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Example of a quasianalytic contraction whose spectrum is a proper subarc of the unit circle

by

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Abstract. A partial answer to a question of Kérchy and Szalai (2015) is given. Namely, it is proved that there exists a quasianalytic contraction whose quasianalytic spectral set is equal to its spectrum and is a proper subarc of the unit circle, but no estimates of the norm of its inverse are given.

1. Introduction. Let $\mathcal{H}$ be a (complex, separable) Hilbert space, and let T be a (bounded linear) operator on $\mathcal{H}$. The spectrum of T is denoted by $\sigma(T)$. The *commutant* $\{T\}'$ is the set of all operators A on $\mathcal{H}$ such that $AT = TA$. Recall that $\{T\}'$ is an algebra closed in the weak operator topology. The lattice of all (closed) subspaces $\mathcal{E}$ of $\mathcal{H}$ such that $A\mathcal{E} \subset \mathcal{E}$ for every $A \in \{T\}'$ is called the *hyperinvariant subspace lattice* of T and is denoted by $\operatorname{Hlat} T$; these subspaces $\mathcal{E}$ are called *hyperinvariant subspaces* of T.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Denote by $\mathcal{L}(\mathcal{H},\mathcal{K})$ the space of all (linear, bounded) transformations acting from $\mathcal{H}$ to $\mathcal{K}$. Set $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H},\mathcal{H})$, then $\mathcal{L}(\mathcal{H})$ is the algebra of all (bounded linear) operators acting on $\mathcal{H}$. Let $T \in \mathcal{L}(\mathcal{H})$, $R \in \mathcal{L}(\mathcal{K})$ and $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$ be such that $XT = RX$. If X is *invertible*, that is, $X^{-1} \in \mathcal{L}(\mathcal{K},\mathcal{H})$, then T and R are called *similar*. If, in addition, X is a unitary transformation, then T and R are called *unitarily equivalent*. It is well known and easy to see that similarity preserves many properties of operators. In particular, if T and R are similar, then $\sigma(T) = \sigma(R)$ and $\mathcal{E} \in \operatorname{Hlat} T$ if and only if $X\mathcal{E} \in \operatorname{Hlat} R$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. A power bounded operator T is *of class* $C_{1\cdot}$ if $\inf_{n\in\mathbb{N}} \|T^n x\| > 0$ for every $0 \neq x \in \mathcal{H}$, and is *of class* $C_{0\cdot}$ if $\lim_n \|T^n x\| = 0$ for every $x \in \mathcal{H}$. Moreover,

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T is *of class* $C_{\cdot a}$ if T^* is of class $C_{a\cdot}$, and T is *of class* C_{ab} if it is of classes $C_{a\cdot}$ and $C_{\cdot b}$, $a, b = 0, 1$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *polynomially bounded* if there exists a constant M such that $\|p(T)\| \leq M \max\{|p(z)| : z \in \mathbb{T}\}$ for every (analytic) polynomial p, where $\mathbb{T}$ is the unit circle. For a natural number n any $n \times n$ matrix can be regarded as an operator on ℓ^2_n; its norm is denoted by the symbol $\|\cdot\|_{\mathcal{L}(\ell^2_n)}$. For a family of polynomials $[p_{ij}]_{i,j=1}^n$ put

$$\|[p_{ij}]_{i,j=1}^n\|_{H^\infty(\ell^2_n)} = \max\{\|[p_{ij}(z)]_{i,j=1}^n\|_{\mathcal{L}(\ell^2_n)} : z \in \mathbb{T}\}.$$

For $T \in \mathcal{L}(\mathcal{H})$ and a family of polynomials $[p_{ij}]_{i,j=1}^n$ the operator

$$[p_{ij}(T)]_{i,j=1}^n \in \mathcal{L}\Big(\bigoplus_{j=1}^n \mathcal{H}\Big)$$

is defined. We call T *completely polynomially bounded* if there exists a constant M such that

$$\|[p_{ij}(T)]_{i,j=1}^n\| \leq M \|[p_{ij}]_{i,j=1}^n\|_{H^\infty(\ell^2_n)} \tag{1.1}$$

for every family of polynomials $[p_{ij}]_{i,j=1}^n$ and any $n \in \mathbb{N}$.

An operator T is called a *contraction* if $\|T\| \leq 1$. The following criterion for an operator to be similar to a contraction is proved in [P]:

An operator T is similar to a contraction if and only if T is completely polynomially bounded.

We recall some definitions and results on unitary asymptotes and quasianalytic operators. For references on unitary asymptotes see [NFBK, Ch. IX.1], [Kér1], [Kér3], [Kér4]; for quasianalytic operators see also [E], [KSz1], [KSz2], [Gam1]; see also references therein.

A pair (X, U), where $U \in \mathcal{L}(\mathcal{K})$ is a unitary operator and X is a transformation such that $XT = UX$, is called a *unitary asymptote* of an operator $T \in \mathcal{L}(\mathcal{H})$ if for any other pair (Y, V), where V is a unitary operator and Y is a transformation such that $YT = VY$, there exists a unique transformation Z such that $ZU = VZ$ and $Y = ZX$. The uniqueness of Z is equivalent to the relation $\bigvee_{n\geq 0} U^{-n}X\mathcal{H} = \mathcal{K}$ (that is, the pair (X, U) is minimal). Two pairs (X, U) and (X_1, U_1), where U and U_1 are unitary operators and X and X_1 are transformations such that $XT = UX$ and $X_1T = U_1X_1$, are *similar* if there exists an *invertible* transformation Z such that $ZU = U_1Z$ and $X_1 = ZX$. It follows from the definition that a unitary asymptote of T is defined up to similarity. If two operators are similar and one of them has a unitary asymptote then so does the other, and the two unitary asymptotes are similar. If $ZU = U_1Z$ for an invertible transformation Z and unitary operators U and U_1, then U and U_1 are unitarily equivalent ([NFBK, Proposition II.3.4], [Co, Proposition II.10.6], [RR, Proposition 1.5]).

Let $T \in \mathcal{L}(\mathcal{H})$ have a unitary asymptote (X, U), where $U \in \mathcal{L}(\mathcal{K})$. *We will assume that* $\mathcal{K} \neq \{0\}$ (that is, the unitary asymptote is non-degenerate). Then we can consider the mapping

$$\gamma_T \colon \{T\}' \to \{U\}', \qquad \gamma_T(A) = D,$$

where $D \in \{U\}'$ is the unique operator such that $XA = DX$, and γ_T is a unital algebra-homomorphism. Furthermore,

$$\sigma(\gamma_T(A)) \subset \sigma(A) \quad \text{for every} A \in \{T\}'. \tag{1.2}$$

For $\mathcal{E} \subset \mathcal{K}$ set

$$X^{-1}\mathcal{E} = \{x \in \mathcal{H} : Xx \in \mathcal{E}\}.$$

Then $X^{-1}\mathcal{E} \in \operatorname{Hlat} T$ for every $\mathcal{E} \in \operatorname{Hlat} U$. It is well known that $\operatorname{Hlat} U \neq \{\{0\}, \mathcal{K}\}$ if U is not multiplication by a unimodular constant on $\mathcal{K}$. But it is possible that

$$X^{-1}\mathcal{E} = \{0\} \quad \text{for every } \mathcal{E} \in \operatorname{Hlat} U \text{ such that } \mathcal{E} \neq \mathcal{K}. \tag{1.3}$$

Such an operator T is called *quasianalytic.*

Denote by $\mathbf{m}$ the normalized linear measure on the unit circle $\mathbb{T}$. For an $\mathbf{m}$-measurable set $\sigma \subset \mathbb{T}$ denote by U_σ the operator of multiplication by the independent variable on $L^2(\sigma) := L^2(\sigma, \mathbf{m})$. It is well known that U_σ is cyclic,

$$\{U_\sigma\}' = \{\eta(U_\sigma) : \eta \in L^\infty(\sigma, \mathbf{m})\},$$

where $\eta(U_\sigma)$ is the operator of multiplication by η, and

$$\operatorname{Hlat} U_\sigma = \{L^2(\tau) : \tau \subset \sigma\}$$

(where τ are measurable with respect to $\mathbf{m}$).

Let $\sigma \subset \mathbb{T}$ be an $\mathbf{m}$-measurable set, and let an operator T have a unitary asymptote (X, U_σ). Then for every $A \in \{T\}'$ there exists a function η in $L^\infty(\sigma, \mathbf{m}) =: L^\infty(\sigma)$ such that $\gamma_T(A) = \eta(U_\sigma)$. The mapping

$$\widehat{\gamma}_T \colon \{T\}' \to L^\infty(\sigma), \qquad \widehat{\gamma}_T(A) = \eta, \tag{1.4}$$

is a unital algebra-homomorphism, and $\widehat{\gamma}_T$ does not depend on the choice of X. Furthermore, $\widehat{\gamma}_T(T) = \chi|_\sigma$, where $\chi(z) = z$ for $z \in \mathbb{T}$. The range $\widehat{\gamma}_T(\{T\}')$ of the mapping $\widehat{\gamma}_T$ is called the *functional commutant* of T (see [KSz1] and references therein).

Every power bounded operator T has a unitary asymptote, and if T is of class $C_{1\cdot}$, then γ_T is injective. If, in addition, a unitary operator from the unitary asymptote (*which will also be called the unitary asymptote*) of T is U_σ for some $\sigma \subset \mathbb{T}$, then $\widehat{\gamma}_T$ is injective, too. Therefore, $\{T\}'$ is an abelian algebra, because $L^\infty(\sigma)$ is. Moreover, if $R \in \{T\}'$ and $\{R\}'$ is abelian, then $\{T\}' = \{R\}'$ (see [KSz1, Proposition 11]; quasianalyticity is not used in the proof here).

Let T be a polynomially bounded operator. Then clearly T is power bounded, and therefore T has a unitary asymptote. If in addition T is of class $C_{\cdot 0}$, then the spectral measure of the unitary asymptote of T is absolutely continuous with respect to $\mathbf{m}$ [Kér3, Theorem 13 and Proposition 15]. If T is quasianalytic, then T is of class C_{10} [Kér3, Proposition 33]. For definition of the *quasianalytic spectral set* of T we refer to [Kér3], [Kér4] and [KSz1]. We recall only that for a quasianalytic polynomially bounded operator T the quasianalytic spectral set coincides with the $\mathbf{m}$-measurable set on which the spectral measure of the unitary asymptote of T is concentrated.

Let T be a quasianalytic operator. Then $\sigma(T)$ is a connected set. Indeed, assume that $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are non-empty disjoint closed sets. Then $T = T_1 \dot{+} T_2$, where each T_k acts on a nonzero space, and $\sigma(T_k) = \sigma_k$, $k = 1, 2$ ([RR, Theorem 2.10]; the Riesz–Dunford functional calculus is used). Therefore, T is similar to $T_1 \oplus T_2$. Since T is quasianalytic, so is $T_1 \oplus T_2$, in particular, it has a non-degenerate unitary asymptote (X, U). By [Kér4, Proposition 13], T_k has a unitary asymptote (X_k, U_k), $k = 1, 2$, and (X, U) is similar to $(X_1 \oplus X_2, U_1 \oplus U_2)$. By (1.2), $\sigma(U_k) \subset \sigma_k$, $k = 1, 2$. Therefore, U_k is the restriction of $U_1 \oplus U_2$ to a hyperinvariant subspace, $k = 1, 2$. This contradicts (1.3). On the other hand, if T has a unitary asymptote (X, U) with $\ker X = \{0\}$, and $\sigma(T) \cap \mathbb{T} = \{\lambda\}$ for some $\lambda \in \mathbb{T}$, then $T = \lambda I$. Indeed, (1.2) implies $U = \lambda I$, and the equality $X^* \overline{\lambda} I = T^* X^*$ implies $T = \lambda I$. Thus, if T is a quasianalytic operator such that $\sigma(T) \subset \mathbb{T}$, then $\sigma(T)$ is a subarc of $\mathbb{T}$ and is not a singleton. Examples of quasianalytic contractions T such that $\sigma(T) = \mathbb{T}$ or $\sigma(T) \cap \mathbb{T} \neq \mathbb{T}$ are known. But in all examples known to the author, the interior of the polynomially convex hull of $\sigma(T)$ is nonempty. (Recall that the polynomially convex hull of a compact set $\sigma \subset \mathbb{C}$ is the union of σ and all the bounded components of $\mathbb{C} \setminus \sigma$; for example, the polynomially convex hull of $\mathbb{T}$ is the closed unit disc $\operatorname{clos} \mathbb{D}$.)

In this paper, a quasianalytic operator R similar to a contraction is constructed such that $\sigma(R) = \{e^{it} : t \in [0, \pi]\}$ and the unitary asymptote of R is $U_{\sigma(R)}$. Therefore, the quasianalytic spectral set of R is $\sigma(R)$. First, an appropriate quasianalytic operator T with $\sigma(T) = \mathbb{T}$ is constructed, and then it is proved that there exists a function $\varrho \in \widehat{\gamma}_T(\{T\}')$ such that $\varrho^2 = \chi$ a.e. on $\mathbb{T}$. Therefore, there exists $R \in \{T\}'$ such that $R^2 = T$. The operator R has the required properties. Since R is similar to some contraction $\widetilde{R}$, this contraction $\widetilde{R}$ is quasianalytic, and $\sigma(\widetilde{R}) = \{e^{it} : t \in [0, \pi]\}$ is the quasianalytic spectral set of $\widetilde{R}$. The proof of the quasianalyticity of T is based on a theorem by Beurling. The existence of nontrivial hyperinvariant subspaces of T and R (and hence of $\widetilde{R}$) is based on a result from [E].

The paper is organized as follows. In Sec. 2 some general relations between operators T and R such that $T = R^2$ are studied. In Sec. 3 an operator T with

$\sigma(T) = \mathbb{T}$ is constructed for which the existence of an operator R such that $T = R^2$ will be proved later. In Sec. 4 the functional commutant $\widehat{\gamma}_T(\{T\}')$ of T is considered. Using transition from the unit disc $\mathbb{D}$ to the upper half-plane $\mathbb{C}_+$ and applying Fourier transform it is shown that a function from $L^\infty(\mathbb{T})$ belongs to $\widehat{\gamma}_T(\{T\}')$ if and only if the corresponding convolution operator is bounded on a weighted space L^2 over $\mathbb{R}$. The main result of this section is Theorem 4.3. Secs. 5 and 8 contain auxiliary results. In Secs. 6 and 7 the quasianalyticity and the existence of hyperinvariant subspaces of T, respectively, are studied. In Sec. 9 the boundedness of the convolution operator on a weighted space L^2 over $\mathbb{R}$ which corresponds to a branch of the square root function is proved. This implies the existence of an operator R such that $T = R^2$.

The following notation will be used. We denote by H^p the Hardy spaces (on some domain of $\mathbb{C}$, which will be specified). Let $I_{\mathcal{H}}$ and $P_{\mathcal{E}}$ be the identity operator on a Hilbert space $\mathcal{H}$ and the orthogonal projection on the subspace $\mathcal{E}$. For positive functions $w(t)$ and $\phi(t)$, the notation $w \asymp \phi$ means that $0 < \inf_t w(t)/\phi(t) \leq \sup_t w(t)/\phi(t) < \infty$. The unit constant function and the identity function are denoted by $\mathbf{1}$ and χ respectively: $\mathbf{1}(z) = 1$ and $\chi(z) = z$, $z \in \operatorname{clos}\mathbb{D}$.

