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Hardy's inequalities, Hilbert inequalities and fractional integrals on function spaces of q-integral p-variation

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Abstract. We obtain Hardy's inequalities and the Hilbert inequalities for function spaces of q-integral p-variation. We also establish the mapping properties of Riemann–Liouville integrals, Weyl integrals and Erdélyi–Kober fractional integrals on these function spaces.

1. Introduction. This paper establishes Hardy's inequalities, the Hilbert inequalities and the mapping properties of some fractional integral operators on function spaces of q-integral p-variation.

The function spaces of q-integral p-variation were introduced by Terehin [28]. These function spaces can be used to study the smoothness properties for functions in Sobolev spaces [29, 30]. Later, these function spaces were applied by Borucka-Cieślewicz to study integral operators with the Dirichlet kernel [2, 3]. Recently, the mapping properties of some integral operators for functions with q-integral p-variation were obtained in [4, 5, 9]. These results give us motivations to further investigate the mapping properties of some integral operators on function spaces of q-integral p-variation, particularly, the Hardy operators, the Hilbert operators, the Riemann–Liouville integrals, the Weyl integrals and the Erdélyi–Kober fractional integrals.

We obtain our results by establishing a general principle on the boundedness of integral operators on function spaces of q-integral p-variation. We obtain this principle by using the Minkowski inequalities and the mapping properties of dilation operators on function spaces of q-integral p-variation.

This paper is organized as follows. Section 2 contains the definition of function spaces of q-integral p-variation. The Minkowski inequalities and the

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mapping properties of dilation operators on function spaces of q-integral p-variation are also presented in Section 2. The main result, namely, the boundedness of integral operators and its applications to Hardy's inequalities, the Hilbert inequalities, the mapping properties of Riemann–Liouville integrals, Weyl integrals and Erdélyi–Kober fractional integrals are established in Section 3.

2. Preliminaries and definitions. Let $\mathcal{M}(\mathbb{R}_+)$ denote the set of Lebesgue measurable functions on $\mathbb{R}_+ = [0, \infty)$.

We state the definition of the function space of q-integral p-variation in the following [9, Definitions 1 and 2].

DEFINITION 2.1. Let $1 \le q \le \infty$ and $1 \le p < \infty$. The *L^q modulus of continuity* of a function $f \in \mathcal{M}(\mathbb{R}_+)$ on $[a, b] \subset \mathbb{R}_+$ is

$$\omega_q(f, a, b) = \sup_{0<h<b-a} \|f(\cdot + h) - f(\cdot)\|_{L^q[a,b]}.$$

The *q-integral p-variation* of f is defined as

$$I_{p,q}(f) = \sup_{P\in\mathcal{P}} \Big(\sum_{i=0}^{n-1} \omega_q(f, x_i, x_{i+1})^p\Big)^{1/p}$$

where the supremum is taken over all partitions in the set

$$\mathcal{P} = \{\{x_0, \dots, x_n\} : n \in \mathbb{N},\ 0 \le x_i < x_{i+1} < \infty,\ i = 0, \dots, n-1\}.$$

The *function space of q-integral p-variation* $I_{p,q}(\mathbb{R}_+)$ consists of all functions $f \in \mathcal{M}(\mathbb{R}_+)$ satisfying

$$\|f\|_{I_{p,q}(\mathbb{R}_+)} = I_{p,q}(f) < \infty.$$

In order to study the mapping properties of Riemann–Liouville integrals and Weyl integrals, we introduce function spaces of power weighted q-integral p-variation.

DEFINITION 2.2. Let $\theta \in \mathbb{R}$, $1 \le q \le \infty$ and $1 \le p < \infty$. The *function space of power weighted q-integral p-variation* $I^\theta_{p,q}(\mathbb{R}_+)$ consists of all functions $f \in \mathcal{M}(\mathbb{R}_+)$ satisfying

$$\|f\|_{I^\theta_{p,q}(\mathbb{R}_+)} = I_{p,q}(t^{-\theta} f(t)) < \infty.$$

Notice that $I_{p,q}(f)$ is a seminorm and $I_{p,q}(f) = 0$ if and only if f is a constant function. Therefore, $\|\cdot\|_{I^\theta_{p,q}(\mathbb{R}_+)}$ and $\|\cdot\|_{I_{p,q}(\mathbb{R}_+)}$ are norms on the quotient spaces $I^\theta_{p,q}(\mathbb{R}_+)/\mathcal{C}$ and $I_{p,q}(\mathbb{R}_+)/\mathcal{C}$, respectively, where $\mathcal{C}$ denotes the class of constant functions.

