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A general double sum identity, mock theta functions, and Bailey pairs

by

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1. Introduction. An area of continued development in q-hypergeometric series is obtaining Bailey pairs associated with indefinite quadratic forms. One of the primary motivators is to find further identities for mock theta functions, holomorphic parts of particular weight $\frac{1}{2}$ harmonic Maass forms [20]. Some of the substantial developments in this way include Andrews [1, 5], Zwegers [21], and Hickerson and Mortenson [12], where mock theta functions identities were developed. Crucial in this framework was use of Bailey pairs to associate these functions with indefinite quadratic forms. For example, in [1] we find the fifth order mock theta function identity

$$\sum_{n\geq 0}\frac{q^{n^2}}{(-q;q)_n}=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{n(5n+1)/2}(1-q^{4n+2})\sum_{|j|\leq n}(-1)^j q^{-j^2},$$

where as usual [8] $(w;q)_n=(w)_n:=(1-w)(1-wq)\cdots(1-wq^{n-1})$. Many other authors have utilized Bailey pairs [5, 7, 11, 16], as well as created multisums related to mock theta functions [11, 15, 16, 18].

The goal of the present paper is to consider certain double sums which involve the truncated form of the sum [9, p. 4, eq. (6.2)]

$$\sum_{n\geq 0}\frac{(a;q)_n z^n}{(q;q)_n}=\frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1.1}$$

Specifically, the finite sum considered by Fine [9, p. 17, eq. (15.4)]. The double sums obtained herein produce new identities for mock theta functions similar to the families given in [11, 15, 16, 18]. The list provided in our Theorem 3.2 shows that linear combinations of mixed mock modular forms give rise to certain double sums, which should be compared to the known

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linear identities found in Ramanujan's lost notebook (e.g. [5], [9, pp. 60–62]). To illustrate an interesting motivating example, recall the identity [9, p. 17, eq. (15.52)]

$$(z;q)_\infty \sum_{n\geq 0}(aq;q)_n z^n = \sum_{n\geq 0}(-z)^n q^{n(n+1)/2} \sum_{0\leq k\leq n} \frac{a^{n-k}}{(q)_k}. \tag{1.2}$$

Taking the limit $z \to 1$ and setting $a = 1$ we have

$$\sum_{n\geq 0}(-1)^n q^{n(n+1)/2} \sum_{0\leq k\leq n} \frac{1}{(q)_k} = (q)_\infty^2. \tag{1.3}$$

The product on the right side of (1.3) is a weight one modular form that has a connection to $\mathbb{Q}(\sqrt{3})$ [12], and has an indefinite quadratic form expansion due to L. J. Rogers. To interpret the left hand side as a generating function we need some notation for partitions. Let $\#(\pi)$ be the number of parts of a partition π, $L(\pi) = \pi_1$ be the largest part of a partition π, and we say π is a *partition* of n if $\sum_{1\leq i\leq r} \pi_i = n$ and $\pi_1 \geq \cdots \geq \pi_r$.

For $r \leq N$, consider

$$\frac{q^{N(N+1)/2}}{(q;q)_r}. \tag{1.4}$$

Clearly (1.4) generates a partition pair (λ, μ) where λ is a triangular partition with $\#(\lambda) = L(\lambda) = N$, and μ is a partition with $L(\mu) \leq r$. We take μ and enlarge it by creating a partition τ where $L(\tau) = \tau_1 = L(\lambda) + L(\mu)$, the next part $\tau_2 = N - 1 + \mu_2$, and so on so that $\tau_i = N - (i-1) + \mu_i$ for $1 \leq i \leq N$. (In the case $r = N$, τ would be a partition into N distinct parts.) Taking the conjugate partition, we create a 1-to-1 correspondence to a partition ϖ where $L(\varpi) = \max\{\#(\lambda), \#(\mu)\}$, and $\#(\varpi) = L(\lambda) + L(\mu)$.

