ANNALES
POLONICI MATHEMATICI
Online First version

η-Einstein contact metric manifolds with purely transversal Bach tensor

Amalendu Ghosh (Chandannagar)

Abstract. We prove that every $(2n+1)$-dimensional η-Einstein contact metric manifold (i.e., the Ricci tensor S satisfies $S = \alpha g + \beta\eta \otimes \eta$ for some smooth functions α, β) with purely transversal Bach tensor is Einstein.

1. Introduction. Let (M^m, g) be a Riemannian manifold. The decomposition of the Riemann curvature tensor R of the metric g into the Weyl W, Ricci S and scalar curvature r component is given by

$$W = R - \frac{2}{m-2} S \wedge g + \frac{r}{(m-1)(m-2)} g \wedge g, \tag{1.1}$$

where $\wedge$ is the well known Kulkarni–Nomizu product. It is well known that Einstein metrics are the critical points for the Einstein–Hilbert functional [B08]. From this point of view, if one takes the functional

$$\mathcal{W}(g) = \int_{M^4} |W|^2 \, dv_g$$

for a 4-dimensional compact Riemannian manifold M, then its critical point is characterized by the vanishing of a conformally invariant, symmetric trace free $(0,2)$ type tensor B which is known as the Bach tensor. It was first introduced by R. Bach [B21] to study conformal relativity. On any Riemannian manifold (M^m, g) $(m \geq 4)$ the Bach tensor is defined as

$$\begin{aligned} B_g(X,Y) = {} & \frac{1}{m-3} \sum_{i,j=1}^{m} (\nabla_{e_i} \nabla_{e_j} W)(X, e_i, e_j, Y) \\ & + \frac{1}{m-2} \sum_{i,j=1}^{m} S(e_i, e_j) W(X, e_i, e_j, Y), \end{aligned} \tag{1.2}$$

2020 *Mathematics Subject Classification*: Primary 53C25; Secondary 53D10.
Key words and phrases: Bach tensor, Einstein metric, contact metric manifold, η-Einstein.
Received 7 October 2020; revised 7 January 2021.
Published online 17 May 2021.

where (e_i), $i = 1, \dots, m$, is a local orthonormal frame on (M, g). On the other hand, taking the divergence of (1.1) and using the contraction of the Bianchi second identity it follows that $\operatorname{div} W = \frac{m-3}{m-2} C$ where C is the $(0,3)$-type *Cotton tensor* C defined by

$$C(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) - \frac{1}{2(m-1)}[(Xr)g(Y,Z) - (Yr)g(X,Z)]. \tag{1.3}$$

Thus, in dimension > 3, $C = 0$ and $\operatorname{div} W = 0$ are equivalent. By (1.1) and (1.3), the Bach tensor (1.2) may be written as (Chen and He [CH13])

$$B(X,Y) = \frac{1}{m-2}\Big[\sum_{i=1}^{m} (\nabla_{e_i} C)(e_i, X)Y + \sum_{i,j=1}^{m} \operatorname{Ric}(e_i, e_j) W(X, e_i, e_j, Y)\Big]. \tag{1.4}$$

In dimension 3, we know that the Weyl tensor W vanishes. Hence, the Bach tensor can then be expressed as

$$B_g(X,Y) = \sum_{i=1}^{3} (\nabla_{e_i} C)(e_i, X)Y. \tag{1.5}$$

The metric g is called *Bach flat* if the Bach tensor B vanishes. The most common examples of Bach flat metrics are Einstein metrics and conformally flat metrics and hence Bach flatness may be considered as a natural generalization of being Einstein and of conformal flatness. Further, any 4-dimensional Bach flat metric is conformally invariant, and a metric conformal to an Einstein metric is also Bach flat. Some more examples of Bach flat manifolds of signature $(2,2)$ were constructed by using the modified Riemannian extensions of affine surfaces (for details see [CG$^+$19]).