Next, we show that $I^\theta_{p,q}(\mathbb{R}_+)$ is a non-trivial set. Let g be a Lipschitz function with $\operatorname{supp} g = [0, 1]$, so

$$|g(x) - g(y)| \le C|x - y|, \quad x, y \ge 0. \tag{2.1}$$

For any $0 \le b < c \le 1$, in view of (2.1), we have

$$\begin{aligned}\omega_q(g,b,c) &= \sup_{0<h<c-b} \|g(\cdot+h)-g(\cdot)\|_{L^q[b,c]} \\ &\le C \sup_{0<h<c-b} h(c-b)^{1/q} = C(c-b)^{1+1/q}.\end{aligned}$$

Furthermore, if $0 \le b < 1 \le c$, then

$$\begin{aligned}\omega_q(g,b,c) &= \sup_{0<h<c-b} \|g(\cdot+h)-g(\cdot)\|_{L^q[b,1]} \\ &\le C \sup_{0<h<c-b} h(1-b)^{1/q} = C(1-b)^{1+1/q}\end{aligned}$$

since $\operatorname{supp} g = [0,1]$.

Obviously, if $a > 1$, then $\omega_q(g,a,b) = 0$. Therefore, without loss of generality, we only need to consider $P = \{x_0,\dots,x_n\}$ with $x_n \le 1$. Consequently,

$$\begin{aligned}\Big(\sum_{i=0}^{n-1} \omega_q(f,x_i,x_{i+1})^p\Big)^{1/p} &\le C\Big(\sum_{i=0}^{n-1} (x_{i+1}-x_i)^{p(1+1/q)}\Big)^{1/p} \\ &\le C\Big(\sum_{i=0}^{n-1} (x_{i+1}-x_i)\Big)^{1+1/q} \le C\end{aligned}$$

because $p(1+1/q) \ge 1$ and $\sum_{i=0}^{n-1}(x_{i+1}-x_i) \le 1$. Since P is arbitrary, we conclude that $g \in I_{p,q}(\mathbb{R}_+)$. Furthermore, for any $\theta \in \mathbb{R}$, $t^\theta g(t) \in I^\theta_{p,q}(\mathbb{R}_+)$. The above example is a particular case of [9, Proposition 2].

We now establish the Minkowski inequalities for $I_{p,q}(\mathbb{R}_+)$. It is a special case of the Bochner inequality [6, Theorem 4]. For completeness, we give a proof of the following result.

THEOREM 2.3. *Let $1 \le q \le \infty$ and $1 \le p < \infty$. Let $f(x,s)$ be a Lebesgue measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$. If $f(x,\cdot)$ is integrable for almost all $x \in \mathbb{R}_+$, then*

$$I_{p,q}\Big(\int_0^\infty f(\cdot,s)\,ds\Big) \le \int_0^\infty I_{p,q}(f(\cdot,s))\,ds.$$

Proof. We just present the proof for $1 \le q < \infty$. Even though the L^∞ norm is not expressed by an integral, the proof for $q = \infty$ is similar.

Write $F(\cdot) = \int_0^\infty f(\cdot,s)\,ds$. For any $P = \{x_0,\dots,x_n\}$ belonging to $\mathcal{P}$, we find that

$$|F(x+h)-F(x)| \le \int_0^\infty |f(x+h,s)-f(x,s)|\,ds.$$

Let $[a,b] \subset \mathbb{R}_+$. The Minkowski inequality for Lebesgue spaces [8, Theorem 6.19] gives

$$\begin{aligned}\|F(\cdot+h)-F(\cdot)\|_{L^q[a.b]} &= \Big(\int_a^b |F(x+h)-F(x)|^q\,dx\Big)^{1/q}\\ &\le \Big(\int_a^b\Big(\int_0^\infty |f(x+h,s)-f(x,s)|\,ds\Big)^q dx\Big)^{1/q}\\ &\le \int_0^\infty \|f(\cdot+h,s)-f(\cdot,s)\|_{L^q[a,b]}\,ds\\ &\le \int_0^\infty \omega_q(f(\cdot,s),a,b)\,ds.\end{aligned}$$