THEOREM 1.1. *Let $r \leq N$. Define $p_{r,N}(n)$ to be the number of partitions of n into parts $\leq N$ where parts $\leq r$ appear at least once, and parts $> r$ and $\leq N$ appear exactly once if $r < N$. For $n \equiv 2 \pmod{24}$, let $I(n)$ be the excess of the number of inequivalent solutions of $n = x^2 - 3y^2$ in which $x + 3y \equiv 4 \pmod{12}$ over those in which $x + 3y \equiv 10 \pmod{12}$. Set $\omega(n) = \sum_{N\geq 0,\, 0\leq r\leq N}(-1)^N p_{r,N}(n)$. Then $\omega(n) = I(n)$.*

Proof. The partition ϖ described above tells us that (1.4) is the generating function for $p_{r,N}(n)$. Therefore, combining (1.3) with the expansion found in [13, p. 77], we have

$$\sum_{n\geq 0}\Big(\sum_{N\geq 0,\, 0\leq r\leq N}(-1)^N p_{r,N}(n)\Big) q^{24n+2} = \sum_{k\in\mathbb{Z},\, 2|l|\leq k} (-1)^{l+k} q^{3(2k+1)^2-(6l+1)^2}. \tag{1.5}$$

The right hand side of (1.5) generates excess of the number of solutions of $2(1-12n) = x^2 - 3y^2$ where $x \equiv 1 \pmod 6$, and $-3y < x < 3y$, in which $3y + x \equiv 4 \pmod{12}$, over those in which $3y + x \equiv 10 \pmod{12}$. Note that the minimal positive x_1 and y_1 that solve $x^2 - 3y^2 = 1$ are $(x_1, y_1) = (2, 1)$. Hence, setting $D = 3$, $y_1 = 1$, $x_1 = 2$ in [4, Lemma 3] tells us that each equivalence class of solutions to this equation has a unique (x, y) such that $-3y < x < 3y$. ∎

We mention that identity (1.5) is also related to the work of Kac and Peterson [14, eq. (5.19)]. The proof in [14] involves computation of string functions of low level, while ours is bijective in nature. (See also Cohen's paper [8], which established that identities like (1.5) have connections to Maass waveforms.)

2. Preliminaries for main identities. A pair of sequences $(\alpha_n(a,q), \beta_n(a,q))$ is called a *Bailey pair* [19] relative to (a,q) if

$$\beta_n(a,q) = \sum_{0\le j\le n} \frac{\alpha_j(a,q)}{(q;q)_{n-j}(aq;q)_{n+j}}. \tag{2.1}$$

Lemma 2.1 ([19]). *For a pair $(\alpha_n(a;q), \beta_n(a,q))$ satisfying* (2.1), *we have*

$$\sum_{n\ge0} (X)_n (Y)_n (aq/XY)^n \beta_n = \frac{(aq/X)_\infty (aq/Y)_\infty}{(aq)_\infty (aq/XY)_\infty} \sum_{n\ge0} \frac{(X)_n (Y)_n (aq/XY)^n \alpha_n}{(aq/X)_n (aq/Y)_n}. \tag{2.2}$$

We also require [6, eq. (S2)], which says that if $(\alpha_n(a;q), \beta_n(a,q))$ is a Bailey pair, then so is $(\alpha'_n(a;q), \beta'_n(a,q))$ where

$$\alpha'_n(a,q) = a^{n/2} q^{n^2/2} \alpha_n(a,q), \tag{2.3}$$

$$\beta'_n(a,q) = \frac{1}{(-\sqrt{aq};q)_n} \sum_{0\le k\le n} \frac{(-\sqrt{aq};q)_k}{(q)_{n-k}} a^{k/2} q^{k^2/2} \beta_k(a,q). \tag{2.4}$$

A related cousin that will be equally useful is [6, eq. (S5)], which tells us that $(\alpha'_n(a;q), \beta'_n(a,q))$ is a Bailey pair where

$$\alpha'_n(a,q) = \frac{(-a^{1/2}q)_n}{(-a^{1/2})_n} a^{n/2} q^{(n^2-n)/2} \alpha_n(a,q), \tag{2.5}$$

$$\beta'_n(a,q) = \frac{1}{(-\sqrt{a};q)_n} \sum_{0\le k\le n} \frac{(-\sqrt{a}\,q;q)_k}{(q)_{n-k}} a^{k/2} q^{(k^2-k)/2} \beta_k(a,q). \tag{2.6}$$

Generalizing [3, p. 139, Lemma 6.5.1], we can obtain another identity that will be key in obtaining our general identity.