On a contact metric manifold the Bach tensor is said to be *purely transversal* if it has components only along the contact subbundle D. This is equivalent to $B(\xi, \cdot) = 0$. In [GS17], the author and Sharma initiated the study of Sasakian manifolds with purely transversal Bach tensor. Later on, the author and Sharma [GS19] studied Bach flat (k,μ)-contact manifolds, which provide contact-theoretic examples of Bach flat metrices in any odd dimension > 3, which are neither Einstein nor conformally flat. Recently, the author proved that a Kenmotsu manifold [K72] of dimension > 3 has a purely transversal Bach tensor if and only if it is η-Einstein. This showed that there is a nice connection between Bach flat metrics and η-Einstein metrics on a certain class of almost contact metric manifolds (for details see [G20]). Since η-Einstein contact metrics are a natural generalization of Einstein metrics within the framework of contact metric manifolds, we are interested in studying η-Einstein metrics under a weaker assumption than Bach flatness and prove

THEOREM 1.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be an η-Einstein contact metric manifold with α and β smooth functions. If M has purely transversal Bach tensor, then α, β are constant and M is Einstein.*

Next, we prove the following

COROLLARY 1.2. *Let M^{2n+1} be an η-Einstein K-contact manifold with purely transversal Bach tensor. If M is complete, then it must be Sasakian and Einstein.*

2. Contact metric geometry. The study of contact geometry has been motivated by classical mechanics, a contact space corresponding to the odd-dimensional extended phase space that includes the time variable. A $(2n+1)$-dimensional smooth manifold M is said to be a *contact manifold* if it carries a global 1-form η such that $\eta \wedge (d\eta)^n$ is non-vanishing everywhere on M. To this 1-form η there corresponds a unique vector field ξ, called the *Reeb vector field*, such that $\eta(\xi) = 1$ and $(d\eta)(\xi, X) = 0$. Let TM be the tangent bundle of M and $D \subset TM$ be its subundle defined as $D = \ker \eta$. Then the tangent bundle of TM splits as $TM = D \oplus L_\xi$, where L_ξ denotes the line tangent to ξ. By polarizing $d\eta$ on D, one can define a Riemannian metric g and a $(1,1)$-tensor field φ satisfying

$$(d\eta)(X,Y) = g(X, \varphi Y), \qquad \eta(X) = g(X, \xi), \qquad \varphi^2 = -I + \eta \otimes \xi.$$

The metric g is called the associated metric and satisfies

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y).$$

A Riemannian manifold together with the structure (φ, ξ, η, g) is said to be a *contact metric manifold*.

Following Blair [B02] we now define two self-adjoint operators h and l by $h = \frac{1}{2}(\mathcal{L}_\xi \varphi)$ and $l = R(\cdot, \xi)\xi$ respectively. These operators also satisfy

$$h\xi = 0 = l\xi, \qquad h\varphi = -\varphi h, \qquad \operatorname{Tr} h = \operatorname{Tr} h\varphi = 0.$$

The covariant derivative of the Reeb vector field and the Ricci operator Q along the Reeb vector field satisfy (see Blair [B02])

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{2.1}$$

$$S(\xi, \xi) = g(Q\xi, \xi) = 2n - \operatorname{Tr} h^2. \tag{2.2}$$

Further, the following formula is also valid on a contact metric manifold [O79]:

$$\sum_{i=1}^{2n+1} (\nabla_{e_i} \varphi) e_i = 2n\xi. \tag{2.3}$$

A contact metric manifold is said to be K-*contact* if ξ is Killing (i.e. $h = 0$).

Sasakian manifolds are an odd-dimensional analogue of Kaehler manifolds. These are contact manifolds satisfying a normality (or integrability) condition. A Sasakian structure may also be characterized in terms of the metric cone. Consider the metric cone $(C(M), \bar{g})$ over a Riemannian manifold (M, g), that is, $C(M) = \mathbb{R}^+ \times M$ with metric given by

$$\bar{g} = dt^2 + t^2 g,$$

where t is a coordinate on $\mathbb{R}^+$. Then we say that M is a *Sasakian manifold* if its metric cone $(C(M), \bar{g})$ is Kaehler [BG00]. Every Sasakian manifold is K-contact; the converse is true only in dimension 3. An *Einstein–Sasakian manifold* is a contact metric manifold that is both Sasakian and Einstein. It is interesting to note that a Sasakian manifold M is Einstein if and only if the metric cone $C(M)$ is Ricci flat Kaehler. Recently Einstein–Sasakian metrics have received a lot of attention in connection with the supersymmetric background relevant to the AdS/CFT correspondence (see [M98] and [CHS85]). For details about contact geometry, we refer to Blair [B02] and Boyer–Galicki [BG00].

Recall that a contact metric manifold is said to be η-*Einstein* if the $(0, 2)$-Ricci tensor S can be expressed as

$$S(Y, Z) = \alpha g(Y, Z) + \beta\eta(Y)\eta(Z) \tag{2.4}$$

for some smooth functions α and β on M (see [O62]). If β vanishes identically, then M is Einstein. Further, it is known that if M is K-contact and $\dim M \geq 5$, then α and β are necessarily constants (see [YK84]). On the other hand, it is easy to verify that an η-Einstein contact metric is invariant under a D-homothetic deformation [T68] defined as

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\varphi} = \varphi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive constant.