2. Square root of an operator

Theorem 2.1. *Let T and R be operators on a Hilbert space such that $R^2 = T$. Then R is power bounded, of class $C_{1\cdot}$, of class $C_{\cdot 0}$, polynomially bounded or similar to a contraction if and only if T has the same property.*

Proof. The proofs of statements concerning power boundedness and belonging to $C_{1\cdot}$ and $C_{\cdot 0}$ are easy and therefore omitted. The proofs of the "only if" parts of the statements concerning polynomial boundedness and similarity to contractions are easy, too. Suppose that T is similar to a contraction. Then T is completely polynomially bounded [P]; see (1.1). We will prove that R is completely polynomially bounded. Then it will be proved that R is similar to a contraction [P].

Let p be a polynomial. Then $p(z) = \sum_{n=0}^N c_n z^n$, $z \in \mathbb{C}$, for some $N \in \mathbb{N}$. For convenience, set $c_n = 0$ for $n \in \mathbb{N}$, $n \geq N+1$. Set

$$p_0(z) = \sum_{n\geq 0} c_{2n} z^n \quad \text{and} \quad p_1(z) = \sum_{n\geq 0} c_{2n+1} z^n, \qquad z \in \mathbb{C}.$$

Clearly,

$$p_0(z^2) = \frac{p(z)+p(-z)}{2} \quad \text{and} \quad zp_1(z^2) = \frac{p(z)-p(-z)}{2}, \qquad z \in \mathbb{C}.$$

For a family of polynomials $[p_{ij}]_{i,j=1}^n$ we have

$$[(p_{ij})_0(z^2)]_{i,j=1}^n = \tfrac{1}{2}\big([p_{ij}(z)]_{i,j=1}^n + [p_{ij}(-z)]_{i,j=1}^n\big),$$

$$z[(p_{ij})_1(z^2)]_{i,j=1}^n = \tfrac{1}{2}\big([p_{ij}(z)]_{i,j=1}^n - [p_{ij}(-z)]_{i,j=1}^n\big), \qquad z \in \mathbb{C}.$$

Therefore,

$$\begin{aligned}\|[(p_{ij})_0(z^2)]_{i,j=1}^n\|_{\mathcal{L}(\ell^2_n)} &\le \tfrac{1}{2}\big(\|[p_{ij}(z)]_{i,j=1}^n\|_{\mathcal{L}(\ell^2_n)} + \|[p_{ij}(-z)]_{i,j=1}^n\|_{\mathcal{L}(\ell^2_n)}\big)\\ &\le \|[p_{ij}]_{i,j=1}^n\|_{H^\infty(\ell^2_n)},\\ |z|\|[(p_{ij})_1(z^2)]_{i,j=1}^n\|_{\mathcal{L}(\ell^2_n)} &\le \tfrac{1}{2}\big(\|[p_{ij}(z)]_{i,j=1}^n\|_{\mathcal{L}(\ell^2_n)} + \|[p_{ij}(-z)]_{i,j=1}^n\|_{\mathcal{L}(\ell^2_n)}\big)\\ &\le \|[p_{ij}]_{i,j=1}^n\|_{H^\infty(\ell^2_n)}, \qquad z \in \mathbb{T}.\end{aligned}$$

Clearly, for every $\zeta \in \mathbb{T}$ there exists $z \in \mathbb{T}$ such that $z^2 = \zeta$. Therefore,

$$\begin{aligned}\|[(p_{ij})_0]_{i,j=1}^n\|_{H^\infty(\ell^2_n)} &\le \|[p_{ij}]_{i,j=1}^n\|_{H^\infty(\ell^2_n)},\\ \|[(p_{ij})_1]_{i,j=1}^n\|_{H^\infty(\ell^2_n)} &\le \|[p_{ij}]_{i,j=1}^n\|_{H^\infty(\ell^2_n)}.\end{aligned}$$

We have

$$[p_{ij}(R)]_{i,j=1}^n = [(p_{ij})_0(T)]_{i,j=1}^n + \Big(\bigoplus_{j=1}^n R\Big) \cdot [(p_{ij})_1(T)]_{i,j=1}^n.$$

Since T is completely polynomially bounded, (1.1) is fulfilled for T with some constant M. Therefore,

$$\begin{aligned}\|[p_{ij}(R)]_{i,j=1}^n\| &\le \|[(p_{ij})_0(T)]_{i,j=1}^n\| + \|R\|\,\|[(p_{ij})_1(T)]_{i,j=1}^n\|\\ &\le M\|[(p_{ij})_0]_{i,j=1}^n\|_{H^\infty(\ell^2_n)} + \|R\|M\|[(p_{ij})_1]_{i,j=1}^n\|_{H^\infty(\ell^2_n)}\\ &= (1+\|R\|)M\|[p_{ij}]_{i,j=1}^n\|_{H^\infty(\ell^2_n)}.\end{aligned}$$

Thus, R is completely polynomially bounded.

If we only suppose that T is polynomially bounded, then the proof of polynomial boundedness of R is similar. ■

LEMMA 2.2. *Set $\varrho(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}t/2}$ for $t \in (0, 2\pi)$. Suppose that $T, R \in \mathcal{L}(\mathcal{H})$ are such that T is polynomially bounded of class $C_{\cdot 0}$, $(X, U_{\mathbb{T}})$ is a unitary asymptote of T, $R^2 = T$ and $XR = \varrho(U_{\mathbb{T}})X$. Then $(X, \varrho(U_{\mathbb{T}}))$ is a unitary asymptote of R.*

Proof. By Theorem 2.1, R is polynomially bounded of class $C_{\cdot 0}$. Therefore, R has a unitary asymptote (Y, V), and the spectral measure of V is absolutely continuous with respect to **m** [Kér3, Theorem 13 and Proposition 15]. By [Kér2, Theorem 6], (Y, V^2) is a unitary asymptote of T. Therefore, there exists an invertible transformation Z_1 such that $Y = Z_1X$ and $Z_1U_{\mathbb{T}} = V^2Z_1$.

Since $\varrho(U_{\mathbb{T}})$ is unitary, there exists a transformation Z_2 such that $X = Z_2Y$ and $Z_2V = \varrho(U_{\mathbb{T}})Z_2$. Therefore, $X = Z_2Z_1X$ and $Z_2Z_1U_{\mathbb{T}} = U_{\mathbb{T}}Z_2Z_1$. It follows from the definition of a unitary asymptote (see the Introduction and the references therein) that

$$\bigvee_{n\geq 0} U_{\mathbb{T}}^{-n} X\mathcal{H} = L^2(\mathbb{T}).$$

Therefore, $Z_2 Z_1 = I_{L^2(\mathbb{T})}$. Thus, $Z_2 = Z_1^{-1}$, and so the pairs $(X, \varrho(U_{\mathbb{T}}))$ and (Y, V) are similar. ■

Corollary 2.3. *Under the assumptions of Lemma* 2.2, *T is quasianalytic if and only if R is quasianalytic.*

Proof. It is well known and easy to see that $\varrho(U_{\mathbb{T}})$ is unitarily equivalent to U_σ with $\sigma = \{\mathrm{e}^{\mathrm{i}t} : t \in (0, \pi)\}$. Therefore, $\{\varrho(U_{\mathbb{T}})\}'$ is an abelian algebra. Since $\varrho(U_{\mathbb{T}}) \in \{U_{\mathbb{T}}\}'$ and $\{U_{\mathbb{T}}\}'$ is abelian, we conclude that

$$\{\varrho(U_{\mathbb{T}})\}' = \{U_{\mathbb{T}}\}'$$

by [KSz1, Proposition 11] (quasianalyticity is not used in the proof there). Thus,

$$\operatorname{Hlat} \varrho(U_{\mathbb{T}}) = \operatorname{Hlat} U_{\mathbb{T}} = \{L^2(\tau) : \tau \subset \mathbb{T}\}. \tag{2.1}$$

Suppose that T is quasianalytic. It follows from the definition (1.3) of quasianalyticity and (2.1) that

$$X^{-1} L^2(\tau) = \{0\} \tag{2.2}$$

for every measurable set $\tau \subset \mathbb{T}$ such that $\mathbf{m}(\tau) < 1$. Since $(X, \varrho(U_{\mathbb{T}}))$ is a unitary asymptote of R, we infer that R is quasianalytic by (2.2) and (2.1).

Conversely, if R is quasianalytic, then T is quasianalytic by the same reasoning.

Note that the "if" part is a particular case of [Kér2, Corollary 13] (applied to R). ■

3. Construction of T. Recall that $\mathbf{1}(z) = 1$ and $\chi(z) = z$ for $z \in \operatorname{clos} \mathbb{D}$, and $H^2(\mathbb{D})$ denotes the Hardy space in $\mathbb{D}$. Set $S = U_{\mathbb{T}}|_{H^2(\mathbb{D})}$ and $H^2_-(\mathbb{D}) = L^2(\mathbb{T}) \ominus H^2(\mathbb{D})$.

Recall that $\widehat{\gamma}_T$ is defined in (1.4).

The following proposition combines [Gam1, Proposition 3.1] (based on [Ca]), [Kér1, Theorem 3] (applied to T^*) and [KSz1, Sec. 5]; its proof is omitted.

Proposition 3.1. *Suppose that $\mathcal{H}_0$ is a Hilbert space, $T_0 \in \mathcal{L}(\mathcal{H}_0)$ is a contraction of class C_{00}, and $X_0 \in \mathcal{L}(\mathcal{H}_0, H^2_-(\mathbb{D}))$ is such that* $\ker X_0 = \{0\}$, $\operatorname{clos} X_0\mathcal{H}_0 = H^2_-(\mathbb{D})$ *and $X_0 T_0 = P_{H^2_-(\mathbb{D})} U_{\mathbb{T}}|_{H^2_-(\mathbb{D})} X_0$. Put*

$$T = \begin{pmatrix} S & (\cdot, X_0^* \overline{\chi})\mathbf{1} \\ \mathbb{O} & T_0 \end{pmatrix}.$$

Then T is similar to a contraction and T is of class C_{10}. Therefore, T admits an H^∞-functional calculus. Furthermore, $(I_{H^2(\mathbb{D})} \oplus X_0, U_{\mathbb{T}})$ is a unitary asymptote of T. Therefore, $\mathbb{T} \subset \sigma(T) \subset \operatorname{clos}\mathbb{D}$, and T is quasianalytic if and only if

$$P_{H^2_-(\mathbb{D})} L^2(\tau) \cap X_0\mathcal{H}_0 = \{0\} \tag{3.1}$$

for every measurable set $\tau \subset \mathbb{T}$ such that $\mathbf{m}(\tau) < 1$.

Let $\eta \in L^\infty(\mathbb{T})$. Then $\eta \in \widehat{\gamma}_T(\{T\}')$ if and only if the mapping

$$(I_{H^2(\mathbb{D})} \oplus X_0)^{-1}\eta(U_{\mathbb{T}})(I_{H^2(\mathbb{D})} \oplus X_0) \tag{3.2}$$

is defined and bounded, and then $\widehat{\gamma}_T^{-1}(\eta)$ is equal to the operator in (3.2).

Let ν be a positive finite Borel measure on $\mathbb{D}$. Clearly, multiplication by the independent variable on $L^2(\nu)$ is a contraction of class C_{00}. Denote by $P^2(\nu)$ the closure of the set of (analytic) polynomials in $L^2(\nu)$, and by S_ν the operator of multiplication by the independent variable in $P^2(\nu)$, i.e.

$$S_\nu \in \mathcal{L}(P^2(\nu)), \quad (S_\nu f)(z) = zf(z), \qquad f \in P^2(\nu),\ z \in \mathbb{D}.$$

Since S_ν is the restriction on an invariant subspace of a contraction of class C_{00}, S_ν is a contraction of class C_{00}, too. Furthermore, if $H^2(\mathbb{D}) \subset L^2(\nu)$, then the natural imbedding of $H^2(\mathbb{D})$ into $L^2(\nu)$ is bounded and

$$P^2(\nu) = \operatorname{clos}_{L^2(\nu)} H^2(\mathbb{D}).$$

Suppose that ν is such that every $f \in P^2(\nu)$ is analytic in $\mathbb{D}$. More precisely, suppose that for every $\lambda \in \mathbb{D}$ there exists $k_\lambda \in P^2(\nu)$ such that $(f, k_\lambda) = f(\lambda)$ for every (analytic) polynomial f, and the mapping $\lambda \mapsto (f, k_\lambda)$, $\mathbb{D} \to \mathbb{C}$, is analytic for every $f \in P^2(\nu)$. Then $f(\lambda) = (f, k_\lambda)$ for ν-a.e. $\lambda \in \mathbb{D}$. For every $\lambda \in \mathbb{D}$ and every $f \in P^2(\nu)$ define $f_\lambda(z) = \frac{f(z)-f(\lambda)}{z-\lambda}$, $z \in \mathbb{D}$. Then $f_\lambda \in P^2(\nu)$. (For the proof, see, for example, [ARS, Introduction], see also [Co, Sec. II.7].) The mapping $f \mapsto f_\lambda$ is a bounded left inverse of $S_\nu - \lambda I$. Therefore, $P^2(\nu) = (S_\nu - \lambda I)P^2(\nu) \dotplus \mathbb{C}\mathbf{1}$, and $(S_\nu - \lambda I)^* P^2(\nu) = P^2(\nu)$. Furthermore, $S_\nu^* k_\lambda = \overline{\lambda} k_\lambda$ and $\dim\ker(S_\nu - \lambda I)^* = 1$ for every $\lambda \in \mathbb{D}$.

Set

$$(Wh)(z) = \overline{z}h(\overline{z}), \qquad h \in L^2(\mathbb{T}),\ z \in \mathbb{T}. \tag{3.3}$$

Clearly, $W \in \mathcal{L}(L^2(\mathbb{T}))$ is unitary, $W = W^{-1}$, and $WH^2(\mathbb{D}) = H^2_-(\mathbb{D})$. If $f \in H^2_-(\mathbb{D})$, then f has the analytic continuation on $\mathbb{C} \setminus \operatorname{clos}\mathbb{D}$ denoted by the same letter f and defined by the formula $f(z) := \frac{1}{z}(Wf)\big(\frac{1}{z}\big)$, $|z| > 1$. Nontangential boundary values of f outside of $\mathbb{T}$ exist and are equal to f a.e. on $\mathbb{T}$, because $Wf \in H^2(\mathbb{D})$ has the analogous property inside of $\mathbb{T}$.

Proposition 3.2. *Let ν be a positive finite Borel measure on $\mathbb{D}$ such that every $f \in P^2(\nu)$ is analytic in $\mathbb{D}$, and $H^2(\mathbb{D}) \subset P^2(\nu)$. Let*

$$J_\nu \in \mathcal{L}(H^2(\mathbb{D}), P^2(\nu))$$

be the natural imbedding. Set $\mathcal{H}_0 = P^2(\nu)$, $T_0 = S_\nu^*$, $X_0 = WJ_\nu^*$ *and define* T *as in Proposition* 3.1. *Then* $\sigma(T) = \mathbb{T}$.

Proof. Let $\lambda \in \mathbb{D}$. There exists $k_\lambda \in P^2(\nu)$ such that $(f, k_\lambda) = f(\lambda)$ for every $f \in P^2(\nu)$.

