_p^m (S_p^*)^n U^b. \end{aligned}$$

If $|i| \geq p^m < p^n$, then $i = p^m a + b$ (with $|b| < p^m$) and

$$\begin{aligned} U^i S_p^m (S_p^*)^n U^j &= U^{p^m a + b} S_p^m (S_p^*)^n U^j = U^b S_p^m U^a (S_p^*)^n U^j \\ &= U^b S_p^m (S_p^*)^n U^{j + p^n a} \end{aligned}$$

where we have used $US_p^* = S_p^* U^p$. If $p^m = p^n \leq |j|$, then $j = p^m a + b$ (with $|b| < p^m$) and

$$U^i S_p^m (S_p^*)^m U^j = U^i S_p^m (S_p^*)^m U^{p^m a + b} = U^i U^{p^m a} S_p^m (S_p^*)^m U^b$$

where we have used $U^{p^m} S_p^m (S_p^*)^m = S_p^m (S_p^*)^m U^{p^m}$. ∎

We will repeatedly use the natural identification between $M_n(\mathbb{C}) \otimes \mathcal{Q}_p$ and $M_n(\mathcal{Q}_p)$.

In the following lemma we single out an isomorphism between the p-adic ring C^*-algebra $\mathcal{Q}_p$ and its tensor product with the $p^h \times p^h$ matrices. This will be useful in some of the subsequent computations.

Lemma 3.2. *For any $p \geq 2$ and $h \geq 1$, the map $\Psi_h : \mathcal{Q}_p \to M_{p^h}(\mathbb{C}) \otimes \mathcal{Q}_p$ given by*

$$\Psi_h(x) := \sum_{i,j=0}^{p^h - 1} e_{i,j} \otimes (S_p^*)^h U^{-i} x U^j S_p^h, \qquad x \in \mathcal{Q}_p,$$

is an isomorphism.

Proof. It is enough to check that the map is multiplicative:

$$\begin{aligned} \Psi_h(x)\Psi_h(y) &= \Big(\sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^h U^{-i} x U^j S_p^h\Big)\Big(\sum_{m,n=0}^{p^h-1} e_{m,n} \otimes (S_p^*)^h U^{-m} y U^n S_p^h\Big) \\ &= \sum_{i,j,m,n=0}^{p^h-1} e_{i,j} e_{m,n} \otimes (S_p^*)^h U^{-i} x U^j S_p^h (S_p^*)^h U^{-m} y U^n S_p^h \\ &= \sum_{i,j,m,n=0}^{p^h-1} \delta_{j,m} e_{i,j} e_{m,n} \otimes (S_p^*)^h U^{-i} x U^j S_p^h (S_p^*)^h U^{-m} y U^n S_p^h \\ &= \sum_{i,j,n=0}^{p^h-1} e_{i,n} \otimes (S_p^*)^h U^{-i} x U^j S_p^h (S_p^*)^h U^{-j} y U^n S_p^h \end{aligned}$$

$$= \sum_{i,n=0}^{p^h-1} e_{i,n} \otimes (S_p^*)^h U^{-i} x \Big(\sum_{j=0}^{p^h-1} U^j S_p^h (S_p^*)^h U^{-j} \Big) y U^n S_p^h$$
$$= \sum_{i,n=0}^{p^h-1} e_{i,n} \otimes (S_p^*)^h U^{-i} x y U^n S_p^h = \Psi_h(xy).$$

Injectivity follows from the simplicity of $\mathcal{Q}_p$. As for the surjectivity, it suffices to show that, for all $x \in \mathcal{Q}_p$ and i, j, the element $e_{i,j} \otimes x$ is in the image of Ψ_h. Indeed,

$$\Psi_h(U^i S_p^h x (S_p^h)^* U^{-j}) = \sum_{i',j'=0}^{p^h-1} e_{i',j'} \otimes (S_p^*)^h U^{-i'} (U^i S_p^h x (S_p^h)^* U^{-j}) U^{j'} S_p^h$$
$$= e_{i,j} \otimes x$$

where in the last step we have used the fact that $\{U^i S_p^h : i=0,1,\ldots,p^h-1\}$ is a family of mutually orthogonal isometries, as can be checked in the canonical representation. Indeed, for any $k \in \mathbb{Z}$ we have

$$(S_p^*)^h U^{-j} U^i S_p^h e_k = (S_p^*)^h e_{kp^h+(i-j)},$$

which means $(S_p^*)^h U^{-j} U^i S_p^h e_k = 0$ if $i \neq j$ because $i - j$ is never a multiple of p^h. ∎

In the following lemma we point out an inequality which will come in useful later.

LEMMA 3.3. *Let d be an integer and let $\{Q_i\}_{i=1}^5, \{R_j\}_{j=1}^3 \subset M_d(\mathbb{C})$. For any integers $m > n$ define $A, B \in M_d(\mathcal{Q}_p)$ by*

$$A := Q_1 \otimes S_p^{m-n} + Q_2 \otimes S_p^{m-n} U + Q_3 \otimes S_p^{m-n} U^* + Q_4 \otimes U S_p^{m-n}$$
$$+ Q_5 \otimes U^* S_p^{m-n},$$
$$B := R_1 \otimes 1 + R_2 \otimes U + R_3 \otimes U^*.$$

Then $\|Q_i\| \leq \|A\|$ for any $i = 1,2,3,4,5$ and $\|R_j\| \leq \|B\|$ for any $j = 1,2,3$.

Proof. We will only treat A, since B can be dealt with even more easily. We think of $M_d(\mathcal{Q}_p) \cong M_d(\mathbb{C}) \otimes \mathcal{Q}_p$ as a concrete C^*-algebra acting on the tensor Hilbert space $\mathbb{C}^d \otimes \ell^2(\mathbb{Z})$. For any $x, x' \in \mathbb{C}^d$ and $y, y' \in \ell^2(\mathbb{Z})$ with $\|x\|, \|x'\|, \|y\|, \|y'\| \leq 1$ we have

$$\|A\| \geq |(A(x \otimes y), x' \otimes y')|$$
$$= |(Q_1 x, x')(S_p^{m-n} y, y') + (Q_2 x, x')(S_p^{m-n} U y, y')$$
$$+ (Q_3 x, x')(S_p^{m-n} U^* y, y') + (Q_4 x, x')(U S_p^{m-n} y, y')$$
$$+ (Q_5 x, x')(U^* S_p^{m-n} y, y')|.$$

There are now five cases to consider.

- For $y = e_1$ and $y' = S_p^{m-n} e_1 = e_{p^{m-n}}$ the inequality simply becomes $\|A\| \geq |(Q_1 x, x')|$ as the remaining four terms are each zero as the product of two factors, the second vanishing by construction. Taking the sup over x, x' in the unit ball of $\mathbb{C}^d$ we get the inequality in the statement.
- For $y = e_1$ and $y' = S_p^{m-n} U e_1 = e_{2p^{m-n}}$ we find $\|A\| \geq |(Q_2 x, x')|$ and the conclusion follows.
- For $y = e_1$ and $y' = S_p^{m-n} U^* e_1 = e_0$ we find $\|A\| \geq |(Q_3 x, x')|$ and the conclusion follows.
- For $y = e_1$ and $y' = U S_p^{m-n} e_1 = e_{p^{m-n}+1}$ we find $\|A\| \geq |(Q_4 x, x')|$ and the conclusion follows.
- For $y = e_1$ and $y' = U^* S_p^{m-n} e_1 = e_{p^{m-n}-1}$ we find $\|A\| \geq |(Q_5 x, x')|$ and the conclusion follows. ∎

The following lemma is one of the main ingredients in the proof of an upper bound for the entropy of a winding endomorphism.

