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TREE REPRESENTATIONS OF THE QUIVER $\widetilde{\mathbb{D}}_m$

BY

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Abstract. We explicitly describe field independent tree representations of the canonically oriented quiver $\widetilde{\mathbb{D}}_m$. Recall that matrices of tree representations involve only the elements 0 and 1 and the total number of ones is exactly $d-1$ where d is the length of the module. Due to a result of Ringel (1998) the existence of tree representations is guaranteed when the module is exceptional (indecomposable and without self-extensions). In this paper we give a complete and general list corresponding to exceptional modules over the path algebra of the canonically oriented Euclidean quiver $\widetilde{\mathbb{D}}_6$ and a method to obtain tree representations for exceptionals in the canonically oriented general case $\widetilde{\mathbb{D}}_m$ from that list. The proof (involving induction and symbolic computation with block matrices) was partially generated by a purposefully developed computer software and is available on arXiv as an appendix to this paper. All the representations given here remain valid over any base field, answering an open question raised/suggested by Ringel also in the $\widetilde{\mathbb{D}}_m$ case.

1. Theoretical background

1.1. Basic notions of representation theory of algebras. Let $Q = (Q_0, Q_1, s, t)$ be a *quiver*, that is, a directed graph, where Q_0 is the set of vertices, Q_1 is the set of arrows and $s, t : Q_1 \to Q_0$ are functions which attach to an arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$. We often write briefly $Q = (Q_0, Q_1)$. Let k be a field and consider the path algebra kQ. The category mod-kQ of finite-dimensional right modules over kQ can be identified with the category rep-kQ of the finite-dimensional k-representations of the quiver Q (therefore we will use the terms "module" and "representation" interchangeably).

Recall that a *k-representation* $M = (M_i, M_\alpha)$ of Q is defined as a set $\{M_i \mid i \in Q_0\}$ of finite-dimensional k-spaces corresponding to the vertices together with k-linear maps $\{M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)} \mid \alpha \in Q_1\}$ corresponding to the arrows. Given two representations $M = (M_i, M_\alpha)$ and $N = (N_i, N_\alpha)$ of Q, a *morphism of representations* $f : M \to N$ is a family of k-linear maps (corresponding to the vertices) $f_i : M_i \to N_i$ such that $N_\alpha f_{s(\alpha)} = f_{t(\alpha)} M_\alpha$

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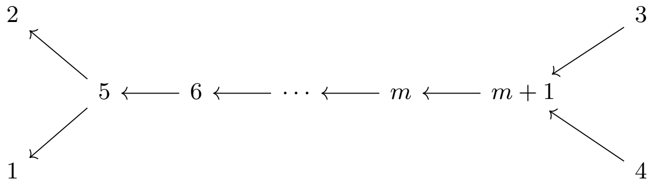
for all $\alpha \in Q_1$. The *dimension vector* of a representation $M = (M_i, M_\alpha)$ is

$$\underline{\dim}\, M = (d_i)_{i \in Q_0} \in \mathbb{Z}Q_0 \quad \text{where} \quad d_i = \dim_k M_i,$$

which is treated as an n-dimensional row vector where $n = |Q_0|$. In this case the *length* of M is $\ell(M) = \sum_{i \in Q_0} d_i$.

There are five types of *Euclidean* (or *tame*) *quivers*: $\widetilde{\mathbb{A}}_m$, $\widetilde{\mathbb{D}}_m$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$. We will consider the canonically oriented quiver of type $\widetilde{\mathbb{D}}_m$ with $m \geq 4$ (having $m+1$ vertices), denoted by $\Delta(\widetilde{\mathbb{D}}_m)$, having the following shape:



We identified the vertices with their labels so $\Delta(\widetilde{\mathbb{D}}_m)_0 = \{1, \dots, m, m+1\}$, and since we have at most one arrow connecting two different vertices, the set of arrows is

$$\Delta(\widetilde{\mathbb{D}}_m)_1 = \{(5 \to 1),\, (5 \to 2),\, (3 \to m+1),\, (4 \to m+1),\\ (6 \to 5),\, (7 \to 6),\, \dots,\, (m+1 \to m)\}.$$

The *Euler form* of an arbitrary quiver Q is the bilinear form defined on $\mathbb{Z}Q_0$ as

$$\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}.$$

Its quadratic form q_Q (called the *Tits form*) is independent of the orientation of Q and in the tame case it is positive semi-definite with radical $\mathbb{Z}\delta$, where δ is a minimal positive imaginary root of the corresponding Kac–Moody root system. A vector $x \in \mathbb{Z}Q_0$ is called a *real root* if $q_Q(x) = 1$, an *imaginary root* if $q_Q(x) = 0$, and it is *positive* if $x_i \in \mathbb{N}$ for all $i \in Q_0$. The *defect* of $x \in \mathbb{Z}Q_0$ is then $\partial x = \langle \delta, x \rangle$. For two (dimension) vectors $d, d' \in \mathbb{Z}Q_0$ we write $d \leq d'$ if $d_i \leq d'_i$ for all $i \in Q_0$.

In the $\widetilde{\mathbb{D}}_m$ case (using the vertex numbering above) the Tits form is

$$q_{\widetilde{\mathbb{D}}_m}(x) = \frac{1}{4}\Big((2x_1 - x_5)^2 + (2x_2 - x_5)^2 + (x_{m+1} - 2x_3)^2 + (x_{m+1} - 2x_4)^2 \\ + 2\sum_{i=5}^{m}(x_i - x_{i+1})^2\Big).$$

Its minimal positive imaginary root is $\delta = (1,1,1,1,2,\dots,2)$.

For an arbitrary acyclic quiver Q let $P(i)$ and $I(i)$ be the indecomposable projective respectively injective module corresponding to the vertex i. The

Cartan matrix C_Q is the matrix whose jth column is $\underline{\dim}\, P(j)$. The *Coxeter matrix* is defined as $\Phi_Q = -C_Q^t C_Q^{-1}$. Then $\Phi_Q \delta = \delta$ and the Euler form satisfies $\langle a, b\rangle = a(C_Q^{-1})^t b^t = -\langle b, \Phi_Q a\rangle$, where $a, b \in \mathbb{Z}Q_0$ (treated as row vectors). Moreover, because our algebra is hereditary, for $M, N \in$ mod-kQ we get

$$\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle = \dim_k \operatorname{Hom}_{kQ}(M, N) - \dim_k \operatorname{Ext}^1_{kQ}(M, N). \tag{1.1}$$

The *Auslander–Reiten translate*s are defined as

$$\tau = D \operatorname{Ext}^1_{kQ}(-, kQ) \quad \text{and} \quad \tau^{-1} = \operatorname{Ext}^1_{kQ}(D(kQ), -)$$

where $D = \operatorname{Hom}_k(-, k)$.

An indecomposable module M is *preprojective* (resp. *preinjective*) if there exists a positive integer s such that $\tau^s(M) = 0$ (resp. $\tau^{-s}(M) = 0$). The indecomposable M is *regular* if it is neither preinjective nor preprojective.

From now on let Q be a tame quiver. The structure of the category mod-kQ and its Auslander–Reiten quiver are well-known. Up to isomorphism, the indecomposable preprojective modules are $\tau^{-s}P(i)$, while the indecomposable preinjectives are $\tau^s I(i)$, where $s \in \mathbb{N}$ and $i \in Q_0$. In what follows we use the somewhat more convenient notation $P(s, i)$ for $\tau^{-s}P(i)$ and $I(s, i)$ for $\tau^s I(i)$. Their dimension vectors satisfy

$$\underline{\dim}\, P(s, i) = \Phi_Q^{-s} \cdot \underline{\dim}\, P(i) \quad \text{and} \quad \underline{\dim}\, I(s, i) = \Phi_Q^{s} \cdot \underline{\dim}\, I(i). \tag{1.2}$$

The category of regular modules is an abelian, exact subcategory which decomposes into a direct sum of serial categories with Auslander–Reiten quivers of the form $\mathbb{Z}\mathbb{A}_\infty/r$, called *tube*s of rank r. A tube of rank 1 is called *homogeneous*, other tubes are *non-homogeneous*.

In the case of $\widetilde{\mathbb{D}}_m$ there are three non-homogeneous tubes, all the other tubes are homogeneous. The non-homogeneous tubes are usually labeled by elements of the set $\{0, 1, \infty\}$, hence $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_m)}$, $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_m)}$ and $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_m)}$ denote the non-homogeneous tubes. We denote by $R_e^l(t)$ a non-homogeneous regular indecomposable, where $e \in \{0, 1, \infty\}$ with $t \in \mathbb{N}^*$ being its regular length and $l \in \{1, \ldots, r\}$, where r is the rank of the tube $\mathcal{T}_e^{\Delta(\widetilde{\mathbb{D}}_m)}$. A module $R_e^l(1)$ is regular simple and it is the quasi-socle of $R_e^l(t)$ for $t \geq 1$. We have $\tau R_e^l(t) = R_e^{l-1}(t)$ for $l \geq 2$ and $\tau R_e^1(t) = R_e^r(t)$. The tubes $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_m)}$ and $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_m)}$ have rank 2, while $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_m)}$ has rank $m - 2$. Hence there are two regular simples in the cases of $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_m)}$ and $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_m)}$, and $m-2$ regular simples in the case of $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_m)}$ (here regular simple means "simple in the subcategory of non-homogeneous regulars", so these are the modules of the form $R_e^l(1)$, with regular length 1). In the case of the quiver $\widetilde{\mathbb{D}}_m$ (with any orientation) the structure of the Auslander–Reiten quiver remains the same over every field (see [23]).

An important fact is that mod-kQ is a *Krull–Schmidt category*, meaning that every module can be written as a direct sum of indecomposables in a unique way (up to order and isomorphism).

Since Q is tame, it is well-known that the dimension vector x of an indecomposable is either a positive real root (i.e. $q_Q(x) = 1$) or a positive imaginary root (i.e. $q_Q(x) = 0$). It is also known that for every positive real root x there is a unique (up to isomorphism) indecomposable M with $\underline{\dim}\, M = x$ (in fact these indecomposables are all the preprojectives, all the preinjectives and the non-homogeneous regular indecomposables with dimension vector different from a multiple of δ).

An indecomposable module M is called *exceptional* if it has no self-extensions (i.e. $\dim_k \mathrm{Ext}^1_{kQ}(M, M) = 0$). This means that its dimension vector is a positive real root (called an *exceptional root*) and $\dim_k \mathrm{End}_{kQ}(M) = 1$. We know that the exceptional indecomposable modules are all the preprojectives, all the preinjectives and the regular non-homogeneous indecomposables with dimension vector falling below δ (see [4]). Thus a positive real root x is exceptional if and only if either $\partial x \neq 0$ or $\partial x = 0$ with $x < \delta$. Note that an indecomposable module M whose dimension vector is an exceptional root, is an exceptional module.

For more details concerning the notions presented in this subsection we refer to [2, 1, 21, 23].

1.2. Tree representations. An indecomposable module $M = (M_i, M_\alpha)$ is called a *tree module* if there is a basis B such that the matrices of the linear maps M_α, written in the basis B, consist only of elements 0 and 1, and the total number of non-zero elements is $\ell(M) - 1$, where $\ell(M) = \sum_{i \in Q_0} d_i$ with $\underline{\dim}\, M = (d_i)_{i \in Q_0}$. Equivalently, M is a tree module if there exists a basis B such that the associated coefficient quiver is a tree (for details see [17]).

In [17] Ringel proves, for arbitrary quivers, that exceptional modules are tree modules. The proof is based on the result by Schofield [19] stating that if M is an exceptional module that is not simple, then there are exceptional modules X, Y with $\mathrm{Hom}_{kQ}(X, Y) = \mathrm{Hom}_{kQ}(Y, X) = \mathrm{Ext}^1_{kQ}(Y, X) = 0$ and an exact sequence $0 \to vY \to M \to uX \to 0$, where u and v are positive integers and uY means $Y \oplus \cdots \oplus Y$ (u times). There are precisely $s(M) - 1$ such sequences where $s(M)$ is the number of non-zero components in $\underline{\dim}\, M$; we call these sequences *Schofield sequence*s and the pair (X, Y) a *Schofield pair* (associated to M). Note that Schofield's original proof assumes an algebraically closed field, but Ringel [18] gives a proof which works over an arbitrary field k. Proposition 6 from [22] states that if X, Y, M are exceptional indecomposables such that $u\,\underline{\dim}\, X + v\,\underline{\dim}\, Y = \underline{\dim}\, M$, then we have a Schofield sequence $0 \to vY \to M \to uX \to 0$ if and only if

$\langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle = 0$. This means that Schofield sequences and pairs depend only on the dimensions of indecomposables, thus their existence is field independent. Also note that although the short exact sequences used in our proofs are Schofield sequences (as above, with $v = u = 1$), we do not use the results from [19] or [18] to construct them, but every short exact sequence used throughout the proofs in [11] is written (and verified) using Lemma 16 (valid over an arbitrary field k).

Although tree representations for some particular quivers are known, the proof in [17] does not give an explicit method for constructing them in general.

In [7] Gabriel gave a full list of indecomposable representations for Dynkin quivers using 0-1-matrices. All but four of the representations given were tree representations. Tree representations in these four cases were given by Crawley-Boevey [3].

Regarding the Euclidean case, Mróz [14] gave a full list of indecomposable tree representations for the quiver of type $\widetilde{\mathbb{D}}_4$ with four subspace orientation. His results were generalized by Lőrinczi and Szántó, giving a full list of tree representations for the quiver of type $\widetilde{\mathbb{D}}_6$ with a particular non-canonical orientation (see [13]). We note that these representations were proved for path algebras over algebraically closed fields only, and in [13] indecomposability was checked only for some random representations from the list (so the checking was incomplete). Analogous problems are considered for canonical algebras in [5], for nilpotent operators in [6] and for poset representations in [8].

Concerning the $\widetilde{\mathbb{D}}_m$ and $\widetilde{\mathbb{E}}_8$ cases, indecomposable representations for preinjectives and preprojectives were given by Kussin, Kędzierski and Meltzer in [9] and [10], respectively (however, those representations are not tree representations). Their aim was not to give explicit representations, but to describe a general method for obtaining indecomposable (not necessarily tree) representations in tame cases.

In [12], using a computer generated proof, the authors managed to describe explicitly, in a field independent manner, tree representations for all possible exceptional indecomposables in the case of the canonically oriented quiver $\widetilde{\mathbb{E}}_6$. In the present paper we will use the same route, but we also have to figure out how to reduce the generic $\Delta(\widetilde{\mathbb{D}}_m)$ case (which cannot be considered in a computer generated proof context) to the $\Delta(\widetilde{\mathbb{D}}_6)$ case. Tree representations for $\Delta(\widetilde{\mathbb{D}}_6)$ (listed in Section 2) were obtained by experimentation in the GAP computer algebra system, followed by guessing the general formula. These matrices were then fed as input to a purposefully developed proof assistant software, performing symbolic computation and carrying out a proof by induction for all of these formulas. The exact method (as well as some theoretical results) is fully described in [12], so we refer the reader to that paper for explanations and details.

The importance of knowing explicit formulas for tree representations stems from a number of advantageous properties. In the case of tree representations, the matrices involved are the "sparsest possible" (i.e. containing the minimal number of non-zero elements), thus reducing the storage and running time complexity in computer implementations. As mentioned before, the exceptional modules are determined by their dimension vectors up to isomorphism, so having a formula for each of them gives a "nice" representative of each isomorphism class. In fact, we could say that tree representations are the "canonical" forms of these modules, analogously to the canonical form of matrix pencils or canonical forms of matrices (for example the Jordan normal form). An example of nice consequences of knowing such sparse forms is the paper [15] by Mróz, where such matrix forms of modules were applied to obtain formulas for the multiplicities of the preprojective and preinjective indecomposables appearing in the decomposition of an arbitrary (four subspace oriented) $\widetilde{\mathbb{D}}_4$-module.

Since the elements involved are only 0 and 1, some natural questions arise regarding the validity of certain (or all) tree representations over arbitrary fields. These questions are stated in [17]. The best-case scenario would be to give tree representations independently of the underlying field.

It is important to realize that *the tree representations given in this paper remain valid independently of the underlying field* (as was also the case for $\widetilde{\mathbb{E}}_6$ in [12]). That is, the 1-0 matrices listed in this paper withstand a replacement of the base field k in mod-$k\Delta(\widetilde{\mathbb{D}}_m)$ by another field k', so that if $M \in$ rep-$k\Delta(\widetilde{\mathbb{D}}_m)$ is an exceptional tree representation, then $M' \in$ rep-$k'\Delta(\widetilde{\mathbb{D}}_m)$ is also an exceptional tree representation whenever $\underline{\dim}\, M = \underline{\dim}\, M'$ and every matrix M_α from the first representation is formally the same as the corresponding matrix M'_α from the second one. So our conjecture as formulated in [12] still holds: we strongly believe that all tree representations of all tame quivers (independently of orientation) are field independent.