In particular, the study of η-Einstein manifolds is chosely related to the Sasakian Calabi problem. Many examples and interesting geometric properties associated with η-Einstein metrics are presented in [BGM06]. Regarding K-contact manifolds, Boyer and Galicki [BG00] proved that a compact Einstein K-contact manifolds is Sasakian. This is also true for compact K-contact η-Einstein manifolds with $\alpha > -2$. We also highlighted that the above result is also true if one relaxes the hypothesis of compactness to completeness (for details we refer to [HS81]).

3. Main result. Before we prove our main result, we prove the following

LEMMA 3.1. *On any η-Einstein contact metric manifold M of dimension $2n+1$ the transversal Bach tensor can be expressed as*

$$B(X,\xi) = \sum_{i=1}^{2n+1} (\nabla_{e_i} C)(e_i, Y)\xi. \tag{3.1}$$

Proof. If we use $m = 2n+1$, equation (1.4) becomes

$$B(X,Y) = \frac{1}{2n-1}\Big[\sum_i (\nabla_{e_i} C)(e_i, X)Y + \sum_{i,j} g(Qe_i, e_j) g(W(X,e_i)e_j, Y)\Big]. \tag{3.2}$$

We notice that the last term in (3.2) can be written as

$$g(Qe_i, e_j)g(W(X,e_i)e_j, Y) = -g(W(X,e_i)Y, Qe_i) = -g(QW(X,e_i)Y, e_i)$$

and hence (3.2) transforms into

$$B(X,Y) = \frac{1}{2n-1}\Big[\sum_i (\nabla_{e_i} C)(e_i, X)Y - \sum_i g(QW(X,e_i)Y, e_i)\Big]. \tag{3.3}$$

Finally, since the Weyl tensor is trace free and $QX = \alpha X + \beta\eta(X)\xi$, the last term of the foregoing equation is

$$\sum_i g(QW(X,e_i)Y, e_i) = \alpha \sum_i g(W(X,e_i)Y, e_i) + \beta \sum_i g(W(X,e_i)Y, \xi) g(\xi, e_i) = \beta g(W(X,\xi)Y, \xi).$$

Taking ξ instead of Y in (3.3) and using the preceding equation yields the required result. ■

Proof of Theorem 1.1. By hypothesis, M is η-Einstein. So taking the trace of (2.4) we obtain

$$r = (2n+1)\alpha + \beta. \tag{3.4}$$

Next, taking the covariant derivative of (2.4) along an arbitrary vector field X and using (2.1) yields

$$(\nabla_X S)(Y,Z) = (X\alpha)g(Y,Z) + (X\beta)\eta(Y)\eta(Z) - \beta[\eta(Y)g(Z, \varphi X + \varphi hX) + \eta(Z)g(Y, \varphi X + \varphi hX)]. \tag{3.5}$$

Since $m = 2n+1$ the Cotton tensor (1.3) reduces to

$$C(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) - \frac{1}{4n}[(Xr)g(Y,Z) - (Yr)g(X,Z)]. \tag{3.6}$$

By virtue of (3.5) and (3.4), (3.6) can be written as

$$
\begin{aligned}
(3.7)\qquad C(X,Y)Z = {} & \frac{2n-1}{4n}\{(X\alpha)g(Y,Z)-(Y\alpha)g(X,Z)\} \\
& + (X\beta)\left\{\eta(Y)\eta(Z)-\frac{1}{4n}g(Y,Z)\right\} \\
& - (Y\beta)\left\{\eta(X)\eta(Z)-\frac{1}{4n}g(X,Z)\right\} \\
& - \beta[\eta(Y)g(Z,\varphi X+\varphi hX) \\
& - \eta(X)g(Z,\varphi Y+\varphi hY)+2\eta(Z)g(Y,\varphi X)].
\end{aligned}
$$