Lemma 3.4. *Let $h, l, m, n \in \mathbb{N}$, with $h > \max\{m, n\}$, $l < p^h$, and let $x \in \mathcal{B}_{l,m,n}$.*

- *If $m > n$, then $\Psi_h(x) = \sum_{j=1}^{p^{m-n}-1} R_j \otimes U^j S_p^{m-n} + R_0 \otimes S_p^{m-n} + \tilde{R}_0 \otimes S_p^{m-n} U + \hat{R}_0 \otimes U^* S_p^{m-n}$ where $R_j, R_0, \tilde{R}_0, \hat{R}_0 \in M_{p^h}(\mathbb{C})$ with $\|R_j\| \leq \|x\|$, $\|R_0\| \leq \|x\|$, $\|\tilde{R}_0\| \leq \|x\|$, $\|\hat{R}_0\| \leq \|x\|$.*
- *If $m = n$, then $\Psi_h(x) = R_1 \otimes 1 + R_2 \otimes U + R_3 \otimes U^*$ where $R_1, R_2, R_3 \in M_{p^h}(\mathbb{C})$ with $\|R_i\| \leq \|x\|$ for all i.*
- *If $m < n$, then $\Psi_h(x) = \sum_{j=1}^{p^{n-m}-1} R_j \otimes (S_p^{n-m})^* U^{-j} + R_0 \otimes S_p^{n-m} + \tilde{R}_0 \otimes S_p^{m-n} U + \hat{R}_0 \otimes U^* S_p^{m-n}$ where $R_j, R_0, \tilde{R}_0, \hat{R}_0 \in M_{p^h}(\mathbb{C})$ with $\|R_j\| \leq \|x\|$, $\|R_0\| \leq \|x\|$, $\|\tilde{R}_0\| \leq \|x\|$, $\|\hat{R}_0\| \leq \|x\|$.*

Proof. Without loss of generality, we may suppose that $x = U^a S_p^m (S_p^*)^n U^d$. We start from the first case, $m > n$. We will have to settle four subcases depending on the signs of a and d.

Suppose that $a \geq 0$, $d \leq 0$, $|d| < p^n$. We have $a = p^m b + r$ (with $0 \leq r < p^m$). Note that $0 \leq b < p^{h-m}$. Then

$$\begin{aligned}
\Psi_h(x) &= \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^h U^{-i} U^a S_p^m (S_p^*)^n U^d U^j S_p^h \\
&= \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^{h-m} ((S_p^*)^m U^{-i} U^r U^{p^m b} S_p^m)(S_p^*)^n U^d U^j S_p^h \\
&= \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^{h-m} ((S_p^*)^m U^{-i} U^r S_p^m) U^b (S_p^*)^n U^d U^j S_p^h
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{p^h-1}\sum_{i_1=0}^{p^m-1}\sum_{i_2=0}^{p^{h-m}-1} e_{i_1+i_2p^m,j} \\
&\qquad\otimes (S_p^*)^{h-m}((S_p^*)^m U^{-(i_1+i_2p^m)}U^r S_p^m)U^b(S_p^*)^n U^d U^j S_p^h \\
&= \sum_{j=0}^{p^h-1}\sum_{i_1=0}^{p^m-1}\sum_{i_2=0}^{p^{h-m}-1} e_{i_1+i_2p^m,j} \\
&\qquad\otimes (S_p^*)^{h-m}U^{-i_2}((S_p^*)^m U^{-i_1}U^r S_p^m)U^b(S_p^*)^n U^d U^j S_p^h \\
&= \sum_{j=0}^{p^h-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j}\otimes (S_p^*)^{h-m}U^{-i_2+b}(S_p^*)^n U^d U^j S_p^h \\
&= \sum_{j=0}^{p^h-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j}\otimes (S_p^*)^{h-m}U^{-i_2+b}((S_p^*)^n U^d U^j S_p^n)S_p^{h-n} \\
&= \sum_{j_1=0}^{p^n-1}\sum_{j_2=0}^{p^{h-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j_1+p^nj_2} \\
&\qquad\otimes (S_p^*)^{h-m}U^{-i_2+b}((S_p^*)^n U^d U^{j_1+p^nj_2} S_p^n)S_p^{h-n} \\
&= \sum_{j_1=0}^{p^n-1}\sum_{j_2=0}^{p^{h-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j_1+p^nj_2} \\
&\qquad\otimes (S_p^*)^{h-m}U^{-i_2+b}((S_p^*)^n U^d U^{j_1} S_p^n)U^{j_2}S_p^{h-n} \\
&= \sum_{j_2=0}^{p^{h-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^nj_2}\otimes (S_p^*)^{h-m}U^{-i_2+b}U^{j_2}S_p^{h-n} \\
&= \sum_{j_2=0}^{p^{h-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^nj_2}\otimes (S_p^*)^{h-m}U^{-i_2+b+j_2}S_p^{h-m}S_p^{m-n} \\
&= \sum_{j_3=0}^{p^{h-m}-1}\sum_{j_4=0}^{p^{m-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^n(j_3+j_4p^{h-m})} \\
&\qquad\otimes (S_p^*)^{h-m}U^{-i_2+b}U^{j_3+j_4p^{h-m}}S_p^{h-m}S_p^{m-n} \\
&= \sum_{j_3=0}^{p^{h-m}-1}\sum_{j_4=0}^{p^{m-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^n(j_3+j_4p^{h-m})} \\
&\qquad\otimes (S_p^*)^{h-m}U^{-i_2+b}U^{j_3}S_p^{h-m}U^{j_4}S_p^{m-n}.
\end{aligned}
$$

Now since $-p^{h-m}+1 \le -i_2+b+j_3 \le 2p^{h-m}-2$, there is only one non-trivial multiple of p^{h-m} among the values taken by $-i_2+b+j_3$, namely p^{h-m} itself. Therefore, the last expression can be rewritten as

$$
\begin{aligned}
&\sum_{j_4=0}^{p^{m-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^n(i_2-b+j_4p^{h-m})}\otimes U^{j_4}S_p^{m-n} \\
&\qquad + \sum_{j_4=0}^{p^{m-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^n(i_2-b+p^{h-m}+j_4p^{h-m})}\otimes U^{j_4+1}S_p^{m-n}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^n(i_2-b+j_4p^{h-m})} \otimes U^{j_4} S_p^{m-n} \\
&\quad + \sum_{j_4=1}^{p^{m-n}} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^n(i_2-b+p^{h-m}+j_4p^{h-m}-p^{h-m})} \otimes U^{j_4} S_p^{m-n} \\
&= \sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^n(i_2-b+j_4p^{h-m})} \otimes U^{j_4} S_p^{m-n} \\
&\quad + \sum_{j_4=1}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^n(i_2-b+p^{h-m}+j_4p^{h-m}-p^{h-m})} \otimes U^{j_4} S_p^{m-n} \\
&\quad + \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^n(i_2-b+p^{h-m}+p^{m-n}p^{h-m}-p^{h-m})} \otimes U^{p^{m-n}} S_p^{m-n},
\end{aligned}
$$

and in the last sum we easily recognize a term of the form $\tilde{R}_0 \otimes S_p^{m-n}U$.