1.3. Constructing tree representations for $\Delta(\widetilde{\mathbb{D}}_m)$ with $m \geq 7$, $\Delta(\widetilde{\mathbb{D}}_4)$ and $\Delta(\widetilde{\mathbb{D}}_5)$ from trees in $\Delta(\widetilde{\mathbb{D}}_6)$. In this section we present an explicit method for solving the following problem: given an exceptional root x in $\Delta(\widetilde{\mathbb{D}}_m)$ where $m \geq 4$, construct an (exceptional) tree representation $M \in$ rep-$k\Delta(\widetilde{\mathbb{D}}_m)$ such that $\underline{\dim}\, M = x$. Recall that a positive real root x is exceptional if either $\partial x \neq 0$, or $\partial x = 0$ and $x < \delta$. Throughout this subsection we denote the identity matrix by I_n (in case $n = 0$ we take I_0 to be the null morphism).

We begin with two lemmas on the form of real roots of the quiver $\Delta(\widetilde{\mathbb{D}}_m)$, where $m \geq 6$. For brevity we write $a, \overset{(i)}{\ldots}, a$ for $\overbrace{a, \ldots, a}^{i \text{ times}}$.

LEMMA 1. *Let x be a real root of the quiver $\Delta(\widetilde{\mathbb{D}}_m)$. Then the form of x is one of the following:*

- $x^{(1)} = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a)$, *where* $i = m-3$;
- $x^{(2)} = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a, b, \overset{(j)}{\ldots}, b)$, *where* $i, j \in \mathbb{N}^*$, $i + j = m-3$ *and* $a \neq b$;
- $x^{(3)} = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a, b, \overset{(j)}{\ldots}, b, a, \overset{(k)}{\ldots}, a)$, *where* $i, j, k \in \mathbb{N}^*$, $i+j+k = m-3$ *and* $a \neq b$;
- $x^{(4)} = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a, b, \overset{(j)}{\ldots}, b, c, \overset{(k)}{\ldots}, c)$, *where* $i, j, k \in \mathbb{N}^*$, $i+j+k = m-3$ *and a, b, c are pairwise different.*

Proof. Since $x = (x_1, \ldots, x_{m+1})$ is a real root, we have $q_{\Delta(\widetilde{\mathbb{D}}_m)}(x) = 1$. But this means

$$\begin{aligned} 4 &= 4q_{\Delta(\widetilde{\mathbb{D}}_m)}(x) \\ &= (2x_1 - x_5)^2 + (2x_2 - x_5)^2 + (x_{m+1} - 2x_3)^2 \\ &\quad + (x_{m+1} - 2x_4)^2 + 2\sum_{i=5}^{m} (x_i - x_{i+1})^2. \end{aligned}$$

Because the components of x are integers, we conclude that $\sum_{i=5}^{m}(x_i - x_{i+1})^2$ is either $0, 1$ or 2. If this sum equals 0 then x has the form $x^{(1)}$. If it equals 1, then x is of the form $x^{(2)}$, while in the last case it may have the form $x^{(3)}$ or $x^{(4)}$. ∎

Combining these four possibilities for x we get the following (alternative) form:

LEMMA 2. *Let x be a real root of the quiver $\Delta(\widetilde{\mathbb{D}}_m)$. Then x has the form*

$$x = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a, b, \overset{(j)}{\ldots}, b, c, \overset{(k)}{\ldots}, c)$$

with a, b and c not necessarily distinct, $i, j, k \in \mathbb{N}^*$ and $i + j + k = m - 3$.

Let us denote by $\mathfrak{R}_m$ the set of exceptional roots over $\Delta(\widetilde{\mathbb{D}}_m)$.

For $m \geq 7$ we introduce $\mathfrak{p}_m : \mathfrak{R}_m \to \mathfrak{R}_6$, where $\mathfrak{p}_m(x) = x'$ with $x \in \mathfrak{R}_m$ constructed according to the following cases (as specified in Lemma 1):

- if $x = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a)$, where $i = m-3$, then $x' = (x_1, x_2, x_3, x_4, a, a, a)$;
- if $x = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a, b, \overset{(j)}{\ldots}, b)$, where $i, j \in \mathbb{N}^*$, $i + j = m - 3$ and $a \neq b$, then $x' = (x_1, x_2, x_3, x_4, a, a, b)$ in case $i \geq 2$, else $x' = (x_1, x_2, x_3, x_4, a, b, b)$;
- if $x = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a, b, \overset{(j)}{\ldots}, b, a, \overset{(k)}{\ldots}, a)$, where $i, j, k \in \mathbb{N}^*$, $i+j+k = m-3$ and $a \neq b$, then $x' = (x_1, x_2, x_3, x_4, a, b, a)$;
- if $x = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a, b, \overset{(j)}{\ldots}, b, c, \overset{(k)}{\ldots}, c)$, where $i, j, k \in \mathbb{N}^*$, $i+j+k = m-3$ and a, b, c are pairwise different, then $x' = (x_1, x_2, x_3, x_4, a, b, c)$.

Lemma 3. *For any $m \geq 7$, the previously introduced $\mathfrak{p}_m : \mathfrak{R}_m \to \mathfrak{R}_6$ is a well-defined surjective function. Moreover, defects are also preserved (i.e. for all $x \in \mathfrak{R}_m$, $\partial_{k\Delta(\widetilde{\mathbb{D}}_m)} x = \partial_{k\Delta(\widetilde{\mathbb{D}}_6)} \mathfrak{p}_m(x)$).*

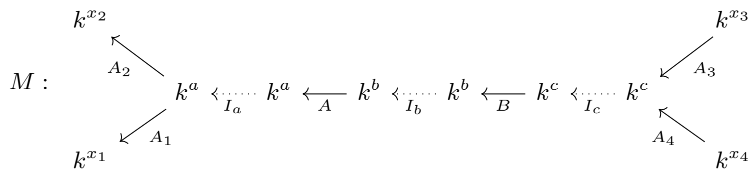
Proof. Let $x \in \mathfrak{R}_m$ be an exceptional root over $\Delta(\widetilde{\mathbb{D}}_m)$ written as in Lemma 2 and $x' = (x_1, x_2, x_3, x_4, a, b, c)$. Using these notations we can write

$$
\begin{aligned}
(1.3)\qquad 4 &= 4q_{\Delta(\widetilde{\mathbb{D}}_m)}(x)\\
&= (2x_1 - a)^2 + (2x_2 - a)^2 + (c - 2x_3)^2 + (c - 2x_4)^2\\
&\quad + 2((a-a)^2 + \cdots + (a-b)^2 + (b-b)^2 + \cdots + (b-c)^2 + \cdots + (c-c)^2)\\
&= (2x_1 - a)^2 + (2x_2 - a)^2 + (c - 2x_3)^2 + (c - 2x_4)^2 + 2((a-b)^2 + (b-c)^2)\\
&= 4q_{\Delta(\widetilde{\mathbb{D}}_6)}(x').
\end{aligned}
$$

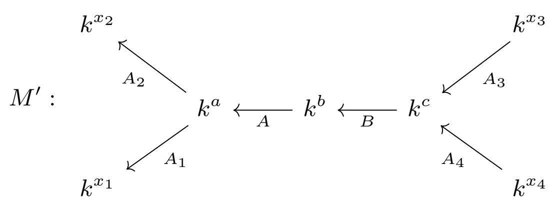
This shows that $x' = \mathfrak{p}_m(x)$ is also a positive real root over $\Delta(\widetilde{\mathbb{D}}_6)$. For the defects we have $\partial_{k\Delta(\widetilde{\mathbb{D}}_m)} x = x_3 + x_4 - x_1 - x_2 = \partial_{k\Delta(\widetilde{\mathbb{D}}_6)} x'$ (see [1, p. 150]), so defects are preserved by $\mathfrak{p}_m$. Moreover, one can easily see that x is an exceptional root over $\Delta(\widetilde{\mathbb{D}}_m)$ if and only if x' is an exceptional root over $\Delta(\widetilde{\mathbb{D}}_6)$: this is because $\partial_{k\Delta(\widetilde{\mathbb{D}}_6)} x' = \partial_{k\Delta(\widetilde{\mathbb{D}}_m)} x$ and $x < \delta_{\Delta(\widetilde{\mathbb{D}}_m)} \Leftrightarrow x' < \delta_{\Delta(\widetilde{\mathbb{D}}_6)}$. Surjectivity is also clear by (1.3). ∎

In the following drawings the dotted arrows represent zero or more arrows of the form $k^d \xleftarrow[I_d]{} k^d$ ($d \in \{a, b, c\}$) connecting vertices with the same dimension, with suitable identity matrices associated to them.

Lemma 4. *Let $m \geq 7$, $x \in \mathfrak{R}_m$ (as in Lemma 2), $x' = (x_1, x_2, x_3, x_4, a, b, c) \in \mathfrak{R}_6$ such that $\mathfrak{p}_m(x) = x'$, and let $M = (M_\alpha, M_i) \in$ rep-$k\Delta(\widetilde{\mathbb{D}}_m)$ and $M' = (M'_\alpha, M'_i) \in$ rep-$k\Delta(\widetilde{\mathbb{D}}_6)$ with $\underline{\dim} M = x$ and $\underline{\dim} M' = x'$ have the following matrices:*



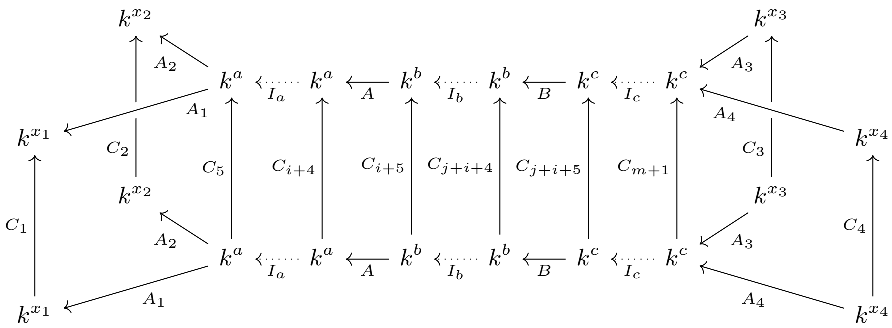
and



Then M is exceptional if and only if M' is exceptional.

Proof. One can see that we have the following correspondence between the matrices of these representations: $M_{5\to1} = M'_{5\to1}$, $M_{5\to2} = M'_{5\to2}$, $M_{3\to(m+1)} = M'_{3\to7}$, $M_{4\to(m+1)} = M'_{4\to7}$, $M_{(4+i+1)\to(4+i)} = M'_{6\to5}$, $M_{(4+i+j+1)\to(4+i+j)} = M'_{7\to6}$. To all other arrows in the representation M we assign suitable identity matrices.

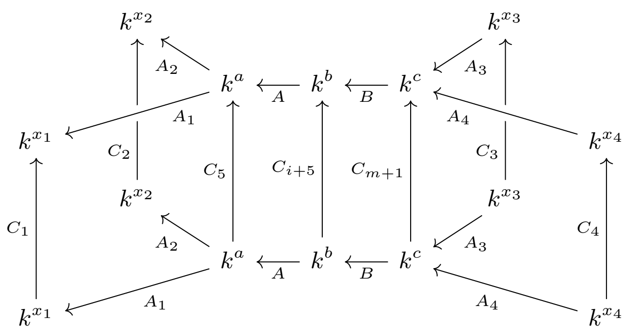
First we show that M is indecomposable if and only if M' is. Consider an endomorphism of M, that is, a collection of matrices $(C_i)_{i\in\Delta(\widetilde{\mathbb{D}}_m)_0}$ compatible with the representation M, making the following diagram commutative:



It follows that

$$\left.\begin{aligned} A_1C_5 &= C_1A_1\\ A_2C_5 &= C_2A_2\\ C_5 = C_6 &= \cdots = C_{i+4}\\ C_{i+4}A &= AC_{i+5}\\ C_{i+5} &= \cdots = C_{j+i+4}\\ C_{j+i+4}B &= BC_{j+i+5}\\ C_{j+i+5} &= \cdots = C_{m+1}\\ C_{m+1}A_3 &= A_3C_3\\ C_{m+1}A_4 &= A_4C_4 \end{aligned}\right\} \iff \left\{\begin{aligned} A_1C_5 &= C_1A_1\\ A_2C_5 &= C_2A_2\\ C_5A &= AC_{i+5}\\ C_{i+5}B &= BC_{m+1}\\ C_{m+1}A_3 &= A_3C_3\\ C_{m+1}A_4 &= A_4C_4 \end{aligned}\right.$$

making the following diagram commutative:



So $(C_i)_{i\in\Delta(\widetilde{\mathbb{D}}_m)_0}$ is an endomorphism of M if and only if $(C_1, C_2, C_3, C_4, C_5, C_{i+5}, C_{m+1})$ is an endomorphism of M'. But $\dim_k \operatorname{End}_{k\Delta(\widetilde{\mathbb{D}}_m)}(M) = 1$ if and only if

$$(C_i)_{i\in\Delta(\widetilde{\mathbb{D}}_m)_0} = \beta \cdot (I_{x_1}, I_{x_2}, I_{x_3}, I_{x_4}, I_a, \dots, I_a, I_b, \dots, I_b, I_c, \dots, I_c),$$

if and only if

$$(C_1, C_2, C_3, C_4, C_5, C_{i+5}, C_{m+1}) = \beta \cdot (I_{x_1}, I_{x_2}, I_{x_3}, I_{x_4}, I_a, I_b, I_c)$$

(with $\beta \in k$), if and only if $\dim_k \operatorname{End}_{k\Delta(\widetilde{\mathbb{D}}_6)}(M') = 1$. Therefore M is indecomposable if and only if M' is. But since their dimensions are exceptional roots, we are done. ∎

We are going to explicitly construct a function $T_m : \mathfrak{R}_m \to \text{rep-}k\Delta(\widetilde{\mathbb{D}}_m)$ such that $T_m(x)$ with $\underline{\dim}\, T_m(x) = x$ is a tree representation for any exceptional root x ($m \geq 4$). Here we treat rep-$k\Delta(\widetilde{\mathbb{D}}_m)$ as a set consisting of only "matrix representations" of Q, where a "matrix representation" is just a collection of matrices of suitable dimensions together with induced vector spaces of the form k^s, encoding a representation of Q.

Constructing tree representations of $\Delta(\widetilde{\mathbb{D}}_6)$. We begin with the $m=6$ case, since by construction the lists in Section 2 define exactly such a function T_6. One can take any exceptional root x over $\Delta(\widetilde{\mathbb{D}}_6)$, identify the corresponding family of representations (based on ∂x and the general forms of the dimension vectors) and apply the right formula to obtain the matrices of the representations. So we can state the following:

Proposition 5. *For any exceptional root x over $\Delta(\widetilde{\mathbb{D}}_6)$ the formulas listed in Section 2 define a function $T_6 : \mathfrak{R}_6 \to \text{rep-}k\Delta(\widetilde{\mathbb{D}}_6)$ with $T_6(x)$ a tree representation.*

Constructing tree representations of $\Delta(\widetilde{\mathbb{D}}_m)$ *with* $m \geq 7$. For the $m \geq 7$ case we define $T_m : \mathfrak{R}_m \to \text{rep-}k\Delta(\widetilde{\mathbb{D}}_m)$ as follows: for $x \in \mathfrak{R}_m$ let $T_m(x) = M$ where the representation $M \in \text{rep-}k\Delta(\widetilde{\mathbb{D}}_m)$ is constructed from $M' = T_6(\mathfrak{p}_m(x)) \in \text{rep-}k\Delta(\widetilde{\mathbb{D}}_6)$; specific matrices of the representation M are $M_{5\to1} = M'_{5\to1}$, $M_{5\to2} = M'_{5\to2}$, $M_{3\to(m+1)} = M'_{3\to7}$, $M_{4\to(m+1)} = M'_{4\to7}$; the other matrices depend on the possible forms of x (see Lemma 1):

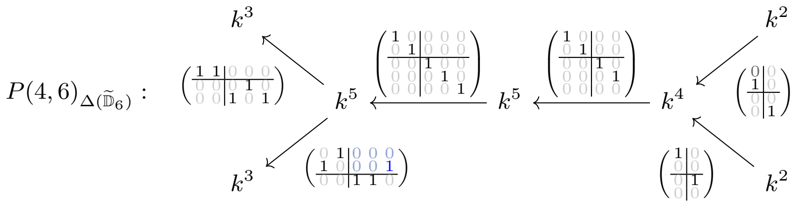
- if $x = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\dots}, a)$, where $i = m-3$, then $M_{m\to(m-1)} = M'_{6\to5}$, $M_{(m+1)\to m} = M'_{7\to6}$ and to all other arrows we assign identity matrices I_a;
- if $x = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\dots}, a, b, \overset{(j)}{\dots}, b)$, where $i, j \in \mathbb{N}^*$, $i + j = m-3$ and $a \neq b$, then $M_{(4+i)\to(3+i)} = M'_{6\to5}$, $M_{(4+i+1)\to(4+i)} = M'_{7\to6}$ in case $i \geq 2$, else (if $i = 1$) $M_{6\to5} = M'_{6\to5}$, $M_{7\to6} = M'_{7\to6}$; to all other arrows we assign appropriate identity matrices (either I_a or I_b);

- if $x = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a, b, \overset{(j)}{\ldots}, b, a, \overset{(k)}{\ldots}, a)$, where $i, j, k \in \mathbb{N}^*$, $i + j + k = m - 3$ and $a \neq b$, then $M_{(4+i+1)\to(4+i)} = M'_{6\to5}$, $M_{(4+i+j+1)\to(4+i+j)} = M'_{7\to6}$ and we assign identity matrices to all other arrows;
- if $x = (x_1, x_2, x_3, x_4, a, \overset{(i)}{\ldots}, a, b, \overset{(j)}{\ldots}, b, c, \overset{(k)}{\ldots}, c)$, where $i, j, k \in \mathbb{N}^*$, $i+j+k = m - 3$ and a, b, c are pairwise different, then $M_{(4+i+1)\to(4+i)} = M'_{6\to5}$, $M_{(4+i+j+1)\to(4+i+j)} = M'_{7\to6}$ and we assign identity matrices to all other arrows.

