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QUOTIENT-TRANSITIVITY AND CYCLIC SUBMODULE-TRANSITIVITY FOR p-ADIC MODULES

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Abstract. Two new notions of transitivity, which we have named quotient-transitivity and transitivity with respect to cyclic submodules for p-adic modules, are introduced. Unlike the classical notions that derive from Abelian group theory, this approach is based on isomorphism of quotients and makes no use of height sequences. The two new notions lead to a sufficiently large class of interesting modules. Our principal result is that finitely generated p-adic modules are both quotient-transitive and transitive with respect to cyclic submodules.

1. Introduction. The notion of transitivity, and also of full transitivity, in the context of modules over a complete discrete valuation ring is well known from Kaplansky's early work [8, Section 18]. The idea is: if M is a module over a complete discrete valuation ring S and ϕ is an automorphism of M with $\phi(x) = y$ for some $x, y \in M$, then certainly the Ulm or height sequences $U_M(x), U_M(y)$ must be equal; transitivity is essentially the converse notion: M is transitive if for all $x, y \in M$ with $U_M(x) = U_M(y)$, there is an automorphism of M mapping $x \mapsto y$. Kaplansky showed that large classes of S-modules are in fact transitive, but subsequent work by, among others, Corner [2] and Megibben [9], showed that, contrary to Kaplansky's initial feeling, not all S-modules are transitive.

In this note, we study another type of transitivity, which in some sense is contrary to the well-known notion from Kaplansky's work [8, Section 18]. As we will see, there are still many modules with this property to investigate. However, many of our arguments in this context will have an arithmetical content that means we are not able at present to handle modules over arbitrary complete discrete valuation rings and we have restricted our attention to modules over the ring of p-adic integers.

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Throughout, R will denote the ring of p-adic integers (for an arbitrary prime p), Q its field of fractions, and the word *module* will refer to a left module over R; note that torsion R-modules are precisely the p-primary Abelian groups and we shall use both terminologies freely.

DEFINITION 1.1. An R-module G is called *transitive with respect to cyclic submodules* if when X, Y are cyclic submodules of G satisfying:

(i) $X \cong Y$, and
(ii) $G/X \cong G/Y$,

there exists an automorphism ϕ of G such that $\phi(X) = Y$.

For simplicity, we say G is *CS-transitive* instead of the full phrase. The definition of CS-transitive modules is, in some sense, related to the concept of submodules being equivalent, however we emphasize that are not making use of a 'valuated approach' here; see [7] for an excellent discussion of this approach in the context of Abelian groups.

There are situations in which condition (i) of Definition 1.1 may be dropped: for example, if G is a torsion-free R-module or if the module G is finite then (i) holds automatically or is an immediate consequence of (ii). More generally it follows from [6, Proposition 2.2] that the same is true if G is an arbitrary homocyclic p-group. But (i) does not hold in general and we find it convenient to make a formal distinction by defining:

DEFINITION 1.2. An R-module M is said to be *quotient-transitive* if, for any pair of non-zero elements $x, y \in M$ such that $M/Rx \cong M/Ry$, there is an automorphism ϕ of M that maps x onto y.

Thus it is immediate that for torsion-free modules and finite modules, the notions of quotient-transitivity and CS-transitivity coincide. Furthermore, a quotient-transitive module is always transitive with respect to cyclic submodules. However, there are modules which are CS-transitive but not quotient-transitive, as we shall see in the next section.

The main objective of the paper is to establish (in Theorem 3.1) that finitely generated p-adic modules are quotient-transitive and hence CS-transitive; a first step in this argument has already been achieved in [6, Theorem 3.2]: finite p-groups have both properties. In addition we discuss the properties in the situation where the modules have non-trivial divisible submodules and we also consider the situation for torsion-free p-adic modules.

Finally, we point out that our notation and terminology are standard; see the texts [3, 4, 5, 8].