Suppose that $a \leq 0$, $d \leq 0$, $|d| < p^n$. We have $a = p^m b + r$ (with $0 \leq r < p^m$). Note that $0 \leq -b \leq p^{h-m}$. Then

$$
\begin{aligned}
\Psi_h(x) &= \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^h U^{-i} U^a S_p^m (S_p^*)^n U^d U^j S_p^h \\
&= \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^h U^{-i} U^r S_p^m U^b (S_p^*)^n U^d U^j S_p^h \\
&= \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^{h-m} ((S_p^*)^m U^{-i} U^r S_p^m) U^b (S_p^*)^n U^d U^j S_p^h \\
&= \sum_{j=0}^{p^h-1} \sum_{i_1=0}^{p^m-1} \sum_{i_2=0}^{p^{h-m}-1} e_{i_1+i_2p^m,j} \otimes (S_p^*)^{h-m} ((S_p^*)^m U^{-(i_1+i_2p^m)} U^r S_p^m) U^b (S_p^*)^n U^d U^j S_p^h \\
&= \sum_{j=0}^{p^h-1} \sum_{i_1=0}^{p^m-1} \sum_{i_2=0}^{p^{h-m}-1} e_{i_1+i_2p^m,j} \otimes (S_p^*)^{h-m} U^{-i_2} ((S_p^*)^m U^{-i_1} U^r S_p^m) U^b (S_p^*)^n U^d U^j S_p^h \\
&= \sum_{j=0}^{p^h-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j} \otimes (S_p^*)^{h-m} U^{-i_2+b} (S_p^*)^n U^d U^j S_p^h \\
&= \sum_{j=0}^{p^h-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j} \otimes (S_p^*)^{h-m} U^{-i_2+b} ((S_p^*)^n U^d U^j S_p^n) S_p^{h-n} \\
&= \sum_{j_1=0}^{p^n-1} \sum_{j_2=0}^{p^{h-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j_1+p^nj_2} \otimes (S_p^*)^{h-m} U^{-i_2+b} ((S_p^*)^n U^d U^{j_1+p^nj_2} S_p^n) S_p^{h-n}
\end{aligned}
$$

$$= \sum_{j_1=0}^{p^n-1} \sum_{j_2=0}^{p^{h-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m, j_1+p^n j_2} \otimes (S_p^*)^{h-m} U^{-i_2+b}((S_p^*)^n U^d U^{j_1} S_p^n) U^{j_2} S_p^{h-n}$$

$$= \sum_{j_2=0}^{p^{h-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m, -d+p^n j_2} \otimes (S_p^*)^{h-m} U^{-i_2+b+j_2} S_p^{h-m} S_p^{m-n}$$

$$= \sum_{j_3=0}^{p^{h-m}-1} \sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m, -d+p^n(j_3+j_4p^{h-m})} \otimes (S_p^*)^{h-m} U^{-i_2+b} U^{j_3+j_4p^{h-m}} S_p^{h-m} S_p^{m-n}$$

$$= \sum_{j_3=0}^{p^{h-m}-1} \sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m, -d+p^n(j_3+j_4p^{h-m})} \otimes (S_p^*)^{h-m} U^{-i_2+b} U^{j_3} S_p^{h-m} U^{j_4} S_p^{m-n}.$$

Now since $-2p^{h-m}+1 \le b+j_3-i_2 \le p^{h-m}-1$, there is only one non-trivial multiple of p^{h-m} among the values taken by $-i_2+b+j_3$, namely $-p^{h-m}$ itself. Therefore, the last expression can be rewritten as

$$\sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m, -d+p^n(i_2-b+j_4p^{h-m})} \otimes U^{j_4} S_p^{m-n}$$
$$+ \sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m, -d+p^n(i_2-b-p^{h-m}+j_4p^{h-m})} \otimes U^{j_4-1} S_p^{m-n}.$$

In the second term the summand corresponding to $j_4 = 0$ accounts for the presence of a term of the type $\hat{R}_0 \otimes U^* S_p^{m-n}$, as in the statement.

Now we assume $a \ge 0$, $d \ge 0$, $|d| < p^n$. We have $a = p^m b + r$ (with $0 \le r < p^m$). Note that $0 \le b < p^{h-m}$. Then

$$\Psi_h(x) = \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^h U^{-i} U^a S_p^m (S_p^*)^n U^d U^j S_p^h$$

$$= \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^{h-m}((S_p^*)^m U^{-i} U^r S_p^m) U^b (S_p^*)^n U^d U^j S_p^h$$

$$= \sum_{j=0}^{p^h-1} \sum_{i_1=0}^{p^m-1} \sum_{i_2=0}^{p^{h-m}-1} e_{i_1+i_2p^m, j} \otimes (S_p^*)^{h-m}((S_p^*)^m U^{-(i_1+i_2p^m)} U^r S_p^m) U^b (S_p^*)^n U^d U^j S_p^h$$

$$= \sum_{j=0}^{p^h-1} \sum_{i_1=0}^{p^m-1} \sum_{i_2=0}^{p^{h-m}-1} e_{i_1+i_2p^m, j} \otimes (S_p^*)^{h-m} U^{-i_2}((S_p^*)^m U^{-i_1} U^r S_p^m) U^b (S_p^*)^n U^d U^j S_p^h$$

$$= \sum_{j=0}^{p^h-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m, j} \otimes (S_p^*)^{h-m} U^{-i_2+b} (S_p^*)^n U^d U^j S_p^h$$

$$= \sum_{j=0}^{p^h-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m, j} \otimes (S_p^*)^{h-m} U^{-i_2+b}((S_p^*)^n U^d U^j S_p^n) S_p^{h-n}$$

$$
\begin{aligned}
&= \sum_{j_1=0}^{p^n-1}\sum_{j_2=0}^{p^{h-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j_1+p^nj_2}\otimes (S_p^*)^{h-m}U^{-i_2+b}((S_p^*)^nU^dU^{j_1+p^nj_2}S_p^n)S_p^{h-n}\\
&= \sum_{j_1=0}^{p^n-1}\sum_{j_2=0}^{p^{h-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j_1+p^nj_2}\otimes (S_p^*)^{h-m}U^{-i_2+b}((S_p^*)^nU^dU^{j_1}S_p^n)U^{j_2}S_p^{h-n}\\
&= \sum_{j_2=0}^{p^{h-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^nj_2}\otimes (S_p^*)^{h-m}U^{-i_2+b}U^{j_2+1}S_p^{h-n}\\
&= \sum_{j_2=0}^{p^{h-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^nj_2}\otimes (S_p^*)^{h-m}U^{-i_2+b+j_2+1}S_p^{h-m}S_p^{m-n}\\
&= \sum_{j_3=0}^{p^{h-m}-1}\sum_{j_4=0}^{p^{m-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^n(j_3+j_4p^{h-m})}\\
&\qquad\qquad\otimes (S_p^*)^{h-m}U^{-i_2+b+1}U^{j_3+j_4p^{h-m}}S_p^{h-m}S_p^{m-n}\\
&= \sum_{j_3=0}^{p^{h-m}-1}\sum_{j_4=0}^{p^{m-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^n(j_3+j_4p^{h-m})}\\
&\qquad\qquad\otimes (S_p^*)^{h-m}U^{-i_2+b+1}U^{j_3}S_p^{h-m}U^{j_4}S_p^{m-n}.
\end{aligned}
$$