Let $h_\diamond \in H^2(\mathbb{D})$, and let $f_\diamond \in P^2(\nu)$. There exists $f \in P^2(\nu)$ such that $(S_\nu - \lambda I)^* f = f_\diamond$. Set $h(z) = \frac{h_\diamond(z) - h_\diamond(\overline{\lambda})}{z - \overline{\lambda}}$, $z \in \mathbb{D}$. Then $h \in H^2(\mathbb{D})$. Taking into account that

$$X_0^* \overline{\chi} = J_\nu W^{-1} \overline{\chi} = J_\nu \mathbf{1} = \mathbf{1},$$

we obtain

$$(T - \overline{\lambda} I)(h \oplus (f + (h_\diamond(\overline{\lambda}) - (f, \mathbf{1}))k_\lambda)) = h_\diamond \oplus f_\diamond.$$

If $\lambda \in \mathbb{D}$, $h \in H^2(\mathbb{D})$, and $f \in P^2(\nu)$ are such that $(T - \overline{\lambda} I)(h \oplus f) = 0$, then there exists $c \in \mathbb{C}$ such that $f = ck_\lambda$ and $(z - \overline{\lambda})h(z) = -c$ for every $z \in \mathbb{D}$. Therefore, $c = 0$. Thus, $\mathbb{D} \cap \sigma(T) = \emptyset$. By Proposition 3.1, we have $\mathbb{T} \subset \sigma(T) \subset \operatorname{clos} \mathbb{D}$. ∎

For $0 \le r < 1$ set

$$D_r = \{z \in \mathbb{C} : |z - r| < 1 - r\}, \qquad \Gamma_r = \partial D_r = \{z \in \mathbb{C} : |z - r| = 1 - r\}, \tag{3.4}$$

and denote by ν_r the arc length measure on Γ_r. (Of course, $D_0 = \mathbb{D}$, $\Gamma_0 = \mathbb{T}$, and $\nu_0 = 2\pi \mathbf{m}$.) Using a linear change of variable and well-known properties of $H^2(\mathbb{D})$, it is easily seen that every $f \in P^2(\nu_r)$ is analytic in D_r, and for every $\lambda \in D_r$ the mapping $f \mapsto f(\lambda)$, $P^2(\nu_r) \to \mathbb{C}$, is bounded.

Lemma 3.3. *Let* $\{a_n\}_{n=1}^\infty$ *and* $\{r_n\}_{n=1}^\infty$ *be families of numbers such that* $a_n > 0$, $0 < r_{n+1} < r_n < 1$ *for every* n, $\sum_{n=1}^\infty a_n \le 1$, *and* $r_n \to 0$. *Set*

$$\nu = \frac{1}{2\pi} \sum_{n=1}^\infty a_n \nu_{r_n}.$$

Then $H^2(\mathbb{D}) \subset L^2(\nu)$, *and if* $f \in P^2(\nu)$, *then* f *is analytic in* $\mathbb{D}$.

Proof. Let $h \in H^2(\mathbb{D})$. Since $\frac{1}{2\pi} \int_{\Gamma_r} |h|^2 \, \mathrm{d}\nu_r \le 2\|h\|^2_{H^2(\mathbb{D})}$ (see, for example, [N, Lemma I.A.6.3.3] or [HJ, Sec. II.1.5.1]), we conclude that $H^2(\mathbb{D}) \subset L^2(\nu)$. On the other hand, $P^2(\nu) \subset P^2(\nu_{r_n})$ for every n, $D_{r_n} \subset D_{r_{n+1}}$ and $\bigcup_{n=1}^\infty D_{r_n} = \mathbb{D}$. Therefore, every $f \in P^2(\nu)$ is analytic in $\mathbb{D}$. ∎

Remark 3.4. Of course, the conclusion of Lemma 3.3 holds true if $\sum_{n=1}^\infty a_n < \infty$. The assumption $\sum_{n=1}^\infty a_n \le 1$ is added for the convenience of the subsequent considerations.

Remark 3.5. The construction of the measure ν from Lemma 3.3 is close to [KT1], [KT2].

4. Transfer to the half-plane and Fourier transform. Set $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ and

$$\varpi(z) = \frac{z - \mathrm{i}}{z + \mathrm{i}}, \qquad z \in \mathbb{C}. \tag{4.1}$$

It is well known and easy to see that $\varpi|_{\mathbb{C}_+}$ is a conformal mapping of $\mathbb{C}_+$ onto $\mathbb{D}$, and for every $0 \le r < 1$ and every f for which the integrals below are defined we have

$$\frac{1}{2\pi} \int\limits_{\Gamma_r} f \,\mathrm{d}\nu_r = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} (f \circ \varpi)\Big(t + \mathrm{i}\frac{r}{1-r}\Big) \frac{\mathrm{d}t}{t^2 + \big(\frac{1}{1-r}\big)^2}, \tag{4.2}$$

where Γ_r and ν_r are defined by (3.4) and just after it. Set

$$\mathcal{J} f(z) = \frac{1}{\sqrt{\pi}} \frac{1}{z + \mathrm{i}} (f \circ \varpi)(z) \tag{4.3}$$

for all functions f and $z \in \mathbb{C}$ for which the definition (4.3) makes sense. Then $\mathcal{J}$ is a unitary transformation from $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$, and for $\eta \in L^\infty(\mathbb{T})$ the operator $\mathcal{J}\eta(U_{\mathbb{T}})\mathcal{J}^{-1}$ is multiplication by $\eta \circ \varpi$ acting on $L^2(\mathbb{R})$. Furthermore, $\mathcal{J}H^2(\mathbb{D}) = H^2(\mathbb{C}_+)$ (see for example [N, Sec. I.A.6.3.1] or [HJ, Sec. II.1.5.1]). Since $f \in H^2(\mathbb{D})$ if and only if $Wf \in H^2_-(\mathbb{D})$, we have

$$(\mathcal{J}Wf)(z) = -(\mathcal{J}f)(-z), \qquad z \in \mathbb{C}_-$$

(see the definition (3.3) and the explanation that follows), and

$$(\mathcal{J}W\mathcal{J}^{-1}h)(z) = -h(-z) \quad \text{for } h \in H^2(\mathbb{C}_+) \text{ and } z \in \mathbb{C}_-. \tag{4.4}$$

For a measure ν defined as in Lemma 3.3 set

$$\mathrm{d}\mu = \sum_{n=1}^{\infty} a_n \mathrm{d}t|_{\mathbb{R}+\mathrm{i}v_n} \quad \text{with} \quad v_n = \frac{r_n}{1 - r_n}, \qquad n \ge 1. \tag{4.5}$$

A straightforward calculation based on (4.2) shows that

(4.6) $\mathcal{J}$ is a unitary transformation from $L^2(\mathbb{D}, \nu)$ onto $L^2(\mathbb{C}_+, \mu)$.

Since $P^2(\nu) = \operatorname{clos}_{L^2(\nu)} H^2(\mathbb{D})$ and $\mathcal{J}H^2(\mathbb{D}) = H^2(\mathbb{C}_+)$, we conclude that $H^2(\mathbb{C}_+) \subset L^2(\mathbb{C}_+, \mu)$ and

$$\mathcal{J}P^2(\nu) = \operatorname{clos}_{L^2(\mu)} H^2(\mathbb{C}_+). \tag{4.7}$$

Denote by J_μ the natural imbedding of $H^2(\mathbb{C}_+)$ into $\operatorname{clos}_{L^2(\mu)} H^2(\mathbb{C}_+)$.

Let $\mathcal{D}(\mathbb{R})$ be the space of test functions, that is, the space of functions from $C^\infty(\mathbb{R})$ with compact support. Let $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ be the spaces of rapidly decreasing smooth functions and of tempered distributions, respectively. Recall that $\mathcal{S}'(\mathbb{R})$ is the dual space of $\mathcal{S}(\mathbb{R})$, $\mathcal{D}(\mathbb{R})$ is contained and dense in $\mathcal{S}(\mathbb{R})$, and $L^p(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ for $1 \le p \le \infty$. The *Fourier transform* $\mathcal{F}$

of a function f defined on $\mathbb{R}$ and its inverse $\mathcal{F}^{-1}$ act by the formulas

(4.8)
$$(\mathcal{F}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ts} f(s)\,\mathrm{d}s, \quad (\mathcal{F}^{-1}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ts} f(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}.$$

It is well known that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous linear mutually inverse bijections on $\mathcal{S}(\mathbb{R})$ and on $\mathcal{S}'(\mathbb{R})$, and $\mathcal{F}$ is unitary on $L^2(\mathbb{R})$. It follows from (4.4) that

$$\mathcal{F}\mathcal{J}W\mathcal{J}^{-1} = \mathcal{J}W\mathcal{J}^{-1}\mathcal{F}. \tag{4.9}$$

For $\Psi \in \mathcal{S}'(\mathbb{R})$ multiplication $\mathcal{M}_\Psi$ by Ψ and convolution $\mathcal{C}_\Psi$ with Ψ are continuous linear mappings from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$. If $\Psi \in L^p(\mathbb{R})$, $1 \le p \le \infty$, then $\mathcal{M}_\Psi$ and $\mathcal{C}_\Psi$ act in the usual way: $(\mathcal{M}_\Psi f)(t) = \Psi(t)f(t)$ and

$$(\mathcal{C}_\Psi f)(t) = \int_{\mathbb{R}} f(t-s)\Psi(s)\,\mathrm{d}s, \qquad t \in \mathbb{R},\ f \in \mathcal{S}(\mathbb{R}). \tag{4.10}$$

It is well known that

$$\mathcal{F}\mathcal{M}_\Psi\mathcal{F}^{-1}f = \frac{1}{\sqrt{2\pi}}\mathcal{C}_{\mathcal{F}\Psi}f, \qquad f \in \mathcal{S}(\mathbb{R}). \tag{4.11}$$

If $\eta \in L^\infty(\mathbb{T})$, then $\mathcal{C}_{\mathcal{F}(\eta\circ\varpi)}$ has an extension from $\mathcal{S}(\mathbb{R})$ onto $L^2(\mathbb{R})$ defined by (4.11), which is a (bounded linear) operator on $L^2(\mathbb{R})$ (that is, $\mathcal{C}_{\mathcal{F}(\eta\circ\varpi)} \in \mathcal{L}(L^2(\mathbb{R}))$.

For $\alpha > 0$ set

$$\theta_\alpha(z) = \mathrm{e}^{\mathrm{i}\alpha z}, \qquad z \in \mathbb{C}. \tag{4.12}$$

Then

$$\big(\mathcal{F}(\theta_\alpha^n f)\big)(t) = (\mathcal{F}f)(t - n\alpha), \quad t \in \mathbb{R},\ n \in \mathbb{Z},\ f \in L^2(\mathbb{R}), \tag{4.13}$$

$$\frac{1}{\sqrt{2\pi}}(\mathcal{C}_{\mathcal{F}\theta_\alpha} f)(t) = f(t-\alpha), \qquad t \in \mathbb{R},\ f \in L^2(\mathbb{R}). \tag{4.14}$$

Set $\mathcal{K}_\alpha = H^2(\mathbb{C}_+) \ominus \theta_\alpha H^2(\mathbb{C}_+)$. By the Paley–Wiener theorem,

$$H^2(\mathbb{C}_+) = \mathcal{F}^{-1}L^2(0,\infty) \tag{4.15}$$

and

$$\theta_\alpha^n \mathcal{K}_\alpha = \theta_\alpha^n \mathcal{F}^{-1}L^2(0,\alpha) = \mathcal{F}^{-1}L^2(n\alpha,(n+1)\alpha) \quad \text{for every } n \in \mathbb{Z}. \tag{4.16}$$

For references see, for example, [Ka, Ch. VI] or [R, Ch. 7].

For $-\infty \le b_1 < b_2 \le \infty$ and measurable $w\colon (b_1,b_2) \to (0,\infty)$ set

$$L^2((b_1,b_2),w) = \Big\{ f\colon (b_1,b_2) \to \mathbb{C} : \int_{b_1}^{b_2} |f(t)|^2 w(t)\,\mathrm{d}t < \infty \Big\}.$$

Proposition 4.1. *Let ν be as in Lemma 3.3, and let μ be defined by ν as in (4.5). Put*

$$\phi(t)=\sum_{k=1}^{\infty}a_k\mathrm{e}^{-2v_kt},\qquad t\in(0,\infty).$$

For $\alpha>0$ set

$$\frac{1}{\omega_\alpha^2(-n-1)}=\sum_{k=1}^{\infty}a_k\mathrm{e}^{-2\alpha nv_k},\qquad n\geq 0,\tag{4.17}$$

and

$$\phi_{\alpha,n}(t)=\sum_{k=1}^{\infty}a_k\mathrm{e}^{-2\alpha v_kn}\mathrm{e}^{-2v_kt}=\phi(t+n\alpha),\qquad t\in(0,\alpha),\ n\geq 0.$$

Then

(4.18) *the Fourier transform $\mathcal{F}$ is well-defined and is a unitary transformation from* $\operatorname{clos}_{L^2(\mu)}H^2(\mathbb{C}_+)$ *onto* $L^2((0,\infty),\phi)$,

$$J_\mu^*\operatorname{clos}_{L^2(\mu)}H^2(\mathbb{C}_+)=\Big\{\bigoplus_{n=0}^{\infty}\theta_\alpha^n\mathcal{F}^{-1}f_n: f_n\in L^2(0,\alpha),\ \sum_{n=0}^{\infty}\|f_n\|^2_{L^2(0,\alpha)}\omega_\alpha^2(-n-1)<\infty\Big\},\tag{4.19}$$

$$(\mathcal{F}\mathcal{J}W\mathcal{J}^{-1}J_\mu^*\mathcal{F}^{-1}f)(t)=-\phi(-t)f(-t),\qquad t\in(-\infty,0),\ f\in L^2((0,\infty),\phi).\tag{4.20}$$

Proof. Set $\mathcal{L}_{\alpha,n}=\theta_\alpha^n\mathcal{F}^{-1}L^2(0,\alpha)$ for $n\geq 0$. By (4.16),

$$H^2(\mathbb{C}_+)=\bigoplus_{n=0}^{\infty}\mathcal{L}_{\alpha,n}\quad\text{and}\quad\|\theta_\alpha^n\mathcal{F}^{-1}f\|_{H^2(\mathbb{C}_+)}=\|f\|_{L^2(0,\alpha)},\qquad f\in L^2(0,\alpha).$$

For $v>0$ define $A_{\alpha,v}\in\mathcal{L}(L^2(0,\alpha))$ by $(A_{\alpha,v}f)(t)=\mathrm{e}^{-vt}f(t)$ for $f\in L^2(0,\alpha)$, $t\in(0,\alpha)$. Then $(\mathcal{F}^{-1}f)(t+\mathrm{i}v)=(\mathcal{F}^{-1}A_{\alpha,v}f)(t)$, $t\in(0,\alpha)$.