2. Divisible and torsion-free modules. Observe first that the situation in regard to divisible torsion p-adic modules is quite straightforward: if x, y are elements of $Q/R = \mathbb{Z}(p^\infty)$ having different order, then the quotients

$\mathbb{Z}(p^\infty)/\langle x\rangle$ and $\mathbb{Z}(p^\infty)/\langle y\rangle$ are isomorphic—they are isomorphic to $\mathbb{Z}(p^\infty)$, but clearly no automorphism of $\mathbb{Z}(p^\infty)$ can map $x \mapsto y$, as automorphisms always preserve orders. Note that a non-trivial torsion divisible p-adic module G can always be written as a direct sum of copies of $\mathbb{Z}(p^\infty)$ and thus $G/\langle x\rangle$ is isomorphic to G itself for any $x \in G$, and hence a non-trivial divisible torsion p-adic module is never quotient-transitive. Looking at this argument one sees that the situation for CS-transitivity is completely the opposite: every divisible torsion p-adic module D is CS-transitive. This follows easily from the observation that if $o(x) = o(y)$ for $x, y \in D$, then there is an automorphism of D mapping $x \mapsto y$.

We record this as

Proposition 2.1. *If D is an arbitrary divisible torsion p-adic module then D is CS-transitive but not quotient-transitive.*

The situation for divisible torsion-free modules is also straightforward: a torsion-free divisible R-module is just a vector space over the field of p-adic numbers and it is clear that such modules are quotient-transitive and hence CS-transitive as well. We also have a strong result for reduced torsion-free R-modules:

Proposition 2.2. *If G is a reduced torsion-free R-module of arbitrary rank, then G is quotient-transitive and hence CS-transitive.*

Proof. Suppose that x, y are non-zero elements of G. Then $x = p^r e$, $y = p^s f$ for some elements e, f of G which have zero height in G. By standard properties of torsion-free J_p-modules, the elements e, f each generate a direct summand of G. Thus we have decompositions $G = Re \oplus G_1 = Rf \oplus G_2$ and $G_1 \cong G_2$ since free modules of finite rank have the cancellation property [1, Corollary 8.8]. Since we are assuming $G/Rx \cong G/Ry$, we see immediately that if $r > 0$ then so too is s and then $r = s$. If $r = 0$ so that $x = e$, then G/Rx is torsion-free, being isomorphic to G_1. Hence G/Ry must also be torsion-free, which forces $s = 0$ or equivalently $y = f$. In either situation the map defined as the direct sum of an isomorphism between G_1 and G_2 and the map sending $e \mapsto f$ is an automorphism ϕ of G with $\phi(x) = y$. ∎

Note that the argument in Proposition 2.2 shows

Corollary 2.3. *If H is a reduced torsion-free R-module, then H/Rx is reduced for any $x \in H$.*

Our next result extends the last two propositions to encompass arbitrary torsion-free R-modules. The proof is based on a careful case-by-case analysis.

Theorem 2.4. *If $G = D \oplus H$ is a torsion-free R-module where D is divisible and H is reduced, then G is quotient-transitive.*

Proof. Note that if $x \in D$ then D decomposes as $D = Q \oplus D_1$ with $x \in Q$; similarly if $y \in H$ then H decomposes as $H = Re \oplus H_1$ where $y \in Re$.

We first consider the possibilities for the structure of quotients of G modulo a cyclic submodule. There are essentially three possibilities:

(i) if $0 \neq x \in D$ then $G/Rx \cong (D/Rx) \oplus H$ and it is easy to see that D/Rx is then divisible of the form $D_1 \oplus Q/Rx$;
(ii) if $0 \neq y \in H$ then $G/Ry \cong D \oplus (H/Ry)$ and H/Ry is of the form $H_1 \oplus Re/Ry$;
(iii) if $z = d + p^t e$ with $d \neq 0$ then we note that $G/Rz \geq (D + Rz)/Rz \cong D/(D \cap Rz) \cong D$; since D is divisible, this means that G/Rz splits as $G/Rz = (D + Re + H_1)/Rz \cong H_1 \oplus (D \oplus Re)/Rz \cong H_1 \oplus D \oplus W$, where $W \cong (D + Re)/(D + Rz)$, whence $W \cong Re/Rp^t e$.

Suppose now that a, b are arbitrary non-zero elements of G with $G/Ra \cong G/Rb$. If $a \in D$, then a check on cases (i)–(iii) above shows that b must also belong to D. Furthermore, it follows from the isomorphism $G/Ra \cong G/Rb$ that the corresponding maximal divisible submodules are isomorphic, i.e., $D/Ra \cong D/Rb$. Appealing to Proposition 2.1 we find an automorphism of D mapping $a \mapsto b$. Since D is a direct summand of G, this automorphism extends to an automorphism of G which still sends $a \mapsto b$.