Since $-p^{h-m}+2\le -i_2+b+1+j_3\le 2p^{h-m}-1$, there is only one non-trivial multiple of p^{h-m} among the values taken by $-i_2+b+j_3$, namely p^{h-m} itself. Therefore, the last expression can be rewritten as

$$
\begin{aligned}
&\sum_{j_4=0}^{p^{m-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^n(i_2-b-1+j_4p^{h-m})}\otimes U^{j_4}S_p^{m-n}\\
&\qquad + \sum_{j_4=0}^{p^{m-n}-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^n(i_2-b-1+p^{h-m}+j_4p^{h-m})}\otimes U^{j_4+1}S_p^{m-n}.
\end{aligned}
$$

Finally, we discuss the fourth subcase: $a\le 0$, $d\ge 0$, $d<p^n$. We have $a=p^mb+r$ (with $0\le r<p^m$). Note that $0\le -b\le p^{h-m}$. We have

$$
\begin{aligned}
\Psi_h(x) &= \sum_{i,j=0}^{p^h-1} e_{i,j}\otimes (S_p^*)^hU^{-i}U^aS_p^m(S_p^*)^nU^dU^jS_p^h\\
&= \sum_{i,j=0}^{p^h-1} e_{i,j}\otimes (S_p^*)^{h-m}((S_p^*)^mU^{-i}U^rS_p^m)U^b(S_p^*)^nU^dU^jS_p^h\\
&= \sum_{j=0}^{p^h-1}\sum_{i_1=0}^{p^m-1}\sum_{i_2=0}^{p^{h-m}-1} e_{i_1+i_2p^m,j}\otimes (S_p^*)^{h-m}((S_p^*)^mU^{-(i_1+i_2p^m)}U^rS_p^m)U^b(S_p^*)^nU^dU^jS_p^h\\
&= \sum_{j=0}^{p^h-1}\sum_{i_1=0}^{p^m-1}\sum_{i_2=0}^{p^{h-m}-1} e_{i_1+i_2p^m,j}\otimes (S_p^*)^{h-m}U^{-i_2}((S_p^*)^mU^{-i_1}U^rS_p^m)U^b(S_p^*)^nU^dU^jS_p^h\\
&= \sum_{j=0}^{p^h-1}\sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j}\otimes (S_p^*)^{h-m}U^{-i_2+b}(S_p^*)^nU^dU^jS_p^h
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{p^h-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j} \otimes (S_p^*)^{h-m} U^{-i_2+b}((S_p^*)^n U^d U^j S_p^n) S_p^{h-n} \\
&= \sum_{j_1=0}^{p^n-1} \sum_{j_2=0}^{p^{h-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j_1+p^nj_2} \otimes (S_p^*)^{h-m} U^{-i_2+b}((S_p^*)^n U^d U^{j_1+p^nj_2} S_p^n) S_p^{h-n} \\
&= \sum_{j_1=0}^{p^n-1} \sum_{j_2=0}^{p^{h-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j_1+p^nj_2} \otimes (S_p^*)^{h-m} U^{-i_2+b}((S_p^*)^n U^d U^{j_1} S_p^n) U^{j_2} S_p^{h-n} \\
&= \sum_{j_2=0}^{p^{h-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^nj_2} \otimes (S_p^*)^{h-m} U^{-i_2+b+1} U^{j_2} S_p^{h-n} \\
&= \sum_{j_2=0}^{p^{h-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^nj_2} \otimes (S_p^*)^{h-m} U^{-i_2+b+j_2+1} S_p^{h-m} S_p^{m-n} \\
&= \sum_{j_3=0}^{p^{h-m}-1} \sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^n(j_3+j_4p^{h-m})} \\
&\qquad\qquad \otimes (S_p^*)^{h-m} U^{-i_2+b+1} U^{j_3+j_4p^{h-m}} S_p^{h-m} S_p^{m-n} \\
&= \sum_{j_3=0}^{p^{h-m}-1} \sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^n(j_3+j_4p^{h-m})} \\
&\qquad\qquad \otimes (S_p^*)^{h-m} U^{-i_2+b+1} U^{j_3} S_p^{h-m} U^{j_4} S_p^{m-n}.
\end{aligned}
$$

Since $-2p^{h-m}+2 \le -i_2+b+1+j_3 \le p^{h-m}$, there are two non-trivial multiples of p^{h-m} among the values taken by $-i_2+b+j_3$, namely $\pm p^{h-m}$. Therefore, the last expression can be rewritten as

$$
\begin{aligned}
&\sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^n(i_2-b-1+j_4p^{h-m})} \otimes U^{j_4} S_p^{m-n} \\
&\quad + \sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^n(i_2-b-1-p^{h-m}+j_4p^{h-m})} \otimes U^{j_4-1} S_p^{m-n} \\
&\quad + \sum_{j_4=0}^{p^{m-n}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,p^n-d+p^n(i_2-b-1+p^{h-m}+j_4p^{h-m})} \otimes U^{j_4+1} S_p^{m-n}.
\end{aligned}
$$

We now move on to the second case, $m=n$. There is no loss of generality to suppose $|a|,|d|<p^h$ and $|d|<p^m$. As in the first case, there are four subcases to consider depending on the signs of a and d. As they are very similar to one another, we treat only one: $a\ge 0$ and $d\le 0$. We observe that $a=p^mb+r$ (with $0\le r<p^m$). Note also that $0\le b<p^{h-m}$. Then

$$
\begin{aligned}
\Psi_h(x) &= \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^h U^{-i} U^a S_p^m (S_p^*)^m U^d U^j S_p^h \\
&= \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^h U^{-i} U^r S_p^m U^b (S_p^*)^m U^d U^j S_p^h
\end{aligned}
$$