By taking the supremum over $h\in(0,b-a)$, we get

$$\omega_q(F,a,b)\le\int_0^\infty \omega_q(f(\cdot,s),a,b)\,ds.$$

Let $P=\{x_0,\ldots,x_n\}\in\mathcal{P}$. By applying the Minkowski inequality for Lebesgue spaces [8, Theorem 6.19] with counting measure, we find that

$$\begin{aligned}\Big(\sum_{i=0}^{n-1}\omega_q(F,x_i,x_{i+1})^p\Big)^{1/p} &\le \Big(\sum_{i=0}^{n-1}\Big(\int_0^\infty \omega_q(f(\cdot,s),x_i,x_{i+1})\,ds\Big)^p\Big)^{1/p}\\ &\le \int_0^\infty\Big(\sum_{i=0}^{n-1}\omega_q(f(\cdot,s),x_i,x_{i+1})^p\Big)^{1/p} ds\\ &\le \int_0^\infty I_{p,q}(f(\cdot,s))\,ds.\end{aligned}$$

By taking the supremum over $P\in\mathcal{P}$, we obtain

$$I_{p,q}(F)\le\int_0^\infty I_{p,q}(f(\cdot,s))\,ds. \blacksquare \tag{2.2}$$

To end this section, we present a result on the operator norm of the dilation $D_\lambda f(\cdot)=f(\lambda\,\cdot)$, $\lambda>0$, on $I_{p,q}(\mathbb{R}_+)$. It is easy to see that for any $\lambda>0$,

$$\begin{aligned}\omega_q(D_\lambda f,a,b) &= \sup_{0<h<b-a}\|f(\lambda\cdot+\lambda h)-f(\lambda\cdot)\|_{L^q[a,b]}\\ &=\lambda^{-1/q}\sup_{0<h<\lambda b-\lambda a}\|f(\cdot+h)-f(\cdot)\|_{L^q[\lambda a,\lambda b]}\\ &=\lambda^{-1/q}\omega_q(f,\lambda a,\lambda b).\end{aligned}$$

Consequently, for any $P = \{x_0, \ldots, x_n\} \in \mathcal{P}$,

$$\Big(\sum_{i=0}^{n-1} \omega_q(D_\lambda f, x_i, x_{i+1})^p\Big)^{1/p} = \lambda^{-1/q}\Big(\sum_{i=0}^{n-1} \omega_q(f, \lambda x_i, \lambda x_{i+1})^p\Big)^{1/p} \le \lambda^{-1/q} I_{p,q}(f)$$

because $\{\lambda x_0, \ldots, \lambda x_n\} \in \mathcal{P}$. Therefore, by taking the supremum over $P = \{x_0, \ldots, x_n\} \in \mathcal{P}$, we get

$$I_{p,q}(D_\lambda f) \le \lambda^{-1/q} I_{p,q}(f). \tag{2.3}$$

Notice that $D_{1/\lambda} D_\lambda g = g$. By applying (2.3) to $D_{1/\lambda}$ and $f = D_\lambda g$, we obtain

$$I_{p,q}(g) = I_{p,q}(D_{1/\lambda} D_\lambda g) \le \lambda^{1/q} I_{p,q}(D_\lambda g).$$

In conclusion, we obtain the following result.

THEOREM 2.4. *Let $\lambda > 0$, $1 \le q \le \infty$ and $1 \le p < \infty$. Then*

$$\|D_\lambda f\|_{I_{p,q}(\mathbb{R}_+)} = \lambda^{-1/q} \|f\|_{I_{p,q}(\mathbb{R}_+)}. \tag{2.4}$$

In particular, the above result yields

$$\|D_\lambda\|_{I_{p,q}(\mathbb{R}_+)\to I_{p,q}(\mathbb{R}_+)} = \lambda^{-1/q}, \qquad \lambda > 0,$$

where the left hand side is the operator norm of $D_\lambda : I_{p,q}(\mathbb{R}_+) \to I_{p,q}(\mathbb{R}_+)$.

3. Main results. This section presents the main results of this paper. We establish Hardy's inequalities, the Hilbert inequalities, and the mapping properties of Riemann–Liouville integrals, Weyl integrals and Erdélyi–Kober fractional integrals on $I_{p,q}(\mathbb{R}_+)$.

We obtain the main results by using a general principle on the boundedness of integral operators on $I_{p,q}(\mathbb{R}_+)$. Therefore, we start with the definition of the class of integral operators used in our studies.