Now suppose that $a \in H$. There are then two possibilities for b: either $b \in H$ or $b = d + p^t e$ where $d \neq 0$ and $H = Re \oplus H_1$. In the former situation, the quotients H/Ra and H/Rb are the 'reduced parts' of $G/Ra, G/Rb$ respectively and consequently they are isomorphic. So by Proposition 2.2 there is an automorphism of H (and hence an automorphism of G) mapping $a \mapsto b$.

Suppose now that $a \in H$, say $a = p^r e$ for some summand of $H = Re \oplus H_1$ and $b = d + p^t f$, where $d \neq 0$ and Rf is some summand of $H = Rf \oplus H_2$. Cancellation of free summands of finite rank implies that $H_1 \cong H_2$ and it is then easy to find an automorphism of H mapping $b \mapsto d + p^t e$. In other words, there is no loss in assuming $f = e$ and $b = d + p^t e$. Now $G/Ra \cong D \oplus H_1 \oplus Re/Rp^r e \cong G/Rb \cong D \oplus H_1 \oplus Re/Rp^t e$, the last isomorphism being a simple application of (iii) above. Equating torsion submodules we deduce that $r = t$. Since D is divisible, we can find $w \in D$ with $d = p^t w$; the mapping $\delta : Re \to D$ given by $e \mapsto w$ is a well-defined homomorphism and if $\Delta = \left(\begin{smallmatrix} 1_D & \delta \\ 0 & 1_H \end{smallmatrix}\right)$, then Δ represents an automorphism of $G = D \oplus H$ which maps $a \mapsto b$.

Finally suppose both a, b are of the form given in case (iii) above. Again there is no loss in assuming $a = d + p^r e$, $b = d_1 + p^t e$. Now equating the torsion submodules of $G/Ra, G/Rb$, we see that $r = t$. Set, as we may, $d_1 - d = p^t w$ and define $\delta : Re \to D$ by $\delta(e) = w$. Exactly as in the previous case the matrix Δ represents an automorphism of G mapping $a \mapsto b$.

Since these are all the possibilities for a, b, we conclude that G is quotient-transitive, as required. ■

2.1. Non-reduced torsion modules. The structure of non-reduced torsion modules is vastly more complicated and shows a marked divergence between the concepts of quotient-transitivity and transitivity with respect to cyclic submodules.

We begin with a technical result which will be useful in other contexts so we state it in a more general form than is needed for the current subsection.

PROPOSITION 2.5. *If $G = D \oplus F$, where D is fully invariant in G, then if G is [quotient-transitive] CS-transitive, both D and F are [quotient-transitive] CS-transitive.*

Proof. We give the proof in the CS-transitivity case; the proof for quotient-transitivity is essentially identical. Since D is fully invariant in G, every endomorphism of G has a matrix representation of the form $\left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \beta \end{smallmatrix}\right)$ where $\alpha \in \mathrm{End}_R(D)$, $\beta \in \mathrm{End}_R(F)$, and $\gamma \in \mathrm{Hom}_R(F, D)$. Consequently, if ϕ is an automorphism of G, it must have a matrix representation as above but with the additional restriction that α, β are automorphisms of D, F respectively.

Suppose now that $d, d_1 \in D$ with $o(d) = o(d_1)$ and $D/\langle d \rangle \cong D/\langle d_1 \rangle$. Set $g = (d, 0)$, $g_1 = (d_1, 0)$ and note that $o(g) = o(g_1)$ and $G/\langle g \rangle \cong G/\langle g_1 \rangle$. Since G is CS-transitive, there is an automorphism of G mapping $g \mapsto g_1$ and it follows easily that there is a corresponding automorphism α of D with $\alpha(d) = d_1$. So D is CS-transitive.

In a similar way, if f, f_1 are elements of F with $o(f) = o(f_1)$ and $F/\langle f \rangle \cong F/\langle f_1 \rangle$, then there is an automorphism of G mapping $(0, f) \mapsto (0, f_1)$ and an identical argument to that above gives an automorphism β of F with $\beta(f) = f_1$ and F is also CS-transitive, as required. ■

An application of this last result and Proposition 2.1 yields

COROLLARY 2.6. *If G is a quotient-transitive torsion p-adic module, then G is reduced.*

However, non-reduced torsion modules which are CS-transitive exist, but it seems difficult to formulate criteria which determine when, for example, a module of the form $Q/R \oplus T$ with T torsion is CS-transitive. We illustrate this with two examples, but first a technical lemma.