Let $n,m\geq 0$, and let $f,g\in L^2(0,\alpha)$. We have

$$\begin{aligned}
&(\theta_\alpha^n\mathcal{F}^{-1}f,\theta_\alpha^m\mathcal{F}^{-1}g)_{L^2(\mu)}\\
&\quad=\sum_{k=1}^{\infty}a_k\int_{\mathbb{R}}(\theta_\alpha^n\mathcal{F}^{-1}f)(t+\mathrm{i}v_k)\overline{(\theta_\alpha^m\mathcal{F}^{-1}g)(t+\mathrm{i}v_k)}\,\mathrm{d}t\\
&\quad=\sum_{k=1}^{\infty}a_k\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\alpha(t+\mathrm{i}v_k)n}\overline{\mathrm{e}^{\mathrm{i}\alpha(t+\mathrm{i}v_k)m}}(\mathcal{F}^{-1}A_{\alpha,v_k}f)(t)\overline{(\mathcal{F}^{-1}A_{\alpha,v_k}g)(t)}\,\mathrm{d}t\\
&\quad=\sum_{k=1}^{\infty}a_k\mathrm{e}^{-\alpha v_k(n+m)}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}(n-m)\alpha t}(\mathcal{F}^{-1}A_{\alpha,v_k}f)(t)\overline{(\mathcal{F}^{-1}A_{\alpha,v_k}g)(t)}\,\mathrm{d}t\\
&\quad=\sum_{k=1}^{\infty}a_k\mathrm{e}^{-\alpha v_k(n+m)}(\theta_\alpha^n\mathcal{F}^{-1}A_{\alpha,v_k}f,\theta_\alpha^m\mathcal{F}^{-1}A_{\alpha,v_k}g)_{L^2(\mathbb{R})}.
\end{aligned}$$

If $n \neq m$, then

$$(\theta_\alpha^n \mathcal{F}^{-1} A_{\alpha,v_k} f, \theta_\alpha^m \mathcal{F}^{-1} A_{\alpha,v_k} g)_{L^2(\mathbb{R})} = 0,$$

because $\mathcal{F}^{-1} L^2(0,\alpha) = \mathcal{K}_\alpha$ by (4.16). If $n = m$, then

$$\begin{aligned}(4.21)\qquad & (\theta_\alpha^n \mathcal{F}^{-1} f, \theta_\alpha^n \mathcal{F}^{-1} g)_{L^2(\mu)} \\ &= \sum_{k=1}^\infty a_k \mathrm{e}^{-2\alpha v_k n} (\theta_\alpha^n \mathcal{F}^{-1} A_{\alpha,v_k} f, \theta_\alpha^n \mathcal{F}^{-1} A_{\alpha,v_k} g)_{L^2(\mathbb{R})} \\ &= \sum_{k=1}^\infty a_k \mathrm{e}^{-2\alpha v_k n} (A_{\alpha,v_k} f, A_{\alpha,v_k} g)_{L^2(0,\alpha)} \\ &= \int_0^\alpha \sum_{k=1}^\infty a_k \mathrm{e}^{-2\alpha v_k n} \mathrm{e}^{-2v_k t} f(t)\overline{g(t)}\,\mathrm{d}t = \int_0^\alpha f(t)\overline{g(t)}\phi_{\alpha,n}(t)\,\mathrm{d}t.\end{aligned}$$

Clearly,

(4.22)
$$\frac{\mathrm{e}^{-2v_1\alpha}}{\omega_\alpha^2(-n-1)} \leq \frac{1}{\omega_\alpha^2(-n-2)} \leq \phi_{\alpha,n}(t) \leq \frac{1}{\omega_\alpha^2(-n-1)}, \qquad t \in (0,\alpha),\ n \geq 0.$$

It is proved that $\mathcal{L}_{\alpha,n}$ is orthogonal to $\mathcal{L}_{\alpha,m}$ for $n, m \geq 0$ and $n \neq m$, and $\mathcal{L}_{\alpha,n}$ is closed in $L^2(\mu)$ for every $n \geq 0$. Consequently,

$$(4.23)\qquad \operatorname{clos}_{L^2(\mu)} H^2(\mathbb{C}_+) = \Big\{ \bigoplus_{n=0}^\infty h_n : h_n \in \mathcal{L}_{\alpha,n},\ \sum_{n=0}^\infty \|h_n\|^2_{L^2(\mu)} < \infty \Big\}.$$

Let $\{f_n\}_{n=0}^\infty \subset L^2(0,\alpha)$. Set $h_n = \theta_\alpha^n \mathcal{F}^{-1} f_n$, $n \geq 0$, and

$$f(t) = f_n(t - n\alpha), \qquad t \in (n\alpha, (n+1)\alpha),\ n \geq 0.$$

By (4.13), $\mathcal{F}(\bigoplus_{n=0}^\infty h_n) = f$. The relation (4.18) follows from the last equality, (4.21), (4.23) and the definition of $\phi_{\alpha,n}$.

Let $J_{\mu,\alpha,n}$ be the natural imbedding of $\mathcal{L}_{\alpha,n}$ as a subspace of $H^2(\mathbb{C}_+)$ into $\mathcal{L}_{\alpha,n}$ as a subspace of $L^2(\mu)$. Since the spaces $\mathcal{L}_{\alpha,n}$, $n \geq 0$, are orthogonal and dense in both spaces $H^2(\mathbb{C}_+)$ and $\operatorname{clos}_{L^2(\mu)} H^2(\mathbb{C}_+)$, we conclude that $J_\mu = \bigoplus_{n=0}^\infty J_{\mu,\alpha,n}$. Consequently,

$$J_\mu^* = \bigoplus_{n=0}^\infty J_{\mu,\alpha,n}^*.$$

Let f, $g \in L^2(0,\alpha)$. By (4.21),

$$\begin{aligned}&(J_{\mu,\alpha,n}^* \theta_\alpha^n \mathcal{F}^{-1} f, \theta_\alpha^n \mathcal{F}^{-1} g)_{H^2(\mathbb{C}_+)} = (\theta_\alpha^n \mathcal{F}^{-1} f, \theta_\alpha^n \mathcal{F}^{-1} g)_{L^2(\mu)} \\ &\quad = \int_0^\alpha f(t)\overline{g(t)}\phi_{\alpha,n}(t)\,\mathrm{d}t = (\phi_{\alpha,n} f, g)_{L^2(0,\alpha)} = (\theta_\alpha^n \mathcal{F}^{-1} \phi_{\alpha,n} f, \theta_\alpha^n \mathcal{F}^{-1} g)_{H^2(\mathbb{C}_+)}.\end{aligned}$$

Thus,

$$J^*_{\mu,\alpha,n}\theta^n_\alpha\mathcal{F}^{-1}f=\theta^n_\alpha\mathcal{F}^{-1}\phi_{\alpha,n}f,\qquad f\in L^2(0,\alpha). \tag{4.24}$$

By (4.24) and (4.23),

$$J^*_\mu\operatorname{clos}_{L^2(\mu)}H^2(\mathbb{C}_+)=\Big\{\bigoplus_{n=0}^\infty\theta^n_\alpha\mathcal{F}^{-1}\phi_{\alpha,n}f_n : f_n\in L^2(0,\alpha),\ \sum_{n=0}^\infty\|\theta^n_\alpha\mathcal{F}^{-1}f_n\|^2_{L^2(\mu)}<\infty\Big\}.$$

Set $g_n=\phi_{\alpha,n}f_n$. Then

$$J^*_\mu\operatorname{clos}_{L^2(\mu)}H^2(\mathbb{C}_+)=\Big\{\bigoplus_{n=0}^\infty\theta^n_\alpha\mathcal{F}^{-1}g_n : g_n\in L^2(0,\alpha),\ \sum_{n=0}^\infty\Big\|\theta^n_\alpha\mathcal{F}^{-1}\frac{g_n}{\phi_{\alpha,n}}\Big\|^2_{L^2(\mu)}<\infty\Big\}.$$

By (4.21),

$$\Big\|\theta^n_\alpha\mathcal{F}^{-1}\frac{g_n}{\phi_{\alpha,n}}\Big\|^2_{L^2(\mu)}=\int_0^\alpha\frac{|g_n(t)|^2}{\phi_{\alpha,n}(t)^2}\phi_{\alpha,n}(t)\,\mathrm{d}t=\int_0^\alpha\frac{|g_n(t)|^2}{\phi_{\alpha,n}(t)}\,\mathrm{d}t.$$

It follows from the latter equality and (4.22) that

$$\sum_{n=0}^\infty\Big\|\theta^n_\alpha\mathcal{F}^{-1}\frac{g_n}{\phi_{\alpha,n}}\Big\|^2_{L^2(\mu)}<\infty\quad\text{if and only if}\ \sum_{n=0}^\infty\|g_n\|^2_{L^2(0,\alpha)}\omega^2_\alpha(-n-1)<\infty.$$

The equality (4.19) is proved.

Let $f\in L^2((0,\infty),\phi)$. Then $f=\bigoplus_{n=0}^\infty f|_{(n\alpha,(n+1)\alpha)}$. Set

$$f_n(t)=f(t+n\alpha),\qquad t\in(0,\alpha),\ n\geq 0.$$

By (4.13) and (4.24),

$$J^*_\mu\mathcal{F}^{-1}f|_{(n\alpha,(n+1)\alpha)}=J^*_\mu\theta^n_\alpha\mathcal{F}^{-1}f_n=\theta^n_\alpha\mathcal{F}^{-1}(\phi_{\alpha,n}f_n)=\mathcal{F}^{-1}((\phi f)|_{(n\alpha,(n+1)\alpha)}).$$

The equality (4.20) follows from (4.9) and (4.4). ∎

Remark 4.2. The idea of Proposition 4.1 is from [FR1], [FR2].

Recall that $\widehat{\gamma}_T$ is defined in (1.4).

Theorem 4.3. *Let T be defined as in Proposition 3.2 with ν as in Lemma 3.3. Define μ as in (4.5). Set*

$$\widetilde{\phi}\colon\mathbb{R}\to(0,\infty),\quad \widetilde{\phi}(t)=\frac{1}{\phi(-t)},\quad t\in(-\infty,0),\quad \widetilde{\phi}(t)=1,\quad t\in(0,\infty),$$

where ϕ is defined as in Proposition 4.1.

Let $\eta\in L^\infty(\mathbb{T})$. Then $\eta\in\widehat{\gamma}_T(\{T\}')$ if and only if $\mathcal{C}_{\mathcal{F}(\eta\circ\varpi)}\in\mathcal{L}(L^2(\mathbb{R},\widetilde{\phi}))$, and then $\widehat{\gamma}_T^{-1}(\eta)$ is unitarily equivalent to $\frac{1}{\sqrt{2\pi}}\mathcal{C}_{\mathcal{F}(\eta\circ\varpi)}$.

Proof. By Proposition 3.1, $\eta \in \widehat{\gamma}_T(\{T\}')$ if and only if the mapping from (3.2) is defined and bounded. By (4.6), (4.7), and (4.18),

$$\mathcal{F}\mathcal{J}\colon H^2(\mathbb{D}) \oplus P^2(\nu) \to L^2(0,\infty) \oplus L^2((0,\infty),\phi) \text{ is unitary.}$$

Set

$$Y = (\mathcal{F}\mathcal{J})_{H^2_-(\mathbb{D}) \to L^2(-\infty,0)} X_0 (\mathcal{F}\mathcal{J})^{-1}_{L^2((0,\infty),\phi) \to P^2(\nu)},$$

where the lower indices of $\mathcal{F}\mathcal{J}$ and $(\mathcal{F}\mathcal{J})^{-1}$ indicate the spaces between which they act. Taking into account the definition of X_0, the equality

$$J_\nu^* = \mathcal{J}^{-1} J_\mu^* \mathcal{J}, \tag{4.25}$$

and applying (4.20) and (4.7), we find that Y acts by the formula

$$Y\colon L^2((0,\infty),\phi) \to L^2(-\infty,0), (Yf)(t) = -\phi(-t)f(-t),$$
$$t \in (-\infty,0),\ f \in L^2((0,\infty),\phi).$$

By (4.11), the mapping from (3.2) is defined and bounded if and only if the mapping

$$(I_{L^2(0,\infty)} \oplus Y^{-1}) \frac{1}{\sqrt{2\pi}} \mathcal{C}_{\mathcal{F}(\eta\circ\varpi)} (I_{L^2(0,\infty)} \oplus Y) \tag{4.26}$$

is defined and bounded.

Define $V\colon L^2((-\infty,0),\widetilde{\phi}) \to L^2((0,\infty),\phi)$ by the formula

$$(Vf)(t) = -\frac{1}{\phi(t)} f(-t), \qquad t \in (0,\infty),\ f \in L^2((-\infty,0),\widetilde{\phi}).$$

Then V is unitary and YV is the natural imbedding of $L^2((-\infty,0),\widetilde{\phi})$ into $L^2(-\infty,0)$. Thus, $I_{L^2(0,\infty)} \oplus YV$ is the natural imbedding of $L^2(\mathbb{R},\widetilde{\phi})$ into $L^2(\mathbb{R})$. Multiplying the mapping (4.26) by $I_{L^2(0,\infty)} \oplus V^{-1}$ from the left and by $I_{L^2(0,\infty)} \oplus V$ from the right, we obtain the conclusion of the theorem. ∎

5. Properties of weights. Let $w, \phi\colon \mathbb{R} \to (0,\infty)$ be measurable functions. If $w \asymp \phi$, then the natural imbedding

$$L^2(\mathbb{R}, w) \to L^2(\mathbb{R}, \phi)$$

is a (bounded) transformation with bounded inverse. Let $\widetilde{\phi}$ be defined in Theorem 4.3. Let $\alpha > 0$, and let $\omega_\alpha^2(-n-1)$ for $n \geq 0$ be defined by (4.17). Since $\sum_{k=1}^\infty a_k \leq 1$, we have $\omega_\alpha^2(-n-1) \geq 1$ for $n \geq 0$. Set

$$w_\alpha(t) = \begin{cases} \omega_\alpha^2(-n-1), & t \in (-(n+1)\alpha, -n\alpha),\ n \geq 0, \\ 1, & t \in (0,\infty). \end{cases} \tag{5.1}$$

By (4.22),

$$w_\alpha \asymp \widetilde{\phi} \tag{5.2}$$

(the estimate in (5.2) depends on α). Therefore, we can consider $L^2(\mathbb{R}, w_\alpha)$ instead of $L^2(\mathbb{R}, \widetilde{\phi})$.