LEMMA 2.2. *For $-1 \le b \le 1$, we have*

$$b^{N+1}\sum_{n\ge 0}\frac{(aq)_n}{(bq)_n}q^{(N+1)n}=\frac{1-b}{1-q^{N+1}}\frac{(q)_{N+1}}{\left(\frac{a}{b}q\right)_{N+1}}\left(\frac{(aq)_\infty}{(b)_\infty}-\sum_{N\ge n\ge 0}\frac{\left(\frac{a}{b}q\right)_n b^n}{(q)_n}\right).$$

Proof. From Fine's text [9, p. 5, eq. (6.3), $t=q^{N+1}$] we have

$$\sum_{n\ge 0}\frac{(aq)_n}{(bq)_n}q^{(N+1)n}=\frac{1-b}{1-q^{N+1}}\sum_{n\ge 0}\frac{\left(\frac{a}{b}q^{N+2}\right)_n}{(q^{N+2})_n}b^n \tag{2.7}$$
$$=\frac{1-b}{1-q^{N+1}}\frac{(q)_{N+1}}{\left(\frac{a}{b}q\right)_{N+1}b^{N+1}}\sum_{n\ge 0}\frac{\left(\frac{a}{b}q\right)_{n+N+1}b^{n+N+1}}{(q)_{n+N+1}b^{N+1}}$$
$$=\frac{1-b}{1-q^{N+1}}\frac{(q)_{N+1}}{\left(\frac{a}{b}q\right)_{N+1}b^{N+1}}\left(\sum_{n\ge 0}\frac{\left(\frac{a}{b}q\right)_n b^n}{(q)_n}-\sum_{N\ge n\ge 0}\frac{\left(\frac{a}{b}q\right)_n b^n}{(q)_n}\right)$$
$$=\frac{1-b}{1-q^{N+1}}\frac{(q)_{N+1}}{\left(\frac{a}{b}q\right)_{N+1}b^{N+1}}\left(\frac{(aq)_\infty}{(b)_\infty}-\sum_{N\ge n\ge 0}\frac{\left(\frac{a}{b}q\right)_n b^n}{(q)_n}\right).$$

The result is valid for $b=\pm 1$ by analytic continuation. ∎

We are now ready to exhibit our main Bailey pairs.

THEOREM 2.3. *Define the A_n by (2.8) below. Then pair $(\alpha_n(q,q), \beta_n(q,q))$ is a Bailey pair where*

$$\alpha_n(q,q)=q^{-n(n+1)/2}A_n(q,q,b),$$
$$\beta_n(q,q)=\frac{1}{(-q)_n(bq)_n}\sum_{0\le k\le n}\frac{(b)_k}{(q)_k}(-1)^k.$$

Further, $(\alpha_n(q,q),\beta_n(q,q))$ is a Bailey pair where

$$\alpha_n(q,q)=\frac{1+q^{1/2}}{1+q^{n+1/2}}q^{-n^2/2}A_n(q,q,b),$$
$$\beta_n(q,q)=\frac{1}{(-q^{3/2})_n(bq)_n}\sum_{0\le k\le n}\frac{(b)_k}{(q)_k}(-q^{1/2})^k.$$

