STUDIA MATHEMATICA
Online First version

Necessary conditions for two-weight inequalities for singular integral operators

by

David Cruz-Uribe, OFS,
and John-Oliver MacLellan (Tuscaloosa, AL)

In memory of Benjamin Muckenhoupt and Richard Wheeden, who pioneered the study of two-weight inequalities

Abstract. We prove necessary conditions on pairs (μ, ν) of measures for a singular integral operator T to satisfy weak (p, p) inequalities, $1 \leq p < \infty$, provided the kernel of T satisfies a weak non-degeneracy condition first introduced by Stein (1993), and the measure μ satisfies a weak doubling condition related to the non-degeneracy of the kernel. We also show similar results for pairs (μ, σ) of measures for the operator $T_\sigma f = T(f d\sigma)$, which has come to play an important role in the study of weighted norm inequalities. Our major tool is a careful analysis of the strong type inequalities for averaging operators; these results are of interest in their own right. Finally, as an application of our techniques, we show that in general a singular operator does not satisfy the endpoint strong type inequality $T : L^1(\nu) \to L^1(\mu)$. Our results unify and extend a number of known results.

1. Introduction. The goal of this paper is to establish necessary conditions for two-weight, weak type inequalities for Calderón–Zygmund operators. This problem has a long history. In the one-weight case it is well known that if each of the Riesz transforms is of weak type (p, p) with respect to a weight w, then $w \in A_p$. See for example [4, Theorem 3.7, p. 417]. Stein [12, p. 210] showed that if any convolution type singular integral operator whose kernel satisfies a weak non-degeneracy condition is bounded on $L^p(w)$, then $w \in A_p$. The necessity of two-weight A_p for the weak (p, p) inequality for the Hilbert transform was established by Muckenhoupt and Wheeden [7].

In this paper we consider two versions of this problem. First suppose (μ, ν) is a pair of positive regular Borel measures on $\mathbb{R}^n$, where μ satisfies a weak doubling condition (see Definition 2.13) and T is a Calderón–Zygmund

2020 *Mathematics Subject Classification*: 42B20, 42B35.
Key words and phrases: singular integrals, averaging operators, A_p weights.
Received 30 April 2020; revised 1 February 2021.
Published online 7 June 2021.

DOI: 10.4064/sm200430-12-3

operator whose kernel satisfies a weak non-degeneracy condition (see Definition 2.4). We first prove the following result on the necessity of the two-weight A_p condition (see Definition 2.5).

THEOREM 1.1. *Let T be a Calderón–Zygmund operator with a non-degenerate kernel in the direction u_0. Suppose that for some $1 \le p < \infty$, and a pair (μ, ν) of positive regular Borel measures, with μ directionally doubling in the direction u_0,*

$$\|Tf\|_{L^{p,\infty}(\mu)} \le C\|f\|_{L^p(\nu)}. \tag{1.1}$$

Then:

(1) *$d\nu = d\nu_s + v dx$ where $v \in L^1_{\mathrm{loc}}$ and ν_s is singular;*
(2) *$\mu \ll \nu$, and $\mu \ll dx$, so $d\mu = u dx$ where $u \in L^1_{\mathrm{loc}}$;*
(3) *$(u, v) \in A_p$ and $u(x) \le Cv(x)$ a.e.*

The second version of the problem is to let (μ, σ) be a pair of positive regular Borel measures on $\mathbb{R}^n$, and consider the singular integral operator T_σ defined (formally) by

$$T_\sigma f(x) = T(f d\sigma)(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, d\sigma(y).$$

This approach to weighted norm inequalities first appeared implicitly in [8] for the maximal operator. We establish necessary conditions on (μ, σ) for T_σ to satisfy the weak type inequality

$$\mu(\{x : |T_\sigma f(x)| > \lambda\}) \le \frac{C}{\lambda^p} \int |f|^p\, d\sigma$$

for $1 < p < \infty$. (See Definition 2.2 for a more careful definition of this operator and the meaning of this inequality.) This problem was considered in [6], where the necessity of the A_p condition was proved for measures (see Definition 2.6) assuming a strong ellipticity condition. More precisely, it was assumed that there exists a family $\{K_j\}_{j=1}^N$ of kernels such that given any unit direction vector u, there exists j such that K_j satisfies a non-degeneracy condition in the direction u (see Definition 2.4). The authors of [9] were able to prove that the stronger PA_p condition (see Definition 2.10) is necessary with a similar hypothesis assuming a strong (p, p) inequality.

Our results are similar but we only assume that we have a single operator T_σ whose kernel is non-degenerate in one direction. We obtain the necessity of A_p under the additional hypothesis that μ satisfies a weak doubling condition related to the non-degeneracy condition (see Definition 2.13). We prove this result for completeness since it is a simple application of the techniques used to prove Theorem 1.1.

THEOREM 1.2. *Let (μ, σ) be a pair of positive regular Borel measures with μ directionally doubling in the direction u_0. Suppose the operator T_σ has a* *non-degenerate kernel in the direction* u_0*, and that for some* $1 < p < \infty$,

$$\|T_\sigma f\|_{L^{p,\infty}(\mu)} \leq C\|f\|_{L^p(\sigma)}. \tag{1.2}$$

Then $(\mu, \sigma) \in A_p$.

More importantly, we also establish the necessity of the PA_p condition with the additional assumption that σ is doubling.

THEOREM 1.3. *Let* (μ, σ) *be a pair of positive regular Borel measures with* μ *directionally doubling in the direction* u_0 *and* σ *doubling. Suppose* T_σ *has a non-degenerate kernel in the direction* u_0*, and that for some* $1 < p < \infty$,

$$\|T_\sigma f\|_{L^{p,\infty}(\mu)} \leq C\|f\|_{L^p(\sigma)}. \tag{1.3}$$

Then $(\mu, \sigma) \in PA_p$.

Finally as an application of our techniques we consider the question of whether a Calderón–Zygmund operator can be bounded from $L^1(\mu)$ to $L^1(\nu)$ for a pair (μ, ν) of positive Borel measures. If the operator under consideration is translation invariant and the measures μ, ν are regular, it is known that this is impossible. See [4, p. 468]. In [7, Theorem 4] Muckenhoupt and Wheeden derived a necessary condition for the Hilbert transform to be bounded from $L^1(v)$ to $L^1(u)$, where u and v are weights. We obtain an analogous estimate (see (5.3) in the proof of Theorem 1.4), but give a more complete characterization in terms of measures.

THEOREM 1.4. *Let* T *be a Calderón–Zygmund operator with a non-degenerate kernel in the direction* u_0*, and let* μ, ν *be positive Borel measures on* $\mathbb{R}^n$.