$$= \sum_{i,j=0}^{p^h-1} e_{i,j} \otimes (S_p^*)^{h-m}((S_p^*)^m U^{-i} U^r S_p^m) U^b (S_p^*)^m U^d U^j S_p^h$$
$$= \sum_{j=0}^{p^h-1} \sum_{i_1=0}^{p^m-1} \sum_{i_2=0}^{p^{h-m}-1} e_{i_1+i_2p^m,j} \otimes (S_p^*)^{h-m}((S_p^*)^m U^{-(i_1+i_2p^m)} U^r S_p^m) U^b (S_p^*)^m U^d U^j S_p^h$$
$$= \sum_{j=0}^{p^h-1} \sum_{i_1=0}^{p^m-1} \sum_{i_2=0}^{p^{h-m}-1} e_{i_1+i_2p^m,j} \otimes (S_p^*)^{h-m} U^{-i_2}((S_p^*)^m U^{-i_1} U^r S_p^m) U^b (S_p^*)^m U^d U^j S_p^h$$
$$= \sum_{j=0}^{p^h-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j} \otimes (S_p^*)^{h-m} U^{-i_2+b} (S_p^*)^m U^d U^j S_p^h$$
$$= \sum_{j=0}^{p^h-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j} \otimes (S_p^*)^{h-m} U^{-i_2+b} ((S_p^*)^m U^d U^j S_p^m) S_p^{h-m}$$
$$= \sum_{j_1=0}^{p^m-1} \sum_{j_2=0}^{p^{h-m}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,j_1+p^mj_2} \otimes (S_p^*)^{h-m} U^{-i_2+b} ((S_p^*)^m U^d U^{j_1+p^mj_2} S_p^m) S_p^{h-m}$$
$$= \sum_{j_2=0}^{p^{h-m}-1} \sum_{i_2=0}^{p^{h-m}-1} e_{r+i_2p^m,-d+p^mj_2} \otimes (S_p^*)^{h-m} U^{-i_2+b+j_2} S_p^{h-m}.$$

Since $-p^{h-m}+1 \leq -i_2+b+j_2 \leq 2p^{h-m}-2$, p^{h-m} is the only non-trivial multiple among the possible values of $-i_2+b+j_2$, which means the above expression can be rewritten as

$$\sum_{j_2=0}^{p^{h-m}-1} e_{r+(j_2+b)p^m,-d+p^mj_2} \otimes 1 + \sum_{j_2=0}^{p^{h-m}-1} e_{r+(j_2+b+p^{h-m})p^m,-d+p^mj_2} \otimes U.$$

Finally, the case $m < n$ requires no work, for it is easily reduced to the first thanks to the fact that Ψ_h is a $*$-homomorphism.

The inequalities involving the norms follow from the formulas above. Indeed, set $A := R_0 \otimes S_p^{m-n} + \tilde{R}_0 \otimes S_p^{m-n} U + \hat{R}_0 \otimes U^* S_p^{m-n}$. When $m > n$,

$$\|\Psi_h(x)\|^2 = \Big\| \Big(\sum_{j_4=1}^{p^{m-n}-1} R_{j_4} \otimes U^{j_4} S_p^{m-n} + A \Big)^* \Big(\sum_{j_4=1}^{p^{m-n}-1} R_{j_4} \otimes U^{j_4} S_p^{m-n} + A \Big) \Big\|$$
$$= \Big\| \sum_{j_4=1}^{p^{m-n}-1} R_{j_4}^* R_{j_4} \otimes 1 + A^*A \Big\|.$$

It follows that

$$\|x\|^2 = \|\Psi_h(x)\|^2 \geq \|R_{j_4}^* R_{j_4}\| = \|R_{j_4}\|^2 \quad \forall j \in \{1, \ldots, p^{m-n}-1\},$$
$$\|x\|^2 \geq \|A^*A\| = \|A\|^2.$$

From Lemma 3.3 we get $\|x\| \geq \max\{\|R_0\|, \|\tilde{R}_0\|, \|\hat{R}_0\|\}$. The cases $m < n$ and $m = n$ are quite analogous. ∎

Going back to the computations of the entropy of winding endomorphisms, the first thing is to provide a lower bound for it. As in [22], this can be done by looking at the restriction of χ_k to a suitable MASA of $\mathcal{Q}_p$. In our case, the MASA is obviously $C^*(U)$ (cf. [2, 8]).

LEMMA 3.5. *For any integer k coprime to p, one has*

$$\mathrm{ht}(\chi_k) \geq \log|k|.$$

Proof. The claim follows by monotonicity:

$$\mathrm{ht}(\chi_k) \geq \mathrm{ht}(\chi_k\restriction_{C^*(U)}) = h^{\mathrm{top}}(T_k) = \log|k|,$$

where $T_k(z) := z^k$ (see e.g. [12]). ∎

We are now in a position to prove the main result of this paper.

Proof of Theorem 3.1. Thanks to Lemma 3.5, all we have to do is to show that $\mathrm{ht}(\chi_k) \leq \log|k|$. For $l \in \mathbb{N}$, we set $\omega_l := \bigcup_{q,r,s=0}^{l} \mathcal{A}_{q,r,s}$, and for $n \in \mathbb{N}$ we denote $\omega_l^{(n)} := \bigcup_{j=0}^{n} \chi_k^j(\omega_l)$. Fix $\delta > 0$. Since $\mathcal{Q}_p$ is nuclear, there exists $(\phi_0, \psi_0, M_{C_l}(\mathbb{C})) \in \mathrm{CPA}\big(\mathcal{Q}_p, \omega_l, \frac{\delta}{16\cdot p^l}\big)$. We want to find an m such that the exponents of U appearing in the elements in $\Psi_m(\omega_l^{(n)}) \subset M_{p^m}(\mathcal{Q}_p)$ are smaller than p^m. This is certainly the case provided that $|k|^n l < p^m$, as follows from a straightforward application of Remark 3.1. The inequality can also be rewritten as $n\log_p|k| + \log_p(l) < m$, which is more suited to our purposes. For instance, we can simply choose $m = [n\log_p|k| + \log_p(l)] + 1$, where $[\cdot]$ denotes integer part.

Again, by nuclearity of $\mathcal{Q}_p$ there exists $d \in \mathbb{N}$ and a u.c.p. map $\gamma : \Psi_m(\mathcal{Q}_p) = M_{p^m}(\mathcal{Q}_p) \to M_d(\mathbb{C})$ and $\eta : M_d(\mathbb{C}) \to \mathcal{Q}_p$ such that for all $a \in \omega_l^{(n)}$ we have $\|\eta\circ\gamma(\Psi_m(a)) - a\| < \delta/2$.

Set $\psi := (\mathrm{id}\otimes\psi_0)\circ\Psi_m$ and $\phi := \eta\circ\gamma\circ(\mathrm{id}\otimes\phi_0)$. Now for any $x \in \omega_l$ and $h \in \mathbb{N}$ with $h \leq n$, we have

$$\begin{aligned}
(1)\qquad &\|\phi\circ\psi(\chi_k^h(x)) - \chi_k^h(x)\| \\
&= \|\eta\circ\gamma\circ(\mathrm{id}\otimes\phi_0\circ\psi_0)\circ\Psi_m(\chi_k^h(x)) - \chi_k^h(x)\| \\
&= \|\eta\circ\gamma\circ(\mathrm{id}\otimes\phi_0\circ\psi_0)\circ\Psi_m(\chi_k^h(x)) - \eta\circ\gamma\circ\Psi_m(\chi_k^h(x))\| \\
&\quad + \|\eta\circ\gamma\circ\Psi_m(\chi_k^h(x)) - \chi_k^h(x)\| \\
&\leq \|\eta\circ\gamma\circ(\mathrm{id}\otimes\phi_0\circ\psi_0)\circ\Psi_m(\chi_k^h(x)) - \eta\circ\gamma\circ\Psi_m(\chi_k^h(x))\| + \delta/2 \\
&\leq \|(\mathrm{id}\otimes(\phi_0\circ\psi_0))\circ\Psi_m(\chi_k^h(x)) - \Psi_m(\chi_k^h(x))\| + \delta/2.
\end{aligned}$$