Proof. To obtain this pair, recall the one-parameter Bailey pair $(A_n(aq,q,b),B_n(aq,q,b))$ (relative to (q,q)) due to Andrews [1, Lemma 6], where

$$A_n(aq,q,b)=\frac{(-1)^n(1-aq^{2n+1})a^nq^{n(3n-1)/2}b^n(aq/b)_n}{(1-aq)(bq)_n} \tag{2.8}$$
$$\times\left(1+\sum_{1\le j\le n}\frac{(aq)_{j-1}(1-aq^{2j})(b)_ja^{-j}q^{-j^2}b^{-j}}{(q)_j(aq/b)_j}\right),$$

$$B_n(aq,q,b) = \frac{1}{(bq)_n}. \tag{2.9}$$

Setting $a = q$ in (2.3)–(2.4), if we suppose that the left side is given by (2.8)–(2.9) with $a = 1$, we find that we would require the first pair in Theorem 2.3, by the uniqueness of Bailey pairs in conjunction with the identity of Fine [9, p. 17, eq. (15.5)]

$$(t)_\infty \sum_{n\ge 0} \frac{(aq)_n}{(bq)_n} t^n = \sum_{n\ge 0} \frac{(-at)^n q^{n(n+1)/2}}{(bq)_n} \sum_{0\le k\le n} \frac{(b)_k}{(q)_k} a^{-k}. \tag{2.10}$$

To see this, multiply both sides of (2.10) by $(t)_\infty^{-1}$ and equate coefficients of t^n to obtain

$$\frac{(aq)_N}{(bq)_N} = \sum_{0\le n\le N} \frac{(-a)^n q^{n(n+1)/2}}{(q)_{N-n}(bq)_n} \sum_{0\le k\le n} \frac{(b)_k}{(q)_k} a^{-k} \tag{2.11}$$

and then set $a = -1$. The second Bailey pair in Theorem 2.3 follows in the same way, with the difference being that we apply (2.5)–(2.6) (with $a = q$) and then put $a = -q^{-1/2}$ in (2.11). ∎

3. Main identities. Having constructed a sufficient number of lemmas in the previous section, we are now ready to state and prove our main identities.

THEOREM 3.1. *We have*

$$\begin{aligned}\frac{(-x)_\infty}{2(-q)_\infty} \sum_{n\ge 0} \frac{(X)_n(Y)_n}{(-q)_n(xq)_n} (q^2/XY)^n \\ + (1-x)\frac{1}{2} \sum_{n\ge 0} \frac{(X)_n(Y)_n}{(q^2;q^2)_n} (-q^2/XY)^n \sum_{k\ge 0} \frac{(-x)_k}{(-q)_k} q^{(n+1)k} \\ = \frac{(q^2/X)_\infty (q^2/Y)_\infty}{(q^2)_\infty (q^2/XY)_\infty} \sum_{n\ge 0} \frac{(X)_n(Y)_n (q^2/XY)^n q^{-n(n+1)/2} A_n(q,q,x)}{(q^2/X)_n (q^2/Y)_n}.\end{aligned} \tag{3.1}$$

Further,

$$\begin{aligned}\frac{(-xq^{1/2})_\infty}{(-q^{1/2})_\infty} \sum_{n\ge 0} \frac{(X)_n(Y)_n}{(-q^{3/2})_n (xq)_n} (q^2/XY)^n \\ + \sum_{n\ge 0} \frac{(X)_n(Y)_n (xq^{1/2})_{n+1}}{(-q^{3/2})_n (q)_n (xq)_n} (-q^{5/2}/XY)^n \sum_{k\ge 0} \frac{(-xq^{1/2})_k}{(-q^{1/2})_k} q^{(n+1)k} \\ = (1+q^{1/2}) \frac{(q^2/X)_\infty (q^2/Y)_\infty}{(q^2)_\infty (q^2/XY)_\infty} \sum_{n\ge 0} \frac{(X)_n(Y)_n (q^2/XY)^n q^{-n^2/2} A_n(q,q,x)}{(1+q^{n+1/2})(q^2/X)_n (q^2/Y)_n}.\end{aligned} \tag{3.2}$$