(1) *If* ν *is singular with respect to Lebesgue measure and* $T : L^1(\nu) \to L^1(\mu)$*, then* $\mu = 0$.
(2) *If* μ *is a regular measure with* $d\mu = d\mu_s + u\,dx$ *where* μ_s *is singular with respect to Lebesgue measure,* $u \not\equiv 0$*,* $d\nu = d\nu_s + v\,dx$ *where* ν_s *is singular with respect to Lebesgue measure, and* v *is a non-negative measurable function such that* $v(x) < \infty$ *a.e., then* T *is not bounded from* $L^1(\nu)$ *to* $L^1(\mu)$.
(3) *If* μ *is a regular measure that is singular with respect to Lebesgue measure, and directionally doubling in the direction* u_0*, and* ν *is a positive regular Borel measure, then* T *is not bounded from* $L^1(\nu)$ *to* $L^1(\mu)$.

REMARK 1.5. The following example shows that the hypothesis in (2) that $v(x) < \infty$ a.e. is needed. Let $d\mu = \chi_{[-1,1]}dx$ and $d\nu = \chi_{\mathbb{R}\setminus[-2,2]}dx + \infty \cdot \chi_{[-2,2]}dx$, and let $Tf(x) = Hf(x)$. Then (μ, ν) satisfies the key estimate

(5.3) below, and for f with $\operatorname{supp}(f) \subset \{x : |x| > 2\}$ we have

$$\int_{\mathbb{R}} |Hf(x)|\,d\mu(x) = \int_{-1}^{1} |Hf(x)|\,dx \le \int_{-1}^{1} \int_{|y|>2} \frac{|f(y)|}{|x-y|}\,dy\,dx$$
$$= \int_{|y|>2} \int_{-1}^{1} \frac{1}{|x-y|}\,dx\,|f(y)|\,dy \le 2 \int_{|y|>2} |f(y)|\,dy = 2\int_{\mathbb{R}} |f(y)|\,d\nu(y).$$

Remark 1.6. The following example shows that the hypothesis in (2) that μ is not totally singular with respect to Lebesgue measure is needed. Let $\mu = \delta_0$, $\nu = \frac{1}{x}dx$ and let $Tf(x) = Hf(x)$. Then for any $f \in L^1(\nu)$,

$$\int_{\mathbb{R}} |Hf(x)|\,d\mu(x) = |Hf(0)| = \left| \int_{\mathbb{R}} \frac{f(y)}{y}\,dy \right| \le \int_{\mathbb{R}} |f(y)|\,d\nu(y).$$

The main idea in our proofs is to reduce the problem of obtaining necessary conditions for the L^p boundedness of singular integrals to that of averaging operators (see Definitions 3.1 and 4.1). For singular integrals T, we work with the averaging operator A_Q. When $\mu = u dx$, $\nu = v dx$, it is well known that the A_p condition characterizes the strong type inequality for A_Q; see Jawerth [5]. For completeness we prove this result (see Theorem 3.2). Furthermore, we obtain a characterization of the strong type inequality for A_Q when μ, ν are positive regular Borel measures (see Theorem 3.3). This result is new and is interesting in its own right.

To study the singular integrals T_σ we introduce the analogous averaging operator $A_{Q,\sigma} f = A_Q(f d\sigma)$. We show that the A_p condition for measures is also necessary and sufficient (see Theorem 4.2) for these operators to be bounded, for $1 < p < \infty$.

The rest of the paper is organized as follows. In Section 2 we give preliminary definitions and notation used in this paper. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorems 1.2 and 1.3. Finally, in Section 5 we prove Theorem 1.4.

2. Preliminaries. Throughout this paper we will use the following notation. The symbol n will denote the dimension of the Euclidean space $\mathbb{R}^n$; $Q(x, r)$ denotes the cube with center $x \in \mathbb{R}^n$ and sidelength $2r$; and $B(x, r)$ denotes the ball with center $x \in \mathbb{R}^n$ and radius r. For a cube Q, rQ is the cube with the same center as Q and with side length r times the length of Q. Positive constants C, c may change value at each appearance. Sometimes we will indicate the dependence on certain parameters by writing for instance $C(n, p)$ etc. We will work extensively with average integrals and use the notation

$$\fint_Q u\,dx = \frac{1}{|Q|}\int_Q u\,dx.$$

We now define the singular integral operators we are interested in. For further details, see [2].

Definition 2.1. We say that an operator T defined on measurable functions is a *Calderón–Zygmund operator* if T is bounded on $L^2(\mathbb{R}^n)$, and for any $f \in L^2_c(\mathbb{R}^n)$ we have the representation

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy, \qquad x \notin \operatorname{supp}(f).$$

Here $K(x,y)$ is a kernel, defined for all $x \neq y$ in $\mathbb{R}^n \times \mathbb{R}^n$, that satisfies the standard estimates

$$|K(x,y)| \le \frac{C_0}{|x-y|^n} \tag{2.1}$$

and

$$|K(x+h,y)-K(x,y)| + |K(x,y+h)-K(x,y)| \le C_0\frac{|h|^\delta}{|x-y|^{n+\delta}} \tag{2.2}$$

for all $|h| < \frac{1}{2}|x-y|$ and some fixed $\delta > 0$.

We want to define the operators T_σ more carefully; to do so we follow the treatment given in [10]. Let (μ,σ) be a pair of regular Borel measures. Fix a Calderón–Zygmund operator T with kernel K. Let $\{\eta_{\epsilon,R}\}_{0<\epsilon<R<\infty}$ be a family of non-negative truncation functions such that, for every $0 < \epsilon < R < \infty$, the function $\eta_{\epsilon,R}$ has support in the annulus $\epsilon < |x| < R$, and $\eta_{\epsilon,R}(x) = 1$ if $2\epsilon < |x| < R/2$. For example, we can take $\eta_{\epsilon,R} = \chi_{\{\epsilon<|x|<R\}}$, but other choices are possible. Define the family of truncated kernels $K_{\epsilon,R}(x,y) = \eta_{\epsilon,R}(x-y)K(x,y)$. These are bounded with compact support for a fixed x or y. Thus, the truncated operators defined by

$$T_\sigma^{\epsilon,R}f(x) = \int_{\mathbb{R}^n} K_{\epsilon,R}(x,y)f(y)\,d\sigma(y), \quad x \in \mathbb{R}^n,$$

are pointwise well defined for $f \in L^1_{\mathrm{loc}}(\sigma)$. Hereafter, we will assume that each of the truncated kernels $\{K_{\epsilon,R}\}_{0<\epsilon<R<\infty}$ satisfies the standard kernel estimates (2.1) and (2.2) with uniform constants.