LEMMA 2.7. *If $G = D \oplus H$, where $D = \mathbb{Z}(p^\infty)$ and $H = \langle h \rangle$ is cyclic of order p^n, then for each $0 \leq k \leq n$, the quotient $G/\langle d + p^k h \rangle$ is isomorphic $D \oplus \mathbb{Z}(p^k)$, where $\mathbb{Z}(p^k) = 0$ if $k = 0$.*

Proof. If $k = n$ then $d + p^k h = d$ and the quotient is then easily seen to be isomorphic to G itself.

If $k < n$ then $G/\langle d + p^k h\rangle \geq (D + \langle d + p^k h\rangle)/\langle d + p^k h\rangle \cong D/(D \cap \langle d + p^k h\rangle)$. Since $\langle d + p^k h\rangle$ is cyclic, the intersection $D \cap \langle d + p^k h\rangle$ is a submodule, not equal to D, of D and hence $D/(D \cap \langle d + p^k h\rangle) \cong D$. Thus $(D + \langle d+p^k h\rangle)/\langle d+p^k h\rangle = (D+p^k H)/\langle d+p^k h\rangle$ is a divisible submodule of $G/\langle d+p^k h\rangle$ and so the latter splits as $G/\langle d+p^k h\rangle = (D+p^k H)/\langle d+p^k h\rangle \oplus Y$, where $Y \cong (D \oplus H)/(D \oplus p^k H) \cong \mathbb{Z}(p^k)$, with the last term being the zero submodule if $k = 0$. ∎

Example 2.8. If $G = D \oplus H$, where $H = \langle h\rangle$ is cyclic of order p^n and $D = \mathbb{Z}(p^\infty)$, then G is CS-transitive.

Proof. Suppose that $x = d + p^k h$, $y = d_1 + p^{k_1} h$, where $d, d_1 \in \mathbb{Z}(p^\infty)$, $o(x) = o(y)$ and $G/\langle x\rangle \cong G/\langle y\rangle$. It follows from Lemma 2.7 that $D \oplus \mathbb{Z}(p^k) \cong D \oplus \mathbb{Z}(p^{k_1})$, whence $k = k_1$. Now equality of the orders of x, y implies that $\max\{o(d), p^{n-k}\} = \max\{o(d_1), p^{n-k}\} = p^\lambda$, say; we consider two cases:

Case 1: $\lambda > n - k$. Then $o(x) = p^\lambda = o(d)$ and similarly $o(y) = p^\lambda = o(d_1)$. Hence there is a p-adic unit u with $d_1 = ud$ and the mapping $u \oplus 1_H$ is an automorphism of G mapping $x \mapsto y$.

Case 2: $\lambda = n - k$. Thus $o(d) \leq p^{n-k}$ and also $o(d_1) \leq p^{n-k}$. Hence $o(d - d_1) = p^t$ for some $t \leq \lambda$. Now D is divisible, so there is a $z \in D$ with $d_1 - d = p^k z$; note that $0 = p^t(d_1 - d) = p^{t+k} z$, so that $o(z) \leq p^{k+t} \leq p^n$. It follows that the map $\delta : H \to D$ with $\delta(h) = z$ is a well-defined homomorphism. Set $\Delta = \begin{pmatrix} 1_D & \delta \\ 0 & 1_H \end{pmatrix}$ and note that Δ represents an automorphism of G. Furthermore, a straightforward calculation yields $\Delta(x) = y$.

Thus in either case we have the desired automorphism mapping $x \mapsto y$ and G is then CS-transitive. ∎

Our next example shows that the converse of Proposition 2.5 does not hold.

Example 2.9. If $G = D \oplus H$, where $D = \mathbb{Z}(p^\infty)$ and H is homocyclic of infinite rank and exponent $n \geq 1$, then G is not CS-transitive.

Proof. Choose $d \in D$ with $o(d) = p^n$ and let $\langle h\rangle$ be a canonical direct summand of H. Then certainly $o(d) = o(h)$ and $G/\langle d\rangle \cong \mathbb{Z}(p^\infty) \oplus H \cong G$, while $G/\langle h\rangle \cong D \oplus H$ also, since H is of infinite rank. However, since D is fully invariant in G, no automorphism of G can map $d \mapsto h$. Thus G is not CS-transitive. ∎

3. Finitely generated p-adic modules. In this section our objective is to extend [6, Theorem 3.2] to finitely generated R-modules. In particular we wish to establish:

Theorem 3.1. *If M is a finitely generated p-adic module, then M is quotient-transitive and hence CS-transitive.*

We prove Theorem 3.1 by a series of steps which reduce the complexity of the problem. Before proceeding to these reductions we list a number of technical results which will aid the simplifications sought.