Proof. Setting $b=-1$ and $a=-x/q$ in Lemma 2.2, we obtain

$$\sum_{N\geq n\geq 0}\frac{(x)_n(-1)^n}{(q)_n}=\frac{(-x)_\infty}{2(-q)_\infty}+\frac{(x)_{N+1}}{2(q)_N}(-1)^N\sum_{n\geq 0}\frac{(-x)_n}{(-q)_n}q^{(N+1)n}. \tag{3.3}$$

Then, inserting the first Bailey pair in Theorem 2.3 into Lemma 2.1, and using (3.3), we find (3.1). Similarly, putting $b=-q^{1/2}$ and $a=-x/q^{1/2}$ in Lemma 2.2, we obtain

$$\begin{aligned}\sum_{N\geq n\geq 0}&\frac{(x)_n(-q^{1/2})^n}{(q)_n}\\&=\frac{(-xq^{1/2})_\infty}{(-q^{1/2})_\infty}+q^{1/2}\frac{(xq^{1/2})_{N+1}}{(1+q^{1/2})(q)_N}(-q^{1/2})^N\sum_{n\geq 0}\frac{(-xq^{1/2})_n}{(-q^{1/2})_n}q^{(N+1)n}.\end{aligned} \tag{3.4}$$

Then, inserting the second Bailey pair in Theorem 2.3 into Lemma 2.1, and using (3.4), we find (3.2). ■

As a nice consequence of Theorem 3.1 we obtain identities for mock theta functions.

THEOREM 3.2. *For $-1\leq x\leq 1$, we have*

$$\begin{aligned}&\frac{1}{2}\frac{(-x)_\infty}{(-q)_\infty}\sum_{n\geq 0}\frac{q^{n(n+1)}}{(-q)_n(xq)_n}+(1-x)\frac{1}{2}\sum_{n\geq 0}\frac{(-1)^nq^{n(n+1)}}{(q^2;q^2)_n}\sum_{k\geq 0}\frac{(-x)_k}{(-q)_k}q^{(n+1)k}\\&\qquad=\frac{1}{(q)_\infty}\sum_{n\geq 0}q^{n^2/2+n/2}A_n(q,q,x)=\frac{1}{(-q)_\infty}\sum_{n\geq 0}\frac{(-q)_n}{(xq)_n}q^{n(n+1)/2},\end{aligned} \tag{3.5}$$

$$\frac{1}{2(-q)_\infty}\sum_{n\geq 0}\frac{q^{n(n+1)}}{(-q)_n}+\frac{1}{2}\sum_{n\geq 0}\frac{q^n(q^{n+2};q^2)_\infty}{(-q)_n}=\frac{1}{(-q)_\infty}\sum_{n\geq 0}(-q)_nq^{n(n+1)/2}. \tag{3.6}$$

$$\begin{aligned}&\frac{1}{2}\frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n(n+1)}}{(-q)_{2n+1}}+\frac{1}{2}\sum_{n\geq 0}\frac{(-1)^nq^{2n(n+1)}}{(q^4;q^4)_n}\sum_{k\geq 0}\frac{(-q;q^2)_k}{(-q^2;q^2)_k}q^{2(n+1)k}\\&\qquad=\frac{1}{(-q^2;q^2)_\infty}\sum_{n\geq 0}\frac{(-q^2;q^2)_n}{(-q;q^2)_{n+1}}q^{n(n+1)},\end{aligned} \tag{3.7}$$

$$(3.8)\quad \frac{(-xq^{1/2})_\infty}{(-q^{1/2})_\infty}\sum_{n\geq 0}\frac{q^{n(n+1)}}{(-q^{1/2})_{n+1}(xq)_n}$$

$$+\,q^{1/2}\sum_{n\geq 0}\frac{(xq^{1/2})_{n+1}}{(-q^{1/2})_{n+1}(q)_n(xq)_n}(-1)^n q^{n(n+3/2)}\sum_{k\geq 0}\frac{(-xq^{1/2})_k}{(-q^{1/2})_k}q^{(n+1)k}$$

$$=\frac{1}{(q^2)_\infty}\sum_{n\geq 0}\frac{q^{n^2/2+n}A_n(q,q,x)}{(1+q^{n+1/2})}=\frac{1}{(-q^{1/2})_\infty}\sum_{n\geq 0}\frac{(-q^{1/2})_n}{(xq)_n}q^{n(n+2)/2},$$