Definition 2.2. Given a Calderón–Zygmund operator T with kernel K, we say that T_σ satisfies the *weak* (p,p) *inequality*, $1 < p < \infty$, provided that there exists a family $\{\eta_{\epsilon,R}\}_{0<\epsilon<R<\infty}$ of truncations such that for all $f \in L^p(\sigma)$ and $0 < \epsilon < R < \infty$,

$$\|T_\sigma^{\epsilon,R}f\|_{L^{p,\infty}(\mu)} \le C\|f\|_{L^p(\sigma)} \tag{2.3}$$

with constant independent of ϵ and R. In this case we write

$$\|T_\sigma f\|_{L^{p,\infty}(\mu)} \leq C\|f\|_{L^p(\sigma)}.$$

Given this definition, in our proofs below we will need to fix particular values of ϵ and R and apply inequality (2.3). We will, however, generally write T_σ instead of $T_\sigma^{\epsilon,R}$ when there is no possibility of confusion.

Remark 2.3. While we need to fix a family of truncations to define T_σ, the choice is less important than it might seem at first. In [10], it is shown that if the pair (μ,σ) satisfies the A_2 condition for measures, (2.5) below, then the corresponding strong $(2,2)$ inequality for T_σ holds independent of the choice of truncations used. A similar argument shows the same is true for weak (p,p) inequalities.

Definition 2.4. For a Calderón–Zygmund operator T with kernel $K(x,y)$, we say T has a *non-degenerate kernel* if there exists $a>0$ and a unit vector u_0 such that for $x,y\in\mathbb{R}^n$, $x-y=tu_0$, $t\in\mathbb{R}$,

$$|K(x,y)| \geq \frac{a}{|x-y|^n}. \tag{2.4}$$

For example, (2.4) holds for the Hilbert transform as well as for any of the Riesz transforms in the direction e_j. However, not all singular integrals satisfy this property. See for example [1, Lemma 1.4] where a "one-sided" Calderón–Zygmund kernel with support in $(0,\infty)$ is constructed; it is established that a sufficient condition for this operator to be bounded is a "one-sided" A_p condition that is strictly weaker than the conditions we consider.

Definition 2.5. Let u, v be non-negative, measurable functions. We say the pair (u,v) is in A_p, $1<p<\infty$, if

$$[u,v]_{A_p} = \sup_Q \left(\fint_Q u\,dx\right)\left(\fint_Q v^{1-p'}\,dx\right)^{p-1} < \infty, \tag{2.5}$$

and in A_1 if

$$\fint_Q u\,dx \leq [u,v]_{A_1} \operatorname*{ess\,inf}_{x\in Q} v(x). \tag{2.6}$$

Definition 2.6. If μ,σ are positive Borel measures, we say that (μ,σ) is in A_p, $1<p<\infty$, if

$$[\mu,\sigma]_{A_p} = \sup_Q \frac{\mu(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^{p-1} < \infty. \tag{2.7}$$

Remark 2.7. If $d\mu = udx$, $d\sigma = v^{1-p'}dx$, then the condition (2.7) is equivalent to (2.5).

Remark 2.8. It is straightforward to see that if (2.5), (2.6), or (2.7) hold for any cube $Q \subset \mathbb{R}^n$, then they also hold for any ball $B \subset \mathbb{R}^n$. We will use this fact below.

Remark 2.9. Inequality (2.7) implies that μ, σ do not share a common point mass: if there exists a point a such that $\sigma\{a\}\mu\{a\} > 0$, then the expression in (2.7) blows up as Q shrinks to $\{a\}$.

Definition 2.10. We say the pair (μ, σ) is in PA_p, $1 < p < \infty$, if for any cube $Q(y_0, r)$,

$$\frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\left(\int_{\mathbb{R}^n}\left(\frac{r^{p'-1}}{(|x-y_0|+r)^{p'}}\right)^n d\sigma(x)\right)^{p-1} \leq C. \tag{2.8}$$

Remark 2.11. It is not difficult to show that inequality (2.8) can be written in an equivalent form that is closer in spirit to the A_p condition:

$$\sup_Q |Q|^{-1}\|\chi_Q\|_{L^p(\mu)}\|M\chi_Q\|_{L^{p'}(\sigma)} < \infty. \tag{2.9}$$

We thank the referee for calling our attention to this fact.

This condition first appeared in [7] in one dimension where those authors proved that it was necessary for the strong type inequality for the Hilbert transform to hold. The n-dimensional version first appeared in [11] in the context of the fractional integral operator. When $p = 2$, this condition is sometimes called "Poisson A_2". This is because the second term on the left-hand side of (2.8) is approximately the Poisson extension of σ evaluated at a point in the upper half-plane given by y_0 and r. It is straightforward to see that the PA_p condition implies the A_p condition.

Definition 2.12. A positive measure μ is said to be *doubling* if there exists a constant $C > 0$ such that for any cube Q, $\mu(2Q) \leq C\mu(Q)$.

Equivalently μ is doubling if $\mu(P) \leq C\mu(Q)$ whenever P, Q are adjacent cubes with $|Q| = |P|$. (We say the cubes P, Q are *adjacent* if the boundaries of P and Q share a point in common.)

For our results we do not need to assume the full doubling condition, but rather a "directional" doubling condition.

Definition 2.13. Let μ be a positive Borel measure, and fix a unit vector u_0. We say μ is *directionally doubling* in the direction u_0 if there exists a constant $C_\mu > 0$ such that given adjacent cubes $P(x_0, r)$, $Q(y_0, r)$ whose centers satisfy $x_0 - y_0 = tu_0$, $t \in \mathbb{R}$, we have

$$\mu(P(x_0,r)) \leq C_\mu \mu(Q(y_0,r)). \tag{2.10}$$

Remark 2.14. Definition 2.13 is weaker than the doubling condition. For example, for $E \subset \mathbb{R}^2$, define $\mu(E) = \iint_E e^{-|x|}\,dx\,dy$. Then it is straightforward to show that μ is directionally doubling in the direction e_2 but is not doubling.

3. Proof of Theorem 1.1. In order to proceed with the proof of our first main result we will need to prove some preliminary results about averaging operators.

Definition 3.1. Given a cube Q, define the *averaging operator* A_Q on a function $f \in L^1_{\text{loc}}$ by

$$A_Q f(x) = \fint_Q f(y)\, dy\, \chi_Q(x).$$

The following result first appeared in [5] but to the best of our knowledge a proof does not appear in the literature. We provide the proof for completeness.

Theorem 3.2. *Given a cube Q, $1 \le p < \infty$, and $(u,v) \in A_p$, for all $f \in L^p(v)$ we have*

$$\|A_Q f\|_{L^p(u)} \le [u,v]_{A_p}^{1/p} \|f\|_{L^p(v)}.$$

Conversely, given $1 \le p < \infty$, if (u,v) is a pair of weights such that for every cube Q,

$$\|A_Q f\|_{L^p(u)} \le K \|f\|_{L^p(v)}, \tag{3.1}$$

then $(u,v) \in A_p$. Moreover, $[u,v]_{A_p} \le K^p$.