PROPOSITION 6. *For $m \geq 7$ the previously defined function $T_m : \mathfrak{R}_m \to \text{rep-}k\Delta(\widetilde{\mathbb{D}}_m)$ gives tree representations for all exceptional roots, i.e. for any $x \in \mathfrak{R}_m$ the representation $T_m(x) \in \text{rep-}k\Delta(\widetilde{\mathbb{D}}_m)$ is a tree representation.*

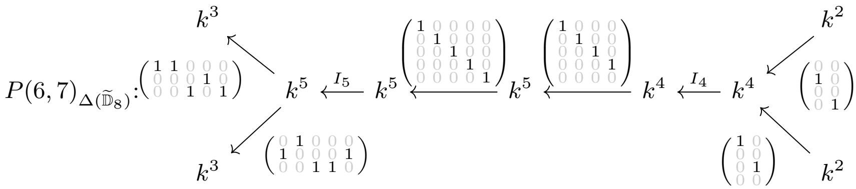
Proof. Just apply Lemma 4 (and the notations from Lemma 2) to see that $M = T_m(x)$ is exceptional. For the number of non-zero elements, let $M' = T_6(\mathfrak{p}_m(x))$. We know that M' is a tree representation, so the total number of ones in the matrices is $\ell(M') - 1 = x_1 + x_2 + x_3 + x_4 + a + b + c - 1$. By construction of the function $\mathfrak{p}_m$, we remove $i - 1$ vertices of dimension a, $j - 1$ vertices of dimension b, and $k - 1$ vertices of dimension c. Since the matrices associated to the arrows connecting them are I_a, I_b, respectively I_c, the total number of ones in the matrices of M will be $x_1 + x_2 + x_3 + x_4 + a + b + c - 1 + (i - 1)a + (j - 1)b + (k - 1)c = x_1 + x_2 + x_3 + x_4 + ia + jb + kc - 1 = \ell(M) - 1$, showing that M is also a tree representation. ∎

EXAMPLE 7. Suppose we need a tree representation for the preprojective indecomposable $P(6, 7)_{\Delta(\widetilde{\mathbb{D}}_8)} \in \text{rep-}k\Delta(\widetilde{\mathbb{D}}_8)$. Using equation (1.2) we find that $\underline{\dim}\, P(6, 7)_{\Delta(\widetilde{\mathbb{D}}_8)} = (3, 3, 2, 2, 5, 5, 5, 4, 4) \in \mathfrak{R}_8$. We compute its corresponding exceptional root over $\Delta(\widetilde{\mathbb{D}}_6)$: $\mathfrak{p}_8(3, 3, 2, 2, 5, 5, 5, 4, 4) = (3, 3, 2, 2, 5, 5, 4) \in \mathfrak{R}_6$. Due to Lemma 3 we know that defects are preserved by $\mathfrak{p}_8$, so we have to search for the corresponding representation among the list of preprojective families in Subsection 2.1. We identify the family $P(8n + 4, 6)_{\Delta(\widetilde{\mathbb{D}}_6)}$ with dimension vector $\underline{\dim}\, P(8n + 4, 6)_{\Delta(\widetilde{\mathbb{D}}_6)} = (4n + 3, 4n + 3, 4n + 2, 4n + 2, 8n + 5, 8n + 5, 8n + 4)$, which for $n = 0$ gives exactly our root. Using the formula given there, we construct the tree representation of $T_6(\mathfrak{p}_8(3, 3, 2, 2, 5, 5, 5, 4, 4)) = P(4, 6)_{\Delta(\widetilde{\mathbb{D}}_6)}$.



As indicated, this representation may be constructed by first forming a direct sum of $P(4,1)_{\Delta(\widetilde{\mathbb{D}}_6)}$ with $P(6,2)_{\Delta(\widetilde{\mathbb{D}}_6)}$ and then inserting the matrix block $\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ into the upper right corner of the matrix associated to the arrow $(5 \to 1)$, thus bringing in an extra 1 into the matrix.

Now we are ready to construct our initial representation $T_8(3,3,2,2,5,5,5,4,4) = P(6,7)_{\Delta(\widetilde{\mathbb{D}}_8)}$ using the above method, by taking the matrices associated to the arrows $(5 \to 1)$, $(5 \to 2)$, $(3 \to 7)$, $(4 \to 7)$, $(6 \to 5)$ and $(7 \to 6)$ from $P(4,6)_{\Delta(\widetilde{\mathbb{D}}_6)}$, associating them to the arrows $(5 \to 1)$, $(5 \to 2)$, $(3 \to 9)$, $(4 \to 9)$, $(7 \to 6)$ respectively $(8 \to 7)$ in $P(6,7)_{\Delta(\widetilde{\mathbb{D}}_8)}$ and putting identity matrices on the remaining arrows:



For the $m = 4$ and $m = 5$ cases we first state some analogous lemmas and then give an explicit construction for $T_4 : \mathfrak{R}_4 \to \text{rep-}k\Delta(\widetilde{\mathbb{D}}_4)$ and $T_5 : \mathfrak{R}_5 \to \text{rep-}k\Delta(\widetilde{\mathbb{D}}_5)$.

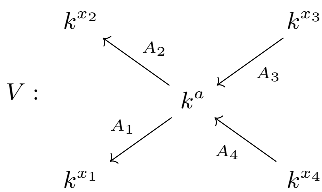
For $m = 4$ we introduce $\mathfrak{i}_4 : \mathfrak{R}_4 \to \mathfrak{R}_6$, where $\mathfrak{i}_4(x_1,x_2,x_3,x_4,a) = (x_1,x_2,x_3,x_4,a,a,a)$ and for $m = 5$ we introduce $\mathfrak{i}_5 : \mathfrak{R}_5 \to \mathfrak{R}_6$, where $\mathfrak{i}_5(x_1,x_2,x_3,x_4,a,b) = (x_1,x_2,x_3,x_4,a,a,b)$.

Lemma 8. *$\mathfrak{i}_4 : \mathfrak{R}_4 \to \mathfrak{R}_6$ and $\mathfrak{i}_5 : \mathfrak{R}_5 \to \mathfrak{R}_6$ are well-defined injective functions. Moreover, they preserve defects (i.e. for all $x \in \mathfrak{R}_4$, $\partial_{k\Delta(\widetilde{\mathbb{D}}_4)} x = \partial_{k\Delta(\widetilde{\mathbb{D}}_6)} \mathfrak{i}_4(x)$ and for all $x \in \mathfrak{R}_5$, $\partial_{k\Delta(\widetilde{\mathbb{D}}_5)} x = \partial_{k\Delta(\widetilde{\mathbb{D}}_6)} \mathfrak{i}_5(x)$).*

Proof. Similar to that of Lemma 3. ∎

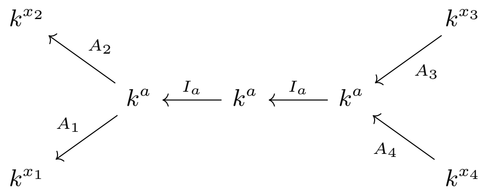
Lemma 9. *The following statements are true:*

(a) *Let $x = (x_1,x_2,x_3,x_4,a) \in \mathfrak{R}_4$, $\mathfrak{i}_4(x) = x' = (x_1,x_2,x_3,x_4,a,a,a) \in \mathfrak{R}_6$ and suppose $V \in \text{rep-}k\Delta(\widetilde{\mathbb{D}}_4)$ and $V' \in \text{rep-}k\Delta(\widetilde{\mathbb{D}}_6)$ with $\underline{\dim} V = x$ and $\underline{\dim} V' = x'$ have the following matrices:*



and

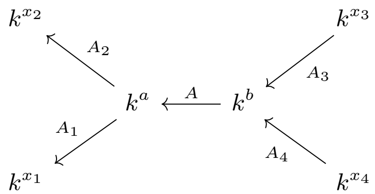
V' :



Then V is exceptional if and only if V' is exceptional.

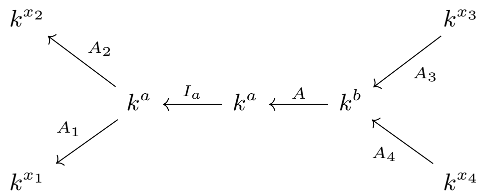
(b) *Let $x = (x_1, x_2, x_3, x_4, a, b) \in \mathfrak{R}_5$, $\mathfrak{i}_5(x) = x' = (x_1, x_2, x_3, x_4, a, a, b) \in \mathfrak{R}_6$ and suppose $Z \in$ rep-$k\Delta(\widetilde{\mathbb{D}}_5)$ and $Z' \in$ rep-$k\Delta(\widetilde{\mathbb{D}}_6)$ with $\underline{\dim}\, Z = x$ and $\underline{\dim}\, Z' = x'$ have the following matrices:*

Z :



and

Z' :



Then Z is exceptional if and only if Z' is exceptional.

Proof. Similar to that of Lemma 4. ■

As one can see, the representations $V', Z' \in$ rep-$k\Delta(\widetilde{\mathbb{D}}_6)$ are somewhat special, in the sense that they must have identity matrices associated to arrows connecting vertices of equal dimension on the central axis of the quiver. Upon inspection of the lists given in Section 2 it can be seen that all representations constructed fulfill this requirement. So we can state:

LEMMA 10. *All the (exceptional) tree representations in the case $\Delta(\widetilde{\mathbb{D}}_6)$ (listed in Section 2) have identity matrices associated to the arrows on the central axis that connect vertices of equal dimension.*

Constructing tree representations of $\Delta(\widetilde{\mathbb{D}}_4)$ and $\Delta(\widetilde{\mathbb{D}}_5)$. We are now ready to give the functions $T_4 : \mathfrak{R}_4 \to$ rep-$k\Delta(\widetilde{\mathbb{D}}_4)$ and $T_5 : \mathfrak{R}_5 \to$ rep-$k\Delta(\widetilde{\mathbb{D}}_5)$. For any $x \in \mathfrak{R}_4$ let $T_4(x) = V$ be constructed from $V' = T_6(\mathfrak{i}_4(x)) \in$ rep-$k\Delta(\widetilde{\mathbb{D}}_6)$ in the following way: $V_{5\to 1} = V'_{5\to 1}$, $V_{5\to 2} = V'_{5\to 2}$, $V_{3\to 5} = V'_{3\to 7}$ and $V_{4\to 5} = V'_{4\to 7}$. Similarly, for any $x \in \mathfrak{R}_5$ let $T_5(x) = Z$ be constructed from $Z' = T_6(\mathfrak{i}_5(x)) \in$ rep-$k\Delta(\widetilde{\mathbb{D}}_6)$ in the following way: $Z_{5\to 1} = Z'_{5\to 1}$, $Z_{5\to 2} = Z'_{5\to 2}$, $Z_{3\to 6} = Z'_{3\to 7}$, $Z_{4\to 6} = Z'_{4\to 7}$ and $Z_{6\to 5} = Z'_{7\to 6}$.

Proposition 11.

(a) *The function* $T_4 : \mathfrak{R}_4 \to \text{rep-}k\Delta(\widetilde{\mathbb{D}}_4)$ *gives tree representations for all exceptional roots, i.e. for any* $x \in \mathfrak{R}_4$ *the representation* $T_4(x) \in \text{rep-}k\Delta(\widetilde{\mathbb{D}}_4)$ *is a tree representation.*

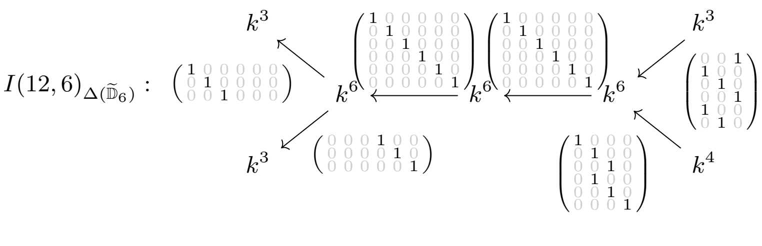
(b) *The function* $T_5 : \mathfrak{R}_5 \to \text{rep-}k\Delta(\widetilde{\mathbb{D}}_5)$ *gives tree representations for all exceptional roots, i.e. for any* $x \in \mathfrak{R}_5$ *the representation* $T_5(x) \in \text{rep-}k\Delta(\widetilde{\mathbb{D}}_5)$ *is a tree representation.*

Proof. Similar to the proof of Proposition 6, using the special forms of the representations in Section 2 (the statement of Lemma 10). ∎

Example 12. Suppose we need a tree representation for the preinjective indecomposable $I(6,4)_{\Delta(\widetilde{\mathbb{D}}_4)} \in \text{rep-}k\Delta(\widetilde{\mathbb{D}}_4)$. Using equation (1.2) we find that $\underline{\dim}\, I(6,4)_{\Delta(\widetilde{\mathbb{D}}_4)} = (3,3,3,4,6) \in \mathfrak{R}_4$. We compute the corresponding exceptional root over $\Delta(\widetilde{\mathbb{D}}_6)$: $\mathfrak{i}_4(3,3,3,4,6) = (3,3,3,4,6,6,6) \in \mathfrak{R}_6$. From Lemma 8 we know that the function $\mathfrak{i}_4$ preserves defects, so we have to search for the corresponding representation among the list of preinjective families in Subsection 2.2. We identify the family $I(8n+4,4)_{\Delta(\widetilde{\mathbb{D}}_6)}$—obtained from $I(8n+4,3)_{\Delta(\widetilde{\mathbb{D}}_6)}$ via the permutation $\tau = (3,4)$ as explained there—with dimension vector

$$\underline{\dim}\, I(8n+4,4)_{\Delta(\widetilde{\mathbb{D}}_6)} = (2n+1, 2n+1, 2n+1, 2n+2, 4n+2, 4n+2, 4n+2),$$

which for $n = 1$ gives exactly our root. Using the formula given there, we construct the tree representation of $T_6(\mathfrak{i}_4(3,3,3,4,6)) = I(12,6)_{\Delta(\widetilde{\mathbb{D}}_6)}$.



Now we are ready to construct our initial representation $T_4(3,3,3,4,6) = I(6,4)_{\Delta(\widetilde{\mathbb{D}}_4)}$ using the above method, by taking the matrices associated to the arrows $(5 \to 1)$, $(5 \to 2)$, $(3 \to 7)$ and $(4 \to 7)$ from $I(12,6)_{\Delta(\widetilde{\mathbb{D}}_6)}$ and associating them to the arrows $(5 \to 1)$, $(5 \to 2)$, $(3 \to 5)$ and $(4 \to 5)$ in $I(6,4)_{\Delta(\widetilde{\mathbb{D}}_4)}$.

$$I(6,4)_{\Delta(\widetilde{\mathbb{D}}_4)}: \quad \begin{array}{ccccc}
k^3 & & & & k^3 \\
& \nwarrow^{\left(\begin{smallmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{smallmatrix}\right)} & & \swarrow^{\left(\begin{smallmatrix} 0&0&1\\ 1&0&0\\ 0&1&0\\ 0&0&1\\ 1&0&0\\ 0&1&0 \end{smallmatrix}\right)} & \\
& & k^6 & & \\
& \swarrow_{\left(\begin{smallmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{smallmatrix}\right)} & & \nwarrow_{\left(\begin{smallmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{smallmatrix}\right)} & \\
k^3 & & & & k^4
\end{array}$$

Remark 13. None of the proofs in this section uses any assumptions about the underlying field k, so all these proofs work exactly the same way in any field. Consequently, since the tree representations given in Section 2 are field independent (in the precise sense given in Subsection 1.4), the representations constructed using the functions T_4, T_5 and T_m ($m \geq 7$) are field independent as well.

1.4. Proving the field independent tree module property. In this part we give a short overview of the method used to prove the tree module property for every representation given in the lists of Section 2. The method presented here has already been used in the case of the canonically oriented $\widetilde{\mathbb{E}}_6$. The method is general (in the sense that it could be applied to any tame quiver) and a more detailed description as well as the proofs and additional information may be found in [12, Section 1.3].

Throughout this subsection, Q denotes an arbitrary tame quiver and k an arbitrary field.