LEMMA 3.2. *Let $M = T \oplus G$ be a p-adic module where T is finite and G is torsion-free. If $0 \neq x, y \in M$ are such that $M/\langle x\rangle \cong M/\langle y\rangle$, then if $x \in T$, we also have $y \in T$.*

Proof. Suppose for a contradiction that $y \notin T$, so that y has infinite order and hence $T \cap \langle y\rangle = 0$. However, $M/\langle x\rangle = (T/\langle x\rangle) \oplus G \cong M/\langle y\rangle$ is impossible since the torsion submodule of $M/\langle x\rangle$ has order $|T/\langle x\rangle| < |T|$, while the cosets $t+\langle y\rangle$ $(t \in T)$ are a set of distinct torsion elements of $M/\langle y\rangle$ of order $|T|$. ∎

LEMMA 3.3. *Suppose that $M = T \oplus \langle f\rangle$ and $x = t + p^s f$, $y = t_1 + p^{s_1} f$. Then if $M/\langle x\rangle \cong M/\langle y\rangle$, we have $s = s_1$.*

Proof. We have $M/\langle x\rangle = (T \oplus \langle f\rangle)/\langle t+p^s f\rangle \geq (T \oplus \langle t+p^s f\rangle)/\langle t+p^s f\rangle \cong T$ and hence $M/\langle x\rangle$ is an extension of T by the module

$$(T \oplus \langle f\rangle)/\langle t + p^s f\rangle/(T \oplus \langle p^s f\rangle)/\langle t + p^s f\rangle \cong \mathbb{Z}(p^s);$$

a similar result holds for $M/\langle y\rangle$ with s replaced by s_1. But then we have $|M/\langle x\rangle| = |T| \times p^s = |M/\langle y\rangle| = |T| \times p^{s_1}$ and since all the group T is finite, we conclude that $s = s_1$, as required. ∎

LEMMA 3.4. *Suppose $M = T \oplus \langle f\rangle$, where T is a finite p-adic module and $\langle f\rangle$ is a free p-adic module. If $x = t+p^s f$ and t decomposes as $t = t_0+p^k t'$ where $k \geq s$, $t_0, t' \in T$, then there is an automorphism of M mapping $x \mapsto x' = t_0 + p^s f$.*

Proof. Define $\theta : \langle f\rangle \to T$ by $\theta(f) = -p^{k-s}t'$. Then the matrix $\Delta = \begin{pmatrix} 1_T & \theta \\ 0 & 1 \end{pmatrix}$ represents an automorphism of M with $\Delta(x) = t_0 + p^s f = x'$. ∎

LEMMA 3.5. *Suppose $M = T \oplus \langle f\rangle$, where T is a finite p-adic module and $\langle f\rangle$ is a free p-adic module. If $x = t + p^s f$ and t decomposes as $t = t_0 + p^k t'$ where $k \geq s$, $t_0, t' \in T$, then $M/\langle x\rangle \cong M/\langle x'\rangle$*

Proof. As $M = T \oplus \langle f\rangle = T \oplus \langle p^{k-s}t' + f\rangle$, the lemma follows. ∎

Suppose then that M is a finitely generated p-adic module so that $M = T \oplus F$, where T is a finite p-adic module and F is a free p-adic module of finite rank. Let $0 \neq x, y \in M$ be such that $M/\langle x\rangle \cong M/\langle y\rangle$; we will show, by a series of reductions, that there is an automorphism ϕ of M with $\phi(x) = y$.

Our first reduction is:

- We may assume that neither x nor y belongs to T.

Proof. If $x \in T$ then it follows from Lemma 3.2 above that y is also an element of T. Therefore

$$T/\langle x\rangle \oplus F = M/\langle x\rangle \cong M/\langle y\rangle = T/\langle y\rangle \oplus F$$

and since finite rank free modules have the cancellation property—see, for example, the argument in [1, Corollary 8.8]—we find that $T/\langle x\rangle \cong T/\langle y\rangle$. Thus [6, Theorem 3.2] implies that there is an automorphism θ of T with $\theta(x) = y$. Set $\phi = \theta \oplus 1_F$ and note that ϕ is an automorphism of M with $\phi(x) = y$. ∎

So we may assume that $x, y \notin T$. Thus the general form of these elements is $x = t + p^s f$, $y = t_1 + p^{s_1} f_1$ for some non-negative integers s, s_1, with $t, t_1 \in T$, $f, f_1 \in F$ such that both f and f_1 are of height 0 in F; note that t, t_1 may be 0.