Proof. Let $Q \subset \mathbb{R}^n$. We first prove the sufficiency of the A_1 condition when $p = 1$. Indeed,

$$\|A_Q f\|_{L^1(u)} = \int_{\mathbb{R}^n} \left| \fint_Q f\, dy\, \chi_Q(x) \right| u\, dx \le \int_Q \frac{u(Q)}{|Q|} |f|\, dy \le [u,v]_{A_1} \int_{\mathbb{R}^n} |f| v\, dy.$$

If $p > 1$, then by Hölder's inequality,

$$\begin{aligned} \|A_Q\|^p_{L^p(u)} &= \int_{\mathbb{R}^n} \left| \fint_Q f\, dy\, \chi_Q(x) \right|^p u\, dx \le \Big(\fint_Q |f| v^{1/p} v^{-1/p}\, dy \Big)^p u(Q) \\ &\le \Big(\int_Q |f|^p v\, dy \Big) \Big(\fint_Q u\, dy \Big) \Big(\fint_Q v^{1-p'}\, dy \Big)^{p-1} \le [u,v]_{A_p} \int_{\mathbb{R}^n} |f|^p v\, dy. \end{aligned}$$

To prove necessity we again first consider the case $p = 1$. Fix a cube Q, and let $S \subset Q$ be measurable. Then applying (3.1) to the function χ_S we get

$$u(Q) \Big(\frac{|S|}{|Q|} \Big) \le K v(S).$$

This can be rewritten as

$$\frac{1}{|Q|}\int_Q u\,dy \le K\frac{1}{|S|}\int_S v\,dy. \tag{3.2}$$

Let $a = \operatorname{ess\,inf}_{x\in Q} v(x)$. Then for each $\epsilon > 0$ there exists $S_\epsilon \subset Q$ with $|S_\epsilon| > 0$ such that $v(x) < a + \epsilon$ for any $x \in S_\epsilon$. Moreover, (3.2) holds with $S = S_\epsilon$. Therefore,

$$⨍_Q u\,dy \le K(a+\epsilon) = K\Big(\operatorname*{ess\,inf}_{x\in Q} v(x) + \epsilon\Big). \tag{3.3}$$

Since $\epsilon > 0$ was arbitrary, (3.3) holds with $\epsilon = 0$. It then follows that $(u, v) \in A_1$ and $[u, v]_{A_1} \le K$.

Now suppose that $p > 1$. In this case we fix a cube Q and let $f = v^{1-p'}\chi_Q$. A priori, f need not be not locally integrable, so we consider the sequence $f_n = \min(v^{1-p'}\chi_Q, n)$. If $f_n = 0$ for all n, then $\int_Q v^{1-p'}\,dy = 0$, and the A_p condition holds. Otherwise, if $\int_Q f_n^p v\,dy > 0$, then we apply (3.1) to get

$$u(Q)\Big(⨍_Q f_n\,dy\Big)^p \le K^p \int_Q {f_n}^p v\,dy.$$

Since the right-hand side is finite and positive, we get

$$\Big(⨍_Q u\,dy\Big)\Big(⨍_Q f_n\,dy\Big)^p\Big(⨍_Q {f_n}^p v\,dy\Big)^{-1} \le K^p.$$

By the monotone convergence theorem,

$$\Big(⨍_Q u\,dy\Big)\Big(⨍_Q v^{1-p'}\,dy\Big)^{p-1} \le K^p.$$

It follows that $(u, v) \in A_p$ and $[u, v]_{A_p} \le K^p$. ■

We now prove an analogue of Theorem 3.2 for measures.

THEOREM 3.3. *Let (μ,ν) be a pair of positive regular Borel measures. Given $1 \le p < \infty$, suppose that there exists a constant C such that for every cube Q,*

$$\|A_Q f\|_{L^p(\mu)} \le C\|f\|_{L^p(\nu)}. \tag{3.4}$$

Then:

(1) *$d\nu = d\nu_s + v dx$ where $v \in L^1_{\mathrm{loc}}$ and ν_s is singular;*
(2) *$\mu \ll \nu$, and $\mu \ll dx$, so $d\mu = u\,dx$ where $u \in L^1_{\mathrm{loc}}$;*
(3) *$(u, v) \in A_p$ and $u(x) \le Cv(x)$ a.e.*

Proof. Fix a cube $Q \subset \mathbb{R}^n$ and let $S \subset Q$ be measurable. Applying (3.4) to the function χ_S we obtain

$$\left(\frac{|S|}{|Q|}\right)^p \mu(Q) \le C\nu(S). \tag{3.5}$$

Suppose ν were singular with respect to Lebesgue measure and $|S| > 0$. Then there exists a set $A \subset \mathbb{R}^n$ such that $|A| = 0$ and $\nu(S) = \nu(S \cap A)$. If we replace S with $S \setminus A$ in (3.5), we find $\mu(Q) = 0$ for any cube $Q \supset S$. This implies that $\mu = 0$. Hence, $d\nu = d\nu_s + v\,dx$ where $v \in L^1_{\text{loc}}$ $v \neq 0$, and ν_s is singular.

Now fix any set $S \subset Q$ with $\nu(S) = 0$. Since ν is regular, for any $\epsilon > 0$ there exists an open set $E \supset S$ such that $\nu(E) < \epsilon$. Since E is open, $E = \bigcup_j Q_j$ where $\{Q_j\}$ is a disjoint collection of dyadic cubes. If we let $Q = S = Q_j$ in (3.5), we have

$$\mu(S) \le \mu(E) = \sum_j \mu(Q_j) \le C \sum_j \nu(Q_j) = C\nu(E) < C\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $\mu(S) = 0$, and so $\mu \ll \nu$.

We can now write (3.4) as

$$\mu(Q)\left|\fint_Q f\,dx\right|^p \le C\Big(\int_{\mathbb{R}^n} |f|^p\,d\nu_s + \int_{\mathbb{R}^n} |f|^p v\,dx\Big). \tag{3.6}$$

Let $A = \operatorname{supp}(\nu_s)$. Since $|A| = 0$, if we set $f = \chi_{S\setminus A}$, we have

$$\left(\frac{|S|}{|Q|}\right)^p \mu(Q) \le Cv(S). \tag{3.7}$$

Using the same argument that showed $\mu \ll \nu$, replacing ν with v, we can see that $\mu \ll vdx \ll dx$. Hence, $d\mu = udx$ for some $u \in L^1_{\text{loc}}$. Then (3.7) can be written as

$$\left(\frac{|S|}{|Q|}\right)^p u(Q) \le Cv(S). \tag{3.8}$$

Let $S = Q$ in (3.8). By the Lebesgue differentiation theorem we deduce that $u(x) \le Cv(x)$ a.e.

Finally notice that the left-hand side of (3.6) is unchanged if we modify f on a set of Lebesgue measure zero. Indeed, we have

$$u(Q)\left|\fint_Q f\,dx\right|^p \le C\Big(\int_{\mathbb{R}^n} |f|^p v\,dx\Big). \tag{3.9}$$

Hence $(u, v) \in A_p$ by Theorem 3.2. ■

Proof of Theorem 1.1. We will show that given any cube Q, the averaging operator satisfies $A_Q : L^p(\nu) \to L^p(\mu)$. The desired conclusion then follows from Theorem 3.3. Choose a constant $t \ge 4$ such that $2C_0(1 + 2^{n+\delta})t^{-\delta} \le a$.