We will use the "field independent" qualifier in relation to representations and short exact sequences in the following precise manner:

Definition 14 ([12]). Let $M \in$ mod-kQ be an (exceptional) indecomposable module. We say that:

(1) The module M is *field independent* (*exceptional*) *indecomposable* if in the corresponding representation $M = (M_i, M_\alpha)$ all the elements in the matrices M_α are either 0 or 1, and for any field k', whenever $M' \in$ mod-$k'Q$ is such that $\underline{\dim}\, M = \underline{\dim}\, M'$ and every matrix M'_α from the corresponding representation $M' = (M'_i, M'_\alpha)$ is formally the same as M_α (for all arrows α), then M' is also (exceptional) indecomposable in mod-$k'Q$.

(2) The module M has the *field independent tree property* if it is a tree module in mod-kQ and it is also a *field independent* (exceptional) indecomposable module (i.e. whenever we consider the corresponding representation with formally the same matrices over any other field k', we still get an exceptional indecomposable tree module in mod-$k'Q$).

(3) A short exact sequence of the form $0 \to Y \xrightarrow{f} Z \xrightarrow{g} X \to 0$ is *field independent* (with $X, Y, Z \in \text{mod-}kQ$) if all the elements in the matrices of the representations X, Y and Z are either 0 or 1, all the elements in the matrices f_i and g_i of the embedding $f = (f_i)_{i \in Q_0}$ respectively the projection $g = (g_i)_{i \in Q_0}$ are either 0 or 1 or -1 and in any field k' the sequence $0 \to Y' \xrightarrow{f'} Z' \xrightarrow{g'} X' \to 0$ is also exact, where $X', Y', Z' \in \text{mod-}k'Q$, $f' : Y' \to Z'$, $g' : Z' \to X'$ correspond in order to X, Y, Z, $f : Y \to Z$, $g : Z \to X$ with the respective dimension vectors unchanged and with all matrices (both from the representations and from the morphisms) being formally the same when considering them over k' instead of k.

The technique used to obtain and prove the formulas in Section 2 (in a field independent way) consists of a mixture of computer experimentation using the computer algebra system GAP [25] followed by a computer aided proof performed by a proof assistant software developed in the purely functional programming language Clean [24], specifically for this purpose. The proof uses our prior knowledge on the existence of certain Schofield sequences (see [22]) and it is based on the following proposition (proved in [12]):

Proposition 15 ([12]). *Let $X, Y, X', Y' \in \text{mod-}kQ$ be indecomposable modules. If $M \in \text{mod-}kQ$ is such that*

(a) *there is an exceptional $Z \in \text{mod-}kQ$ such that (X, Y) and (X', Y') are Schofield pairs associated to Z,*
(b) *there exist two short exact sequences*

$$0 \to Y \to M \to X \to 0$$

and

$$0 \to Y' \to M \to X' \to 0,$$

(c) *$X \ncong X'$ or $Y \ncong Y'$,*
(d) *$\dim_k \operatorname{Ext}^1_{kQ}(X, Y) = \dim_k \operatorname{Ext}^1_{kQ}(X', Y') = 1$,*

then M is exceptional indecomposable.

The formulas for the matrices listed in Section 2 were obtained after extensive experimentation and testing in GAP, working over small finite fields (for details see [12, Remarks 8–10]). Then the "guessed" formulas were introduced into an input LaTeX document which in turn was processed by the proof assistant. The computer aided proof is basically induction on the dimensions of the representations (detailed in [12, Subsection 1.3]). For given input data (supposedly) defining short exact sequences (the two different Schofield sequences required by Proposition 15) the proof assistant verifies using Lemma 16 that indeed, two short exact sequences may be constructed using the given matrices (in a field independent way). To complete the proof for the tree module property, it also counts the total number of ones in matrices.

LEMMA 16 ([12]). *Let* $X, Y, Z \in \text{mod-}kQ$ *and* $f = (f_i)_{i\in Q_0}$, $g = (g_i)_{i\in Q_0}$ *families of* k*-linear maps* $f_i : Y_i \to Z_i$, $g_i : Z_i \to X_i$. *Then there is a short exact sequence* $0 \to Y \xrightarrow{f} Z \xrightarrow{g} X \to 0$ *if and only if the following conditions hold* (*we identify the maps* f_i *and* g_i *with their matrices in the canonical basis*)*:*

(a) *the matrices* f_i (*respectively* g_i) *have maximal column* (*respectively row*) *ranks,*
(b) $f_{t(\alpha)}Y_\alpha = Z_\alpha f_{s(\alpha)}$ *and* $g_{t(\alpha)}Z_\alpha = X_\alpha g_{s(\alpha)}$, *for all* $\alpha \in Q_1$,
(c) $g_i f_i = 0$ *for all* $i \in Q_0$,
(d) $\underline{\dim}\, Z = \underline{\dim}\, X + \underline{\dim}\, Y$.

The output of the proof assistant is the quite lengthy Appendix [11] to this paper. Every step performed by the proof assistant (including the elementary operations and the details of computing the block matrix sums and products) is output in a detailed step-by-step fashion as if written "by hand". In this way one does not have to believe in the correctness of the implementation, because the complete proof is "on paper" and every single step may be crosschecked and verified by a human mathematician.

We have developed the software in the purely functional programming language Clean (see [24, 16]). Clean influenced (and is influenced by) Haskell, so it has a somewhat similar syntax. It possesses some distinctive features such as uniqueness types, which allow destructive transformation of state information within a purely functional framework (see [20]). Clean allowed us to implement block-matrix operations safely and very efficiently (the complete process, including parsing the input document, performing symbolic block-matrix operations and other checks, finally generating the output in LaTeX takes less than one minute on an ordinary desktop PC, with the application running in single threaded mode).

REMARK 17. Field independence is a major result of our current work, so to be absolutely sure that every operation performed by the proof assistant is field independent, we implemented it in such a way that it does not use any field at all. Rather it treats 0, 1 and -1 just as some symbols obeying the identity, commutative and distributive laws. Sums such as $1 + 1 + \cdots + 1$ are kept unevaluated and can be reduced to 0, 1 or -1 only by a suitable number of -1 symbols. In this way we never encounter a value like 2 which may or may not be considered 0 depending on the characteristic. If a resulting matrix held such a "value", the proof assistant would stop short with an error message. Matrix rank computations are also performed "with care", avoiding situations when a matrix could have different ranks over different fields (see the description of method (1) from *The process of proving the field independent tree property* in [12, Subsection 1.3]).

For some more details on block-matrix arithmetic, rank computation and other steps performed by the proof assistant software see [11].

1.5. Notations. The matrices given in Section 2 are written using blocks of various sizes. The row and column sizes of blocks are given by expressions of the form $an+b$, where $n \in \mathbb{N}$ is a parameter, a is a given non-negative integer, b is a given integer. Every matrix here is composed either of identity blocks or rectangular zero blocks. We denote an identity block simply by 1 and a zero block by 0. The row and column sizes will be written as "decorations" along the border of the matrix, as in the following example:

$$\begin{array}{r} \\ 2n \\ n{+}1 \\ n \\ 2n{+}2 \end{array}\begin{array}{c} \begin{array}{cc} 2n & n \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

where the matrix is of size $(6n+3) \times 3n$ and is composed of two identity blocks (of sizes $2n \times 2n$ and $n \times n$) and six zero blocks with various compatible sizes.

The matrices may be given using arithmetic expressions containing symbolic block matrices and identifiers referencing other matrices. Possible operations are: addition, direct sum defined as $A \oplus B = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$ and a special kind of "sum" denoted by $\boxplus$ which adds the right hand matrix into the upper right corner of the left hand matrix. Formally: if $A \in \mathcal{M}_{m,n}(k)$ and $B \in \mathcal{M}_{m',n'}(k)$ with $m' \leq m$ and $n' \leq n$, then

$$A \boxplus B = A + \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix},$$

where $\left[\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{M}_{m,n}(k)$ is obtained by adding as many zero columns to the left of B and as many zero rows beneath it to make the resulting matrix of the same size as A. This operation is useful to insert non-zero elements into the upper right part of a matrix obtained by direct sum.

Representations are given as families having similar block matrices. For example $P(8n+4,5)$ denotes such a family of representations (where $n \in \mathbb{N}$). Sometimes one needs the previous or next value of n when writing matrices in terms of others, therefore we need to replace n. Such a substitution is denoted like $P(8n+3,5)[n \mapsto n-1]$, which in this case is the module $P(8n-5,5)$ for any fixed value of n.

For a representation $Z = (Z_i, Z_\alpha)$ we only give the matrices Z_α. For a module Z and an arrow $\alpha \in Q_1$ we denote the matrix Z_α by M_α^Z. When we give all the matrices "by value", a representation will be written like this (with $d_i \in \mathbb{N}$ for $i \in Q_0$ and with the matrices M_α^Z in this specific order):

$$\begin{aligned} \underline{\dim}\, Z &= (d_1,\, d_2,\, d_3,\, d_4,\, d_5,\, d_6,\, d_7), \\ Z &= (M^Z_{5\to1},\, M^Z_{5\to2},\, M^Z_{6\to5},\, M^Z_{7\to6},\, M^Z_{3\to7},\, M^Z_{4\to7}). \end{aligned}$$

There is another notation, when the matrices are written with the use of expressions involving the operations $\oplus$ and $\boxplus$, referencing other matrices of representations. In this case there are always two other representations Y, X and a specific arrow α' such that the matrices of Z can be given as $M_\alpha^Z = M_\alpha^Y \oplus M_\alpha^X$ for all $\alpha \neq \alpha'$, and $M_{\alpha'}^Z = (M_{\alpha'}^Y \oplus M_{\alpha'}^X) \boxplus M$ for a matrix M with exactly one element equal to 1 and all other elements zero. Therefore we give the representation Z in the following form (specifying the matrix M):

$$\begin{aligned} \underline{\dim}\, Z &= (d_1,\, d_2,\, d_3,\, d_4,\, d_5,\, d_6,\, d_7), \\ M_\alpha^Z &= M_\alpha^Y \oplus M_\alpha^X \quad \text{for } \alpha \neq \alpha', \\ M_{\alpha'}^Z &= (M_{\alpha'}^Y \oplus M_{\alpha'}^X) \boxplus M. \end{aligned}$$

For small values of n we may give some representations concretely (the general formula may work only for $n > 0$ in some cases).

2. Tree representations of the quiver $\Delta(\widetilde{\mathbb{D}}_6)$. In this section we list the formulas describing the matrices of the representations corresponding to exceptional modules: the preprojective indecomposables (Subsection 2.1), the preinjective indecomposables (Subsection 2.2) and the regular non-homogeneous indecomposables with dimension vector below δ (Subsection 2.3). For convenience, at the beginning of each subsection, we present a graphical representation of the corresponding part of the Auslander–Reiten quiver. Blue arrows (see the pdf file) indicate the existence of a so-called irreducible monomorphism, while red arrows represent irreducible epimorphisms between suitable indecomposable modules (for details see [1]).

In the case of preprojectives and preinjectives the representations can be grouped in families of the form $P(8n + r, i)$ respectively $I(8n + r, i)$, where $i \in \{1, \dots, 7\}$ and $r \in \{0, \dots, 7\}$. Representations belonging to the same family have similar dimension vectors and matrices, depending only on $n \in \mathbb{N}$. The formulas for matrices listed here are rigorously proved to be correct—i.e. they give a field independent tree representation of the respective family in the sense of Definition 14(2)—in the Appendix to this article ([11]) using the method described in [12]. The Appendix also contains a more detailed presentation of some of the representations from the lists (e.g. matrices written out explicitly for small values of $n = 0, 1, 2, \dots$).

2.1. The preprojective indecomposable representations. The preprojective indecomposable modules correspond to the vertices of the preprojective part of the Auslander–Reiten quiver, as shown in Figure 2.1 (at each vertex we provide the dimension vector in the graphical form corresponding to the shape of $\widetilde{\mathbb{D}}_6$).

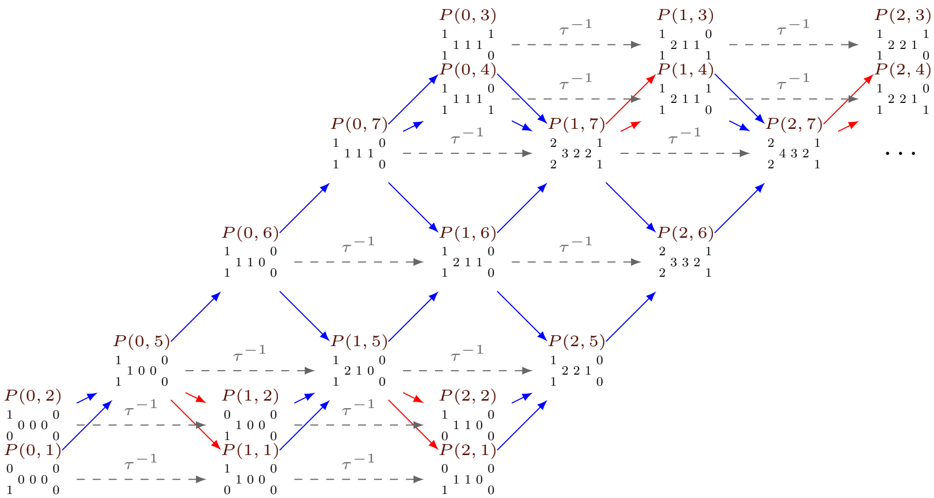


Fig. 2.1. Preprojective part of the Auslander–Reiten quiver $\Delta(\widetilde{\mathbb{D}}_6)$

Due to the symmetry of the quiver $\Delta(\widetilde{\mathbb{D}}_6)$ we only give the families of representations of the form $P(s,1)$, $P(s,3)$, $P(s,5)$, $P(s,6)$ and $P(s,7)$. For $P(s,2)$ and $P(s,4)$ we can use the permutations $\sigma = (1,2)$ and $\tau = (3,4)$ to write them in terms of $P(s,1)$ and $P(s,3)$ in the following way ($s \geq 0$):

$$\underline{\dim}\, P(s,2) = (d_{\sigma(i)})_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}, \quad \text{where} \quad \underline{\dim}\, P(s,1) = (d_i)_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0},$$
$$\underline{\dim}\, P(s,4) = (d_{\tau(i)})_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}, \quad \text{where} \quad \underline{\dim}\, P(s,3) = (d_i)_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}$$

for the dimension vectors, and

$$P(s,2) = (M_{\sigma(i)\to\sigma(j)})_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}, \quad \text{where} \quad P(s,1) = (M_{i\to j})_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1},$$
$$P(s,4) = (M_{\tau(i)\to\tau(j)})_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}, \quad \text{where} \quad P(s,3) = (M_{i\to j})_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1},$$

for the matrices.

In what follows we list the tree representations for preprojective families of the form $P(s,1)$, $P(s,3)$, $P(s,5)$, $P(s,6)$ and $P(s,7)$:

$\underline{\dim}\, P(8n,1) = (2n+1, 2n, 2n, 2n, 4n, 4n, 4n)$,
$P(0,1) = (0,0,0,0,0,0)$,

$$P(8n,1) = \left(\begin{array}{c|cccc} & 2n-1 & 1 & 1 & 2n-1 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2n-1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{array},\ \begin{array}{c|cc} & 2n & 2n \\ \hline 2n & 1 & 1 \end{array},\ \begin{array}{c|c} & 4n \\ \hline 4n & 1 \end{array},\ \begin{array}{c|c} & 4n \\ \hline 4n & 1 \end{array},\ \begin{array}{c|c} & 2n \\ \hline 2n & 0 \\ 2n & 1 \end{array},\ \begin{array}{c|c} & 2n \\ \hline 2n & 1 \\ 2n & 0 \end{array} \right), \quad n>0;$$

$\underline{\dim}\, P(8n+1,1) = (2n, 2n+1, 2n, 2n, 4n+1, 4n, 4n)$,

$$P(8n+1,1) = \left(\begin{array}{c|ccc} & 2n & 2n & 1 \\ \hline 2n & 1 & 1 & 0 \end{array},\ \begin{array}{c|ccc} & 2n & 1 & 2n \\ \hline 1 & 0 & 1 & 0 \\ 2n & 1 & 0 & 1 \end{array},\ \begin{array}{c|c} & 4n \\ \hline 4n & 1 \\ 1 & 0 \end{array},\ \begin{array}{c|c} & 4n \\ \hline 4n & 1 \end{array},\ \begin{array}{c|c} & 2n \\ \hline 2n & 0 \\ 2n & 1 \end{array},\ \begin{array}{c|c} & 2n \\ \hline 2n & 1 \\ 2n & 0 \end{array} \right);$$

$\underline{\dim}\, P(8n+2,1) = (2n+1, 2n, 2n, 2n, 4n+1, 4n+1, 4n)$,

$$P(8n+2,1) = \left(\begin{array}{c|ccc} & 2n & 1 & 2n \\ \hline 1 & 0 & 1 & 0 \\ 2n & 1 & 0 & 1 \end{array},\ \begin{array}{c|ccc} & 2n & 2n & 1 \\ \hline 2n & 1 & 1 & 0 \end{array},\ \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array},\ \begin{array}{c|c} & 4n \\ \hline 4n & 1 \\ 1 & 0 \end{array},\ \begin{array}{c|c} & 2n \\ \hline 2n & 0 \\ 2n & 1 \end{array},\ \begin{array}{c|c} & 2n \\ \hline 2n & 1 \\ 2n & 0 \end{array} \right);$$