By Remark 3.1, $\chi_k^h(x)$ is in $\mathcal{B}_{q,r,s}$ where $q \leq |k|^h l$ (which is smaller than p^m by construction), and $r, s \leq l$. As of now we will also assume $r > s$ since the case $r \leq s$ can be handled by similar computations. By (1) and Lemma 3.4

we get

$$
\begin{aligned}
&\|\phi\circ\psi(\chi_k^h(x))-\chi_k^h(x)\|\\
&<\Big\|\sum_{j_4=1}^{p^{r-s}-1} R_{j_4}\otimes(\phi_0\circ\psi_0)(U^{j_4}S_p^{r-s})+R_0\otimes(\phi_0\circ\psi_0)(S_p^{r-s})\\
&\quad+\tilde{R}_0\otimes(\phi_0\circ\psi_0)(S_p^{r-s}U)+\hat{R}_0\otimes(\phi_0\circ\psi_0)(U^*S_p^{r-s})\\
&\quad-\sum_{j_4=1}^{p^{r-s}-1} R_{j_4}\otimes U^{j_4}S_p^{r-s}-R_0\otimes S_p^{r-s}-\tilde{R}_0\otimes S_p^{r-s}U-\hat{R}_0\otimes U^*S_p^{r-s}\Big\|+\delta/2\\
&=\Big\|\sum_{j_4=1}^{p^{r-s}-1}(R_{j_4}\otimes(\phi_0\circ\psi_0)(U^{j_4}S_p^{r-s})-U^{j_4}S_p^{r-s})\\
&\quad+R_0\otimes((\phi_0\circ\psi_0)(S_p^{r-s})-S_p^{r-s})\\
&\quad+\tilde{R}_0\otimes((\phi_0\circ\psi_0)(S_p^{r-s}U)-S_p^{r-s}U)+\\
&\quad+\hat{R}_0\otimes((\phi_0\circ\psi_0)(U^*S_p^{r-s})-U^*S_p^{r-s})\Big\|+\delta/2\\
&\le\sum_{j_4=1}^{p^{r-s}-1}\|(R_{j_4}\otimes(\phi_0\circ\psi_0)(U^{j_4}S_p^{r-s})-U^{j_4}S_p^{r-s})\|\\
&\quad+\|R_0\otimes((\phi_0\circ\psi_0)(S_p^{r-s})-S_p^{r-s})+\tilde{R}_0\otimes((\phi_0\circ\psi_0)(S_p^{r-s}U)-S_p^{r-s}U)\\
&\quad+\hat{R}_0\otimes((\phi_0\circ\psi_0)(S_p^{r-s}U)-S_p^{r-s}U)\|+\delta/2\\
&\le\frac{\delta}{16\cdot p^l}\sum_{j_4=1}^{p^{r-s}-1}1+\frac{\delta}{16\cdot p^l}3+\frac{\delta}{2}\le\frac{\delta}{16\cdot p^l}(p^{r-s}-1)+\frac{3\delta}{16\cdot p^l}+\frac{\delta}{2}\\
&\le\frac{\delta}{16}+\frac{3\delta}{32}+\frac{\delta}{2}=\frac{19\delta}{32}<\delta
\end{aligned}
$$

where we have used $\|R_{j_i}\|\le 1$ for all i, $\|R_0\|\le 1$, $\|\tilde{R}_0\|\le 1$, $\|\hat{R}_0\|\le 1$. The above shows that the triple $(\phi,\psi,M_{p^m}(\mathbb{C})\otimes M_{C_l}(\mathbb{C}))$ is in $\mathrm{CPA}(\mathcal{Q}_p,\omega_l^{(n)},\delta)$. Therefore, $\mathrm{rcp}(\omega_l^{(n)},\delta)\le C_l p^m$. Accordingly,

$$
\begin{aligned}
\log\mathrm{rcp}(\omega_l^{(n)},\delta)&\le\log(C_l)+m\log(p)\\
&=\log(C_l)+\log(p)\big([n\log_p|k|+\log_p(l)]+1\big)\\
&=\log(C_l)+\log(p)\left(\frac{\log|k|}{\log(p)}n+\log_p(l)\right)+2\log(p)\\
&\le\log(C_l)+n\log|k|+\log(l)+2\log(p),
\end{aligned}
$$

from which we find

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{rcp}(\omega_l^{(n)}, \delta) \leq \log |k|.$$

The conclusion is thus due to the Kolmogorov–Sinai property of non-commutative entropy. ■

It is worth noting that all winding endomorphisms leave the Bunce–Deddens algebra $\mathcal{Q}_p^{\mathbb{T}}$ invariant, which means one can also compute the entropy of the restriction of a winding endomorphism to this subalgebra. It turns out that the index of the restriction does not decrease:

Corollary 3.1. *For any integer k coprime to p, one has*

$$\mathrm{ht}(\chi_k\restriction_{\mathcal{Q}_p^{\mathbb{T}}}) = \log |k|.$$

Proof. The claim follows directly from the monotonicity of the entropy:

$$\log |k| = \mathrm{ht}(\chi_k\restriction_{C^*(U)}) \leq \mathrm{ht}(\chi_k\restriction_{\mathcal{Q}_p^{\mathbb{T}}}) \leq \mathrm{ht}(\chi_k) = \log |k|. \ ■$$

4. Watatani index of winding endomorphisms. Motivated by the work done in [16] on quadratic permutation endomorphisms of the Cuntz algebra $\mathcal{O}_2$, in this section we study the relation between the entropy and index of the restriction to the Bunce–Deddens subalgebras of our winding endomorphisms.

We are going to show that, just as entropy, the Watatani index (see [25] for the definition and the main properties) of the restriction of χ_k to the Bunce–Deddens algebra $\mathcal{Q}_p^{\mathbb{T}}$ can also be computed exactly. More precisely, for any integer k coprime to p, this index turns out to be $|k|$. Rather interestingly, the entropy of $\chi_k\restriction_{\mathcal{Q}_p^{\mathbb{T}}}$ is then the natural logarithm of the index of $\chi_k\restriction_{\mathcal{Q}_p^{\mathbb{T}}}$.

For the reader's convenience we recall some basic definitions. We start with an inclusion of unital C^*-algebras $\mathcal{A} \subset \mathcal{B}$ with a common unit I such that there exists a faithful conditional expectation $E : \mathcal{B} \to \mathcal{A}$.

Definition 4.1. A finite family $\{u_1, \dots, u_n\} \subset \mathcal{B}$ is said to be a *quasi-basis* for E if for any $x \in \mathcal{B}$ one has

$$x = \sum_{i=1}^{n} u_i E(u_i^* x) = \sum_{i=1}^{n} E(x u_i) u_i^*.$$

Now our conditional expectation $E : \mathcal{B} \to \mathcal{A}$ has finite index if there exists a quasi-basis for E. The index is then defined as $\mathrm{Ind}(E) := \sum_{i=1}^{n} u_i u_i^* \in \mathcal{B}$. In fact, the index does not depend on the quasi-basis chosen. Moreover, $\mathrm{Ind}(E)$ is in the center of B. In particular, if B has trivial center, then $\mathrm{Ind}(E)$ is a positive real number greater than or equal to 1. The Watatani index of the inclusion $\mathcal{A} \subset \mathcal{B}$ is then defined as the infimum of the set of indices obtained as above corresponding to any conditional expectation of finite index. It turns out that this infimum is actually a minimum provided that both $\mathcal{B}$ and $\mathcal{A}$ have trivial center [25].