Here, a is the constant in (2.4), and δ, C_0 are as in (2.2). We further require $t = NC_2/\sqrt{n}$, where $1/\sqrt{n} \le C_2 \le 1$ and N is an integer. The exact choice of the constant C_2 will be made clear below. Let x_0, y_0 be two points satisfying $x_0 - y_0 = tr\sqrt{n}\, u_0$, $r > 0$, and consider the cubes $Q(x_0, r)$, $Q(y_0, r)$. Given any point $x \in Q(x_0, r)$ we can write $x = x_0 + h$, where $|h| < r\sqrt{n}$. Similarly, given $y \in Q(y_0, r)$, we have $y = y_0 + k$ where $|k| < r\sqrt{n}$. We claim that for such x and y,

$$|K(x,y) - K(x_0,y_0)| \le \frac{1}{2}|K(x_0,y_0)|. \tag{3.10}$$

To prove this we will apply (2.2), which is possible since $|h| < r\sqrt{n} \le \frac{1}{2}|x_0 - y_0|$, and

$$|x_0 + h - y_0| \ge |x_0 - y_0| - |h| \ge tr\sqrt{n} - r\sqrt{n} \ge \frac{t}{2} r\sqrt{n} \ge 2|k|.$$

Thus, we can estimate as follows:

$$\begin{aligned} |K(x,y) - K(x_0,y_0)| &\le |K(x_0+h, y_0+k) - K(x_0+h, y_0)| \\ &\quad + |K(x_0+h, y_0) - K(x_0,y_0)| \\ &\le \frac{C_0|k|^\delta}{|x_0+h-y_0|^{n+\delta}} + \frac{C_0|h|^\delta}{|x_0-y_0|^{n+\delta}} = I_1 + I_2. \end{aligned} \tag{3.11}$$

We can bound I_2 immediately:

$$I_2 \le \frac{C_0(r\sqrt{n})^\delta}{(tr\sqrt{n})^\delta|x_0-y_0|^n} = C_0\frac{t^{-\delta}}{|x_0-y_0|^n}.$$

To estimate I_1, note that

$$|x_0 + h - y_0| \ge \frac{t}{2} r\sqrt{n} = \frac{1}{2}|x_0 - y_0|.$$

Hence,

$$I_1 \le \frac{C_0 2^{n+\delta}(r\sqrt{n})^\delta}{(tr\sqrt{n})^\delta|x_0-y_0|^n} = C_0\frac{2^{n+\delta}t^{-\delta}}{|x_0-y_0|^n}.$$

If we combine these estimates, by our choice of t and (2.4) we have

$$I_1 + I_2 \le \frac{a}{2}\,\frac{1}{|x_0-y_0|^n} \le \frac{1}{2}|K(x_0,y_0)|,$$

which proves (3.10).

It now follows that for any $x \in Q(x_0, r)$, $y \in Q(y_0, r)$, the kernel $K(x,y)$ always has the same sign. Therefore, if we fix a non-negative function f with

$\operatorname{supp}(f) \subset Q(x_0, r)$, then

$$|Tf(y)| = \Big| \int_{Q(x_0,r)} K(x,y) f(x)\, dx \Big| = \int_{Q(x_0,r)} |K(x,y)| f(x)\, dx \tag{3.12}$$
$$\geq \int_{Q(x_0,r)} |K(x_0,y_0)| f(x)\, dx - \int_{Q(x_0,r)} |K(x,y) - K(x_0,y_0)| f(x)\, dx,$$

again by (3.10) and (2.4),

$$\geq \frac{1}{2}|K(x_0,y_0)| \int_{Q(x_0,r)} f(x)\, dx \geq \frac{a}{2|x_0 - y_0|^n} \int_{Q(x_0,r)} f(x)\, dx$$
$$\geq \frac{a}{2(tr\sqrt{n})^n} \int_{Q(x_0,r)} f(x)\, dx = c(a,t,n) \fint_{Q(x_0,r)} f(x)\, dx.$$

Given this inequality and the assumption that T satisfies a weak (p,p) inequality, for any $0 < \lambda < c(a,t,n) \fint_{Q(x_0,r)} f\, dx$ we have

$$\mu(Q(y_0,r)) \leq \mu(\{x : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda^p} \int_{Q(x_0,r)} |f|^p\, d\nu.$$

If we take the supremum over all such λ, we get

$$\mu(Q(y_0,r)) \Big(\fint_{Q(x_0,r)} f\, dx \Big)^p \leq c(a,t,n,p) \int_{Q(x_0,r)} |f|^p\, d\nu. \tag{3.13}$$

Now fix a value of C_2, depending only on u_0, so that starting from $Q(y_0,r)$ we can form a chain of adjacent cubes $Q(x_j,r)$, $j = 1, \dots, N$, in the direction $-u_0$ such that $x_1 = y_0$ and $x_N = x_0$. Each $Q(x_j,r)$ satisfies $\mu(Q(x_{j+1},r)) \leq C_\mu \mu(Q(x_j,r))$, where C_μ is the directional doubling constant from (2.10). The number N of cubes, lying between $Q(y_0,r)$ and $Q(x_0,r)$ depends only on t and n. Thus, there exists a constant $C = C(C_\mu, t, n)$ such that $\mu(Q(x_0,r)) \leq C\mu(Q(y_0,r))$. Hence,

$$\mu(Q(x_0,r)) \Big(\fint_{Q(x_0,r)} f\, dx \Big)^p \leq C\mu(Q(y_0,r)) \Big(\fint_{Q(x_0,r)} f\, dx \Big)^p \leq C \int_{Q(x_0,r)} |f|^p\, d\nu.$$

Since the resulting constant depends only on C_2, C_μ, p, t, n, a and not on $Q(x_0,r)$, we have shown that the averaging operators A_Q map $L^p(\nu)$ to $L^p(\mu)$ uniformly for all Q. Therefore, by Theorem 3.3 we get the desired conclusion. ■

4. Proofs of Theorems 1.2 and 1.3. Before proceeding with the proof of Theorem 1.2 we first define the related averaging operator.

DEFINITION 4.1. Given a positive measure σ and a cube Q, define the *averaging operator* $A_{Q,\sigma}$ acting on a function $f \in L^1_{\mathrm{loc}}(\sigma)$ by

$$A_{Q,\sigma}f(x) = \frac{1}{|Q|}\int_Q f(y)\,d\sigma(y)\,\chi_Q(x). \tag{4.1}$$

The following result characterizes the L^p boundedness of $A_{Q,\sigma}$.