Our next reduction is:

- There is no loss in assuming $f = f_1$.

Proof. Since $h_F(f) = 0 = h_F(f_1)$, we have direct sum decompositions $F = \langle f\rangle \oplus F_1 = \langle f_2\rangle \oplus F_2$, where F_2, F_1 are free of rank $\mathrm{rk}(F) - 1$. Let θ be an isomorphism of $\langle f_1\rangle$ onto $\langle f\rangle$ mapping $f_1 \mapsto f$ and ϕ an isomorphism $\phi : F_2 \to F_1$. Then $\psi = \theta \oplus \phi$ is an automorphism of F and it extends to an automorphism of M of the form $1_T \oplus \psi$ which maps $y \mapsto y_1 = t_1 + p^{s_1} f$. Consequently, there is an automorphism of M mapping $x \mapsto y$ if, and only if, there is an automorphism mapping $x \mapsto y_1$. ∎

We can actually say a little more: if $x = t + p^s f$, $y = t_1 + p^{s_1} f$ then $M/\langle x\rangle = ((T \oplus \langle f\rangle)/\langle x\rangle) \oplus F_1 \cong ((T \oplus \langle f\rangle)/\langle y\rangle) \oplus F_1 = M/\langle y\rangle$ and again cancellation gives $T \oplus \langle f\rangle/\langle x\rangle \cong T \oplus \langle f\rangle/\langle y\rangle$. Since an automorphism of $T \oplus \langle f\rangle$ extends trivially to an automorphism of M, we have a further reduction:

- There is no loss in assuming $M = T \oplus \langle f\rangle$, $x = t + p^s f$, $y = t_1 + p^{s_1} f$.

Now a direct appeal to Lemma 3.3 gives us a significant reduction:

- We may assume that $M = T \oplus \langle f\rangle$, $x = t + p^s f$, $y = t_1 + p^s f$.

In fact we may assume a little more: if t, t_1 are both zero, then $x = y$ and the identity automorphism of M will map $x \mapsto y$. In addition, we may assume $s > 0$. For if not, let $\theta : \langle f\rangle \to T$ be the mapping given by $\theta(f) = t_1 - t$ and set Δ to be the representation of the automorphism of M given by $\Delta = \begin{pmatrix} 1_T & \theta \\ 0 & 1 \end{pmatrix}$. Then clearly $\Delta(x) = y$.

So we have a further reduction:

- There is no loss in assuming $M = T \oplus \langle f\rangle$, $x = t + p^s f$, $y = t_1 + p^s f$, $s > 0$ and at least one of t, t_1 is non-zero.

From now on we will assume that $t \neq 0$.

An immediate consequence of Lemma 3.5 is that if we express the element t as a sum of multiples of a basis of T, then each component can be taken to have height $< s$. So we arrive at our final reduction, where for reasons that will become clear shortly, we have changed the notation for the torsion elements in the expression of x, y:

- There is no loss in assuming that $M = T \oplus \langle f \rangle$, $x = c + p^s f$, $y = d + p^s f$ and $c \neq 0$ is such that in its representation as a sum of basis elements of T, each component has height less than s.

We now apply the Baer–Fuchs lemma (see [5, Lemma 10.1.4] or [4, Lemma 65.4]) to the torsion element c. Thus we find a summand C of T, say $T = C \oplus H$, with $C = \langle c_1 \rangle \oplus \cdots \oplus \langle c_t \rangle$, where $o(c_i) = p^{n_i + k_i}$, $c = p^{k_1} c_1 + \cdots + p^{k_t} c_t$, $0 \leq k_1 < \cdots < k_t$ and $0 < n_1 < \cdots < n_t$.