Let $K : (0,\infty) \times (0,\infty) \to \mathbb{R}$ be a Lebesgue measurable function. We consider the integral operator

$$Tf(t) = \int_0^\infty K(s,t) f(s)\, ds, \qquad t \ge 0.$$

THEOREM 3.1. *Let $1 \le q \le \infty$, $1 \le p < \infty$, $\theta \in \mathbb{R}$ and $K : (0,\infty)\times(0,\infty) \to \mathbb{R}$ be a Lebesgue measurable function. Suppose that for any $\lambda > 0$,*

$$K(\lambda s, \lambda t) = \lambda^{-1+\theta} K(s,t) \tag{3.1}$$

and

$$\int_0^\infty |K(u,1)| u^{-1/q}\, du < \infty \tag{3.2}$$

where $u^{-1/q} = 1$ for $u \in (0,\infty)$ when $q = \infty$. There exists a constant $C > 0$ such that for any $f \in I_{p,q}(\mathbb{R}_+)$,

$$\|Tf\|_{I^\theta_{p,q}(\mathbb{R}_+)} \le C\Big(\int_0^\infty |K(u,1)|u^{-1/q}\,du\Big)\|f\|_{I_{p,q}(\mathbb{R}_+)}. \tag{3.3}$$

Proof. We just give the proof for $1 \le q < \infty$. For $q = \infty$, by using the convention $u^{-1/q} = 1$ for $u \in (0,\infty)$, the proof is similar.

It suffices to show that $T_\theta : I_{p,q}(\mathbb{R}_+) \to I_{p,q}(\mathbb{R}_+)$ with

$$\|T_\theta f\|_{I_{p,q}(\mathbb{R}_+)} \le \Big(\int_0^\infty |K(u,1)|u^{-1/q}\,du\Big)\|f\|_{I_{p,q}(\mathbb{R}_+)}$$

where

$$T_\theta f(t) = t^{-\theta} Tf(t), \quad t > 0.$$

Notice that

$$T_\theta f(t) = \int_0^\infty K_\theta(s,t) f(s)\,ds$$

where

$$K_\theta(s,t) = t^{-\theta}K(s,t), \qquad s \ge 0,\ t > 0.$$

Let $f \in I^\theta_{p,q}(\mathbb{R}_+)$. By using the substitution $s = tu$, we have

$$\begin{aligned} T_\theta f(t) &= \int_0^\infty K_\theta(s,t)f(s)\,ds = \int_0^\infty t^{-\theta}K(s,t)f(s)\,ds \\ &= \int_0^\infty t^{-\theta}K(tu,t)f(tu)t\,du = \int_0^\infty K(u,1)f(tu)\,du \end{aligned}$$

where we use (3.1) in the last equality.

Theorems 2.3 and 2.4 yield

$$\begin{aligned} \|T_\theta f\|_{I_{p,q}(\mathbb{R}_+)} &= \Big\|\int_0^\infty K(u,1)D_u f(\cdot)\,du\Big\|_{I_{p,q}(\mathbb{R}_+)} \\ &\le \int_0^\infty |K(u,1)|\,\|D_u f\|_{I_{p,q}(\mathbb{R}_+)}\,du \\ &= \Big(\int_0^\infty |K(u,1)|u^{-1/q}\,du\Big)\|f\|_{I_{p,q}(\mathbb{R}_+)}. \ \blacksquare \end{aligned}$$

The previous theorem yields Hardy's inequalities, the Hilbert inequalities, and the mapping properties of Riemann–Liouville integrals, Weyl integrals and Erdélyi–Kober fractional integrals on $I_{p,q}(\mathbb{R}_+)$ in the following subsections.

3.1. Hardy's inequality. Hardy's inequality is one of the most famous inequalities in analysis. For the history and recent developments of Hardy's inequalities, the reader is referred to [24, 25].

For any locally integrable function f on $\mathbb{R}_+$, the *Hardy operator* H is defined as

$$Hf(t) = \frac{1}{t}\int_0^t f(s)\,ds, \quad t \geq 0.$$

Let f be a function integrable on each interval (t,∞) for $t > 0$. The Hardy operator H^* is defined as

$$H^*f(t) = \int_t^\infty \frac{f(s)}{s}\,ds, \quad t \geq 0.$$

We see that

$$Hf(t) = \int_0^\infty K(s,t)f(s)\,ds \quad \text{where} \quad K(s,t) = \frac{1}{t}\chi_{\{(s,t):\,0<s<t\}}.$$

Similarly,

$$H^*f(t) = \int_0^\infty K(s,t)f(s)\,ds \quad \text{where} \quad K(s,t) = \frac{1}{s}\chi_{\{(s,t):\,0<t<s\}}.$$

Therefore, Theorem 3.1 yields Hardy's inequalities on $I_{p,q}(\mathbb{R}_+)$.