THEOREM 4.2. *Given a cube Q, $1 \le p < \infty$, and a pair (μ,σ) of positive regular Borel measures, suppose that (μ,σ) satisfies the A_p condition* (2.7)*. Then for all $f \in L^p(\sigma)$,*

$$\|A_{Q,\sigma}f\|_{L^p(\mu)} \le [\mu,\sigma]_{A_p}^{1/p}\|f\|_{L^p(\sigma)}.$$

Conversely, given $1 \le p < \infty$, if (μ,σ) is a pair of positive regular Borel measures such that for every cube Q,

$$\|A_{Q,\sigma}f\|_{L^p(\mu)} \le K\|f\|_{L^p(\sigma)}, \tag{4.2}$$

then $(\mu,\sigma) \in A_p$. Moreover $[\mu,\sigma]_{A_p} \le K^p$.

Proof. We first prove necessity. Fix a cube Q and let $1 \le p < \infty$. If $\sigma(Q) = 0$, then (2.7) is immediate. If $\sigma(Q) > 0$, let $f = \chi_Q$ in (4.2). Then

$$\mu(Q)\left(\frac{\sigma(Q)}{|Q|}\right)^p \le K^p\sigma(Q).$$

Dividing by $\sigma(Q)$ and taking the supremum over all cubes Q we find that $(\mu,\sigma) \in A_p$ and $[\mu,\sigma]_{A_p} \le K^p$. The proof of sufficiency is similar to that of Theorem 3.2 so we omit the details. ■

Proof of Theorem 1.2. The proof is a straightforward modification of the proof of Theorem 1.1. Fix the cubes $Q(x_0,r)$ and $Q(y_0,r)$ as before. Then with the same notation as before, if $x \in Q(x_0,r)$ and $y \in Q(y_0,r)$, then

$$|x-y| = |x_0 - y_0 + h - k| < tr\sqrt{n} + 2r\sqrt{n} < 2tr\sqrt{n}.$$

Similarly, we have $|x-y| > \frac{1}{2}tr\sqrt{n}$. Therefore, if we choose $0 < \epsilon < \frac{1}{4}tr\sqrt{n}$ and $R > 4tr\sqrt{n}$, then the kernel $K_{\epsilon,R}(x,y)$ equals $K(x,y)$ and so satisfies the non-degeneracy condition (2.4) with a uniform constant. We also see that it satisfies the standard estimates (2.1) and (2.2).

For simplicity, we now write T_σ instead of $T_\sigma^{\epsilon,R}$ and K for $K_{\epsilon,R}$. If we repeat the previous argument, we find that K satisfies the estimate (3.10). We can then repeat the proof of (3.12), using the fact that T_σ satisfies the weak (p,p) inequality with uniform constant, to get

$$|T_\sigma f(y)| \ge \frac{c(a,t,n)}{|Q(x_0,r)|}\int_{Q(x_0,r)} f(x)\,d\sigma(x).$$

Given this inequality we continue to argue as we did in the proof of Theorem 1.1 to deduce that the averaging operator $A_{Q,\sigma} = A_{Q(x_0,r),\sigma}$ maps

$L^p(\sigma)$ to $L^p(\mu)$. This estimate holds for every cube $Q(x_0,r)$ with constants independent of ϵ and R, and so $(\mu,\sigma)\in A_p$ by Theorem 4.2. ■

Proof of Theorem 1.3. We adapt the proof of Theorem 1.1, exchanging the roles of x_0 and y_0. Fix a cube $Q(y_0,r)$. Choose $t\geq 4$ as in the proof of Theorem 1.1. Rather than considering the cube $Q(x_0,r)$ we replace it with a ball. Fix $S>r$ and for each $r\leq s\leq S$ let $B_s=B(x_s,s\sqrt{n})$, where $x_s=y_0+ts\sqrt{n}\,u_0$. If we now argue as we did in the proof of Theorem 1.2, if we fix $R>4tS\sqrt{n}$ and $\epsilon<\frac{1}{4}tr\sqrt{n}$, then for $y\in Q(y_0,r)$ and $x\in B_s$, $K_{\epsilon,R}$ satisfies the non-degeneracy condition (2.4) with a uniform constant. We also see that it satisfies the standard estimates (2.1) and (2.2). Again, we will write T_σ for $T_\sigma^{\epsilon,R}$ and K for $K_{\epsilon,R}$.

We can now argue as follows: For all $y\in Q(y_0,r)$, $y=y_0+k$ where $|k|\leq r\sqrt{n}$, and for $x\in B_s$, $x=x_s+h$ where $|h|\leq s\sqrt{n}$. As in the proof of Theorem 1.1 we have $|h|\leq s\sqrt{n}\leq\frac{1}{2}|x_s-y_0|$, and

$$|x_s+h-y_0|\geq|x_s-y_0|-|h|\geq ts\sqrt{n}-s\sqrt{n}\geq\frac{t}{2}s\sqrt{n}\geq\frac{t}{2}r\sqrt{n}\geq 2|k|.$$

We can now apply (2.2) as in estimate (3.11) to find that for $y\in Q(y_0,r)$ and $x\in B_s$,

$$|K(x,y)-K(x_s,y_0)|\leq\frac{1}{2}|K(x_s,y_0)|. \tag{4.3}$$

This implies that for any $y\in Q(y_0,r)$ and $x\in B_s$, $K(x,y)$ always has the same sign. Moreover,

$$|K(x,y)|\geq\frac{1}{2}|K(x_s,y_0)|. \tag{4.4}$$

Therefore,

$$\begin{aligned}|K(x,y)|\geq\frac{1}{2}|K(x_s,y_0)|&\geq\frac{a}{2}\frac{1}{|x_s-y_0|^n}\geq\frac{a}{2}\frac{1}{(|x-x_s|+|x-y_0|)^n}\\&\geq c(a,n)\frac{1}{(|x-y_0|)^n}\geq c(a,n)\frac{1}{(|x-y_0|+r)^n};\end{aligned} \tag{4.5}$$

the second-to-last inequality follows since $|x-x_s|\leq s\sqrt{n}\leq ts\sqrt{n}=|x-y_0|$.