LEMMA 5.1. *Let $\{a_k\}_{k=1}^\infty$ and $\{v_k\}_{k=1}^\infty$ be families of numbers such that $a_k > 0$, $0 < v_{k+1} < v_k$ for every k, and $\sum_{k=1}^\infty a_k \le 1$. For $\alpha > 0$ define $\omega_\alpha^2(-n-1)$ for $n \ge 0$ by (4.17). Then*

$$\omega_\alpha^2(-n-m-1) \le \Big(1 + 2\sum_{k=1}^\infty a_k\Big)\omega_\alpha^2(-n-1)\omega_\alpha^2(-m-1), \qquad m, n \ge 0.$$

Proof. We have

$$\begin{aligned}
\frac{1}{\omega_\alpha^2(-n-1)\omega_\alpha^2(-m-1)} &= \sum_{k=1}^\infty a_k \mathrm{e}^{-2\alpha n v_k} \sum_{l=1}^\infty a_l \mathrm{e}^{-2\alpha m v_l} \\
&= \sum_{k=1}^\infty a_k \mathrm{e}^{-2\alpha n v_k} \Big(\sum_{l=1}^{k-1} a_l \mathrm{e}^{-2\alpha m v_l} + a_k \mathrm{e}^{-2\alpha m v_k} + \sum_{l=k+1}^\infty a_l \mathrm{e}^{-2\alpha m v_l}\Big) \\
&= \sum_{k=2}^\infty a_k \mathrm{e}^{-2\alpha n v_k} \sum_{l=1}^{k-1} a_l \mathrm{e}^{-2\alpha m v_l} + \sum_{k=1}^\infty a_k^2 \mathrm{e}^{-2\alpha(n+m)v_k} \\
&\quad + \sum_{k=2}^\infty a_k \mathrm{e}^{-2\alpha m v_k} \sum_{l=1}^{k-1} a_l \mathrm{e}^{-2\alpha n v_l} \\
&\le \sum_{k=2}^\infty a_k \mathrm{e}^{-2\alpha n v_k} \mathrm{e}^{-2\alpha m v_k} \sum_{l=1}^{k-1} a_l + \sum_{k=1}^\infty a_k \mathrm{e}^{-2\alpha(n+m)v_k} \\
&\quad + \sum_{k=2}^\infty a_k \mathrm{e}^{-2\alpha m v_k} \mathrm{e}^{-2\alpha n v_k} \sum_{l=1}^{k-1} a_l \\
&\le 2\sum_{l=1}^\infty a_l \sum_{k=2}^\infty a_k \mathrm{e}^{-2\alpha(n+m)v_k} + \sum_{k=1}^\infty a_k \mathrm{e}^{-2\alpha(n+m)v_k} \\
&= \Big(1 + 2\sum_{k=1}^\infty a_k\Big)\frac{1}{\omega_\alpha^2(-n-m-1)}. \ \blacksquare
\end{aligned}$$

COROLLARY 5.2. *Let $\{a_k\}_{k=1}^\infty$ and $\{v_k\}_{k=1}^\infty$ be families of numbers such that $a_k > 0$, $0 < v_{k+1} < v_k$ for every k, and $\sum_{k=1}^\infty a_k \le 1$. Set*

$$C = 1 + 2\sum_{k=1}^\infty a_k.$$

For $\alpha > 0$ define w_α by (5.1). Then

$$w_\alpha(t+s) \le C^2 \omega_\alpha(-2)^2 w_\alpha(t) w_\alpha(s), \qquad t, s \in \mathbb{R}.$$

Proof. First, consider the case where $t, s < 0$. Then there exist $m, n \ge 0$ such that $t \in (-(n+1)\alpha, -n\alpha)$ and $s \in (-(m+1)\alpha, -m\alpha)$. Therefore

$t+s \in (-(n+m+2)\alpha, -(n+m)\alpha)$. Thus,

$$\begin{aligned} w_\alpha(t+s) &\le \omega_\alpha^2(-n-m-2) \le C\omega_\alpha^2(-n-1)\omega_\alpha^2(-m-2) \\ &\le C\omega_\alpha^2(-n-1)C\omega_\alpha^2(-m-1)\omega_\alpha^2(-2) = C^2\omega_\alpha^2(-2)w_\alpha(t)w_\alpha(s). \end{aligned}$$

In the remaining cases the conclusion follows from the fact that w_α is non-increasing and $w_\alpha \equiv 1$ on $(0,\infty)$. ∎

LEMMA 5.3. *Let $\{a_k\}_{k=1}^\infty$ and $\{v_k\}_{k=1}^\infty$ be families of numbers such that $a_k > 0$, $0 < v_{k+1} < v_k$ for every k, $v_k \to 0$ and $\sum_{k=1}^\infty a_k \le 1$. For $\alpha > 0$ define $\omega_\alpha^2(-n-1)$ for $n \ge 0$ by (4.17). Then for every $\varepsilon > 0$ there exists a finite constant C_ε (which also depends on α) such that $\omega_\alpha^2(-n-1) \le C_\varepsilon \mathrm{e}^{\varepsilon n}$ for all $n \ge 0$.*

Proof. Since $v_k \to 0$, there exists k_ε such that $2\alpha v_{k_\varepsilon} \le \varepsilon$. We have

$$\frac{1}{\omega_\alpha^2(-n-1)} = \sum_{k=1}^\infty a_k \mathrm{e}^{-2\alpha n v_k} \ge \sum_{k=k_\varepsilon}^\infty a_k \mathrm{e}^{-2\alpha n v_k} \ge \sum_{k=k_\varepsilon}^\infty a_k \mathrm{e}^{-\varepsilon n}.$$

Thus, $C_\varepsilon = 1/\sum_{k=k_\varepsilon}^\infty a_k$. ∎

Recall that $\mathcal{C}_\varphi$ denotes convolution with a function φ (see (4.10)).

LEMMA 5.4. *Let $C > 0$, and let $w\colon \mathbb{R} \to (0,\infty)$ and $\psi\colon \mathbb{R} \to \mathbb{C}$ be measurable functions such that $w(t+s) \le Cw(t)w(s)$ for all $s,t \in \mathbb{R}$ and $\psi\sqrt{w} \in L^1(\mathbb{R})$. Then $\mathcal{C}_\psi \in \mathcal{L}(L^2(\mathbb{R},w))$ and $\|\mathcal{C}_\psi\| \le \sqrt{C}\,\|\psi\sqrt{w}\|_{L^1(\mathbb{R})}$.*

Proof. Define $B \in \mathcal{L}(L^2(\mathbb{R},w), L^2(\mathbb{R}))$ by $Bf = \sqrt{w}\,f$ for $f \in L^2(\mathbb{R},w)$. Then B is unitary and

$$(B\mathcal{C}_\psi B^{-1}f)(t) = \int_{\mathbb{R}} \psi(t-s)f(s)\frac{\sqrt{w(t)}}{\sqrt{w(s)}}\,\mathrm{d}s, \qquad t \in \mathbb{R},\ f \in L^2(\mathbb{R}).$$

Therefore,

$$\begin{aligned} |(B\mathcal{C}_\psi B^{-1}f)(t)| &\le \int_{\mathbb{R}} |\psi(t-s)f(s)|\frac{\sqrt{w(t)}}{\sqrt{w(s)}}\,\mathrm{d}s \\ &\le \int_{\mathbb{R}} |\psi(t-s)f(s)|\frac{\sqrt{Cw(t-s)w(s)}}{\sqrt{w(s)}}\,\mathrm{d}s \\ &= \sqrt{C}\int_{\mathbb{R}} |\psi(t-s)|\sqrt{w(t-s)}|f(s)|\,\mathrm{d}s \\ &= \sqrt{C}(\mathcal{C}_{|\psi|\sqrt{w}}|f|)(t), \qquad t \in \mathbb{R},\ f \in L^2(\mathbb{R}). \end{aligned}$$

It is well known (and can be deduced from (4.11)) that if $\varphi \in L^1(\mathbb{R})$, then $\mathcal{C}_\varphi \in \mathcal{L}(L^2(\mathbb{R}))$ and $\|\mathcal{C}_\varphi\| \le \|\varphi\|_{L^1(\mathbb{R})}$. Setting $\varphi = |\psi|\sqrt{w}$, we obtain

$$\|B\mathcal{C}_\psi B^{-1}f\|_{L^2(\mathbb{R})} \le \sqrt{C}\,\big\||\psi|\sqrt{w}\big\|_{L^1(\mathbb{R})}\big\||f|\big\|_{L^2(\mathbb{R})}.$$

Clearly, $\big\||\psi|\sqrt{w}\big\|_{L^1(\mathbb{R})} = \|\psi\sqrt{w}\|_{L^1(\mathbb{R})}$ and $\big\||f|\big\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$. The conclusion of the lemma follows from the unitarity of B. ∎

Recall that ϖ is defined by (4.1) and $\mathcal{F}$ is the Fourier transform (see (4.8)).

Lemma 5.5. *Let $C > 0$, and let $w\colon \mathbb{R} \to [1,\infty)$ be a nonincreasing function such that $w(t+s) \le Cw(t)w(s)$ for all $s,t \in \mathbb{R}$ and*

$$\int_{\mathbb{R}} \frac{\log w(t)}{1+t^2}\,\mathrm{d}t < \infty. \tag{5.3}$$

Then there exists $\eta \in L^\infty(\mathbb{T})$ such that $\eta(\mathrm{e}^{\mathrm{i}t}) = 0$ for $t \in (\pi, 2\pi)$, $\eta \not\equiv 0$, and

$$\mathcal{C}_{\mathcal{F}(\eta\circ\varpi)} \in \mathcal{L}(L^2(\mathbb{R}, w)).$$

Proof. There is a function $h\colon \mathbb{R} \to (0,\infty)$ such that $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $h\sqrt{w} \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \frac{\log h(t)}{1+t^2}\,\mathrm{d}t > -\infty. \tag{5.4}$$

Indeed, take $c, \varepsilon > 0$ and set

$$h(t) = 1, \quad t \in (-c,c), \quad \text{and} \quad h(t) = \frac{1}{|t|^{1+\varepsilon}\sqrt{w(t)}}, \quad t \in (-\infty,-c)\cup(c,\infty).$$

Since w is nonincreasing, it is bounded on $(-c,c)$, and so $h\sqrt{w} \in L^1(\mathbb{R})$. Since $w \ge 1$ on $\mathbb{R}$, we have $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Furthermore,

$$\int_{\mathbb{R}} \frac{\log h(t)}{1+t^2}\,\mathrm{d}t = \Big(\int_{-\infty}^{-c} + \int_{c}^{\infty}\Big)\Big(-\frac{(1+\varepsilon)\log|t| + \frac{1}{2}\log w(t)}{1+t^2}\Big)\,\mathrm{d}t > -\infty$$

by (5.3).

By (5.4), there exists $\psi \in H^1(\mathbb{C}_+) \cap H^2(\mathbb{C}_+)$ such that $|\psi| = h$ a.e. on $\mathbb{R}$ (see, for example, [Gar, Theorem II.4.4] or [HJ, Sec. II.3.1.2]). By Lemma 5.4, $\mathcal{C}_\psi \in \mathcal{L}(L^2(\mathbb{R}, w))$. Set $\eta = (\mathcal{F}^{-1}\psi)\circ\varpi^{-1}$. Since $\psi \in H^2(\mathbb{C}_+)$ and $(\mathcal{F}^{-1}\psi)(-t) = (\mathcal{F}\psi)(t)$ for $t \in \mathbb{R}$, we have $(\mathcal{F}^{-1}\psi) \in L^2(-\infty, 0)$ by (4.15). It remains to note that $\varpi((0,\infty)) = \{\mathrm{e}^{\mathrm{i}t} : t \in (\pi, 2\pi)\}$. ∎

The following lemma will be applied in Sec. 9.

Lemma 5.6. *Suppose that $\delta > 0$, and $B \in \mathcal{L}(L^2(\mathbb{R}))$ is such that for every $-\infty < b_1 < b_2 < \infty$,*

$$BL^2(b_1, b_2) \subset L^2(b_1 - \delta, b_2 + \delta).$$

Suppose that a sequence $\{\omega(n)\}_{n\in\mathbb{Z}}$ of positive numbers is such that

$$\frac{\omega(n)}{\omega(n+1)} \asymp 1, \qquad n \in \mathbb{Z}.$$

For $\alpha > \delta$ set $w_\alpha(t) = \omega^2(n)$ for $t \in (n\alpha, (n+1)\alpha)$ and $n \in \mathbb{Z}$. Then

$B \in \mathcal{L}(L^2(\mathbb{R}, w_\alpha))$, *and*

$$\|B\|^2_{\mathcal{L}(L^2(\mathbb{R},w_\alpha))} \leq 3\|B\|^2_{\mathcal{L}(L^2(\mathbb{R}))}\left(1 + \sup_{n\in\mathbb{Z}} \frac{\omega^2(n)}{\omega^2(n+1)} + \sup_{n\in\mathbb{Z}} \frac{\omega^2(n)}{\omega^2(n-1)}\right).$$

Proof. Let $f \in L^2(\mathbb{R}, w_\alpha)$. Set $f_n = f|_{(n\alpha,(n+1)\alpha)}$ for $n \in \mathbb{Z}$. By assumption, Bf_n is well-defined, and $Bf_n \in L^2((n-1)\alpha, (n+2)\alpha)$. We have

$$\begin{aligned} Bf &= \sum_{n\in\mathbb{Z}} (Bf_n|_{((n-1)\alpha,n\alpha)} + Bf_n|_{(n\alpha,(n+1)\alpha)} + Bf_n|_{((n+1)\alpha,(n+2)\alpha)}) \\ &= \bigoplus_{n\in\mathbb{Z}} (Bf_{n-1} + Bf_n + Bf_{n+1})|_{(n\alpha,(n+1)\alpha)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\|(Bf_{n-1} + Bf_n + Bf_{n+1})|_{(n\alpha,(n+1)\alpha)}\|^2_{L^2(\mathbb{R},w_\alpha)} \\ &\quad= \omega^2(n)\|(Bf_{n-1} + Bf_n + Bf_{n+1})|_{(n\alpha,(n+1)\alpha)}\|^2_{L^2(\mathbb{R})} \\ &\quad\leq 3\omega^2(n)\big(\|(Bf_{n-1})|_{(n\alpha,(n+1)\alpha)}\|^2_{L^2(\mathbb{R})} + \|(Bf_n)|_{(n\alpha,(n+1)\alpha)}\|^2_{L^2(\mathbb{R})} \\ &\qquad\qquad + \|(Bf_{n+1})|_{(n\alpha,(n+1)\alpha)}\|^2_{L^2(\mathbb{R})}\big) \\ &\quad\leq 3\omega^2(n)\|B\|^2_{\mathcal{L}(L^2(\mathbb{R}))}\big(\|f_{n-1}\|^2_{L^2(\mathbb{R})} + \|f_n\|^2_{L^2(\mathbb{R})} + \|f_{n+1}\|^2_{L^2(\mathbb{R})}\big) \\ &\quad= 3\omega^2(n)\|B\|^2_{\mathcal{L}(L^2(\mathbb{R}))}\left(\frac{1}{\omega^2(n-1)}\|f_{n-1}\|^2_{L^2(\mathbb{R},w_\alpha)} + \frac{1}{\omega^2(n)}\|f_n\|^2_{L^2(\mathbb{R},w_\alpha)} \right.\\ &\qquad\qquad \left. + \frac{1}{\omega^2(n+1)}\|f_{n+1}\|^2_{L^2(\mathbb{R},w_\alpha)}\right) \\ &\quad= 3\|B\|^2_{\mathcal{L}(L^2(\mathbb{R}))}\left(\frac{\omega^2(n)}{\omega^2(n-1)}\|f_{n-1}\|^2_{L^2(\mathbb{R},w_\alpha)} + \|f_n\|^2_{L^2(\mathbb{R},w_\alpha)} \right.\\ &\qquad\qquad \left. + \frac{\omega^2(n)}{\omega^2(n+1)}\|f_{n+1}\|^2_{L^2(\mathbb{R},w_\alpha)}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \|Bf\|^2_{L^2(\mathbb{R},w_\alpha)} &= \sum_{n\in\mathbb{Z}} \|(Bf_{n-1} + Bf_n + Bf_{n+1})|_{(n\alpha,(n+1)\alpha)}\|^2_{L^2(\mathbb{R},w_\alpha)} \\ &\leq 3\|B\|^2_{\mathcal{L}(L^2(\mathbb{R}))}\sum_{n\in\mathbb{Z}}\left(\frac{\omega^2(n)}{\omega^2(n-1)}\|f_{n-1}\|^2_{L^2(\mathbb{R},w_\alpha)} + \|f_n\|^2_{L^2(\mathbb{R},w_\alpha)} \right.\\ &\qquad\qquad \left. + \frac{\omega^2(n)}{\omega^2(n+1)}\|f_{n+1}\|^2_{L^2(\mathbb{R},w_\alpha)}\right) \\ &= 3\|B\|^2_{\mathcal{L}(L^2(\mathbb{R}))}\sum_{n\in\mathbb{Z}}\left(\frac{\omega^2(n+1)}{\omega^2(n)} + 1 + \frac{\omega^2(n-1)}{\omega^2(n)}\right)\|f_n\|^2_{L^2(\mathbb{R},w_\alpha)} \\ &\leq 3\|B\|^2_{\mathcal{L}(L^2(\mathbb{R}))}\left(1 + \sup_{n\in\mathbb{Z}} \frac{\omega^2(n)}{\omega^2(n+1)} + \sup_{n\in\mathbb{Z}} \frac{\omega^2(n)}{\omega^2(n-1)}\right)\sum_{n\in\mathbb{Z}}\|f_n\|^2_{L^2(\mathbb{R},w_\alpha)} \\ &= 3\|B\|^2_{\mathcal{L}(L^2(\mathbb{R}))}\left(1 + \sup_{n\in\mathbb{Z}} \frac{\omega^2(n)}{\omega^2(n+1)} + \sup_{n\in\mathbb{Z}} \frac{\omega^2(n)}{\omega^2(n-1)}\right)\|f\|^2_{L^2(\mathbb{R},w_\alpha)}. \ \blacksquare \end{aligned}$$