$\underline{\dim}\, P(8n+3,1) = (2n, 2n+1, 2n, 2n, 4n+1, 4n+1, 4n+1)$,

$$P(8n+3,1) = \left(\begin{array}{c|ccc} & 2n & 2n & 1 \\ \hline 2n & 1 & 1 & 0 \end{array},\ \begin{array}{c|ccc} & 2n & 1 & 2n \\ \hline 1 & 0 & 1 & 0 \\ 2n & 1 & 0 & 1 \end{array},\ \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array},\ \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array},\ \begin{array}{c|c} & 2n \\ \hline 2n & 0 \\ 2n & 1 \\ 1 & 0 \end{array},\ \begin{array}{c|c} & 2n \\ \hline 2n & 1 \\ 2n+1 & 0 \end{array} \right);$$

$\underline{\dim}\, P(8n+4,1) = (2n+2, 2n+1, 2n+1, 2n+1, 4n+2, 4n+2, 4n+2)$,
$P(8n+4,1)$

$$= \left(\begin{array}{c|cccc} & 2n & 1 & 1 & 2n \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2n & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{array},\ \begin{array}{c|cc} & 2n+1 & 2n+1 \\ \hline 2n+1 & 1 & 1 \end{array},\ \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \end{array},\ \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \end{array},\ \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 0 \\ 2n+1 & 1 \end{array},\ \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ 2n+1 & 0 \end{array} \right);$$

$\underline{\dim}\, P(8n+5,1) = (2n+1, 2n+2, 2n+1, 2n+1, 4n+3, 4n+2, 4n+2)$,
$P(8n+5,1)$

$$= \left(\begin{array}{c|ccc} & 2n+1 & 2n+1 & 1 \\ \hline 2n+1 & 1 & 1 & 0 \end{array},\ \begin{array}{c|ccc} & 2n+1 & 1 & 2n+1 \\ \hline 1 & 0 & 1 & 0 \\ 2n+1 & 1 & 0 & 1 \end{array},\ \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \\ 1 & 0 \end{array},\ \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \end{array},\ \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 0 \\ 2n+1 & 1 \end{array},\ \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ 2n+1 & 0 \end{array} \right);$$

$\underline{\dim} P(8n+6,1) = (2n+2, 2n+1, 2n+1, 2n+1, 4n+3, 4n+3, 4n+2)$,

$P(8n+6,1)$

$$= \left(\begin{array}{c|ccc} & 2n{+}1 & 1 & 2n{+}1 \\ \hline 1 & 0 & 1 & 0 \\ 2n{+}1 & 1 & 0 & 1 \end{array}, \begin{array}{c|ccc} & 2n{+}1 & 2n{+}1 & 1 \\ \hline 2n{+}1 & 1 & 1 & 0 \end{array}, \begin{array}{c|c} & 4n{+}3 \\ \hline 4n{+}3 & 1 \end{array}, \begin{array}{c|c} & 4n{+}2 \\ \hline 4n{+}2 & 1 \\ 1 & 0 \end{array}, \begin{array}{c|c} & 2n{+}1 \\ \hline 2n{+}1 & 0 \\ 2n{+}1 & 1 \end{array}, \begin{array}{c|c} & 2n{+}1 \\ \hline 2n{+}1 & 1 \\ 2n{+}1 & 0 \end{array} \right);$$

$\underline{\dim} P(8n+7,1) = (2n+1, 2n+2, 2n+1, 2n+1, 4n+3, 4n+3, 4n+3)$,

$P(8n+7,1)$

$$= \left(\begin{array}{c|ccc} & 2n{+}1 & 2n{+}1 & 1 \\ \hline 2n{+}1 & 1 & 1 & 0 \end{array}, \begin{array}{c|ccc} & 2n{+}1 & 1 & 2n{+}1 \\ \hline 1 & 0 & 1 & 0 \\ 2n{+}1 & 1 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n{+}3 \\ \hline 4n{+}3 & 1 \end{array}, \begin{array}{c|c} & 4n{+}3 \\ \hline 4n{+}3 & 1 \end{array}, \begin{array}{c|c} & 2n{+}1 \\ \hline 2n{+}1 & 0 \\ 2n{+}1 & 1 \\ 1 & 0 \end{array}, \begin{array}{c|c} & 2n{+}1 \\ \hline 2n{+}1 & 1 \\ 2n{+}2 & 0 \end{array} \right);$$

$\underline{\dim} P(8n,3) = (2n+1, 2n+1, 2n+1, 2n, 4n+1, 4n+1, 4n+1)$,

$P(0,3) = ([1], [1], [1], [1], [1], 0)$,

$$P(8n,3) = \left(\begin{array}{c|ccccc} & 2n{-}1 & 1 & 2n{-}1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2n{-}1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{array}, \begin{array}{c|cccc} & 2n{-}1 & 2n{-}1 & 1 & 2 \\ \hline 2n{-}1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n{+}1 \\ \hline 4n{+}1 & 1 \end{array}, \begin{array}{c|c} & 4n{+}1 \\ \hline 4n{+}1 & 1 \end{array}, \right.$$
$$\left. \begin{array}{c|ccc} & 2n{-}1 & 1 & 1 \\ \hline 2n{-}1 & 0 & 0 & 0 \\ 2n{-}1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}, \begin{array}{c|cc} & 2n{-}1 & 1 \\ \hline 2n{-}1 & 1 & 0 \\ 2n{-}1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right), \qquad n > 0;$$

$\underline{\dim} P(8n+1,3) = (2n+1, 2n+1, 2n, 2n+1, 4n+2, 4n+1, 4n+1)$,

$$P(1,3) = \left([1 \;\; 0],\ [1 \;\; 1],\ \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ [1],\ 0,\ [1] \right),$$

$$P(8n+1,3) = \left(\begin{array}{c|cccccc} & 2n{-}1 & 2n{-}1 & 1 & 1 & 1 & 1 \\ \hline 2n{-}1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{c|cccccc} & 2n{-}1 & 1 & 2n{-}1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2n{-}1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}, \begin{array}{c|cc} & 4n & 1 \\ \hline 4n & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{array}, \right.$$
$$\left. \begin{array}{c|c} & 4n{+}1 \\ \hline 4n{+}1 & 1 \end{array}, \begin{array}{c|cc} & 2n{-}1 & 1 \\ \hline 2n{-}1 & 0 & 0 \\ 2n{-}1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array}, \begin{array}{c|ccc} & 2n{-}1 & 1 & 1 \\ \hline 2n{-}1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2n{-}1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right), \qquad n > 0;$$

$$\underline{\dim}\, P(8n+2,3) = (2n+1, 2n+1, 2n+1, 2n, 4n+2, 4n+2, 4n+1),$$

$$P(2,3) = \left([1\ \ 0],\ [1\ \ 1],\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ [1],\ 0 \right),$$

$$P(8n+2,3) = \left(\begin{array}{r|ccccc|} & {\scriptstyle 2n-1} & {\scriptstyle 1} & {\scriptstyle 2n-1} & {\scriptstyle 1} & {\scriptstyle 2} \\ \hline {\scriptstyle 1} & 0 & 1 & 0 & 0 & 0 \\ {\scriptstyle 2n-1} & 1 & 0 & 1 & 0 & 0 \\ {\scriptstyle 1} & 0 & 0 & 0 & 1 & 0 \\ \hline \end{array},\ \begin{array}{r|cccccc|} & {\scriptstyle 2n-1} & {\scriptstyle 2n-1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} \\ \hline {\scriptstyle 2n-1} & 1 & 1 & 0 & 0 & 0 & 0 \\ {\scriptstyle 1} & 0 & 0 & 0 & 1 & 1 & 0 \\ {\scriptstyle 1} & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array},\ \begin{array}{r|c|} & {\scriptstyle 4n+2} \\ \hline {\scriptstyle 4n+2} & 1 \\ \hline \end{array}, \right.$$

$$\left. \begin{array}{r|cc|} & {\scriptstyle 4n} & {\scriptstyle 1} \\ \hline {\scriptstyle 4n} & 1 & 0 \\ {\scriptstyle 1} & 0 & 0 \\ {\scriptstyle 1} & 0 & 1 \\ \hline \end{array},\ \begin{array}{r|ccc|} & {\scriptstyle 2n-1} & {\scriptstyle 1} & {\scriptstyle 1} \\ \hline {\scriptstyle 2n-1} & 0 & 0 & 0 \\ {\scriptstyle 2n-1} & 1 & 0 & 0 \\ {\scriptstyle 1} & 0 & 0 & 1 \\ {\scriptstyle 1} & 0 & 1 & 0 \\ {\scriptstyle 1} & 0 & 0 & 1 \\ \hline \end{array},\ \begin{array}{r|cc|} & {\scriptstyle 2n-1} & {\scriptstyle 1} \\ \hline {\scriptstyle 2n-1} & 1 & 0 \\ {\scriptstyle 2n-1} & 0 & 0 \\ {\scriptstyle 1} & 0 & 1 \\ {\scriptstyle 1} & 0 & 1 \\ {\scriptstyle 1} & 0 & 0 \\ \hline \end{array} \right), \quad n > 0;$$

$$\underline{\dim}\, P(8n+3,3) = (2n+1, 2n+1, 2n, 2n+1, 4n+2, 4n+2, 4n+2),$$

$$P(3,3) = \left([1\ \ 0],\ [1\ \ 1],\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ 0,\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right),$$

$$P(8n+3,3) = \left(\begin{array}{r|cccccc|} & {\scriptstyle 2n-1} & {\scriptstyle 2n-1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} \\ \hline {\scriptstyle 2n-1} & 1 & 1 & 0 & 0 & 0 & 0 \\ {\scriptstyle 1} & 0 & 0 & 0 & 1 & 0 & 0 \\ {\scriptstyle 1} & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array},\ \begin{array}{r|cccccc|} & {\scriptstyle 2n-1} & {\scriptstyle 1} & {\scriptstyle 2n-1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} \\ \hline {\scriptstyle 1} & 0 & 1 & 0 & 0 & 0 & 0 \\ {\scriptstyle 2n-1} & 1 & 0 & 1 & 0 & 0 & 0 \\ {\scriptstyle 1} & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline \end{array},\ \begin{array}{r|c|} & {\scriptstyle 4n+2} \\ \hline {\scriptstyle 4n+2} & 1 \\ \hline \end{array}, \right.$$

$$\left. \begin{array}{r|c|} & {\scriptstyle 4n+2} \\ \hline {\scriptstyle 4n+2} & 1 \\ \hline \end{array},\ \begin{array}{r|cc|} & {\scriptstyle 2n-1} & {\scriptstyle 1} \\ \hline {\scriptstyle 2n-1} & 0 & 0 \\ {\scriptstyle 2n-1} & 1 & 0 \\ {\scriptstyle 1} & 0 & 1 \\ {\scriptstyle 1} & 0 & 1 \\ {\scriptstyle 2} & 0 & 0 \\ \hline \end{array},\ \begin{array}{r|ccc|} & {\scriptstyle 2n-1} & {\scriptstyle 1} & {\scriptstyle 1} \\ \hline {\scriptstyle 2n-1} & 1 & 0 & 0 \\ {\scriptstyle 1} & 0 & 0 & 1 \\ {\scriptstyle 2n-1} & 0 & 0 & 0 \\ {\scriptstyle 1} & 0 & 1 & 0 \\ {\scriptstyle 1} & 0 & 0 & 0 \\ {\scriptstyle 1} & 0 & 0 & 1 \\ \hline \end{array} \right), \quad n > 0;$$

$$\underline{\dim}\, P(8n+4,3) = (2n+2, 2n+2, 2n+2, 2n+1, 4n+3, 4n+3, 4n+3),$$

$$P(4,3) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right),$$

$$P(8n+4,3)=\left(\begin{array}{c|ccccccc} & 2n-1 & 1 & 2n-1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2n-1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}, \begin{array}{c|ccccccc} & 2n-1 & 2n-1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2n-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array},\right.$$

$$\left.\begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array}, \begin{array}{c|cccc} & 2n-1 & 1 & 1 & 1 \\ \hline 2n-1 & 0 & 0 & 0 & 0 \\ 2n-1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{c|ccc} & 2n-1 & 1 & 1 \\ \hline 2n-1 & 1 & 0 & 0 \\ 2n-1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right), \quad n>0;$$

$$\underline{\dim}\, P(8n+5,3)=(2n+2,2n+2,2n+1,2n+2,4n+4,4n+3,4n+3),$$

$$P(8n+5,3)=\left(\begin{array}{c|cccc} & 2n+1 & 2n+1 & 1 & 1 \\ \hline 2n+1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{c|cccc} & 2n+1 & 1 & 2n+1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2n+1 & 1 & 0 & 1 & 0 \end{array}, \begin{array}{c|cc} & 4n+2 & 1 \\ \hline 4n+2 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array},\right.$$

$$\left.\begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 0 \\ 2n+1 & 1 \\ 1 & 0 \end{array}, \begin{array}{c|cc} & 2n+1 & 1 \\ \hline 2n+1 & 1 & 0 \\ 1 & 0 & 1 \\ 2n & 0 & 0 \\ 1 & 0 & 1 \end{array}\right);$$

$$\underline{\dim}\, P(8n+6,3)=(2n+2,2n+2,2n+2,2n+1,4n+4,4n+4,4n+3),$$

$$P(8n+6,3)=\left(\begin{array}{c|cccc} & 2n+1 & 1 & 2n+1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2n+1 & 1 & 0 & 1 & 0 \end{array}, \begin{array}{c|cccc} & 2n+1 & 2n+1 & 1 & 1 \\ \hline 2n+1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n+4 \\ \hline 4n+4 & 1 \end{array}, \begin{array}{c|cc} & 4n+2 & 1 \\ \hline 4n+2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{array},\right.$$

$$\left.\begin{array}{c|c} & 2n+2 \\ \hline 2n+1 & 0 \\ 2n+2 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ 2n+2 & 0 \end{array}\right);$$

$$\underline{\dim}\, P(8n+7,3)=(2n+2,2n+2,2n+1,2n+2,4n+4,4n+4,4n+4),$$

$$P(8n+7,3) = \left(\begin{array}{r|cccc|} & 2n{+}1 & 2n{+}1 & 1 & 1 \\ 2n{+}1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{r|cccc|} & 2n{+}1 & 1 & 2n{+}1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 2n{+}1 & 1 & 0 & 1 & 0 \end{array}, \begin{array}{r|c|} & 4n{+}4 \\ 4n{+}4 & 1 \end{array}, \begin{array}{r|c|} & 4n{+}4 \\ 4n{+}4 & 1 \end{array}, \begin{array}{r|c|} & 2n{+}1 \\ 2n{+}1 & 0 \\ 2n{+}1 & 1 \\ 2 & 0 \end{array}, \begin{array}{r|cc|} & 2n{+}1 & 1 \\ 2n{+}1 & 1 & 0 \\ 1 & 0 & 1 \\ 2n{+}1 & 0 & 0 \\ 1 & 0 & 1 \end{array} \right);$$

$$\underline{\dim}\, P(8n,5) = (4n+1, 4n+1, 4n, 4n, 8n+1, 8n, 8n),$$
$$M_\alpha^{P(8n,5)} = M_\alpha^{P(8n,1)} \oplus M_\alpha^{P(8n+1,1)} \quad \text{for } \alpha \neq (5 \to 1),$$
$$M_{5\to 1}^{P(8n,5)} = (M_{5\to 1}^{P(8n,1)} \oplus M_{5\to 1}^{P(8n+1,1)}) \boxplus \begin{array}{r|cc|} & 4n & 1 \\ 2n & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$$\underline{\dim}\, P(8n+1,5) = (4n+1, 4n+1, 4n, 4n, 8n+2, 8n+1, 8n),$$
$$M_\alpha^{P(8n+1,5)} = M_\alpha^{P(8n+1,1)} \oplus M_\alpha^{P(8n+2,1)} \quad \text{for } \alpha \neq (5 \to 2),$$
$$M_{5\to 2}^{P(8n+1,5)} = (M_{5\to 2}^{P(8n+1,1)} \oplus M_{5\to 2}^{P(8n+2,1)}) \boxplus \begin{array}{r|cc|} & 4n & 1 \\ 2n & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$$\underline{\dim}\, P(8n+2,5) = (4n+1, 4n+1, 4n, 4n, 8n+2, 8n+2, 8n+1),$$
$$M_\alpha^{P(8n+2,5)} = M_\alpha^{P(8n+2,1)} \oplus M_\alpha^{P(8n+3,1)} \quad \text{for } \alpha \neq (7 \to 6),$$
$$M_{7\to 6}^{P(8n+2,5)} = (M_{7\to 6}^{P(8n+2,1)} \oplus M_{7\to 6}^{P(8n+3,1)}) \boxplus \begin{array}{r|cc|} & 4n & 1 \\ 4n & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$$\underline{\dim}\, P(8n+3,5) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+3, 8n+3, 8n+3),$$
$$P(3,5) = \left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right),$$
$$M_\alpha^{P(8n+3,5)} = M_\alpha^{P(8n,3)} \oplus M_\alpha^{P(8n+3,3)} \quad \text{for } \alpha \neq (5 \to 2),$$
$$M_{5\to 2}^{P(8n+3,5)} = (M_{5\to 2}^{P(8n,3)} \oplus M_{5\to 2}^{P(8n+3,3)}) \boxplus \begin{array}{r|ccc|} & 4n{-}1 & 1 & 2 \\ 2n{-}1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}, \quad n > 0;$$