Our next result is key:

Lemma 3.6. *With the notation as above, if $G = C \oplus \langle f \rangle$ and $x = c + p^s f$, then $G/\langle x \rangle \cong \mathbb{Z}(p^{n_t+s}) \oplus (C/\langle c \rangle)$.*

Proof. Note that $p^{n_t+s} f = p^{n_t} x \in \langle x \rangle$ and no multiple of f less than p^{n_t+s} can be in $\langle x \rangle$ since any multiple of c less than p^{n_t} will have a torsion component. Thus, $o(f + \langle x \rangle) = p^{n_t+s}$. Note also that if $g = r_1 c_1 + \cdots + r_t c_t + rf$ is an arbitrary element of G, then $p^{n_t+s} g = p^{n_t+s}(r_1 c_1 + \cdots + r_t c_t + rf) = rp^{n_t} x \in \langle x \rangle$ and so the exponent satisfies $e(G/\langle x \rangle) \leq n_t + s$; since $e(f + \langle x \rangle) = n_t + s$, we have $e(G/\langle x \rangle) = n_t + s$. Hence it follows from [8, Lemma 4] that $f + \langle x \rangle$ generates a direct summand of $G/\langle x \rangle$. For convenience we write $X = \langle x \rangle$.

Let $A = \langle f \rangle + X$ so that $\frac{G}{X} = \frac{A}{X} \oplus \frac{B}{X}$ for some B, also $\frac{G}{A} \cong \frac{B}{X}$ and we have

$$\frac{G}{A} = \frac{(\langle f \rangle + X) + C}{\langle f \rangle + X};$$

then $\frac{G}{A} \cong \frac{C}{((\langle f \rangle + X) \cap C}$. Suppose w is an arbitrary element of $(\langle f \rangle + X) \cap C$. Then $w = uf + v(p^{k_1} c_1 + \cdots + p^{k_t} c_t + p^s f) = w_1 c_1 + \cdots + w_t c_t$. By linear independence $u + vp^s = 0$ and $(w_i - vp^{k_i}) c_i = 0$ for $1 \leq i \leq t$. Hence $w = v(p^{k_1} c_1 + \cdots + p^{k_t} c_t) = vc$, so $w \in \langle c \rangle$. Clearly, though, $c \in (\langle f \rangle + X) \cap C$, so we have the equality $(\langle f \rangle + X) \cap C = \langle c \rangle$. Substitute to get $\frac{G}{X} = \frac{A}{X} \oplus \frac{B}{X} \cong \frac{A}{X} \oplus \frac{G}{A}$. Now

$$\frac{A}{X} = \frac{\langle f \rangle + X}{X} \cong \frac{\langle f \rangle}{\langle f \rangle \cap X} \cong \frac{\langle f \rangle}{\langle p^{n_t+s} f \rangle} \cong \mathbb{Z}(p^{n_t+s}) \quad \text{and} \quad B/X \cong C/\langle c \rangle.$$

Thus, $G/X \cong \mathbb{Z}(p^{n_t+s}) \oplus (C/\langle c \rangle)$. ∎

Remark 3.7. A useful consequence of Lemma 3.6 is the following: Suppose that $s > k_t$. Then the groups C and $G/\langle x \rangle$ have no cyclic summands of the same order. Indeed, it follows from [6, Proposition 3.1] that C and

$C/\langle c\rangle$ have no cyclic summands in common and since the assumption that $s > k_t$ implies that $p^{n_t+s} > p^{n_t+k_t}$, the order of the largest summand of C, the result follows.

We need one further technical result before embarking on the final step in the proof of Theorem 3.1.

LEMMA 3.8. *If $M = T \oplus \langle f\rangle$, $x = c + p^s f$, $y = p^s f$ and s is greater than the heights of the components of c, then $M/\langle x\rangle \cong M/\langle y\rangle$ if, and only if, $c = 0$.*

Proof. Clearly if $c = 0$ then $M/\langle x\rangle \cong M/\langle y\rangle$. If $c \neq 0$, we show that the isomorphism $M/\langle x\rangle \cong M/\langle y\rangle$ can never hold. By Lemma 3.6 we have

$$M/\langle x\rangle = (C_1/\langle x\rangle) \oplus H \cong \mathbb{Z}(p^{n_t+s}) \oplus (C/\langle c\rangle) \oplus H \cong M/\langle y\rangle = \mathbb{Z}(p^s) \oplus C \oplus H.$$

Since finite cyclic groups also have the cancellation property, we conclude that $\mathbb{Z}(p^{n_t+s}) \oplus (C/\langle c\rangle) \cong \mathbb{Z}(p^s) \oplus C$. This is impossible: the order of the first term on the left-hand side above is greater than the maximum order $p^{n_t+k_t}$ of a summand of C since $s > k_t$ and also $n_t + s > s$ since $n_t > 0$. ∎

Since we were in the situation where $c \neq 0$, we conclude that $M/\langle x\rangle$ cannot be isomorphic to $M/\langle y\rangle$ if $d = 0$.