THEOREM 3.2. *Let $1 \leq q \leq \infty$ and $1 \leq p < \infty$.*

(1) *If $1 < q \leq \infty$, then for any $f \in I_{p,q}(\mathbb{R}_+)$, we have*

$$\|Hf\|_{I_{p,q}(\mathbb{R}_+)} \leq C_q\|f\|_{I_{p,q}(\mathbb{R}_+)} \tag{3.4}$$

where $C_q = \frac{q}{q-1}$ when $1 < q < \infty$ and $C_q = 1$ when $q = \infty$.

(2) *If $1 \leq q < \infty$, then for any $f \in I_{p,q}(\mathbb{R}_+)$, we have*

$$\|H^*f\|_{I_{p,q}(\mathbb{R}_+)} \leq q\|f\|_{I_{p,q}(\mathbb{R}_+)}. \tag{3.5}$$

Proof. Since

$$Hf(t) = \int_0^\infty K(s,t)f(s)\,ds$$

with $K(s,t) = \frac{1}{t}\chi_{\{(s,t):\,0<s<t\}}$, it satisfies (3.1) with $\theta = 0$ and

$$\int_0^\infty |K(u,1)|u^{-1/q}\,du = \int_0^1 u^{-1/q}\,du = \frac{q}{q-1}$$

because $1 < q \leq \infty$. Therefore, Theorem 3.1 yields (3.4).

Similarly, we have

$$H^*f(t) = \int_0^\infty K(s,t)f(s)\,ds$$

with $K(s,t) = \frac{1}{s}\chi_{\{(s,t):\,0<t<s\}}$. It fulfills (3.1) with $\theta = 0$ and

$$\int_0^\infty |K(u,1)|u^{-1/q}\,du = \int_1^\infty u^{-1-1/q}\,du = q$$

because $1 \leq q < \infty$. Theorem 3.1 gives (3.5). ∎

For the study of Hardy's inequalities on some other function spaces such as BMO and Hardy type spaces, the reader is referred to [12, 13, 14, 15, 17, 31, 32].

3.2. Hilbert inequality. We present the Hilbert inequality in this section. Recall that the *Hilbert operator* is defined as

$$\mathcal{H}f(t) = \int_0^\infty \frac{f(s)}{t+s}\,ds$$

when $f(\cdot)/(t+\cdot)$ is integrable for almost all $t \in \mathbb{R}_+$.

Thus,

$$\mathcal{H}f(t) = \int_0^\infty K(s,t)f(s)\,ds$$

where $K(s,t) = \frac{1}{s+t}$.

Theorem 3.3. *Let $1 < q < \infty$ and $1 \leq p < \infty$. For any $f \in I_{p,q}(\mathbb{R}_+)$, we have*

$$\|\mathcal{H}f\|_{I_{p,q}(\mathbb{R}_+)} \leq C\|f\|_{I_{p,q}(\mathbb{R}_+)} \tag{3.6}$$

where $C = \int_0^\infty \frac{u^{-1/q}}{1+u}\,du$.

Proof. Obviously, $K(s,t) = \frac{1}{s+t}$ satisfies (3.1) with $\theta = 0$. Moreover,

$$\int_0^\infty |K(u,1)|u^{-1/q}\,du = \int_0^\infty \frac{u^{-1/q}}{1+u}\,du \leq \int_0^1 u^{-1/q}\,du + \int_1^\infty u^{-1-1/q}\,du < \infty$$

because $1 < q < \infty$. Theorem 3.1 guarantees the validity of (3.6). ∎

We also have the Hilbert inequality on Morrey spaces, block spaces and amalgam spaces [10, 11, 16, 17].

3.3. Fractional integrals. In this section we consider Riemann–Liouville integrals, Weyl integrals and Erdélyi–Kober fractional integrals. They arise in the study of fractional calculus [21].