Taking the trace of the foregoing equation over Y, Z and noting that the Cotton tensor is trace free, we obtain

$$
(3.8)\qquad (2n-1)(X\alpha)+(X\beta)-2(\xi\beta)\eta(X)=0.
$$

Writing this as

$$
d\{(2n-1)\alpha+\beta)=2(\xi\beta)\eta,
$$

where d is exterior differentiation, and then operating on it with d and recalling $d^2=0$ provides

$$
d(\xi\beta)\wedge\eta+(\xi\beta)d\eta=0.
$$

In other words, we can write

$$
(\xi\beta)d\eta(X,Y)=Y(\xi\beta)\eta(X)-X(\xi\beta)\eta(Y).
$$

For $X,Y\perp\xi$, the foregoing equation yields $\xi\beta=0$, as $d\eta$ is non-vanishing on M. Hence (3.8) reduces to $X\beta=-(2n-1)X\alpha$. By making use of this, (3.7) reduces to

$$
\begin{aligned}
(3.9)\qquad C(X,Y)Z = {} & \frac{2n-1}{2n}\{(X\alpha)g(Y,Z)-(Y\alpha)g(X,Z)\} \\
& + (2n-1)\{(Y\alpha)\eta(X)\eta(Z)-(X\alpha)\eta(Y)\eta(Z)\} \\
& - \beta[\eta(Y)g(Z,\varphi X+\varphi hX)-\eta(X)g(Z,\varphi Y+\varphi hY) \\
& - 2\eta(Z)g(\varphi X,Y)].
\end{aligned}
$$

Setting $Z=\xi$ in (3.9) and writing $X\alpha=g(D\alpha,X)$, where D is the gradient operator of g, we deduce

$$
\begin{aligned}
(3.10)\qquad C(X,Y)\xi = {} & -\frac{(2n-1)^2}{2n}\{g(X,D\alpha)\eta(Y)-g(Y,D\alpha)\eta(X)\} \\
& + 2\beta g(\varphi X,Y).
\end{aligned}
$$

If we differentiate (3.10) along an arbitrary vector field Z and use (2.1), we obtain

$$\begin{aligned}
&(\nabla_Z C)(X,Y)\xi - C(X,Y)(\varphi Z + \varphi hZ)\\
&\qquad = -\frac{(2n-1)^2}{2n}\{g(X,\nabla_Z D\alpha)\eta(Y) - g(Y,\nabla_Z D\alpha)\eta(X)\\
&\qquad\quad - (X\alpha)g(Y,\varphi Z + \varphi hZ) + (Y\alpha)g(X,\varphi Z + \varphi hZ)\}\\
&\qquad\quad - 2(2n-1)(Z\alpha)g(\varphi X,Y) + 2\beta g((\nabla_Z\varphi)X,Y),
\end{aligned}$$

where we have used $X\beta = -(2n-1)X\alpha$. By (3.9) the foregoing equation transforms to

(3.11)
$$\begin{aligned}
&(\nabla_Z C)(X,Y)\xi - \frac{2n-1}{2n}\{(X\alpha)g(Y,\varphi Z + \varphi hZ) - (Y\alpha)g(X,\varphi Z + \varphi hZ)\}\\
&\qquad + \beta\{\eta(Y)g(\varphi Z + \varphi hZ, \varphi X + \varphi hX)\\
&\qquad - \eta(X)g(\varphi Z + \varphi hZ, \varphi X + \varphi hX)\}\\
&\qquad = -\frac{(2n-1)^2}{2n}\{g(X,\nabla_Z D\alpha)\eta(Y) - g(Y,\nabla_Z D\alpha)\eta(X)\\
&\qquad\quad - (X\alpha)g(Y,\varphi Z + \varphi hZ) + (Y\alpha)g(X,\varphi Z + \varphi hZ)\}\\
&\qquad\quad - 2(2n-1)(Z\alpha)g(\varphi X,Y) + 2\beta g((\nabla_Z\varphi)X,Y).
\end{aligned}$$

Taking $X = Z = e_i$ in (3.11), summing over $i = 1,\dots,2n+1$ and using (3.1), (2.3) we obtain

$$\begin{aligned}
(3.12)\qquad & B(X,\xi) + \frac{2n-1}{2n}g(D\alpha,\varphi Y + h\varphi Y) + \beta\{2n + \mathrm{Tr}(h^2)\}\eta(Y)\\
&= -\frac{(2n-1)^2}{2n}\{(\operatorname{div} D\alpha)\eta(Y) - g(Y,\nabla_\xi D\alpha) + g(D\alpha,\varphi Y + h\varphi Y)\}\\
&\quad + 2(2n-1)g(D\alpha,\varphi Y) + 4n\beta\eta(Y),
\end{aligned}$$