Thus to complete the proof of our main result of this section, Theorem 3.1, it suffices to consider the following situation:

- $M = T \oplus \langle f\rangle$, $T = C \oplus H = D \oplus K$ where C, D are given by:
  - (i) $C = \langle c_1\rangle \oplus \cdots \oplus \langle c_t\rangle$, where $o(c_i) = p^{n_i+k_i}$, $c = p^{k_1}c_1 + \cdots + p^{k_t}c_t$, $0 \leq k_1 < \cdots < k_t$, $0 < n_1 < \cdots < n_t$ and $x = c + p^s f$ where $s > n_t$;
  - (ii) $D = \langle d_1\rangle \oplus \cdots \oplus \langle d_u\rangle$, where $o(d_j) = p^{m_j+\ell_j}$, $d = p^{\ell_1}d_1 + \cdots + p^{\ell_u}d_u$, $0 \leq \ell_1 < \cdots < \ell_u$, $0 < m_1 < \cdots < m_u$ and $y = d + p^s f$ where $s > m_u$;
  - (iii) $M/\langle x\rangle \cong M/\langle y\rangle$.

Proof of Theorem 3.1. Now set $C_1 = C \oplus \langle f\rangle$, $D_1 = D \oplus \langle f\rangle$. Let $f_\alpha(-)$ denote the αth Ulm invariant of $(-)$. Then for all α we have

$$f_\alpha(T) = f_\alpha(C) + f_\alpha(H) = f_\alpha(D) + f_\alpha(K). \tag{1}$$

Also, $M/\langle x\rangle = (C_1 \oplus H)/\langle x\rangle = (C_1/\langle x\rangle) \oplus H \cong M/\langle y\rangle = (D_1/\langle y\rangle) \oplus K$.

So $f_\alpha(C_1/\langle x\rangle) + f_\alpha(H) = f_\alpha(D_1/\langle y\rangle) + f_\alpha(K)$. Since all the Ulm invariants under consideration are finite integers, we may substitute from (1) to obtain

$$f_\alpha(C_1/\langle x\rangle) + f_\alpha(D) = f_\alpha(D_1/\langle y\rangle) + f_\alpha(C). \tag{2}$$

Choose α such that $f_\alpha(C) = 1$. Then, as observed in Remark 3.7, $f_\alpha(C_1/\langle x\rangle) = 0$ and so for each such α, $f_\alpha(D) = f_\alpha(D_1/\langle y\rangle) + 1 \geq 1$. Hence D contains a summand isomorphic to C. Reversing roles and choosing β such that $f_\beta(D) = 1$, we see that C contains a summand isomorphic to D. Since

C, D are finite, we conclude that $C \cong D$. It is well known that finite Abelian groups have the cancellation property and so $H \cong K$. It now follows from (2) above that the Ulm invariants of $C_1/\langle x\rangle$ and $D_1/\langle y\rangle$ are equal and thus $C_1/\langle x\rangle \cong D_1/\langle y\rangle$.

Utilizing Lemma 3.6 we get $\mathbb{Z}(p^{n_t+s}) \oplus C/\langle c\rangle \cong \mathbb{Z}(p^{m_u+s}) \oplus D/\langle d\rangle$. However, by Remark 3.7, $\mathbb{Z}(p^{n_t+s})$, $\mathbb{Z}(p^{m_u+s})$ are, respectively, the maximum order subgroups on each side of the above isomorphism and hence they must be isomorphic. Applying cancellation again gives $C/\langle c\rangle \cong D/\langle d\rangle$. Since $C \cong D$, we can apply the argument in the torsion case, see [6, Theorem 3.2], to conclude that the Ulm sequences $U_T(c), U_T(d)$ are equal and so, as the module T is transitive in the sense of Kaplansky, there is an automorphism ϕ of T with $\phi(c) = d$. Clearly, ϕ extends trivially to an automorphism $\phi \oplus 1$ of M which maps $x \mapsto y$. Since x, y were arbitrary, we have established Theorem 3.1. ∎

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