Define the truncated cone

$$C_r=\bigcup_{s\geq r}B_s.$$

Notice C_r has central axis y_0+su_0 for $s\geq r$. For $S>0$ let

$$f_{r,S}(x)=\left(\frac{1}{|x-y_0|+r}\right)^{n(p'-1)}\chi_{C_r\cap B(y_0,S)}.$$

Then for all $y \in Q(y_0, r)$ we have

$$
(4.6) \qquad |T_\sigma f_{r,S}(y)| = \int_{C_r \cap B(y_0,S)} |K(x,y)| \left(\frac{1}{(|x-y_0|+r)} \right)^{n(p'-1)} d\sigma(x)
$$

$$
\geq c(a,n) \int_{C_r \cap B(y_0,S)} \left(\frac{1}{(|x-y_0|+r)^{p'}} \right)^{n} d\sigma(x).
$$

We observe that T_σ satisfies the weak (p,p) inequality with uniform constant, so we can argue as we did to derive (3.13) in the proof of Theorem 1.1 to get

$$
\mu(Q(y_0,r)) \left(\int_{C_r \cap B(y_0,S)} \left(\frac{1}{(|x-y_0|+r)^{p'}} \right)^{n} d\sigma(x) \right)^{p}
$$

$$
\leq C \int_{C_r \cap B(y_0,S)} \left(\frac{1}{|x-y_0|+r} \right)^{np(p'-1)} d\sigma(x).
$$

Since $p(p'-1) = p'$, we have

$$
\frac{\mu(Q(y_0,r))}{|Q(y_0,r)|} \left(\int_{C_r \cap B(y_0,S)} \left(\frac{r^{p'-1}}{(|x-y_0|+r)^{p'}} \right)^{n} d\sigma(x) \right)^{p-1} \leq C.
$$

Since the constant C is independent of ϵ and R, and so of S, we can take the limit as $S \to \infty$, and by the monotone convergence theorem we get

$$
(4.7) \qquad \frac{\mu(Q(y_0,r))}{|Q(y_0,r)|} \left(\int_{C_r} \left(\frac{r^{p'-1}}{(|x-y_0|+r)^{p'}} \right)^{n} d\sigma(x) \right)^{p-1} \leq C.
$$

We will now extend inequality (4.7) to all of $\mathbb{R}^n$. Let

$$
A_k = B(y_0, 2^{k+1} tr\sqrt{n}) \setminus B(y_0, 2^k tr\sqrt{n}).
$$

Consider the ball $B(x_k, 2^{k+2} tr\sqrt{n})$, where

$$
x_k = y_0 + \frac{2^{k+1} + 2^k}{2} tr\sqrt{n}\, u_0 = y_0 + \tfrac{3}{8} \cdot 2^{k+2} tr\sqrt{n}\, u_0.
$$

This is the ball of radius $2^{k+2} tr\sqrt{n}$ centered at the midpoint of the portion of the central axis of C_r that lies inside A_k. We claim $A_k \subset B(x_k, 2^{k+2} tr\sqrt{n})$. To see this, fix $x \in A_k$; then

$$
|x - x_k| \leq |x - y_0| + |x_k - y_0| \leq 2^{k+1} tr\sqrt{n} + \tfrac{3}{8} \cdot 2^{k+2} tr\sqrt{n} \leq 2^{k+2} tr\sqrt{n}.
$$

Since $B\big(x_k, \frac{3}{8} \cdot 2^{k+2} r\sqrt{n}\big)$ is one of the balls B_s that define C_r, it is immediate that

$$
\bigcup_{k=0}^{\infty} B\big(x_k, \tfrac{3}{8} \cdot 2^{k+2} r\sqrt{n}\big) \subset C_r.
$$

Since σ is doubling, there exists a constant $C = C(t, n, \sigma)$ such that

$$\sigma(B(x_k, 2^{k+2}tr\sqrt{n})) \le C\sigma\big(B\big(x_k, \tfrac{3}{8}\cdot 2^{k+2}r\sqrt{n}\big)\big).$$

Hence, we can estimate as follows:

$$\begin{aligned}
&\frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\Bigg(\int\limits_{\mathbb{R}^n\setminus B(y_0,tr\sqrt{n})}\bigg(\frac{r^{p'-1}}{(|x-y_0|+r)^{p'}}\bigg)^n d\sigma(x)\Bigg)^{p-1}\\
&\quad= \frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\Bigg(\sum_{k=0}^{\infty}\int\limits_{A_k}\bigg(\frac{r^{p'-1}}{(|x-y_0|+r)^{p'}}\bigg)^n d\sigma(x)\Bigg)^{p-1}\\
&\quad\le \frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\Bigg(\sum_{k=0}^{\infty}\bigg(\frac{r^{p'-1}}{(2^k tr\sqrt{n}+r)^{p'}}\bigg)^n \sigma(A_k)\Bigg)^{p-1}\\
&\quad\le \frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\Bigg(\sum_{k=0}^{\infty}\bigg(\frac{r^{p'-1}}{(2^k tr\sqrt{n}+r)^{p'}}\bigg)^n \sigma(B(x_k,2^{k+2}tr\sqrt{n}))\Bigg)^{p-1}\\
&\quad\le C\frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\Bigg(\sum_{k=0}^{\infty}\bigg(\frac{r^{p'-1}}{(2^k tr\sqrt{n}+r)^{p'}}\bigg)^n \sigma(B(x_k,\tfrac{3}{8}\cdot 2^{k+2}r\sqrt{n}))\Bigg)^{p-1}\\
&\quad\le C\frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\Bigg(\sum_{k=0}^{\infty}\int\limits_{B(x_k,\frac{3}{8}(2^{k+2})r\sqrt{n})}\bigg(\frac{r^{p'-1}}{(|x-y_0|+r)^{p'}}\bigg)^n d\sigma(x)\Bigg)^{p-1}\\
&\quad\le C\frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\Bigg(\int\limits_{C_r}\bigg(\frac{r^{p'-1}}{(|x-y_0|+r)^{p'}}\bigg)^n d\sigma(x)\Bigg)^{p-1} \le C.
\end{aligned}$$

The third-to-last inequality holds since $2^k tr\sqrt{n} \ge \frac{1}{2}|x-y_0|$ for any x in $B\big(\frac{3}{8}\cdot 2^{k+2}r\sqrt{n}\big)$.

By Remark 2.8, we can apply the result of Theorem 1.2 to the ball $B(y_0, tr\sqrt{n})$ to get

$$\begin{aligned}
&\frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\Bigg(\int\limits_{B(y_0,tr\sqrt{n})}\bigg(\frac{r^{p'-1}}{(|x-y_0|+r)^{p'}}\bigg)^n d\sigma(x)\Bigg)^{p-1}\\
&\qquad\qquad\le \frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\bigg(\frac{\sigma(B(y_0,tr\sqrt{n}))}{|B(y_0,tr\sqrt{n})|}\bigg)^{p-1}\\
&\qquad\qquad\le \frac{\mu(B(y_0,tr\sqrt{n}))}{|B(y_0,tr\sqrt{n})|}\bigg(\frac{\sigma(B(y_0,tr\sqrt{n}))}{|B(y_0,tr\sqrt{n})|}\bigg)^{p-1} \le C.
\end{aligned}$$

If we combine this inequality with the previous estimate, we get

$$\frac{\mu(Q(y_0,r))}{|Q(y_0,r)|}\Bigg(\int\limits_{\mathbb{R}^n}\bigg(\frac{r^{p'-1}}{(|x-y_0|+r)^{p'}}\bigg)^n d\sigma(x)\Bigg)^{p-1} \le C.$$

Since this holds for every cube $Q(y_0, r)$, it follows that (μ, σ) satisfies the PA_p condition. ■

5. Strong $(1,1)$ inequalities. For the proof of Theorem 1.4 we first give some preliminary lemmas.

LEMMA 5.1. *Let v be a measurable function. Then for a.e. $x \in \mathbb{R}^n$,*

$$\lim_{r\to 0^+}\Big[\operatorname*{ess\,inf}_{y\in Q(x,r)} v(y)\Big] \le v(x). \tag{5.1}$$

The proof of Lemma 5.1 is implicit in [7, Theorem 4] in one dimension; it is the same in higher dimensions.