$$\underline{\dim}\, P(8n+4,5) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+5, 8n+4, 8n+4),$$
$$M_\alpha^{P(8n+4,5)} = M_\alpha^{P(8n+4,1)} \oplus M_\alpha^{P(8n+5,1)} \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to 1}^{P(8n+4,5)} = (M_{5\to 1}^{P(8n+4,1)} \oplus M_{5\to 1}^{P(8n+5,1)}) \boxplus \begin{array}{c} \\ 2n+1 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 4n+2 & 1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, P(8n+5,5) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+6, 8n+5, 8n+4),$$

$$M_\alpha^{P(8n+5,5)} = M_\alpha^{P(8n+5,1)} \oplus M_\alpha^{P(8n+6,1)} \quad \text{for } \alpha \neq (5 \to 2),$$

$$M_{5\to 2}^{P(8n+5,5)} = (M_{5\to 2}^{P(8n+5,1)} \oplus M_{5\to 2}^{P(8n+6,1)}) \boxplus \begin{array}{c} \\ 2n+1 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 4n+2 & 1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, P(8n+6,5) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+6, 8n+6, 8n+5),$$

$$M_\alpha^{P(8n+6,5)} = M_\alpha^{P(8n+6,1)} \oplus M_\alpha^{P(8n+7,1)} \quad \text{for } \alpha \neq (7 \to 6),$$

$$M_{7\to 6}^{P(8n+6,5)} = (M_{7\to 6}^{P(8n+6,1)} \oplus M_{7\to 6}^{P(8n+7,1)}) \boxplus \begin{array}{c} \\ 4n+2 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 4n+2 & 1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, P(8n+7,5) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+7, 8n+7, 8n+7),$$

$$M_\alpha^{P(8n+7,5)} = M_\alpha^{P(8n+7,2)} \oplus M_\alpha^{P(8n,2)[n\mapsto n+1]} \quad \text{for } \alpha \neq (5 \to 2),$$

$$M_{5\to 2}^{P(8n+7,5)} = (M_{5\to 2}^{P(8n+7,2)} \oplus M_{5\to 2}^{P(8n,2)[n\mapsto n+1]}) \boxplus \begin{array}{c} \\ 1 \\ 2n \end{array}\!\!\begin{array}{c} \begin{array}{cc} 4n+3 & 1 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, P(8n,6) = (4n+1, 4n+1, 4n, 4n, 8n+1, 8n+1, 8n),$$

$$M_\alpha^{P(8n,6)} = M_\alpha^{P(8n,1)} \oplus M_\alpha^{P(8n+2,2)} \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to 1}^{P(8n,6)} = (M_{5\to 1}^{P(8n,1)} \oplus M_{5\to 1}^{P(8n+2,2)}) \boxplus \begin{array}{c} \\ 2n \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 4n & 1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, P(8n+1,6) = (4n+1, 4n+1, 4n, 4n, 8n+2, 8n+1, 8n+1),$$

$$M_\alpha^{P(8n+1,6)} = M_\alpha^{P(8n+1,1)} \oplus M_\alpha^{P(8n+3,2)} \quad \text{for } \alpha \neq (6 \to 5),$$

$$M_{6\to 5}^{P(8n+1,6)} = (M_{6\to 5}^{P(8n+1,1)} \oplus M_{6\to 5}^{P(8n+3,2)}) \boxplus \begin{array}{c} \\ 4n \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 4n & 1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, P(8n+2,6) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+3, 8n+3, 8n+2),$$

$$P(2,6) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),$$

$$M_\alpha^{P(8n+2,6)} = M_\alpha^{P(8n,3)} \oplus M_\alpha^{P(8n+2,4)} \quad \text{for } \alpha \neq (5 \to 2),$$

$$M_{5\to 2}^{P(8n+2,6)} = (M_{5\to 2}^{P(8n,3)} \oplus M_{5\to 2}^{P(8n+2,4)}) \boxplus \begin{array}{c|ccc|} & 4n & 1 & 1 \\ 2n-1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}, \quad n>0;$$

$$\underline{\dim}\, P(8n+3,6) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+4, 8n+3, 8n+3),$$

$$M_{\alpha}^{P(8n+3,6)} = M_{\alpha}^{P(8n+1,4)} \oplus M_{\alpha}^{P(8n+3,3)} \quad \text{for } \alpha \neq (6\to 5),$$

$$M_{6\to 5}^{P(8n+3,6)} = (M_{6\to 5}^{P(8n+1,4)} \oplus M_{6\to 5}^{P(8n+3,3)}) \boxplus \begin{array}{c|ccc|} & 4n & 1 & 1 \\ 4n & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array};$$

$$\underline{\dim}\, P(8n+4,6) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+5, 8n+5, 8n+4),$$

$$M_{\alpha}^{P(8n+4,6)} = M_{\alpha}^{P(8n+4,1)} \oplus M_{\alpha}^{P(8n+6,2)} \quad \text{for } \alpha \neq (5\to 1),$$

$$M_{5\to 1}^{P(8n+4,6)} = (M_{5\to 1}^{P(8n+4,1)} \oplus M_{5\to 1}^{P(8n+6,2)}) \boxplus \begin{array}{c|cc|} & 4n+2 & 1 \\ 2n+1 & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$$\underline{\dim}\, P(8n+5,6) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+6, 8n+5, 8n+5),$$

$$M_{\alpha}^{P(8n+5,6)} = M_{\alpha}^{P(8n+5,1)} \oplus M_{\alpha}^{P(8n+7,2)} \quad \text{for } \alpha \neq (6\to 5),$$

$$M_{6\to 5}^{P(8n+5,6)} = (M_{6\to 5}^{P(8n+5,1)} \oplus M_{6\to 5}^{P(8n+7,2)}) \boxplus \begin{array}{c|cc|} & 4n+2 & 1 \\ 4n+2 & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$$\underline{\dim}\, P(8n+6,6) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+7, 8n+7, 8n+6),$$

$$M_{\alpha}^{P(8n+6,6)} = M_{\alpha}^{P(8n+6,1)} \oplus M_{\alpha}^{P(8n,2)[n\mapsto n+1]} \quad \text{for } \alpha \neq (5\to 2),$$

$$M_{5\to 2}^{P(8n+6,6)} = (M_{5\to 2}^{P(8n+6,1)} \oplus M_{5\to 2}^{P(8n,2)[n\mapsto n+1]}) \boxplus \begin{array}{c|cc|} & 1 & 4n+3 \\ 2n & 0 & 0 \\ 1 & 1 & 0 \end{array};$$

$$\underline{\dim}\, P(8n+7,6) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+8, 8n+7, 8n+7),$$

$$M_{\alpha}^{P(8n+7,6)} = M_{\alpha}^{P(8n+5,3)} \oplus M_{\alpha}^{P(8n+7,4)} \quad \text{for } \alpha \neq (6\to 5),$$

$$M_{6\to 5}^{P(8n+7,6)} = (M_{6\to 5}^{P(8n+5,3)} \oplus M_{6\to 5}^{P(8n+7,4)}) \boxplus \begin{array}{c|ccc|} & 4n+2 & 1 & 1 \\ 4n+2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array};$$

$$\underline{\dim}\, P(8n,7) = (4n+1, 4n+1, 4n, 4n, 8n+1, 8n+1, 8n+1),$$

$$M_{\alpha}^{P(8n,7)} = M_{\alpha}^{P(8n,1)} \oplus M_{\alpha}^{P(8n+3,1)} \quad \text{for } \alpha \neq (5\to 1),$$

$$M_{5\to 1}^{P(8n,7)} = (M_{5\to 1}^{P(8n,1)} \oplus M_{5\to 1}^{P(8n+3,1)}) \boxplus \begin{matrix} & {\scriptstyle 4n} & {\scriptstyle 1} \\ {\scriptstyle 2n} & 0 & 0 \\ {\scriptstyle 1} & 0 & 1 \end{matrix};$$

$\underline{\dim}\, P(8n+1,7) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+3, 8n+2, 8n+2),$

$$M_{\alpha}^{P(8n+1,7)} = M_{\alpha}^{P(8n+1,6)} \oplus M_{\alpha}^{R_1^2(1)} \quad \text{for } \alpha \neq (4 \to 7),$$

$$M_{4\to 7}^{P(8n+1,7)} = (M_{4\to 7}^{P(8n+1,6)} \oplus M_{4\to 7}^{R_1^2(1)}) \boxplus \begin{matrix} & {\scriptstyle 1} \\ {\scriptstyle 8n} & 0 \\ {\scriptstyle 1} & 1 \end{matrix};$$

$\underline{\dim}\, P(8n+2,7) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+4, 8n+3, 8n+2),$

$$M_{\alpha}^{P(8n+2,7)} = M_{\alpha}^{P(8n+2,5)} \oplus M_{\alpha}^{R_1^2(2)} \quad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to 7}^{P(8n+2,7)} = (M_{3\to 7}^{P(8n+2,5)} \oplus M_{3\to 7}^{R_1^2(2)}) \boxplus \begin{matrix} & {\scriptstyle 1} \\ {\scriptstyle 8n} & 0 \\ {\scriptstyle 1} & 1 \end{matrix};$$

$\underline{\dim}\, P(8n+3,7) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+4, 8n+4, 8n+3),$

$$P(3,7) = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right),$$

$$M_{\alpha}^{P(8n+3,7)} = M_{\alpha}^{P(8n+2,3)} \oplus M_{\alpha}^{P(8n+3,3)} \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to 1}^{P(8n+3,7)} = (M_{5\to 1}^{P(8n+2,3)} \oplus M_{5\to 1}^{P(8n+3,3)}) \boxplus \begin{matrix} & {\scriptstyle 4n} & {\scriptstyle 1} & {\scriptstyle 1} \\ {\scriptstyle 2n} & 0 & 0 & 0 \\ {\scriptstyle 1} & 0 & 1 & 0 \end{matrix}, \quad n > 0;$$

$\underline{\dim}\, P(8n+4,7) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+5, 8n+5, 8n+5),$

$$M_{\alpha}^{P(8n+4,7)} = M_{\alpha}^{P(8n+4,1)} \oplus M_{\alpha}^{P(8n+7,1)} \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to 1}^{P(8n+4,7)} = (M_{5\to 1}^{P(8n+4,1)} \oplus M_{5\to 1}^{P(8n+7,1)}) \boxplus \begin{matrix} & {\scriptstyle 4n+2} & {\scriptstyle 1} \\ {\scriptstyle 2n+1} & 0 & 0 \\ {\scriptstyle 1} & 0 & 1 \end{matrix};$$

$\underline{\dim}\, P(8n+5,7) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+7, 8n+6, 8n+6),$

$$M_{\alpha}^{P(8n+5,7)} = M_{\alpha}^{P(8n+5,6)} \oplus M_{\alpha}^{R_1^2(1)} \quad \text{for } \alpha \neq (4 \to 7),$$

$$M_{4\to 7}^{P(8n+5,7)} = (M_{4\to 7}^{P(8n+5,6)} \oplus M_{4\to 7}^{R_1^2(1)}) \boxplus \begin{matrix} & {\scriptstyle 1} \\ {\scriptstyle 8n+4} & 0 \\ {\scriptstyle 1} & 1 \end{matrix};$$

$\underline{\dim}\, P(8n+6,7) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+8, 8n+7, 8n+6),$

$$M_\alpha^{P(8n+6,7)} = M_\alpha^{P(8n+6,6)} \oplus M_\alpha^{R_1^3(1)} \quad \text{for } \alpha \neq (5 \to 2),$$

$$M_{5\to 2}^{P(8n+6,7)} = (M_{5\to 2}^{P(8n+6,6)} \oplus M_{5\to 2}^{R_1^3(1)}) \boxplus \begin{array}{c} \\ 4n+3 \\ 1 \end{array}\!\!\begin{array}{c} 1 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, P(8n+7,7) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+8, 8n+8, 8n+7),$$

$$M_\alpha^{P(8n+7,7)} = M_\alpha^{P(8n+7,2)} \oplus M_\alpha^{P(8n+2,2)[n\mapsto n+1]} \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to 1}^{P(8n+7,7)} = (M_{5\to 1}^{P(8n+7,2)} \oplus M_{5\to 1}^{P(8n+2,2)[n\mapsto n+1]}) \boxplus \begin{array}{c} \\ 2n+1 \\ 1 \end{array}\!\!\begin{array}{c} 1 \quad 4n+4 \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{array}.$$

2.2. The preinjective indecomposable modules. The preinjective indecomposable modules correspond to the vertices of the preinjective part of the Auslander–Reiten quiver, as shown in Figure 2.2.

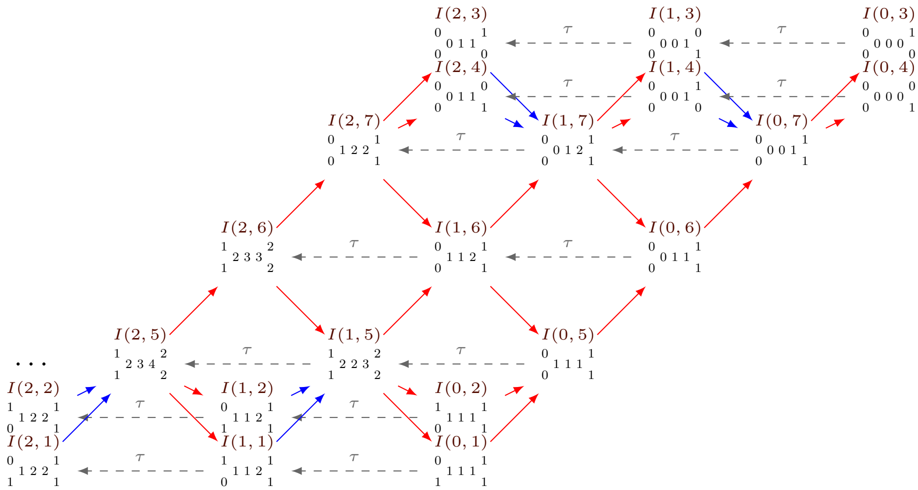


Fig. 2.2. Preinjective part of the Auslander–Reiten quiver of $\Delta(\widetilde{\mathbb{D}}_6)$

Due to the symmetry of the quiver $\Delta(\widetilde{\mathbb{D}}_6)$ we give only the families of representations of the form $I(s,1)$, $I(s,3)$, $I(s,5)$, $I(s,6)$ and $I(s,7)$. For $I(s,2)$ and $I(s,4)$ we can use the permutations $\sigma = (1,2)$ and $\tau = (3,4)$ to write them in terms of $I(s,1)$ and $I(s,3)$ in the following way ($s \geq 0$):

$$\underline{\dim}\, I(s,2) = (d_{\sigma(i)})_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}, \quad \text{where} \quad \underline{\dim}\, I(s,1) = (d_i)_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0},$$

$$\underline{\dim}\, I(s,4) = (d_{\tau(i)})_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}, \quad \text{where} \quad \underline{\dim}\, I(s,3) = (d_i)_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0},$$

for the dimension vectors, and

$$I(s,2) = (M_{\sigma(i)\to\sigma(j)})_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}, \quad \text{where} \quad I(s,1) = (M_{i\to j})_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1},$$

$$I(s,4) = \left(M_{\tau(i)\to\tau(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}, \quad \text{where} \quad I(s,3) = \left(M_{i\to j}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}$$

for the matrices.