We begin with the definitions of Riemann–Liouville integrals and Weyl integrals. Let $0 < \alpha < 1$. For any locally integrable function f, the *Riemann–Liouville integral* and the *Weyl integral* of f are defined as

$$R_\alpha f(t) = \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \qquad t \ge 0,$$

$$W_\alpha f(t) = \int_t^\infty (s-t)^{\alpha-1} f(s)\,ds, \qquad t \ge 0.$$

Riemann–Liouville integrals can be used to study the mean convergence of Laguerre series [23]. The reader is referred to [1, 20] for weighted norm inequalities and modular inequalities for Riemann–Liouville integrals and Weyl integrals, respectively.

The following concerns Riemann–Liouville integrals and Weyl integrals on $I_{p,q}(\mathbb{R}_+)$.

THEOREM 3.4. *Let $0 < \alpha < 1$, $1 \le q \le \infty$ and $1 \le p < \infty$.*

(1) *If $1 < q \le \infty$, then for any $f \in I_{p,q}(\mathbb{R}_+)$, we have*

$$\|R_\alpha f\|_{I^\alpha_{p,q}(\mathbb{R}_+)} \le C_0 \|f\|_{I_{p,q}(\mathbb{R}_+)} \tag{3.7}$$

where $C_0 = \int_0^1 (1-u)^{\alpha-1} u^{-1/q}\,du$.

(2) *If $1 \le q < 1/\alpha$, then for any $f \in I_{p,q}(\mathbb{R}_+)$, we have*

$$\|W_\alpha f\|_{I^\alpha_{p,q}(\mathbb{R}_+)} \le C_1 \|f\|_{I_{p,q}(\mathbb{R}_+)} \tag{3.8}$$

where $C_1 = \int_1^\infty (u-1)^{\alpha-1} u^{-1/q}\,du$.

Proof. We find that

$$R_\alpha f(t) = \int_0^\infty K_0(s,t) f(s)\,ds \quad \text{where} \quad K_0(s,t) = (t-s)^{\alpha-1} \chi_{\{(s,t):\, 0<s<t\}}.$$

It satisfies (3.1) with $\theta = \alpha$ and

$$\int_0^\infty |K_0(u,1)| u^{-1/q}\,du = \int_0^1 (1-u)^{\alpha-1} u^{-1/q}\,du$$

$$\le C\Big(\int_0^{1/2} u^{-1/q}\,du + \int_{1/2}^1 (1-u)^{\alpha-1}\,du \Big) < \infty$$

since $1 < q \le \infty$. Theorem 3.1 gives (3.7).

For the Weyl integral, we have

$$W_\alpha f(t)=\int_0^\infty K_1(s,t)f(s)\,ds \quad\text{where}\quad K_1(s,t)=(s-t)^{\alpha-1}\chi_{\{(s,t):\,0<t<s\}}.$$

It fulfills (3.1) with $\theta=\alpha$. Moreover,

$$\begin{aligned}\int_0^\infty |K_1(u,1)|u^{-1/q}\,du&=\int_1^\infty (u-1)^{\alpha-1}u^{-1/q}\,du\\&\le C\Big(\int_1^2 (u-1)^{\alpha-1}\,du+\int_2^\infty u^{\alpha-1/q-1}\,du\Big)<\infty\end{aligned}$$

because $1\le q<1/\alpha$. Theorem 3.1 yields (3.8). ∎

Notice that the mapping properties for Riemann–Liouville integrals on $I_{p,q}(\mathbb{R}_+)$ were given in [9, Section 6]. The result in [9, Theorem 5] shows that R_α is a bounded operator from Lebesgue spaces to $I_{p,q}(\mathbb{R}_+)$. Thus, Theorem 3.4 gives a complementary result for Riemann–Liouville integrals on $I_{p,q}(\mathbb{R}_+)$.

Erdélyi and Kober generalized Riemann–Liouville integrals and Weyl integrals in [7, 22].

For any locally integrable function f, the *Erdélyi–Kober fractional integrals* of f are given by

$$\begin{aligned}I^\nu_{\alpha,\xi}f(t)&=\frac{\nu t^{-\nu(\xi+\alpha)}}{\Gamma(\alpha)}\int_0^t (t^\nu-s^\nu)^{\alpha-1}s^{\nu\xi+\nu-1}f(s)\,ds,\\ J^\nu_{\alpha,\eta}f(t)&=\frac{\nu t^{\nu\eta}}{\Gamma(\alpha)}\int_t^\infty (s^\nu-t^\nu)^{\alpha-1}s^{-\nu(\alpha+\eta)+\nu-1}f(s)\,ds\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function. We use the definitions of Erdélyi–Kober fractional integrals from [21, (1.1.17) and (1.1.17*)].