(3.9)

$$\frac{1}{(-q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n(n+1)}}{(-q;q^2)_{n+1}}+q\sum_{n\geq 0}\frac{(-1)^n q^{n(2n+3)}}{(-q;q^2)_{n+1}(q^2;q^2)_n}\sum_{k\geq 0}\frac{q^{2(n+1)k}}{(-q;q^2)_k}$$

$$=\frac{1}{(-q;q^2)_\infty}\sum_{n\geq 0}(-q;q^2)_n q^{n(n+2)}.$$

(3.10)

$$\frac{(q;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n(n+1)}}{(-q)_{2n+1}}+q\sum_{n\geq 0}\frac{(-1)^n q^{n(2n+3)}}{(q^4;q^4)_n}\sum_{k\geq 0}\frac{(q;q^2)_k}{(-q;q^2)_k}q^{2(n+1)k}$$

$$=\frac{1}{(-q;q^2)_\infty}\sum_{n\geq 0}\frac{(-q;q^2)_n}{(-q^2;q^2)_n}q^{n(n+2)},$$

$$(3.11)\quad \frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n(n+1)}}{(-q;q^2)_{n+1}^2}$$

$$+\,q\sum_{n\geq 0}\frac{(-q^2;q^2)_{n+1}}{(-q;q^2)_{n+1}^2(q^2;q^2)_n}(-1)^n q^{n(2n+3)}\sum_{k\geq 0}\frac{(-q^2;q^2)_k}{(-q;q^2)_k}q^{2(n+1)k}$$

$$=\frac{1}{(-q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{n(n+2)}}{1+q^{2n+1}}.$$

Proof. For (3.5) let $X,Y\to\infty$ in (3.1), and then invoke the transformation through the indefinite theta expansion [1]. Equation (3.6) is the $x=0$ case of (3.5). Equation (3.7) is the $x=-q^{1/2}$ case of (3.5). For (3.8) let $X,Y\to\infty$ in (3.2), and then invoke the indefinite theta expansion [1]. Equation (3.9) is the $x=0$ case of (3.8). Equation (3.11) is the $x=-q^{1/2}$ case of (3.8). ∎

First, we remark on our

$$(3.12)\quad \sum_{n\geq 0}\frac{q^{n(n+1)}}{(-q)_n(xq)_n}\sum_{0\leq k\leq n}\frac{(x)_k}{(q)_k}(-1)^k$$

and

$$\sum_{n\geq 0}\frac{q^{n(n+1)}}{(-q^{1/2})_{n+1}(xq)_n}\sum_{0\leq k\leq n}\frac{(x)_k}{(q)_k}(-1)^k. \tag{3.13}$$

Equation (3.12) is a mixed mock modular form for $x = 0, -q^{1/2}$, since multiplying by $(-q)_\infty$ gives a mock modular form, and for $x = -1$ we have a modular form, $(q^2; q^2)_\infty$. Similarly, equation (3.13) is a mixed mock modular form for $x = 0, -1$, since multiplying by $(-q^{1/2})_\infty$ gives a mock modular form, and for $x = -q^{-1/2}$ we have a modular form $q^{-1}((q^2; q^2)_\infty/(q; q^2)_\infty - 1)/(-q^{1/2})_\infty$. Equation (3.6) provides a connection between two mock theta functions of order five [1]. Equation (3.7) provides a connection between the eighth order mock theta function $T_1(q)$ [9], and a mock theta function found in [2]. Equation (3.9) relates two mock theta functions of order five [1]. Equation (3.10) provides a connection between the eighth order mock theta function $S_1(q)$ [10], and a mock theta function found in [2]. Equation (3.11) involves a mock theta function of order three [9, p. 55].

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