In what follows we list the tree representations for preinjective families of the form $I(s,1)$, $I(s,3)$, $I(s,5)$, $I(s,6)$ and $I(s,7)$:

$\underline{\dim} I(8n,1) = (2n+1, 2n, 2n+1, 2n+1, 4n+1, 4n+1, 4n+1)$,

$$I(8n,1) = \left(\begin{array}{c|cc} & 2n & 2n+1 \\ \hline 2n+1 & 0 & 1 \end{array},\ \begin{array}{c|cc} & 2n & 2n+1 \\ \hline 2n & 1 & 0 \end{array},\ \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array},\ \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array},\ \begin{array}{c|cc} & 2n & 1 \\ \hline 2n & 1 & 0 \\ 2n & 1 & 0 \\ 1 & 0 & 1 \end{array},\ \begin{array}{c|cc} & 2n & 1 \\ \hline 2n & 1 & 0 \\ 1 & 0 & 1 \\ 2n & 1 & 0 \end{array} \right);$$

$\underline{\dim} I(8n+1,1) = (2n, 2n+1, 2n+1, 2n+1, 4n+1, 4n+1, 4n+2)$,

$$I(1,1) = \left(0,\ [1],\ [1],\ [0 \quad 1],\ \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right),$$

$$I(8n+1,1) = \left(\begin{array}{c|cccc} & 1 & 1 & 2n-1 & 2n \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2n-1 & 0 & 0 & 1 & 0 \end{array},\ \begin{array}{c|ccc} & 1 & 2n & 2n \\ \hline 1 & 1 & 0 & 0 \\ 2n & 0 & 0 & 1 \end{array},\ \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array},\ \begin{array}{c|cc} & 1 & 4n+1 \\ \hline 4n+1 & 0 & 1 \end{array},\right.$$
$$\left. \begin{array}{c|ccc} & 1 & 2n-1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2n-1 & 0 & 1 & 0 \\ 2n-1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array},\ \begin{array}{c|cccc} & 1 & 1 & 2n-2 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2n-2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 2n-2 & 0 & 0 & 1 & 0 \end{array} \right), \quad n > 0;$$

$\underline{\dim} I(8n+2,1) = (2n+1, 2n, 2n+1, 2n+1, 4n+1, 4n+2, 4n+2)$,

$$I(8n+2,1) = \left(\begin{array}{c|cc} & 2n & 2n+1 \\ \hline 2n+1 & 0 & 1 \end{array},\ \begin{array}{c|cc} & 2n & 2n+1 \\ \hline 2n & 1 & 0 \end{array},\ \begin{array}{c|cc} & 1 & 4n+1 \\ \hline 4n+1 & 0 & 1 \end{array},\ \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \end{array},\ \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ 2n+1 & 1 \end{array},\ \begin{array}{c|cc} & 2n & 1 \\ \hline 1 & 0 & 0 \\ 2n & 1 & 0 \\ 2n & 1 & 0 \\ 1 & 0 & 1 \end{array} \right);$$

$\underline{\dim} I(8n+3,1) = (2n, 2n+1, 2n+1, 2n+1, 4n+2, 4n+2, 4n+2)$,

$$I(8n+3,1) = \left(\begin{array}{cc} & \begin{matrix} 1 & 2n & 2n+1 \end{matrix} \\ 2n & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n+1 & 2n+1 \end{matrix} \\ 2n+1 & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 4n+2 \\ 4n+2 & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 4n+2 \\ 4n+2 & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 2n+1 \\ \begin{matrix} 2n+1 \\ 2n+1 \end{matrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n & 1 \end{matrix} \\ \begin{matrix} 1 \\ 2n \\ 2n \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \right);$$

$$\underline{\dim}\, I(8n+4,1) = (2n+2, 2n+1, 2n+2, 2n+2, 4n+3, 4n+3, 4n+3),$$

$$I(8n+4,1) = \left(\begin{array}{cc} & \begin{matrix} 2n+1 & 2n+2 \end{matrix} \\ 2n+2 & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n+1 & 2n+2 \end{matrix} \\ 2n+1 & \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array}, \begin{array}{cc} & 4n+3 \\ 4n+3 & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 4n+3 \\ 4n+3 & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n+1 & 1 \end{matrix} \\ \begin{matrix} 2n+1 \\ 2n+1 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n+1 & 1 \end{matrix} \\ \begin{matrix} 2n+1 \\ 1 \\ 2n+1 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \right);$$

$$\underline{\dim}\, I(8n+5,1) = (2n+1, 2n+2, 2n+2, 2n+2, 4n+3, 4n+3, 4n+4),$$

$$I(8n+5,1) = \left(\begin{array}{cc} & \begin{matrix} 2n+1 & 2n+2 \end{matrix} \\ 2n+1 & \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n+1 & 2n+2 \end{matrix} \\ 2n+2 & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 4n+3 \\ 4n+3 & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 1 & 4n+3 \end{matrix} \\ 4n+3 & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n+1 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 2n+1 \\ 2n+1 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 2n+2 \\ \begin{matrix} 2n+2 \\ 2n+2 \end{matrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array} \right);$$

$$\underline{\dim}\, I(8n+6,1) = (2n+2, 2n+1, 2n+2, 2n+2, 4n+3, 4n+4, 4n+4),$$

$$I(8n+6,1) = \left(\begin{array}{cc} & \begin{matrix} 2n+1 & 2n+2 \end{matrix} \\ 2n+2 & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n+1 & 2n+2 \end{matrix} \\ 2n+1 & \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 1 & 4n+3 \end{matrix} \\ 4n+3 & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 4n+4 \\ 4n+4 & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n+1 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 2n+1 \\ 2n+1 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 2n+2 \\ \begin{matrix} 2n+2 \\ 2n+2 \end{matrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array} \right);$$

$$\underline{\dim}\, I(8n+7,1) = (2n+1, 2n+2, 2n+2, 2n+2, 4n+4, 4n+4, 4n+4),$$

$$I(8n+7,1) = \left(\begin{array}{cc} & \begin{matrix} 1 & 2n+1 & 2n+2 \end{matrix} \\ 2n+1 & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n+2 & 2n+2 \end{matrix} \\ 2n+2 & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 4n+4 \\ 4n+4 & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 4n+4 \\ 4n+4 & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{matrix} 2n+1 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 2n+1 \\ 2n+1 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \begin{array}{cc} & 2n+2 \\ \begin{matrix} 2n+2 \\ 2n+2 \end{matrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array} \right);$$

$\underline{\dim}\, I(8n,3) = (2n, 2n, 2n+1, 2n, 4n, 4n, 4n)$,

$$I(8n,3) = \left(\begin{array}{r@{}c} & \begin{matrix} 2n & 2n \end{matrix} \\ 2n & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 2n & 2n \end{matrix} \\ 2n & \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & 4n \\ 4n & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & 4n \\ 4n & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 1 & 2n-1 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 2n-1 \\ 2n-1 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 2n-1 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 2n-1 \\ 1 \\ 2n-1 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \right);$$

$\underline{\dim}\, I(8n+1,3) = (2n, 2n, 2n, 2n+1, 4n, 4n, 4n+1)$,

$I(1,3) = (0,\ 0,\ 0,\ 0,\ 0,\ [1])$,

$$I(8n+1,3) = \left(\begin{array}{r@{}c} & \begin{matrix} 1 & 1 & 1 & 2n-2 & 2n-1 \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 2n-2 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 1 & 1 & 2n-1 & 2n-1 \end{matrix} \\ \begin{matrix} 1 \\ 2n-1 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & 4n \\ 4n & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 1 & 4n \end{matrix} \\ 4n & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 1 & 2n-1 \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 1 \\ 2n-1 \\ 2n-1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 1 & 1 & 2n-2 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 2n-2 \\ 2n-2 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \right), \quad n > 0;$$

$\underline{\dim}\, I(8n+2,3) = (2n, 2n, 2n+1, 2n, 4n, 4n+1, 4n+1)$,

$I(2,3) = (0,\ 0,\ 0,\ [1],\ [1],\ 0)$,

$$I(8n+2,3) = \left(\begin{array}{r@{}c} & \begin{matrix} 2n & 2n \end{matrix} \\ 2n & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 2n & 2n \end{matrix} \\ 2n & \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 1 & 4n \end{matrix} \\ 4n & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & 4n+1 \\ 4n+1 & \begin{bmatrix} 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 1 & 2n-1 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 2n-1 \\ 2n-1 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \begin{array}{r@{}c} & \begin{matrix} 2n-1 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 2n-1 \\ 1 \\ 2n-1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \right), \quad n > 0;$$

$\underline{\dim}\, I(8n+3,3) = (2n, 2n, 2n, 2n+1, 4n+1, 4n+1, 4n+1)$,

$I(3,3) = (0,\ 0,\ [1],\ [1],\ 0,\ [1])$,

$$I(8n+3,3)$$

$$= \left(\begin{matrix} & \begin{matrix} 1 & 1 & 2n-1 & 2n \end{matrix} \\ \begin{matrix} 1 \\ 2n-1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{matrix}, \begin{matrix} & \begin{matrix} 2n+1 & 2n \end{matrix} \\ 2n & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{matrix}, \begin{matrix} & 4n+1 \\ 4n+1 & \begin{bmatrix} 1 \end{bmatrix} \end{matrix}, \begin{matrix} & 4n+1 \\ 4n+1 & \begin{bmatrix} 1 \end{bmatrix} \end{matrix}, \begin{matrix} & \begin{matrix} 2n-1 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 2n-1 \\ 2n-1 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix}, \begin{matrix} & \begin{matrix} 1 & 2n \end{matrix} \\ \begin{matrix} 1 \\ 2n \\ 2n \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \end{matrix} \right),$$

$$n > 0;$$

$$\underline{\dim}\, I(8n+4,3) = (2n+1, 2n+1, 2n+2, 2n+1, 4n+2, 4n+2, 4n+2),$$

$$I(8n+4,3)$$

$$= \left(\begin{matrix} & \begin{matrix} 2n+1 & 2n+1 \end{matrix} \\ 2n+1 & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{matrix}, \begin{matrix} & \begin{matrix} 2n+1 & 2n+1 \end{matrix} \\ 2n+1 & \begin{bmatrix} 1 & 0 \end{bmatrix} \end{matrix}, \begin{matrix} & 4n+2 \\ 4n+2 & \begin{bmatrix} 1 \end{bmatrix} \end{matrix}, \begin{matrix} & 4n+2 \\ 4n+2 & \begin{bmatrix} 1 \end{bmatrix} \end{matrix}, \begin{matrix} & \begin{matrix} 1 & 2n & 1 \end{matrix} \\ \begin{matrix} 1 \\ 2n \\ 2n \\ 1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}, \begin{matrix} & \begin{matrix} 2n & 1 \end{matrix} \\ \begin{matrix} 1 \\ 2n \\ 1 \\ 2n \end{matrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{matrix} \right);$$

$$\underline{\dim}\, I(8n+5,3) = (2n+1, 2n+1, 2n+1, 2n+2, 4n+2, 4n+2, 4n+3),$$

$$I(5,3) = \left(\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \right),$$

$$I(8n+5,3) = \left(\begin{matrix} & \begin{matrix} 1 & 1 & 1 & 2n-1 & 2n \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 2n-1 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{matrix}, \begin{matrix} & \begin{matrix} 1 & 2n+1 & 2n \end{matrix} \\ \begin{matrix} 1 \\ 2n \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}, \begin{matrix} & 4n+2 \\ 4n+2 & \begin{bmatrix} 1 \end{bmatrix} \end{matrix}, \begin{matrix} & \begin{matrix} 1 & 4n+2 \end{matrix} \\ 4n+2 & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{matrix}, \begin{matrix} & \begin{matrix} 1 & 2n-1 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 2n-1 \\ 2n-1 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}, \begin{matrix} & \begin{matrix} 1 & 1 & 2n \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 1 \\ 2n \\ 2n \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix} \right), \quad n > 0;$$

$$\underline{\dim}\, I(8n+6,3) = (2n+1, 2n+1, 2n+2, 2n+1, 4n+2, 4n+3, 4n+3),$$

$$I(6,3) = \left(\begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right),$$

$I(8n+6,3)$

$$= \left(\begin{array}{c|cc} & 2n{+}1 & 2n{+}1 \\ \hline 2n{+}1 & 0 & 1 \end{array}, \begin{array}{c|cc} & 2n{+}1 & 2n{+}1 \\ \hline 2n{+}1 & 1 & 0 \end{array}, \begin{array}{c|cc} & 1 & 4n{+}2 \\ \hline 4n{+}2 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n{+}3 \\ \hline 4n{+}3 & 1 \end{array}, \begin{array}{c|ccc} & 1 & 2n & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2n & 0 & 1 & 0 \\ 2n & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}, \begin{array}{c|cc} & 2n & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2n & 1 & 0 \\ 1 & 0 & 1 \\ 2n & 1 & 0 \end{array} \right.$$

$$\left. + \begin{array}{c|ccc} & 1 & 1 & 2n{-}1 \\ \hline 1 & 0 & 1 & 0 \\ 4n{+}2 & 0 & 0 & 0 \end{array} \right), \qquad n > 0;$$

$\underline{\dim}\, I(8n+7,3) = (2n+1, 2n+1, 2n+1, 2n+2, 4n+3, 4n+3, 4n+3)$,

$I(8n+7,3)$

$$= \left(\begin{array}{c|cccc} & 1 & 1 & 2n & 2n{+}1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2n & 0 & 0 & 1 & 0 \end{array}, \begin{array}{c|cc} & 2n{+}2 & 2n{+}1 \\ \hline 2n{+}1 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n{+}3 \\ \hline 4n{+}3 & 1 \end{array}, \begin{array}{c|c} & 4n{+}3 \\ \hline 4n{+}3 & 1 \end{array}, \begin{array}{c|c} & 2n{+}1 \\ \hline 1 & 0 \\ 2n{+}1 & 1 \\ 2n{+}1 & 1 \end{array}, \begin{array}{c|ccc} & 1 & 2n & 1 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2n & 0 & 1 & 0 \\ 2n & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right);$$

$\underline{\dim}\, I(8n,5) = (4n, 4n, 4n+1, 4n+1, 8n+1, 8n+1, 8n+1)$,

$I(0,5) = (0, 0, [1], [1], [1], [1])$,

$$M_\alpha^{I(8n,5)} = M_\alpha^{I(8n,1)} \oplus M_\alpha^{I(8n+7,1)[n\mapsto n-1]} \qquad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to7}^{I(8n,5)} = (M_{3\to7}^{I(8n,1)} \oplus M_{3\to7}^{I(8n+7,1)[n\mapsto n-1]}) \boxplus \begin{array}{c|cc} & 2n{-}1 & 1 \\ \hline 1 & 0 & 1 \\ 4n & 0 & 0 \end{array}, \qquad n > 0;$$

$\underline{\dim}\, I(8n+1,5) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+2, 8n+2, 8n+3)$,

$$I(1,5) = \left([0 \quad 1], [1 \quad 0], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right),$$

$$M_\alpha^{I(8n+1,5)} = M_\alpha^{I(8n+4,4)} \oplus M_\alpha^{I(8n+1,4)} \qquad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to7}^{I(8n+1,5)} = (M_{3\to7}^{I(8n+4,4)} \oplus M_{3\to7}^{I(8n+1,4)}) \boxplus \begin{array}{c|ccc} & 1 & 1 & 2n{-}1 \\ \hline 1 & 0 & 1 & 0 \\ 4n{+}1 & 0 & 0 & 0 \end{array}, \qquad n > 0;$$

$\underline{\dim}\, I(8n+2,5) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+2, 8n+3, 8n+4)$,

$$M_\alpha^{I(8n+2,5)} = M_\alpha^{I(8n+2,1)} \oplus M_\alpha^{I(8n+1,1)} \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to1}^{I(8n+2,5)} = (M_{5\to1}^{I(8n+2,1)} \oplus M_{5\to1}^{I(8n+1,1)}) \boxplus \begin{array}{c} \\ 2n \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 4n & 1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n+3,5) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+3, 8n+4, 8n+4),$$

$$M_\alpha^{I(8n+3,5)} = M_\alpha^{I(8n+3,1)} \oplus M_\alpha^{I(8n+2,1)} \quad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to7}^{I(8n+3,5)} = (M_{3\to7}^{I(8n+3,1)} \oplus M_{3\to7}^{I(8n+2,1)}) \boxplus \begin{array}{c} \\ 1 \\ 4n+1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2n \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n+4,5) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+5, 8n+5, 8n+5),$$

$$M_\alpha^{I(8n+4,5)} = M_\alpha^{I(8n+4,1)} \oplus M_\alpha^{I(8n+3,1)} \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to1}^{I(8n+4,5)} = (M_{5\to1}^{I(8n+4,1)} \oplus M_{5\to1}^{I(8n+3,1)}) \boxplus \begin{array}{c} \\ 1 \\ 2n+1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 4n+1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n+5,5) = (4n+3, 4n+3, 4n+4, 4n+4, 8n+6, 8n+6, 8n+7),$$

$$M_\alpha^{I(8n+5,5)} = M_\alpha^{I(8n+5,1)} \oplus M_\alpha^{I(8n+4,1)} \quad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to7}^{I(8n+5,5)} = (M_{3\to7}^{I(8n+5,1)} \oplus M_{3\to7}^{I(8n+4,1)}) \boxplus \begin{array}{c} \\ 1 \\ 4n+3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2n+1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n+6,5) = (4n+3, 4n+3, 4n+4, 4n+4, 8n+6, 8n+7, 8n+8),$$

$$M_\alpha^{I(8n+6,5)} = M_\alpha^{I(8n+6,1)} \oplus M_\alpha^{I(8n+5,1)} \quad \text{for } \alpha \neq (4 \to 7),$$

$$M_{4\to7}^{I(8n+6,5)} = (M_{4\to7}^{I(8n+6,1)} \oplus M_{4\to7}^{I(8n+5,1)}) \boxplus \begin{array}{c} \\ 1 \\ 4n+3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2n+1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n+7,5) = (4n+3, 4n+3, 4n+4, 4n+4, 8n+7, 8n+8, 8n+8),$$

$$M_\alpha^{I(8n+7,5)} = M_\alpha^{I(8n+7,1)} \oplus M_\alpha^{I(8n+6,1)} \quad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to7}^{I(8n+7,5)} = (M_{3\to7}^{I(8n+7,1)} \oplus M_{3\to7}^{I(8n+6,1)}) \boxplus \begin{array}{c} \\ 1 \\ 4n+3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2n+1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n,6) = (4n, 4n, 4n+1, 4n+1, 8n, 8n+1, 8n+1),$$