where $\mathrm{Tr}(h^2)$ is the g-trace of the self adjoint operator h^2 (i.e. $\mathrm{Tr}(h^2) = \sum_i g(he_i,e_i)$) and $\operatorname{div} D\alpha = \sum_i g(\nabla_{e_i} D\alpha, e_i)$. By hypothesis the Bach tensor is purely tranversal. Hence the foregoing equation reduces to

$$\begin{aligned}
(3.13)\qquad & \frac{2n-1}{2n}g(D\alpha,\varphi Y + h\varphi Y) + \beta\{2n + \mathrm{Tr}(h^2)\}\eta(Y)\\
&= -\frac{(2n-1)^2}{2n}\{(\operatorname{div} D\alpha)\eta(Y) - g(Y,\nabla_\xi D\alpha) + g(D\alpha,\varphi Y + h\varphi Y)\}\\
&\quad + 2(2n-1)g(D\alpha,\varphi Y) + 4n\beta\eta(Y).
\end{aligned}$$

Since $X\beta = -(2n-1)X\alpha$ and $\xi\beta = 0$ (derived earlier), we have $\xi\alpha = 0$. Thus, using $\nabla_\xi\xi = 0$ and taking covariant differentiation of $g(\xi, D\alpha) = \xi\alpha$, we deduce $g(\xi,\nabla_\xi D\alpha) = \xi(\xi\alpha) = 0$. Taking ξ instead of Y in (3.13) and making use of the preceding equation yields

$$\frac{(2n-1)^2}{2n}\operatorname{div}(D\alpha) = \beta(\mathrm{Tr}(h^2) - 2n). \tag{3.14}$$

Inserting (3.14) into (3.13) it follows that

$$2ng(D\alpha, h\varphi Y) - 2ng(D\alpha, \varphi Y) = (2n-1)g(Y, \nabla_\xi D\alpha). \tag{3.15}$$

Now, $\xi\alpha = 0$ implies $\mathcal{L}_\xi \alpha = 0$, where $\mathcal{L}$ is the Lie derivative operator. Operating with d provides $\mathcal{L}_\xi\, d\alpha = 0$, from which we obtain $[\xi, D\alpha] = 0$. Therefore using (2.1) one can derive

$$\nabla_\xi D\alpha = \nabla_{D\alpha}\xi = -\varphi D\alpha - \varphi h D\alpha. \tag{3.16}$$

We now use (3.16) in (3.15) to achieve

$$hD\alpha = (4n-1)D\alpha. \tag{3.17}$$

In view of this, (3.16) takes the form

$$\nabla_\xi D\alpha = -4n\varphi D\alpha. \tag{3.18}$$

Next, we take covariant differentiation of (3.17) along ξ and recall (3.18), (3.17) in order to get

$$(\nabla_\xi h)D\alpha = -8n(4n-1)\varphi D\alpha. \tag{3.19}$$

On the other hand, taking covariant differentiation of (3.17) along an arbitrary vector field X and then taking the inner product of the resulting equation with Y shows that

$$((\nabla_X h)Y, D\alpha) + g(\nabla_X D\alpha, hY) = (4n-1)g(\nabla_X D\alpha, Y).$$

Anti-symmetrizing this equation and making use of $g(\nabla_X D\alpha, Y) = g(\nabla_Y D\alpha, X)$ it follows that

$$((\nabla_X h)Y - (\nabla_Y h)X, D\alpha) + g(\nabla_X D\alpha, hY) - g(\nabla_Y D\alpha, hX) = 0.$$

Replacing Y by ξ and recalling (2.1), (3.17), (3.18) and (3.17), we obtain $\varphi D\alpha = 0$. Operating on this with φ and since $\xi\alpha = 0$ it follows that α is constant. Further, since $X\beta = -(2n-1)X\alpha$, β is also constant, and hence the scalar curvature r is also constant. Then from (3.14) it follows that

$$\beta(\mathrm{Tr}(h^2) - 2n) = 0.$$

Since β is constant we have either $\beta = 0$, or $\beta \neq 0$. First, suppose that $\beta \neq 0$. Then $\mathrm{Tr}(h^2) = 2n$. By (2.2) and (2.4) the foregoing equation yields $\alpha + \beta = 0$. Using this we infer from (2.4) that $Q\xi = 0$. Taking covariant differentiation along an arbitrary vector field X and making use of (2.1), (2.4) yields

$$(\nabla_X Q)\xi = \alpha(\varphi X + \varphi h X). \tag{3.20}$$

Since the scalar curvature r is constant, we set $Z = \xi$ in (3.6) and use (3.20) to obtain

$$C(X,Y)\xi = 2\alpha g(\varphi X, Y).$$

On the other hand, (3.10) reduces to $C(X,Y)\xi = 2\beta g(\varphi X, Y)$. Comparing this with the preceding equation provides $(\alpha - \beta)d\eta(X,Y) = 0$. This implies that $\alpha = \beta$. Hence $\beta = 0$ and we arrive at a contradiction. Thus $\beta = 0$ and M is Einstein. This completes the proof. ∎