6. Quasianalyticity. We will apply Beurling's quasianalyticity theorem (see [Ko, Ch. VII.B.5]).

THEOREM 6.1 (Beurling). *Let* $-\infty < b_1 < b_2 < \infty$. *For* $c > 0$ *set*

$$\mathcal{G}(c) = \{t + \mathrm{i}y : b_1 < t < b_2,\ 0 < y < c\}.$$

For a function f *analytic in* $\mathcal{G}(c)$ *set*

$$\varsigma_{\mathcal{G}(c)}(f) = \sup_{0<y<c} \Big(\int_{b_1}^{b_2} |f(t + \mathrm{i}y)|^2 \,\mathrm{d}t\Big)^{1/2}.$$

For $\varphi \in L^2(b_1, b_2)$ *and* $u \in [1, \infty)$ *define* $M(u)$ *by the relation*

$$\mathrm{e}^{-M(u)} = \inf \Big\{ \Big(\int_{b_1}^{b_2} |\varphi(t) - f(t)|^2 \,\mathrm{d}t\Big)^{1/2} : f \text{ is analytic in } \mathcal{G}(c) \text{ and } \varsigma_{\mathcal{G}(c)}(f) \le \mathrm{e}^u \Big\}.$$

If

$$\int_1^\infty \frac{M(u)}{u^2} \,\mathrm{d}u = \infty$$

and $|\{t \in \mathbb{R} : b_1 < t < b_2,\ \varphi(t) = 0\}| > 0$ (*where* $|\cdot|$ *is the linear measure of a subset of* $\mathbb{R}$), *then* $\varphi \equiv 0$.

To apply Theorem 6.1 we need the following simple lemma; its proof is omitted.

LEMMA 6.2. *Let* $M\colon (0, \infty) \to (0, \infty)$ *be a non-decreasing function. Let* $\alpha > 0$. *Then*

$$\sum_{n=1}^\infty \frac{M(n\alpha)}{n^2} = \infty \quad \text{if and only if} \quad \int_1^\infty \frac{M(u)}{u^2} \,\mathrm{d}u = \infty.$$

THEOREM 6.3. *Let* T *be defined as in Proposition* 3.2 *with* ν *as in Lemma* 3.3. *For* $\alpha > 0$ *define* $\omega_\alpha^2(-n - 1)$ *for* $n \ge 0$ *by* (4.17). *Then* T *is quasianalytic if and only if*

$$\sum_{n=0}^\infty \frac{\log \omega_\alpha(-n-1)}{(n+1)^2} = \infty. \tag{6.1}$$

Proof. "If" part. By Proposition 3.1, T is quasianalytic if and only if (3.1) is fulfilled. By the construction of T, $\mathcal{H}_0 = P^2(\nu)$ and $X_0 = WJ_\nu^*$, where W is defined by (3.3). Recall that $\mathcal{J}$ is defined by (4.3) and J_μ is defined after (4.7). Applying $\mathcal{J}$ and taking into account (4.4), (4.7), and (4.25), the relation (3.1) can be rewritten as follows:

$$h \in H^2(\mathbb{C}_+),\ g \in J_\mu^* \operatorname{clos}_{L^2(\mu)} H^2(\mathbb{C}_+),\ |\{t \in \mathbb{R} : h(t) = g(-t)\}| > 0 \implies g \equiv 0.$$

We have $g \in H^2(\mathbb{C}_+)$. Set $g_*(z) = g(-z)$ for $z \in \mathbb{C}_-$. Then $g_* \in L^2(\mathbb{R}) \ominus H^2(\mathbb{C}_+)$. Therefore, it sufficient to prove that $h(t) - g(-t) = 0$ for a.e. $t \in \mathbb{R}$.

By (4.19),

$$g = \bigoplus_{n=0}^{\infty} \theta_\alpha^n \mathcal{F}^{-1} g_n, \quad \text{where} \quad g_n \in L^2(0,\alpha) \quad \text{and}$$

$$C_{1g} := \sum_{n=0}^{\infty} \|g_n\|^2_{L^2(0,\alpha)} \omega_\alpha^2(-n-1) < \infty.$$

Since $\omega_\alpha^2(-n-1) \to \infty$ as $n \to \infty$,

$$C_{2g} := \sum_{n=0}^{\infty} \|g_n\|^2_{L^2(0,\alpha)} < \infty.$$

Let $-\infty < b_1 < b_2 < \infty$ be such that

$$|\{t \in (b_1, b_2) : h(t) = g(-t)\}| > 0. \tag{6.2}$$

For $c > 0$ define $\mathcal{G}(c)$ as in Theorem 6.1.

Set $\varphi(t) = h(t) - g(-t)$ for $t \in (b_1, b_2)$, and

$$f_{1n}(z) = (\theta_\alpha^n \mathcal{F}^{-1} g_n)(-z),$$

$$f_n(z) = h(z) - \bigoplus_{k=0}^{n} f_{1k}(z), \qquad z \in \mathbb{C}_+ \cup \mathbb{R},\ n \geq 0.$$

Clearly, f_{1n} and f_n are analytic in $\mathbb{C}_+$,

$$f_{1n}(z) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}\alpha n z} \int_0^\alpha \mathrm{e}^{-\mathrm{i}zs} g_n(s)\,\mathrm{d}s, \qquad z \in \mathbb{C}_+,$$

and

$$\begin{aligned}
2\pi(\varsigma_{\mathcal{G}(c)}(f_{1n}))^2 &= \sup_{0<y<c} \int_{b_1}^{b_2} |\mathrm{e}^{-\mathrm{i}(t+\mathrm{i}y)n\alpha}|^2 \Big| \int_0^\alpha \mathrm{e}^{-\mathrm{i}(t+\mathrm{i}y)s} g_n(s)\,\mathrm{d}s \Big|^2 \mathrm{d}t \\
&\leq \sup_{0<y<c} \int_{b_1}^{b_2} \mathrm{e}^{2yn\alpha} \Big(\int_0^\alpha |\mathrm{e}^{-\mathrm{i}(t+\mathrm{i}y)s} g_n(s)|\,\mathrm{d}s \Big)^2 \mathrm{d}t \\
&\leq \mathrm{e}^{2cn\alpha} \sup_{0<y<c} \int_{b_1}^{b_2} \Big(\int_0^\alpha \mathrm{e}^{ys} |g_n(s)|\,\mathrm{d}s \Big)^2 \mathrm{d}t \\
&\leq \mathrm{e}^{2c(n+1)\alpha} \sup_{0<y<c} \int_{b_1}^{b_2} \Big(\int_0^\alpha |g_n(s)|\,\mathrm{d}s \Big)^2 \mathrm{d}t \\
&\leq \mathrm{e}^{2c(n+1)\alpha} \int_{b_1}^{b_2} \alpha \Big(\int_0^\alpha |g_n(s)|^2\,\mathrm{d}s \Big) \mathrm{d}t \\
&= \mathrm{e}^{2c(n+1)\alpha} (b_2 - b_1)\alpha \|g_n\|^2_{L^2(0,\alpha)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\varsigma_{\mathcal{G}(c)}(f_n) &\leq \varsigma_{\mathcal{G}(c)}(h) + \sum_{k=0}^{n} \varsigma_{\mathcal{G}(c)}(f_{1k}) \\ &\leq \varsigma_{\mathcal{G}(c)}(h) + (b_2-b_1)^{1/2}\frac{\sqrt{\alpha}}{\sqrt{2\pi}}\sum_{k=0}^{n} \mathrm{e}^{c(k+1)\alpha}\|g_k\|_{L^2(0,\alpha)} \\ &\leq \varsigma_{\mathcal{G}(c)}(h) + (b_2-b_1)^{1/2}\frac{\sqrt{\alpha}}{\sqrt{2\pi}}\Big(\sum_{k=0}^{n} \mathrm{e}^{2c(k+1)\alpha}\Big)^{1/2}\Big(\sum_{k=0}^{n}\|g_k\|^2_{L^2(0,\alpha)}\Big)^{1/2} \\ &\leq \varsigma_{\mathcal{G}(c)}(h) + (b_2-b_1)^{1/2}\frac{\sqrt{\alpha}}{\sqrt{2\pi}}(n+1)^{1/2}\mathrm{e}^{c(n+1)\alpha}C_{2g}^{1/2}.\end{aligned}$$

Set $C_1 = (b_2-b_1)^{1/2}\frac{\sqrt{\alpha}}{\sqrt{2\pi}}$. Since $h \in H^2(\mathbb{C}_+)$, we have $\varsigma_{\mathcal{G}(c)}(h) \leq \|h\|_{H^2(\mathbb{C}_+)}$. Thus,

$$\varsigma_{\mathcal{G}(c)}(f_n) \leq \|h\|_{H^2(\mathbb{C}_+)} + C_1 C_{2g}^{1/2}(n+1)^{1/2}\mathrm{e}^{c(n+1)\alpha}, \qquad n \geq 0.$$

Take $0 < c < 1/\alpha$. Then there exists C_2 (which depends on c) such that

$$\|h\|_{H^2(\mathbb{C}_+)} + C_1 C_{2g}^{1/2}(n+1)^{1/2}\mathrm{e}^{c(n+1)\alpha} \leq C_2\mathrm{e}^n \quad \text{for all } n \in \mathbb{N}.$$

We obtain

$$\varsigma_{\mathcal{G}(c)}\Big(\frac{1}{C_2}f_n\Big) \leq \mathrm{e}^n \quad \text{for all } n \in \mathbb{N}.$$

We have

$$\begin{aligned}&\Big(\int_{b_1}^{b_2}|\varphi(t)-f_n(t)|^2\,\mathrm{d}t\Big)^{1/2} \leq \|\varphi - f_n\|_{L^2(\mathbb{R})} = \Big\|\bigoplus_{k=n+1}^{\infty} f_{1k}\Big\|_{L^2(\mathbb{R})} \\ &= \Big(\sum_{k=n+1}^{\infty}\|g_k\|^2_{L^2(0,\alpha)}\Big)^{1/2} \leq \frac{1}{\omega_\alpha(-n-2)}\Big(\sum_{k=n+1}^{\infty}\omega_\alpha^2(-k-1)\|g_k\|^2_{L^2(0,\alpha)}\Big)^{1/2} \\ &\qquad\leq \frac{1}{\omega_\alpha(-n-2)}C_{1g}\end{aligned}$$

(because $\omega_\alpha(-k-1) \geq \omega_\alpha(-n-2)$ for $k \geq n+1$).

Define $M(n)$ as in Theorem 6.1 applied to $\frac{1}{C_2}\varphi$. Then

$$\mathrm{e}^{-M(n)} \leq \Big(\int_{b_1}^{b_2}\Big|\frac{1}{C_2}\varphi(t) - \frac{1}{C_2}f_n(t)\Big|^2\,\mathrm{d}t\Big)^{1/2} \leq \frac{1}{\omega_\alpha(-n-2)}C_{1g}.$$

Therefore, $M(n) \geq \log\omega_\alpha(-n-2) - \log C_{1g}$. By the assumption (6.1),

$$\sum_{n=1}^{\infty}\frac{M(n)}{n^2} = \infty.$$

By Lemma 6.2, $\frac{1}{C_2}\varphi$ satisfies the conclusion of Theorem 6.1, that is, $\frac{1}{C_2}\varphi \equiv 0$. Therefore, $h(t) - g(-t) = 0$ for a.e. $t \in (b_1, b_2)$.

Since (b_1, b_2) is an arbitrary interval satisfying (6.2), we conclude that $h(t) - g(-t) = 0$ for a.e. $t \in \mathbb{R}$.

"Only if" part. Take $\alpha > 0$. Define w_α by (5.1). If the sum in (6.1) is finite, then (5.3) is fulfilled for w_α. By Lemma 5.5, there exists $\eta \in L^\infty(\mathbb{T})$ such that $\eta(\mathrm{e}^{\mathrm{i}t}) = 0$ for $t \in (\pi, 2\pi)$, $\eta \not\equiv 0$, and $\mathcal{C}_{\mathcal{F}(\eta\circ\varpi)} \in \mathcal{L}(L^2(\mathbb{R}, w_\alpha))$. By (5.2), $\mathcal{C}_{\mathcal{F}(\eta\circ\varpi)} \in \mathcal{L}(L^2(\mathbb{R}, \widetilde{\phi}))$, too. By Theorem 4.3, $\eta \in \widehat{\gamma}_T(\{T\}')$ (where $\widehat{\gamma}_T$ is defined in (1.4)). Since $\eta \not\equiv 0$ and $\eta = 0$ on a set of positive measure, T is not quasianalytic by [KSz1, Proposition 21]. ∎

LEMMA 6.4. *Let $\{a_k\}_{k=1}^\infty$ and $\{v_k\}_{k=1}^\infty$ be families of numbers such that $a_k > 0$, $0 < v_{k+1} < v_k$ for every k, and $\sum_{k=1}^\infty a_k \le 1$. Let $\{k_n\}_{n=1}^\infty$ be a sequence such that*

$$\sum_{n=1}^\infty \frac{v_{k_n}}{n} = \infty,$$

and let $\{c_n\}_{n=1}^\infty$ be defined by the equality

$$\sum_{k=k_n+1}^\infty a_k = \mathrm{e}^{-nc_n}, \qquad n \ge 1.$$

For $\alpha > 0$ define $\omega_\alpha^2(-n-1)$ for $n \ge 0$ by (4.17). If $2\alpha v_{k_n} \le c_n$ for sufficiently large n, then (6.1) is fulfilled.