To accomplish our computation of the index, we will make use of the following key result, where the image of the winding endomorphism (restricted to $\mathcal{Q}_p^{\mathbb{T}}$) is seen to coincide with the fixed-point subalgebra of a finite-order automorphism, which we next define. Fix $z := e^{2\pi i/k}$ and let $\alpha : \mathcal{Q}_p^{\mathbb{T}} \to \mathcal{Q}_p^{\mathbb{T}}$ be the automorphism mapping U to zU and fixing $S_p^h(S_p^*)^h$ for all $h \in \mathbb{N}$ (see e.g. [18]). Note that $\alpha^{|k|} = \mathrm{id}_{\mathcal{Q}_p^{\mathbb{T}}}$, which allows us to define a conditional expectation E from $\mathcal{Q}_p^{\mathbb{T}}$ to $(\mathcal{Q}_p^{\mathbb{T}})^\alpha := \{x \in \mathcal{Q}_p^{\mathbb{T}} : \alpha(x) = x\}$ as

$$E(x) := \frac{1}{|k|}\sum_{l=0}^{|k|-1} \alpha^n(x), \qquad x \in \mathcal{Q}_p^{\mathbb{T}}. \tag{2}$$

Proposition 4.1. *The image $\chi_k(\mathcal{Q}_p^{\mathbb{T}})$ coincides with $(\mathcal{Q}_p^{\mathbb{T}})^\alpha$.*

First of all, we need a preliminary lemma.

Lemma 4.1. *Let $p, k \in \mathbb{Z}$ be coprime. For any $i, h \in \mathbb{N}$, there exist $b, m \in \mathbb{Z}$ such that $i + bp^h = mk$.*

Proof. Since k and p are coprime, so are k and p^h. This means that $1 = ck + dp^h$ for some $c, d \in \mathbb{Z}$. It follows that $i = ick + idp^h$ and so we may choose $b = -id$ and $m = ic$. ∎

Proof of Proposition 4.1. Since $\chi_k(\mathcal{Q}_p^{\mathbb{T}})$ is generated by the monomials of the form $U^{ki}S_p^h(S_p^*)^hU^{kj}$ with $i, j \in \mathbb{Z}$, $h \in \mathbb{N}$, the inclusion $\chi_k(\mathcal{Q}_p^{\mathbb{T}}) \subset (\mathcal{Q}_p^{\mathbb{T}})^\alpha$ is clear.

For the converse inclusion, we need to use the conditional expectation $E : \mathcal{Q}_p^{\mathbb{T}} \to (\mathcal{Q}_p^{\mathbb{T}})^\alpha$ defined in (2). As already mentioned, the algebra $\mathcal{Q}_p^{\mathbb{T}}$ is linearly generated by the elements of the form $U^iS_p^h(S_p^*)^hU^j$, $h \in \mathbb{N}$ and $i, j \in \mathbb{Z}$. So all we have to do is show that the images of these elements under E are in $\chi_k(\mathcal{Q}_p^{\mathbb{T}})$. A straightforward computation shows that $E(U^iS_p^h(S_p^*)^hU^j)$ is zero unless $i + j$ is 0 mod k, that is, $j = -i + mk$ for some $m \in \mathbb{Z}$. If $E(U^iS_p^h(S_p^*)^hU^j)$ is not zero, then it is equal to $U^iS_p^h(S_p^*)^hU^{-i}U^{km}$. Now $U^iS_p^h(S_p^*)^hU^{-i}U^{km}$ lies in $\chi_k(\mathcal{Q}_p^{\mathbb{T}})$ if and only if $U^iS_p^h(S_p^*)^hU^{-i}$ does. Thanks to the chain of equalities

$$U^iS_p^h(S_p^*)^hU^{-i} = U^iS_p^hU^bU^{-b}(S_p^*)^hU^{-i} = U^{i+bp^h}S_p^h(S_p^*)^hU^{-i-bp^h},$$

which hold for any $b \in \mathbb{N}$, our claim finally follows from Lemma 4.1. ∎

Remark 4.1. The inclusion $(\mathcal{Q}_p^{\mathbb{T}})^\alpha \subset \mathcal{Q}_p^{\mathbb{T}}$ is an example of a non-commutative self-covering of the type considered in [9].

We are now in a position to compute the index of χ_k relative to the conditional expectation E considered above.

Theorem 4.1. *The index of $\chi_k(\mathcal{Q}_p^{\mathbb{T}})$ in $\mathcal{Q}_p^{\mathbb{T}}$ relative to the conditional expectation E above is $|k|$.*

Proof. This is a direct consequence of the description of $\chi_k(\mathcal{Q}_p^{\mathbb{T}})$ as the fixed-point algebra under the action of the finite group $\mathbb{Z}_{|k|}$, of order $|k|$. ∎

Our next goal is to show that the conditional expectation E we have used so far is actually unique. This will be a consequence of Corollary 1.4.3 in [25] once we have ascertained that $\chi_k(\mathcal{Q}_p^{\mathbb{T}})' \cap \mathcal{Q}_p^{\mathbb{T}} = \mathbb{C}$.

Proposition 4.2. *For any k coprime to p, the relative commutant $\chi_k(\mathcal{Q}_p^{\mathbb{T}})' \cap \mathcal{Q}_p^{\mathbb{T}}$ is trivial.*

Proof. Since $\mathcal{D}_p \subset \chi_k(\mathcal{Q}_p^{\mathbb{T}})$, we have $\chi_k(\mathcal{Q}_p^{\mathbb{T}})' \cap \mathcal{Q}_p^{\mathbb{T}} \subset \mathcal{D}_p \cap C^*(U^k)'$. We are thus led to prove that any $x \in \mathcal{D}_p$ such that $xU^k = U^k x$ is actually a scalar. We will work in the canonical representation of $\mathcal{Q}_p$, which acts on the Hilbert space $\ell^2(\mathbb{Z})$. Now any $x \in \mathcal{D}_p$ is a diagonal operator with respect to the canonical basis $\{e_i : i \in \mathbb{Z}\}$ of $\ell^2(\mathbb{Z})$, that is, $xe_i = x_i e_i$, $i \in \mathbb{Z}$, for suitable $x_i \in \mathbb{C}$. Since $U^k e_i = e_{i+k}$ for any $i \in \mathbb{Z}$, it is easy to see that any such x commutes with U if and only if $x_i = x_{i+hk}$ for any $i, h \in \mathbb{Z}$. In particular, the set $\{x_i : i \in \mathbb{Z}\}$ is finite. In other terms, the spectrum of x is finite as well, which means its spectral projections belong to $\mathcal{D}_p$. Now write $x = \sum_{i=0}^{k-1} x_i P_i$, where P_i is the orthogonal projection onto the subspace $\overline{\operatorname{span}}\{e_{i+hk} : h, k \in \mathbb{Z}\}$. As there can exist different values of i (in $\{0, 1, \ldots, k-1\}$) giving the same x_i, we rewrite the above sum as $x := \sum_{\lambda \in \sigma(x)} Q_\lambda$, where Q_λ is the spectral projection associated with λ and $Q_\lambda := \sum_{i: x_i = \lambda} P_i$. But because $\sigma(x)$ is finite, each Q_λ can be obtained via the continuous functional calculus of x, which means $Q_\lambda \in \mathcal{D}_p$ for every $\lambda \in \sigma(x)$.