$$M_\alpha^{I(8n,6)} = M_\alpha^{I(8n+2,4)} \oplus M_\alpha^{I(8n,3)} \quad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to7}^{I(8n,6)} = (M_{3\to7}^{I(8n+2,4)} \oplus M_{3\to7}^{I(8n,3)}) \boxplus \begin{array}{c}\\ {\scriptstyle 1} \\ {\scriptstyle 4n}\end{array}\!\!\begin{array}{c}{\scriptstyle 2n \quad\; 1} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\end{array};$$

$$\begin{aligned}
&\underline{\dim}\, I(8n+1,6) = (4n, 4n, 4n+1, 4n+1, 8n+1, 8n+1, 8n+2),\\
&I(1,6) = \left(0, 0, [1], [0 \quad 1], \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right),\\
&M_\alpha^{I(8n+1,6)} = M_\alpha^{I(8n+1,1)} \oplus M_\alpha^{I(8n+7,2)[n\mapsto n-1]} \qquad \text{for } \alpha \neq (5\to2),\\
&M_{5\to2}^{I(8n+1,6)} = (M_{5\to2}^{I(8n+1,1)} \oplus M_{5\to2}^{I(8n+7,2)[n\mapsto n-1]}) \boxplus \begin{array}{c}\\ {\scriptstyle 1} \\ {\scriptstyle 2n}\end{array}\!\!\begin{array}{c}{\scriptstyle 1 \;\; 4n-1} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\end{array}, \qquad n>0;
\end{aligned}$$

$$\begin{aligned}
&\underline{\dim}\, I(8n+2,6) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+2, 8n+3, 8n+3),\\
&M_\alpha^{I(8n+2,6)} = M_\alpha^{I(8n+2,2)} \oplus M_\alpha^{I(8n,1)} \qquad \text{for } \alpha \neq (3\to7),\\
&M_{3\to7}^{I(8n+2,6)} = (M_{3\to7}^{I(8n+2,2)} \oplus M_{3\to7}^{I(8n,1)}) \boxplus \begin{array}{c}\\ {\scriptstyle 1} \\ {\scriptstyle 4n+1}\end{array}\!\!\begin{array}{c}{\scriptstyle 2n \quad\; 1} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\end{array};
\end{aligned}$$

$$\begin{aligned}
&\underline{\dim}\, I(8n+3,6) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+3, 8n+3, 8n+4),\\
&I(3,6) = \left([0 \quad 1 \quad 0], [0 \quad 0 \quad 1], \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\right),\\
&M_\alpha^{I(8n+3,6)} = M_\alpha^{I(8n+5,4)} \oplus M_\alpha^{I(8n+3,3)} \qquad \text{for } \alpha \neq (5\to1),\\
&M_{5\to1}^{I(8n+3,6)} = (M_{5\to1}^{I(8n+5,4)} \oplus M_{5\to1}^{I(8n+3,3)}) \boxplus \begin{array}{c}\\ {\scriptstyle 1} \\ {\scriptstyle 2n}\end{array}\!\!\begin{array}{c}{\scriptstyle 1 \quad 1 \;\; 4n-1} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{array}, \qquad n>0;
\end{aligned}$$

$$\begin{aligned}
&\underline{\dim}\, I(8n+4,6) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+4, 8n+5, 8n+5),\\
&M_\alpha^{I(8n+4,6)} = M_\alpha^{I(8n+6,3)} \oplus M_\alpha^{I(8n+4,4)} \qquad \text{for } \alpha \neq (3\to7),\\
&M_{3\to7}^{I(8n+4,6)} = (M_{3\to7}^{I(8n+6,3)} \oplus M_{3\to7}^{I(8n+4,4)}) \boxplus \begin{array}{c}\\ {\scriptstyle 1} \\ {\scriptstyle 4n}\end{array}\!\!\begin{array}{c}{\scriptstyle 1 \quad 2n} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\end{array};
\end{aligned}$$

$$\begin{aligned}
&\underline{\dim}\, I(8n+5,6) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+5, 8n+5, 8n+6),\\
&M_\alpha^{I(8n+5,6)} = M_\alpha^{I(8n+5,2)} \oplus M_\alpha^{I(8n+3,1)} \qquad \text{for } \alpha \neq (5\to1),\\
&M_{5\to1}^{I(8n+5,6)} = (M_{5\to1}^{I(8n+5,2)} \oplus M_{5\to1}^{I(8n+3,1)}) \boxplus \begin{array}{c}\\ {\scriptstyle 1} \\ {\scriptstyle 2n+1}\end{array}\!\!\begin{array}{c}{\scriptstyle 1 \;\; 4n+1} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\end{array};
\end{aligned}$$

$$\underline{\dim}\, I(8n+6,6) = (4n+3,4n+3,4n+4,4n+4,8n+6,8n+7,8n+7),$$
$$M_\alpha^{I(8n+6,6)} = M_\alpha^{I(8n+6,2)} \oplus M_\alpha^{I(8n+4,1)} \quad \text{for } \alpha \neq (3 \to 7),$$
$$M_{3\to 7}^{I(8n+6,6)} = (M_{3\to 7}^{I(8n+6,2)} \oplus M_{3\to 7}^{I(8n+4,1)}) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & 2n+1 \end{array} \\ \begin{array}{c} 1 \\ 4n+3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n+7,6) = (4n+3,4n+3,4n+4,4n+4,8n+7,8n+7,8n+8),$$
$$M_\alpha^{I(8n+7,6)} = M_\alpha^{I(8n+7,2)} \oplus M_\alpha^{I(8n+5,1)} \quad \text{for } \alpha \neq (3 \to 7),$$
$$M_{3\to 7}^{I(8n+7,6)} = (M_{3\to 7}^{I(8n+7,2)} \oplus M_{3\to 7}^{I(8n+5,1)}) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & 2n+1 \end{array} \\ \begin{array}{c} 1 \\ 4n+3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n,7) = (4n,4n,4n+1,4n+1,8n,8n,8n+1),$$
$$I(0,7) = (0,\ 0,\ 0,\ 0,\ [1],\ [1]),$$
$$M_\alpha^{I(8n,7)} = M_\alpha^{I(8n,1)} \oplus M_\alpha^{I(8n+5,1)[n\mapsto n-1]} \quad \text{for } \alpha \neq (4 \to 7),$$
$$M_{4\to 7}^{I(8n,7)} = (M_{4\to 7}^{I(8n,1)} \oplus M_{4\to 7}^{I(8n+5,1)[n\mapsto n-1]}) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & 2n-1 \end{array} \\ \begin{array}{c} 4n \\ 1 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{array}, \quad n > 0;$$

$$\underline{\dim}\, I(8n+1,7) = (4n,4n,4n+1,4n+1,8n,8n+1,8n+2),$$
$$I(1,7) = \left(0,0,0,[0 \quad 1], \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right),$$
$$M_\alpha^{I(8n+1,7)} = M_\alpha^{I(8n+1,1)} \oplus M_\alpha^{I(8n+6,1)[n\mapsto n-1]} \quad \text{for } \alpha \neq (5 \to 2),$$
$$M_{5\to 2}^{I(8n+1,7)} = (M_{5\to 2}^{I(8n+1,1)} \oplus M_{5\to 2}^{I(8n+6,1)[n\mapsto n-1]}) \boxplus \begin{array}{cc} & \begin{array}{cc} 4n+2 & 1 \end{array} \\ \begin{array}{c} 1 \\ 1 \\ 2n-3 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}, \quad n > 0;$$

$$\underline{\dim}\, I(8n+2,7) = (4n,4n,4n+1,4n+1,8n+1,8n+2,8n+2),$$
$$I(2,7) = \left(0,0,[0 \quad 1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right),$$
$$M_\alpha^{I(8n+2,7)} = M_\alpha^{I(8n+2,1)} \oplus M_\alpha^{I(8n+7,1)[n\mapsto n-1]} \quad \text{for } \alpha \neq (5 \to 1),$$
$$M_{5\to 1}^{I(8n+2,7)} = (M_{5\to 1}^{I(8n+2,1)} \oplus M_{5\to 1}^{I(8n+7,1)[n\mapsto n-1]}) \boxplus \begin{array}{cc} & \begin{array}{cc} 4n-1 & 1 \end{array} \\ \begin{array}{c} 2n \\ 1 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \quad n > 0;$$

$$\underline{\dim}\, I(8n+3,7) = (4n+1,4n+1,4n+2,4n+2,8n+3,8n+3,8n+3),$$
$$M_\alpha^{I(8n+3,7)} = M_\alpha^{I(8n+3,1)} \oplus M_\alpha^{I(8n,1)} \quad \text{for } \alpha \neq (5 \to 2),$$

$$M_{5\to 2}^{I(8n+3,7)} = (M_{5\to 2}^{I(8n+3,1)} \oplus M_{5\to 2}^{I(8n,1)}) \boxplus \begin{array}{c} \\ 2n \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 4n & 1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n+4,7) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+4, 8n+4, 8n+5),$$
$$M_{\alpha}^{I(8n+4,7)} = M_{\alpha}^{I(8n+4,1)} \oplus M_{\alpha}^{I(8n+1,1)} \quad \text{for } \alpha \neq (3 \to 7),$$
$$M_{3\to 7}^{I(8n+4,7)} = (M_{3\to 7}^{I(8n+4,1)} \oplus M_{3\to 7}^{I(8n+1,1)}) \boxplus \begin{array}{c} \\ 1 \\ 4n+2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 2n & 1 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n+5,7) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+4, 8n+5, 8n+6),$$
$$M_{\alpha}^{I(8n+5,7)} = M_{\alpha}^{I(8n+6,4)} \oplus M_{\alpha}^{I(8n+5,4)} \quad \text{for } \alpha \neq (3 \to 7),$$
$$M_{3\to 7}^{I(8n+5,7)} = (M_{3\to 7}^{I(8n+6,4)} \oplus M_{3\to 7}^{I(8n+5,4)}) \boxplus \begin{array}{c} \\ 1 \\ 1 \\ 4n+1 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 1 & 2n \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n+6,7) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+5, 8n+6, 8n+6),$$
$$M_{\alpha}^{I(8n+6,7)} = M_{\alpha}^{I(8n+6,1)} \oplus M_{\alpha}^{I(8n+3,1)} \quad \text{for } \alpha \neq (3 \to 7),$$
$$M_{3\to 7}^{I(8n+6,7)} = (M_{3\to 7}^{I(8n+6,1)} \oplus M_{3\to 7}^{I(8n+3,1)}) \boxplus \begin{array}{c} \\ 1 \\ 1 \\ 4n+2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2n \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim}\, I(8n+7,7) = (4n+3, 4n+3, 4n+4, 4n+4, 8n+7, 8n+7, 8n+7),$$
$$M_{\alpha}^{I(8n+7,7)} = M_{\alpha}^{I(8n+7,1)} \oplus M_{\alpha}^{I(8n+4,1)} \quad \text{for } \alpha \neq (3 \to 7),$$
$$M_{3\to 7}^{I(8n+7,7)} = (M_{3\to 7}^{I(8n+7,1)} \oplus M_{3\to 7}^{I(8n+4,1)}) \boxplus \begin{array}{c} \\ 1 \\ 4n+3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2n+1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}.$$

2.3. The exceptional regular modules. As mentioned in the introductory part, there are only a finite number of exceptional regular modules. These are the non-homogeneous indecomposable regulars with dimension vector falling below $\delta = (1, 1, 1, 1, 2, 2, 2)$, marked green in Figure 2.3 (see the pdf file). Note that $\underline{\dim}\, R_0^l(2) = \underline{\dim}\, R_1^{l'}(4) = \underline{\dim}\, R_\infty^l(2) = \delta$, where $l \in \{1, 2\}$, $l' \in \{1, 2, 3, 4\}$.

Representations of regular simples of $\Delta(\widetilde{\mathbb{D}}_6)$ are also given in [21]; we include them here only for completeness:

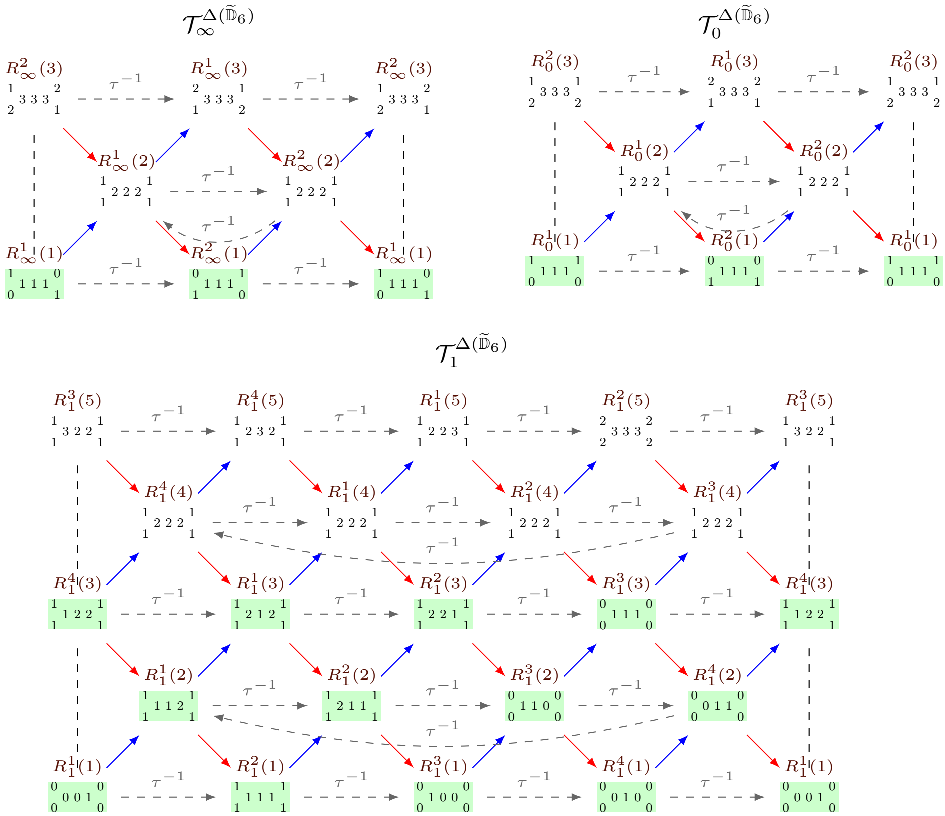


Fig. 2.3. Regular non-homogeneous tubes in the case of $\Delta(\widetilde{\mathbb{D}}_6)$

$\underline{\dim}\, R_\infty^1(1) = (0,1,0,1,1,1,1)$,	$R_\infty^1(1) = (0,[1],[1],[1],0,[1])$;
$\underline{\dim}\, R_\infty^2(1) = (1,0,1,0,1,1,1)$,	$R_\infty^2(1) = ([1],0,[1],[1],[1],0)$;
$\underline{\dim}\, R_0^1(1) = (0,1,1,0,1,1,1)$,	$R_0^1(1) = (0,[1],[1],[1],[1],0)$;
$\underline{\dim}\, R_0^2(1) = (1,0,0,1,1,1,1)$,	$R_0^2(1) = ([1],0,[1],[1],0,[1])$;
$\underline{\dim}\, R_1^1(1) = (0,0,0,0,0,0,1)$,	$R_1^1(1) = (0,0,0,0,0,0)$;
$\underline{\dim}\, R_1^1(2) = (1,1,1,1,1,1,2)$,	$R_1^1(2) = \left([1],[1],[1],[0\quad 1],\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$;
$\underline{\dim}\, R_1^1(3) = (1,1,1,1,2,1,2)$,	$R_1^1(3) = \left([1\quad 0],[1\quad 1],\begin{bmatrix}1\\0\end{bmatrix},[0\quad 1],\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$;

$\underline{\dim}\, R_1^2(1) = (1,1,1,1,1,1,1)$,	$R_1^2(1) = ([1],[1],[1],[1],[1],[1])$;
$\underline{\dim}\, R_1^2(2) = (1,1,1,1,2,1,1)$,	$R_1^2(2) = \left([1 \quad 0],[1 \quad 1],\begin{bmatrix}1\\0\end{bmatrix},[1],[1],[1]\right)$;
$\underline{\dim}\, R_1^2(3) = (1,1,1,1,2,2,1)$,	$R_1^2(3) = \left([1 \quad 0],[1 \quad 1],\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},[1],[1]\right)$;
$\underline{\dim}\, R_1^3(1) = (0,0,0,0,1,0,0)$,	$R_1^3(1) = (0,0,0,0,0,0)$;
$\underline{\dim}\, R_1^3(2) = (0,0,0,0,1,1,0)$,	$R_1^3(2) = (0,0,[1],0,0,0)$;
$\underline{\dim}\, R_1^3(3) = (0,0,0,0,1,1,1)$,	$R_1^3(3) = (0,0,[1],[1],0,0)$;
$\underline{\dim}\, R_1^4(1) = (0,0,0,0,0,1,0)$,	$R_1^4(1) = (0,0,0,0,0,0)$;
$\underline{\dim}\, R_1^4(2) = (0,0,0,0,0,1,1)$,	$R_1^4(2) = (0,0,0,[1],0,0)$;
$\underline{\dim}\, R_1^4(3) = (1,1,1,1,1,2,2)$,	$R_1^4(3) = \left([1],[1],[0 \quad 1],\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$.

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