Proof. Set $a = \sum_{k=1}^\infty a_k$. We have

$$\begin{aligned}\frac{1}{\omega_\alpha^2(-n-1)} &= \sum_{k=1}^\infty a_k \mathrm{e}^{-2\alpha n v_k} = \sum_{k=1}^{k_n} a_k \mathrm{e}^{-2\alpha n v_k} + \sum_{k=k_n+1}^\infty a_k \mathrm{e}^{-2\alpha n v_k} \\ &\le \sum_{k=1}^{k_n} a_k \mathrm{e}^{-2\alpha n v_{k_n}} + \sum_{k=k_n+1}^\infty a_k \le a\mathrm{e}^{-2\alpha n v_{k_n}} + \mathrm{e}^{-nc_n} \\ &= a\mathrm{e}^{-2\alpha n v_{k_n}}\left(1 + \frac{1}{a}\mathrm{e}^{-n(c_n - 2\alpha v_{k_n})}\right).\end{aligned}$$

Therefore,

$$\log \frac{1}{\omega_\alpha^2(-n-1)} \le \log a - 2\alpha n v_{k_n} + \frac{1}{a}\mathrm{e}^{-n(c_n - 2\alpha v_{k_n})}.$$

Consequently,

$$2\sum_{n=0}^\infty \frac{\log \omega_\alpha(-n-1)}{(n+1)^2} \ge -\sum_{n=0}^\infty \frac{\log a}{(n+1)^2} + 2\alpha \sum_{n=0}^\infty \frac{n v_{k_n}}{(n+1)^2} - \frac{1}{a}\sum_{n=0}^\infty \frac{\mathrm{e}^{-n(c_n - 2\alpha v_{k_n})}}{(n+1)^2}.$$

By assumption, $\sum_{n=0}^\infty \frac{n v_{k_n}}{(n+1)^2} = \infty$ and $\sum_{n=0}^\infty \frac{\mathrm{e}^{-n(c_n - 2\alpha v_{k_n})}}{(n+1)^2} < \infty$. ∎

EXAMPLE 6.5. Let $0 < a < 1$. Set $a_n = (1-a)a^{n-1}$ and $v_n = \frac{1}{\log(n+1)}$, $n \geq 1$. Then $\{a_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ satisfy the assumption of Lemma 6.4 with $k_n = n$ and $c_n = -\log a$, $n \geq 1$.

7. Existence of hyperinvariant subspaces. Recall the definition of a bilateral weighted shift (see [E]). Let $\omega\colon \mathbb{Z} \to (0,\infty)$ be a nonincreasing function. Set

$$\ell^2_\omega = \Big\{u = \{u(n)\}_{n\in\mathbb{Z}} : \|u\|^2_\omega = \sum_{n\in\mathbb{Z}} |u(n)|^2\omega^2(n) < \infty\Big\}.$$

The *bilateral weighted shift* $S_\omega \in \mathcal{L}(\ell^2_\omega)$ acts by the formula

$$(S_\omega u)(n) = u(n-1), \qquad n \in \mathbb{Z},\ u \in \ell^2_\omega.$$

THEOREM 7.1. *Let T be defined as in Proposition 3.2 with ν as in Lemma 3.3. For $\alpha > 0$ set*

$$\vartheta_\alpha(z) = \mathrm{e}^{\alpha\frac{z+1}{z-1}}, \qquad z \in \mathbb{D}.$$

Set $\omega_\alpha(n) = 1$ for $n \geq 0$, and define $\omega_\alpha(n)$ for $n \leq -1$ by (4.17). Then $\vartheta_\alpha(T)$ is similar to $\bigoplus_{j\in\mathbb{N}} S_{\omega_\alpha}$.

Proof. Recall that ϖ and θ_α are defined by (4.1) and (4.12), respectively. Clearly, $\vartheta_\alpha \circ \varpi = \theta_\alpha$. By Theorem 4.3, $\vartheta_\alpha(T)$ is unitarily equivalent to $\frac{1}{\sqrt{2\pi}}\mathcal{C}_{\mathcal{F}\theta_\alpha}$ acting on $L^2(\mathbb{R}, \widetilde{\phi})$. Define w_α by (5.1). By (5.2), $\frac{1}{\sqrt{2\pi}}\mathcal{C}_{\mathcal{F}\theta_\alpha}$ is similar to the same operator acting on $L^2(\mathbb{R}, w_\alpha)$. We have

$$L^2(\mathbb{R}, w_\alpha) = \Big\{\bigoplus_{n\in\mathbb{Z}} f_n : f_n \in L^2(n\alpha, (n+1)\alpha), \\ \sum_{n\in\mathbb{Z}} \|f_n\|^2_{L^2(n\alpha,(n+1)\alpha)}\omega^2_\alpha(n) < \infty\Big\}.$$

By (4.14), $\frac{1}{\sqrt{2\pi}}(\mathcal{C}_{\mathcal{F}\theta_\alpha} f)(t) = f(t-\alpha)$ for $t \in \mathbb{R}$.

Therefore, $\frac{1}{\sqrt{2\pi}}\mathcal{C}_{\mathcal{F}\theta_\alpha}$ on $L^2(\mathbb{R}, w_\alpha)$ is unitarily equivalent to the bilateral shift on the weighted space of sequences $\{f_n\}_{n\in\mathbb{Z}}$, where $f_n \in L^2(0,\alpha)$. Since $\dim L^2(0,\alpha) = \infty$, we conclude that $\frac{1}{\sqrt{2\pi}}\mathcal{C}_{\mathcal{F}\theta_\alpha}$ on $L^2(\mathbb{R}, w_\alpha)$ is unitarily equivalent to $\bigoplus_{j\in\mathbb{N}} S_{\omega_\alpha}$. ■

COROLLARY 7.2. *Let T be defined as in Proposition 3.2 with ν as in Lemma 3.3. Then for every $\alpha > 0$ there exists a singular inner function $\eta_\alpha \in H^\infty$ such that the range of $(\eta_\alpha \circ \vartheta_\alpha)(T)$ is not dense.*

Proof. Define the weight $\omega_\alpha = \{\omega_\alpha(n)\}_{n\in\mathbb{Z}}$ by (4.17). It follows from Lemmas 5.1 and 5.3 that ω_α is a dissymmetric weight (see [E] for definition). By [E, Theorem 5.7], there exists a singular inner function $\eta_\alpha \in H^\infty$ (which depends on ω_α) such that the range of $\eta_\alpha(S_{\omega_\alpha})$ is not dense. Therefore, the range of $\eta_\alpha(\bigoplus_{j\in\mathbb{N}} S_{\omega_\alpha})$ is not dense. Taking into account that

$$(\eta_\alpha \circ \vartheta_\alpha)(T) = (\eta_\alpha(\vartheta_\alpha(T))$$

and applying Theorem 7.1 we obtain the conclusion of the corollary. ∎

REMARK 7.3. Let T be an operator which admits an H^∞-functional calculus (see [Kér3, Sec. 5]. Let $\vartheta \in H^\infty$ be a singular inner function with at least two singularities. If $\vartheta(T)$ is invertible and $\sigma(T) = \mathbb{T}$, then T cannot be quasianalytic by [Gam2, Theorems 2.5 and 2.6].

For $\alpha > 0$, let ϑ_α be defined in Theorem 7.1. Then ϑ_α has the only singularity at the point $1 \in \mathbb{T}$. Let T be defined as in Proposition 3.2 with ν as in Lemma 3.3. By Proposition 3.2, $\sigma(T) = \mathbb{T}$. By Theorem 7.1, $\vartheta_\alpha(T)$ is similar to $\bigoplus_{j\in\mathbb{N}} S_{\omega_\alpha}$. By [E], S_{ω_α} is invertible. Thus, $\vartheta_\alpha(T)$ is invertible. By results of Sec. 6, T can be quasianalytic.

8. Convolution and Fourier transform. The results of this section will be applied in Sec. 9. Recall that $\mathcal{F}$ and $\mathcal{C}_\varphi$ denote the Fourier transform and convolution with a function φ (see (4.8) and (4.10)).

LEMMA 8.1. *Suppose that $\delta > 0$, $\psi \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, and*

$$\int_{-\delta}^{\delta} \left| \frac{\psi(t) - \psi(0)}{t} \right| \mathrm{d}t < \infty. \tag{8.1}$$

For $t \in \mathbb{R}$ set

$$\psi_1(t) = \int_{-\delta}^{\delta} \frac{\mathrm{e}^{\mathrm{i}ts} - 1}{s} \psi(s)\,\mathrm{d}s.$$

Then $\psi_1 \in L^\infty(\mathbb{R})$. For $f \in \mathcal{D}(\mathbb{R})$ and $t \in \mathbb{R}$ set

$$(\mathcal{A}_{1\psi} f)(t) = \int_{-\delta}^{\delta} \psi(s) \frac{f(t-s) - f(t)}{s}\,\mathrm{d}s.$$

Then $\mathcal{A}_{1\psi} f \in L^\infty(\mathbb{R})$, and if $-\infty < b_1 < b_2 < \infty$ are such that $f(t) = 0$ for $t \in (-\infty, b_1] \cup [b_2, \infty)$, then $(\mathcal{A}_{1\psi} f)(t) = 0$ for $t \in (-\infty, b_1 - \delta] \cup [b_2 + \delta, \infty)$. Furthermore,

$$\mathcal{F}^{-1} \mathcal{A}_{1\psi} f = \psi_1 \mathcal{F}^{-1} f.$$

Consequently, $\mathcal{A}_{1\psi}$ can be extended from $\mathcal{D}(\mathbb{R})$ onto $L^2(\mathbb{R})$ and

$$\mathcal{A}_{1\psi} \in \mathcal{L}(L^2(\mathbb{R})).$$

Proof. We have

$$|\psi_1(t)| \le \int_{-\delta}^{\delta} \left| \frac{\psi(s) - \psi(0)}{s} \right| |\mathrm{e}^{\mathrm{i}ts} - 1|\,\mathrm{d}s + |\psi(0)| \left| \int_{-\delta}^{\delta} \frac{\mathrm{e}^{\mathrm{i}ts} - 1}{s}\,\mathrm{d}s \right|$$

and

$$\int_{-\delta}^{\delta} \frac{e^{its}-1}{s}\,ds = i\int_{-\delta}^{\delta} \frac{\sin ts}{s}\,ds = i\int_{-\delta t}^{\delta t} \frac{\sin s}{s}\,ds.$$

Since

$$\sup_{c>0}\left|\int_{-c}^{c} \frac{\sin s}{s}\,ds\right| < \infty,$$

we conclude that $\psi_1 \in L^\infty(\mathbb{R})$.

Let $f \in \mathcal{D}(\mathbb{R})$ and $t \in \mathbb{R}$. Then

$$\max_{s\in[-\delta,\delta]}\left|\frac{f(t-s)-f(t)}{s}\right| \le \max_{s\in[t-\delta,t+\delta]} |f'(s)| \le \max_{\mathbb{R}} |f'| < \infty.$$

Since

$$|(\mathcal{A}_{1\psi} f)(t)| \le \max_{s\in[-\delta,\delta]}\left|\frac{f(t-s)-f(t)}{s}\right| \int_{-\delta}^{\delta} |\psi(s)|\,ds,$$

we conclude that $\mathcal{A}_{1\psi} f \in L^\infty(\mathbb{R})$. Let $-\infty < b_1 < b_2 < \infty$ be such that $f(t) = 0$ for $t \in (-\infty, b_1] \cup [b_2, \infty)$. Then $f(t-s) = f(t) = 0$ for $t \in (-\infty, b_1-\delta] \cup [b_2+\delta, \infty)$ and $s \in [-\delta, \delta]$. Let $x \in \mathbb{R}$. By Fubini's theorem,

$$\begin{aligned}(\mathcal{F}^{-1}\mathcal{A}_{1\psi} f)(x) &= \frac{1}{\sqrt{2\pi}} \int_{b_1-\delta}^{b_2+\delta} e^{ixt} (\mathcal{A}_{1\psi} f)(t)\,dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{b_1-\delta}^{b_2+\delta} e^{ixt} \int_{-\delta}^{\delta} \psi(s)\frac{f(t-s)-f(t)}{s}\,ds\,dt \\ &= \int_{-\delta}^{\delta} \frac{\psi(s)}{s} \frac{1}{\sqrt{2\pi}} \int_{b_1-\delta}^{b_2+\delta} e^{ixt}(f(t-s)-f(t))\,dt\,ds \\ &= \int_{-\delta}^{\delta} \frac{\psi(s)}{s}(e^{ixs}-1)\,ds\,(\mathcal{F}^{-1}f)(x) = \psi_1(x)(\mathcal{F}^{-1}f)(x). \ \blacksquare\end{aligned}$$

THEOREM 8.2. *Let $\Psi \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$ be such that $\Psi' \in L^1(\mathbb{R})$. Set $\psi = \mathcal{F}\Psi'$. Suppose that $\psi \in L^1(\mathbb{R})$ and ψ satisfies* (8.1) *for some $\delta > 0$ (and hence for all finite δ). For $f \in \mathcal{D}(\mathbb{R})$ and $t \in \mathbb{R}$ set*

$$(\mathcal{A}_\psi f)(t) = \int_{\mathbb{R}} \psi(s)\frac{f(t-s)-f(t)}{s}\,ds.$$

Then

$$\mathcal{A}_\psi f = i\mathcal{C}_{\mathcal{F}\Psi} f - i\sqrt{2\pi}\,\Psi(0)f, \qquad f \in \mathcal{D}(\mathbb{R}).$$

Proof. Fix $\delta > 0$. Set

$$\psi_2(t) = \chi_{(-\infty,-\delta)\cup(\delta,\infty)}(t)\frac{\psi(t)}{t}, \quad t \in \mathbb{R}, \quad \text{and} \quad c_\psi = \int_{\mathbb{R}} \psi_2(t)\,\mathrm{d}t. \tag{8.2}$$

(Of course, ψ_2 and c_ψ depend on δ.) Clearly, $\psi_2 \in L^1(\mathbb{R})$. Therefore, $\mathcal{C}_{\psi_2} f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for every $f \in \mathcal{D}(\mathbb{R})$. We have

$$\mathcal{A}_\psi f = \mathcal{A}_{1\psi} f + \mathcal{C}_{\psi_2} f - c_\psi f, \qquad f \in \mathcal{D}(\mathbb{R}). \tag{8.3}$$

Set

$$\Phi(t) = \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}ts} - 1}{s}\psi(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}.$$

By Lemma 8.1, $\Phi(t) \in L^\infty(\mathbb{R})$ and $\mathcal{F}^{-1}\mathcal{A}_{1\psi} f = \psi_1 \mathcal{F}^{-1} f$ for $f \in \mathcal{D}(\mathbb{R})$. Since $\psi_2 \in L^1(\mathbb{R})$, we have

$$\mathcal{F}^{-1}\mathcal{C}_{\psi_2} f = \sqrt{2\pi}\,(\mathcal{F}^{-1}\psi_2) \cdot (\mathcal{F}^{-1} f), \qquad f \in \mathcal{D}(\mathbb{R})$$

(see, for example, [Ka, Theorem VI.1.3] or [R, Theorem 7.2]). Thus,

$$\mathcal{F}^{-1}\mathcal{A}_\psi f = \Phi \mathcal{F}^{-1} f, \qquad f \in \mathcal{D}(\mathbb{R}). \tag{8.4}$$

We will show that $\Phi' = \mathrm{i}\sqrt{2\pi}\,\Psi'$. We have

$$\Phi'(t) = \int_{\mathbb{R}} \left(\frac{\mathrm{e}^{\mathrm{i}ts} - 1}{s}\right)'_t \psi(s)\,\mathrm{d}s = \mathrm{i}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ts}\psi(s)\,\mathrm{d}s, \qquad t \in \mathbb{R},$$

because $\psi \in L^1(\mathbb{R})$. Since $L^1(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$, we have $\psi, \Psi' \in \mathcal{S}'(\mathbb{R})$. Taking into account that $\psi = \mathcal{F}\Psi'$, we conclude that

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ts}\psi(s)\,\mathrm{d}s = \sqrt{2\pi}\,(\mathcal{F}^{-1}\psi)(t) = \sqrt{2\pi}\,\Psi'(t).$$

Therefore, $\Phi' = \mathrm{i}\sqrt{2\pi}\,\Psi'$ a.e. on $\mathbb{R}$. (Actually, $\Phi'(t) = \mathrm{i}\sqrt{2\pi}\,\Psi'(t)$ for every $t \in \mathbb{R}$, because $\Psi' \in C(\mathbb{R})$ by assumption and $\Phi' = \mathrm{i}\sqrt{2\pi}\,\mathcal{F}^{-1}\psi$ with $\psi \in L^1(\mathbb{R})$.)