As we next show, this implies that x must be a multiple of the identity. For, if it is not a scalar, then $\sigma(x)$ contains at least two different values, say λ and μ. Now any projection in $\mathcal{D}_p$ is a finite sum of projections of the type $U^m (S_p)^n (S_p^*)^n U^{-m}$ with $m, n \in \mathbb{N}$ [2, Lemma 6.21]. In particular, $Q_\lambda \geq U^m (S_p)^n (S_p^*)^n U^{-m}$ and $Q_\mu \geq U^{m'} (S_p)^n (S_p^*)^n U^{-m'}$ for some $n, m, m' \in \mathbb{N}$ (there is no loss of generality in assuming that the power n of the isometry S_p is the same in the two inequalities).

Now take $a := p^n h_1 + m$ and $b := p^n h_2 + m'$. Observe that by construction if e_n lies in the range of Q_λ, so does e_{n+k}. We claim that $a + lk \equiv b \mod p^n$ for some $l \in \mathbb{Z}$. From this we find $Q_\lambda e_{a+lk} = Q_\mu e_{a+lk} = e_{a+lk}$, which is absurd since $Q_\lambda Q_\mu = 0$.

The claim follows from the fact that k and p^n are coprime: multiplying $ks + p^n t = 1$ by $b - a$ we get $(b-a)sk + t(b-a)p^n = b - a$, that is, $a + (b-a)sk = b - t(b-a)p^n$ and the claim is thus satisfied with $l = (b-a)s$. The proof is complete. ∎

Remark 4.2. From the proof of the foregoing result we can single out the interesting information that all powers T^k of the p-adic odometer T on the Cantor set K with k and p coprime enjoy the following property: any continuous T^k-invariant function is a constant.

As an application of the previous result we also find the following.

THEOREM 4.2. *For any k coprime to p, the Watatani index of $\chi_k(\mathcal{Q}_p^{\mathbb{T}})$ in $\mathcal{Q}_p^{\mathbb{T}}$ is $|k|$.*

Proof. Since the conditional expectation $E : \mathcal{Q}_p^{\mathbb{T}} \to \chi_k(\mathcal{Q}_p^{\mathbb{T}})$ is unique, the index we computed in Theorem 4.1 is actually the Watatani index of the inclusion $\chi_k(\mathcal{Q}_p^{\mathbb{T}}) \subset \mathcal{Q}_p^{\mathbb{T}}$. ■

We believe that the index of χ_k is still $|k|$ at the level of the whole p-adic ring C^*-algebras. However, we have not been able to prove this in full generality. Nevertheless, we know the index is $|k|$ when $k = \pm(p-1)^i$, $i \geq 1$ (note that p and $p-1$ are always coprime).

We start our analysis with $k = p-1$. In this case by universality it is not difficult to see that for any $z \in \mathbb{T}$ such that $z^{p-1} = 1$, $\beta(S_p) := S_p$ and $\beta(U) := zU$ defines an automorphism of $\mathcal{Q}_p$. Again, β has finite order: $\beta^{p-1} = \mathrm{id}_{\mathcal{Q}_p}$. In the following we take z as a primitive root of unity of order $p-1$ so that the corresponding β gives an automorphic action of $\mathbb{Z}_{p-1}$. Let now F be the conditional expectation from $\mathcal{Q}_p$ to $\mathcal{Q}_p^\beta$ given by $F := \frac{1}{p-1}\sum_{i=0}^{p-2}\beta^i$. We have the following result.

THEOREM 4.3. *Let $p \geq 2$ be a fixed integer. If $k = p-1$, then $\chi_k(\mathcal{Q}_p) = \mathcal{Q}_p^\beta$. In particular, the index of $\chi_{p-1}(\mathcal{Q}_p)$ relative to F is $p-1$.*

Proof. The equality $\chi_k(\mathcal{Q}_p) = \mathcal{Q}_p^\beta$ is trivially satisfied. By the same argument as in the proof of Theorem 4.1 it follows that the index of χ_k is equal to $p-1 = k$. ■

REMARK 4.3. The case of a negative k, say $k = -(p-1)$, easily follows from the above result as $\chi_{-k} = \chi_{-1} \circ \chi_k$ and the index of χ_{-1} is 1 because χ_{-1} is an automorphism.

In order to conclude that the value of the index determined above is actually the Watatani index, we need to prove that F is the only conditional expectation from $\mathcal{Q}_p$ onto $\mathcal{Q}_p^\beta$.

PROPOSITION 4.3. *The conditional expectation $F : \mathcal{Q}_p \to \mathcal{Q}_p^\beta$ is unique.*

Proof. Again, we need only prove that $(\mathcal{Q}_p^\beta)' \cap \mathcal{Q}_p = \mathbb{C}$, which is a consequence of the equality $C^*(S_p)' \cap \mathcal{Q}_p = \mathbb{C}$ (see [8, Theorem 4.6]) since $\beta(S_p) = S_p$. ■

As a result, the following is now straightforward.

THEOREM 4.4. *For any $p \geq 2$, the Watatani index of $\chi_k(\mathcal{Q}_p)$ in $\mathcal{Q}_p$ is $|k|$ if $k = p-1$.*

Now iterating the procedure above, it is clear that there exists a conditional expectation F_j from $\chi_{p-1}^j(\mathcal{Q}_p)$ onto $\chi_{p-1}^{j+1}(\mathcal{Q}_p)$ and the index of F_j is still $p-1$. Compounding these conditional expectations, for every $i \geq 1$ one obtains a conditional expectation from $\mathcal{Q}_p$ onto $\chi_{p-1}^i = \chi_{(p-1)^i}$ whose index is obviously $(p-1)^i$. Again, the relative commutant $\chi_{(p-1)^i}(\mathcal{Q}_p)' \cap \mathcal{Q}_p$ is trivial as $\chi_{(p-1)^i}(S_p) = S_p$, and so the conditional expectation is unique. Therefore, collecting everything together, we get the following result.

THEOREM 4.5. *For any $k = \pm(p-1)^i$, $i \in \mathbb{N}$, the Watatani index of $\chi_k(\mathcal{Q}_p)$ in $\mathcal{Q}_p$ is equal to k.*

Acknowledgements. We owe a debt of gratitude to the anonymous referees for their quick but careful reading of the manuscript, which has brought many improvements. The results in Section 4, in particular, originated from a request made by one of the referees.

We also wish to thank Simone Del Vecchio for several useful discussions on the topic of this paper.

Finally, V. A. acknowledges the support by the Swiss National Science foundation through the SNF project no. 178756 (Fibred links, L-space covers and algorithmic knot theory).

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