Erdélyi–Kober fractional integrals have applications in mathematical physics [27] and the studies of solar neutrino physics [26]. The reader is referred to [21] for a detailed account on the use of Erdélyi–Kober fractional integrals in fractional calculus. For the boundedness of Erdélyi–Kober fractional integrals on Hardy spaces, BMO and ball Banach function spaces, the reader is referred to [18, 19].

THEOREM 3.5. *Let $\nu,\xi,\eta,\alpha>0$, $1\le q\le\infty$ and $1\le p<\infty$.*

(1) *If $\nu\xi+\nu>1/q$, then for any $f\in I_{p,q}(\mathbb{R}_+)$, we have*

$$\|I^\nu_{\alpha,\xi}f\|_{I_{p,q}(\mathbb{R}_+)}\le C_0\|f\|_{I_{p,q}(\mathbb{R}_+)} \tag{3.9}$$

where $C_0=\frac{\nu}{\Gamma(\alpha)}\int_0^1(1-u^\nu)^{\alpha-1}u^{\nu\xi+\nu-1-1/q}\,du$.

(2) *For any* $f \in I_{p,q}(\mathbb{R}_+)$, *we have*

$$\|J^{\nu}_{\alpha,\eta} f\|_{I_{p,q}(\mathbb{R}_+)} \le C_1 \|f\|_{I_{p,q}(\mathbb{R}_+)} \tag{3.10}$$

where $C_1 = \frac{\nu}{\Gamma(\alpha)} \int_1^\infty (u^\nu - 1)^{\alpha-1} u^{-\nu(\alpha+\eta)+\nu-1-1/q}\,du$.

Proof. We have

$$I^{\nu}_{\alpha,\xi} f(t) = \int_0^\infty E(s,t) f(s)\,ds$$

where

$$E(s,t) = \frac{\nu t^{-\nu(\xi+\alpha)}}{\Gamma(\alpha)} (t^\nu - s^\nu)^{\alpha-1} s^{\nu\xi+\nu-1} \chi_{\{(s,t):\, 0<s<t\}}.$$

It is easy to see that E satisfies (3.1) with $\theta = 0$. Furthermore, by using the substitution $u^\nu = y$, we find that

$$\begin{aligned}
\int_0^\infty |E(u,1)| u^{-1/q}\,du &= \frac{\nu}{\Gamma(\alpha)} \int_0^1 (1-u^\nu)^{\alpha-1} u^{\nu\xi+\nu-1-1/q}\,du \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-y)^{\alpha-1} y^{\xi-1/q\nu}\,dy \\
&= \frac{B\big(\xi - \frac{1}{q\nu} + 1, \alpha\big)}{\Gamma(\alpha)} < \infty
\end{aligned}$$

where $B(\cdot,\cdot)$ is the Euler Beta function because $\nu\xi + \nu > 1/q$ and $\alpha > 0$. The validity of (3.9) is guaranteed by Theorem 3.1.

For $J^{\nu}_{\alpha,\eta}$, we find that

$$J^{\nu}_{\alpha,\eta} f(t) = \int_0^\infty G(s,t) f(s)\,ds$$

where

$$G(s,t) = \frac{\nu t^{\nu\eta}}{\Gamma(\alpha)} (s^\nu - t^\nu)^{\alpha-1} s^{-\nu(\alpha+\eta)+\nu-1} \chi_{\{(s,t):\, 0<t<s\}}.$$

Then G fulfills (3.1) with $\theta = 0$ and by using the substitution $u^{-\nu} = y$, we find that

$$\begin{aligned}
\int_0^\infty |G(u,1)| u^{-1/q}\,du &= \frac{\nu}{\Gamma(\alpha)} \int_1^\infty (u^\nu - 1)^{\alpha-1} u^{-\nu(\alpha+\eta)+\nu-1-1/q}\,du \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-y)^{\alpha-1} y^{\eta+1/q\nu-1}\,dy \\
&= \frac{B\big(\eta + \frac{1}{q\nu}, \alpha\big)}{\Gamma(\alpha)} < \infty
\end{aligned}$$

since $\alpha, \nu, \eta > 0$. Theorem 3.1 gives (3.10). ∎

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