Proof of Corollary 1.2. For a K-contact manifold we have $h = 0$, and hence from the previous theorem, we can conclude that M is Einstein. But for a K-contact manifold we know that $S(X,\xi) = 2n\eta(X)$. Hence M is Einstein with Einstein constant $2n$. By hypothesis, M is complete and therefore by Myers's theorem [M35], M is compact. Finally, by a result of Boyer and Galicki [BG00], a compact K-contact Einstein manifold is Sasakian. This completes the proof. ∎

Remark 3.2. For dimension 3, the Weyl curvature tensor vanishes. Hence from Theorem 1.1, a 3-dimensional η-Einstein contact metric manifold M with purely transversal Bach tensor is either of constant curvature or Sasakian. But it is known [BS90] that any 3-dimensional contact metric manifold of constant curvature is either flat, or Sasakian of constant curvature 1.

Acknowledgements. The author is grateful to the anonymous referee for some valuable comments.

References

[B21] R. Bach, *Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krummungstensorbegriffs*, Math. Z. 9 (1921), 110–135.

[B08] A. L. Besse, *Einstein Manifolds*, Springer, Berlin, 2008.

[B02] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progr. Math. 203, Birkhäuser, Basel, 2002.

[BS90] D. E. Blair and R. Sharma, *Three dimensional locally symmetric contact metric manifolds*, Boll. Un. Mat. Ital. A (7) 4 (1990), 385–390.

[BG00] C. P. Boyer and K. Galicki, *On Sasakian-Einstein geometry*, Int. J. Math. 11 (2000), 873–909.

[BGM06] C. P. Boyer, K. Galicki and P. Matzeu, *On η-Einstein Sasakian geometry*, Comm. Math. Phys. 262 (2006), 177–208.

[CG+19] E. Calviño-Louzao, E. García-Río, P. Gilkey, I. Gutiérrez-Rodríguez and R. Vázquez-Lorenzo, *Constructing Bach flat manifolds of signature* $(2,2)$ *using the modified Riemannian extension*, J. Math. Phys. 60 (2019), art. 013511, 14 pp.

[CHS85] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings*, Nuclear Phys. B 258 (1985), 46–74.

[CH13] Q. Chen and C. He, *On Bach flat warped product Einstein manifolds*, Pacific J. Math. 265 (2013), 313-326.

[G20] A. Ghosh, *Cotton tensor, Bach tensor and Kenmotsu manifolds*, Afr. Mat. 31 (2020), 1193–1205.

[GS17] A. Ghosh and R. Sharma, *Sasakian manifolds with purely transversal Bach tensor*, J. Math. Phys. 58 (2017), art. 103502, 6 pp.

[GS19] A. Ghosh and R. Sharma, *Classification of* (k, μ)*-contact manifolds with divergence free Cotton tensor and vanishing Bach tensor*, Ann. Polon. Math. 122 (2019), 153–163.

[HS81] I. Hasegawa and M. Seino, *Some remarks on Sasakian geometry-Applications of Myers' theorem and the canonical affine connection*, J. Hokkaido Univ. Ed. Sect. II A 32 (1981/82), 1–7.

[K72] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tôhoku Math. J. (2) 24 (1972), 93–103.

[M98] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2 (1998), 231–252.

[M35] S. B. Myers, *Connections between differential geometry and topology. I. Simply connected surfaces*, Duke Math. J. 1 (1935), 376–391.

[O62] M. Okumura, *On infinitesimal conformal and projective transformations of normal contact spaces*, Tôhoku Math. J. (2) 14 (1962), 398–412.

[O79] Z. Olszak, *On contact metric manifolds*, Tôhoku Math. J. (2) 31 (1979), 247–253.

[T68] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois J. Math. 12 (1968), 700–717.

[YK84] K. Yano and M. Kon, *Structures on Manifolds*, World Sci., 1984.

Amalendu Ghosh
Department of Mathematics
Chandernagore College
Chandannagar, Hooghly 712 136, West Bengal, India
E-mail: aghosh_70@yahoo.com