DEFINITION 5.2. A family $\{E_r\}_{r>0}$ of Borel subsets of $\mathbb{R}^n$ is said to *shrink nicely* to $x \in \mathbb{R}^n$ if

$$E_r \subset B(x,r) \quad \text{for each } r$$

and there exists a constant α independent of r such that

$$|E_r| > \alpha|B(x,r)|.$$

LEMMA 5.3. *Let μ be a regular Borel measure on $\mathbb{R}^n$, and let $d\mu = d\mu_s + u\,dx$ be its Lebesgue Radon–Nikodym decomposition. Then for a.e. $x \in \mathbb{R}^n$,*

$$\lim_{r\to 0}\frac{\mu(E_r)}{|E_r|} = u(x).$$

The proof of Lemma 5.3 can be found in [3, Theorem 3.22, p. 99].

Proof of Theorem 1.4. First suppose that the measure ν is singular with respect to Lebesgue measure. As in the proof of Theorem 1.3 fix a cube $Q(y_0,r)$ and define the truncated cone C_r. Let f be a non-negative function with $\operatorname{supp}(f) \subset Q(y_0,r)$. Then, if we estimate as in the proof of Theorem 1.3 to get (4.6), we deduce for all $x \in C_r$ that

$$|Tf(x)| \ge c(a,n)\int_{Q(y_0,r)}\frac{f(y)}{(r+|x-y_0|)^n}\,dy.$$

By assumption $T : L^1(\nu) \to L^1(\mu)$, so

$$\begin{aligned}\int_{Q(y_0,r)} f(x)\,d\nu(x) &\ge c\int_{\mathbb{R}^n}|Tf(x)|\,d\mu(x)\\ &\ge c\int_{C_r}\int_{Q(y_0,r)}\frac{f(y)}{(r+|x-y_0|)^n}\,dy\,d\mu(x)\\ &= c\int_{Q(y_0,r)} f(y)\,dy\int_{C_r}\frac{1}{(r+|x-y_0|)^n}\,d\mu(x).\end{aligned}\tag{5.2}$$

If $\mu \ne 0$, since μ is a Borel measure, there exists a ball B such that $\mu(B) > 0$. Fix a point y_0 and $r > 0$ such that $B \subset C_r$. Let $f = \chi_{Q(y_0,r)\setminus\operatorname{supp}(\nu)}$ in inequality (5.2); then the left-hand side equals 0. Since ν is singular with respect to Lebesgue measure, $|\operatorname{supp}(\nu)| = 0$, so the first term on the right-hand side is positive. Since the integrand in the second term on the right-hand side is bounded away from 0, the second term is positive unless $\mu(C_r) = 0$, a contradiction. Hence, $\mu = 0$.

Now let μ be a regular measure with Lebesgue decomposition $d\mu = d\mu_s + udx$, where $u \not\equiv 0$, and suppose $d\nu = d\nu_s + vdx$, where v is a non-negative function such that $v(x) < \infty$ a.e. Fix a point y_0 such that $0 < u(y_0) < \infty$. We can further assume that y_0 is a Lebesgue point for μ in the sense of Lemma 5.3, and that the conclusion of Lemma 5.1 holds for the function v at y_0. Let $a = \operatorname{ess\,inf}_{x\in Q(y_0,r)} v(x)$. Given $\epsilon > 0$, let $E = \{x \in Q(y_0, r) : v(x) < a + \epsilon\}$, $A = E \setminus \operatorname{supp}(\nu_s)$, and set $f = |A|^{-1}\chi_A$ in inequality (5.2). By the definition of the essential infimum, $|E| > 0$, and since $|\operatorname{supp}(\nu_s)| = 0$, we have $|A| > 0$. Thus,

$$\int_{C_r} \frac{1}{(r + |x - y_0|)^n}\, d\mu(x) \le C\frac{\nu(A)}{|A|} \le C\frac{v(A)}{|A|}$$
$$\le C(a + \epsilon) = C\Big[\operatorname*{ess\,inf}_{x\in Q(y_0,r)} v(x) + \epsilon\Big].$$

Since $\epsilon > 0$ was arbitrary, this inequality holds with $\epsilon = 0$. As $r \to 0$, C_r converges to the cone C_0 with central axis $y_0 + su_0, s \ge 0$. Therefore, by the monotone convergence theorem and Lemma 5.1 we have

$$\int_{C_0} \frac{1}{|x - y_0|^n}\, d\mu(x) \le Cv(y_0). \tag{5.3}$$

Let $B_j = B(y_0, 2^{-j}), A_j = (C_0 \cap B_j) \setminus B_{j+1}$. Since C_0 has constant aperture, there exists $0 < \alpha < 1$ such that $|A_j| = \alpha|B_j|$, so the collection $\{A_j\}$ shrinks nicely to y_0. Then

$$\lim_{j\to\infty} \frac{\mu(A_j)}{|B_j|} = \alpha u(y_0). \tag{5.4}$$

Fix j_0 such that for all $j \ge j_0$ we have $\mu(A_j)/|B_j| \ge (\alpha/2)u(y_0)$. Hence,

$$v(y_0) \ge c \sum_{j\ge j_0} \int_{A_j} \frac{1}{|x - y_0|^n}\, d\mu(x)$$
$$\ge c \sum_{j\ge j_0} 2^{nj}\mu(A_j) \ge c \sum_{j\ge j_0} \frac{\mu(A_j)}{|B_j|} \ge c \sum_{j\ge j_0} u(y_0) = \infty.$$

Let $E = \{x : 0 < u(x) < \infty\}$, which has positive measure since $u \not\equiv 0$. Then $v(x) = \infty$ for a.e. $x \in E$. This contradicts $v(x) < \infty$ a.e.

Finally, suppose μ is a regular measure that is singular with respect to Lebesgue measure, and is directionally doubling in the direction u_0, and ν is a positive regular Borel measure. If $T : L^1(\nu) \to L^1(\mu)$, then it satisfies a weak $(1, 1)$ inequality, and so by Theorem 1.1, μ is absolutely continuous, a contradiction. ∎

Acknowledgements. The authors would like to thank the anonymous referee for very helpful comments.

The first author is supported by research funds from the Dean of the College of Arts & Sciences, University of Alabama.

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David Cruz-Uribe, OFS, John-Oliver MacLellan
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487, U.S.A.
E-mail: dcruzuribe@ua.edu
jomaclellan@crimson.ua.edu