Since $\Phi(0) = 0$, we conclude that $\Phi = \mathrm{i}\sqrt{2\pi}\,\Psi - \mathrm{i}\sqrt{2\pi}\,\Psi(0)$. The conclusion of the theorem follows from (8.4) and (4.11). ∎

9. Square root again. Denote by ϱ the branch of square root defined in $\mathbb{C} \setminus [0, \infty)$ with $\varrho(-1) = \mathrm{i}$. Set

$$\Psi = \varrho \circ \varpi, \tag{9.1}$$

where ϖ is defined by (4.1). Then Ψ is analytic in $\mathbb{C}\setminus\{\mathrm{i}y : |y| \geq 1\}$, $\Psi(\mathbb{C}\setminus\{\mathrm{i}y : |y| \geq 1\}) = \mathbb{C}_+$, and

$$\Psi'(z) = \frac{\mathrm{i}}{\Psi(z)}\,\frac{1}{(z+\mathrm{i})^2}, \qquad z \in \mathbb{C} \setminus \{\mathrm{i}y : |y| \geq 1\}.$$

For $\lambda \notin \mathbb{C}_+ \cup \mathbb{R}$ set

$$\Psi_\lambda = \frac{1}{\Psi - \lambda}; \tag{9.2}$$

then Ψ_λ is analytic in $\mathbb{C} \setminus \{\mathrm{i}y : |y| \geq 1\}$ and $\Psi_\lambda' = -\frac{1}{(\Psi-\lambda)^2}\Psi'$.

The proof of the following theorem can be found in [Ka, Sec. VI.7.1].

THEOREM 9.1 (Paley–Wiener). *Let $c > 0$, and let a function f be analytic in $\{z \in \mathbb{C} : z = t + \mathrm{i}y,\ t \in \mathbb{R},\ y \in (-c, c)\}$ and such that*

$$\sup_{y\in(-c,c)} \int_{\mathbb{R}} |f(t + \mathrm{i}y)|^2 \,\mathrm{d}t < \infty. \tag{9.3}$$

Then $\int_{\mathbb{R}} \mathrm{e}^{2c|t|}|(\mathcal{F}(f|_{\mathbb{R}}))(t)|^2 \,\mathrm{d}t < \infty$.

LEMMA 9.2. *Let $\lambda \notin \mathbb{C}_+ \cup \mathbb{R}$, and let Ψ_λ be defined by* (9.2). *Then for every $0 < c < 1$, Ψ_λ' satisfies* (9.3).

Proof. We have $|\Psi_\lambda'| \leq \frac{1}{\operatorname{dist}(\lambda,\mathbb{C}_+)^2}|\Psi'|$. Therefore, it is sufficient to prove that Ψ' satisfies (9.3), which follows from the equality

$$|\Psi'(t + \mathrm{i}y)|^2 = \frac{1}{(t^2 + (1 - y)^2)^{1/2}(t^2 + (1 + y)^2)^{3/2}}, \qquad t \in \mathbb{R},\ |y| < 1. \ \blacksquare$$

LEMMA 9.3. *Let $\lambda \notin \mathbb{C}_+ \cup \mathbb{R}$, and let Ψ_λ be defined by* (9.2). *Set $\psi_\lambda = \mathcal{F}\Psi_\lambda'$. Let $\delta > 0$. Then ψ_λ satisfies* (8.1).

Proof. Since $\Psi_\lambda' \in L^1(\mathbb{R})$, we have

$$\frac{\psi_\lambda(t) - \psi_\lambda(0)}{t} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mathrm{i}ts} - 1}{t} \Psi_\lambda'(s) \,\mathrm{d}s.$$

Therefore,

$$\begin{aligned}\int_0^\delta \left|\frac{\psi_\lambda(t) - \psi_\lambda(0)}{t}\right| \mathrm{d}t &\leq \frac{1}{\sqrt{2\pi}} \int_0^\delta\int_{\mathbb{R}} \left|\frac{\mathrm{e}^{-\mathrm{i}ts} - 1}{t}\right| |\Psi_\lambda'(s)| \,\mathrm{d}s\,\mathrm{d}t \\ &\leq \frac{1}{\sqrt{2\pi}} \frac{1}{\operatorname{dist}(\lambda, \mathbb{C}_+)^2} \int_0^\delta\int_{\mathbb{R}} \left|\frac{\mathrm{e}^{-\mathrm{i}ts} - 1}{t}\right| |\Psi'(s)| \,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

We have $|\Psi'(s)| = \frac{1}{1+s^2}$, $s \in \mathbb{R}$, and

$$\begin{aligned}\int_0^\delta\int_{\mathbb{R}} \left|\frac{\mathrm{e}^{-\mathrm{i}ts} - 1}{t}\right| |\Psi'(s)| \,\mathrm{d}s\,\mathrm{d}t &= 2\int_0^\delta\int_{\mathbb{R}} \frac{\left|\sin\frac{ts}{2}\right|}{|t|} \frac{1}{1+s^2} \,\mathrm{d}s\,\mathrm{d}t \\ = 2\int_0^\delta\int_{\mathbb{R}} \frac{\left|\sin\frac{s}{2}\right|}{|t|} \frac{1}{1+\left(\frac{s}{t}\right)^2} \frac{\mathrm{d}s}{|t|} \,\mathrm{d}t &= 2\int_{\mathbb{R}} \left|\sin\frac{s}{2}\right| \int_0^\delta \frac{1}{t^2+s^2} \,\mathrm{d}t\,\mathrm{d}s \\ &= 2\int_{\mathbb{R}} \frac{\left|\sin\frac{s}{2}\right|}{|s|} \arctan\frac{\delta}{|s|} \,\mathrm{d}s < \infty.\end{aligned}$$

Thus,

$$\int_0^{\delta}\left|\frac{\psi_\lambda(t)-\psi_\lambda(0)}{t}\right|\mathrm{d}t<\infty.$$

The estimate for $\int_{-\delta}^{0}\left|\frac{\psi_\lambda(t)-\psi_\lambda(0)}{t}\right|\mathrm{d}t$ is obtained similarly. ∎

The following two lemmas are actually established in the proofs of Lemmas 9.2 and 9.3.

LEMMA 9.4. *Let Ψ be defined by* (9.1). *Then for every* $0<c<1$, Ψ' *satisfies* (9.3).

LEMMA 9.5. *Let Ψ be defined by* (9.1). *Set* $\psi=\mathcal{F}\Psi'$. *Let* $\delta>0$. *Then* ψ *satisfies* (8.1).

Recall that $\widehat{\gamma}_T$ is defined in (1.4).

THEOREM 9.6. *Let T be defined as in Proposition* 3.2 *with ν as in Lemma* 3.3. *Let ϱ be the branch of square root defined in $\mathbb{C}\setminus[0,\infty)$ with $\varrho(-1)=\mathrm{i}$. Then there exists $R\in\{T\}'$ such that $\widehat{\gamma}_T(R)=\varrho|_{\mathbb{T}}$, and $\sigma(R)\subset\mathbb{C}_+\cup\mathbb{R}$.*

Proof. Since $\widehat{\gamma}_T$ is a unital algebra-homomorphism, it is sufficient to prove that $\varrho|_{\mathbb{T}}\in\widehat{\gamma}_T(\{T\}')$ and $\frac{1}{\varrho-\lambda}\big|_{\mathbb{T}}\in\widehat{\gamma}_T(\{T\}')$ for every $\lambda\notin\mathbb{C}_+\cup\mathbb{R}$ (see [KSz1]).

Let Ψ be defined by (9.1), and for $\lambda\notin\mathbb{C}_+\cup\mathbb{R}$ let Ψ_λ be defined as by (9.2). Let $\widetilde{\phi}$ be defined as in Theorem 4.3. By Theorem 4.3, it is sufficient to prove that $\mathcal{C}_{\mathcal{F}\Psi}\in\mathcal{L}(L^2(\mathbb{R},\widetilde{\phi}))$, and $\mathcal{C}_{\mathcal{F}\Psi_\lambda}\in\mathcal{L}(L^2(\mathbb{R},\widetilde{\phi}))$ for every $\lambda\notin\mathbb{C}_+\cup\mathbb{R}$. Take $\alpha>0$. Let w_α be defined by (5.1). By (5.2), $w_\alpha\asymp\widetilde{\phi}$. Therefore, it is sufficient to prove that $\mathcal{C}_{\mathcal{F}\Psi}\in\mathcal{L}(L^2(\mathbb{R},w_\alpha))$, and $\mathcal{C}_{\mathcal{F}\Psi_\lambda}\in\mathcal{L}(L^2(\mathbb{R},w_\alpha))$ for every $\lambda\notin\mathbb{C}_+\cup\mathbb{R}$. Since $\sup_{(b_1,b_2)}w_\alpha<\infty$ for every $-\infty<b_1<b_2<\infty$, we have $\mathcal{D}(\mathbb{R})\subset L^2(\mathbb{R},w_\alpha)$.

Take $0<\delta<\alpha$. Set $\psi=\mathcal{F}\Psi'$. By Theorem 9.1 and Lemmas 9.4 and 9.5, Ψ satisfies the conditions of Theorem 8.2. By Theorem 8.2, it is sufficient to prove that the mapping $\mathcal{A}_\psi$ which is defined on $\mathcal{D}(\mathbb{R})$ can be extended as a (bounded linear) operator onto $L^2(\mathbb{R},w_\alpha)$.

By (8.3), it is sufficient to prove that $\mathcal{A}_{1\psi}$ and $\mathcal{C}_{\psi_2}$, where ψ_2 is defined by (8.2), can be extended as (bounded linear) operators onto $L^2(\mathbb{R},w_\alpha)$. By Lemmas 5.1 and 8.1, w_α and $\mathcal{A}_{1\psi}$ satisfy the assumptions of Lemma 5.6. Thus, $\mathcal{A}_{1\psi}\in\mathcal{L}(L^2(\mathbb{R},w_\alpha))$ by Lemma 5.6. By Theorem 9.1 and Lemma 5.3, $\psi_2\sqrt{w_\alpha}\in L^1(\mathbb{R})$. By Corollary 5.2, w_α satisfies the assumptions of Lemma 5.4. By Lemma 5.4, $\mathcal{C}_{\psi_2}\in\mathcal{L}(L^2(\mathbb{R},w_\alpha))$.

Thus, $\mathcal{C}_{\mathcal{F}\Psi}\in\mathcal{L}(L^2(\mathbb{R},\widetilde{\phi}))$. For Ψ_λ with $\lambda\notin\mathbb{C}_+\cup\mathbb{R}$, the proof is the same. ∎

Corollary 9.7. *There exists a quasianalytic contraction* $\widetilde{R}$ *with* $\sigma(\widetilde{R}) = \{e^{it} : t \in [0,\pi]\}$ *and with a unitary asymptote* $U_{\sigma(\widetilde{R})}$. *Consequently,* $\sigma(\widetilde{R})$ *coincides with the quasianalytic spectral set of* $\widetilde{R}$ *and* $\sigma(\widetilde{R}) \neq \mathbb{T}$.

Proof. Let T and R be from Theorem 9.6. Clearly, $(\varrho|_{\mathbb{T}})^2 = \chi$ (where $\chi(z) = z$, $z \in \mathbb{T}$). Since $\widehat{\gamma}_T(R) = \varrho|_{\mathbb{T}}$ and $\widehat{\gamma}_T(T) = \chi$, we conclude that $R^2 = T$, because $\widehat{\gamma}_T$ is a unital algebra-homomorphism.

Set $\sigma = \{e^{it} : t \in [0,\pi]\}$. By Theorem 9.6, $\sigma(R) \subset \mathbb{C}_+ \cup \mathbb{R}$. By Proposition 3.2, $\sigma(T) = \mathbb{T}$. Since $R^2 = T$, we have $\sigma(T) = \{\lambda^2 : \lambda \in \sigma(R)\}$. Consequently, $\sigma(R) = \sigma$.

By Proposition 3.1, T is similar to a contraction. In particular, T is polynomially bounded. By Lemma 2.2, $\varrho(U_{\mathbb{T}})$ is a unitary asymptote of R. Since $\varrho(U_{\mathbb{T}})$ is unitarily equivalent to U_σ, we find that U_σ is a unitary asymptote of R.

Let ν be chosen such that T is quasianalytic. (This is possible by results of Sec. 6.) By Corollary 2.3, R is quasianalytic.

By Theorem 2.1, R is similar to some contraction $\widetilde{R}$.

Since similarity preserves the required properties of operators, the contraction $\widetilde{R}$ satisfies the conclusion of the corollary. ■

Remark 9.8. Let T and R be operators constructed in the proof of Corollary 9.7. By Proposition 3.1 and Lemma 2.2, unitary asymptotes of T and R are cyclic unitary operators. Therefore, $\{T\}'$ and $\{R\}'$ are abelian algebras. Since $R \in \{T\}'$, we conclude that $\{R\}' = \{T\}'$ by [KSz1, Proposition 11]. Consequently, $\operatorname{Hlat} R = \operatorname{Hlat} T$. By Corollary 7.2, $\operatorname{Hlat} T$ is nontrivial. Therefore, $\operatorname{Hlat} R$ (and hence $\operatorname{Hlat} \widetilde{R}$, where $\widetilde{R}$ is from Corollary 9.7) is nontrivial, too.

Remark 9.9. Let T be defined as in Proposition 3.2 with ν as in Lemma 3.3. For $\alpha > 0$, let ϑ_α be defined in Theorem 7.1. By Remark 7.3, $\vartheta_\alpha(T)$ is invertible.

For every $0 < r < 1$ set $\mathcal{G}_r = \mathbb{D} \setminus D_r$, where D_r is defined in (3.4). Then $\inf_{\mathcal{G}_r} |\vartheta_\alpha| > 0$. Denote by κ_r a conformal mapping of $\mathbb{D}$ onto $\mathcal{G}_r$. Let Q be an operator which admits an H^∞-functional calculus (see [Kér3, Theorem 23]). By [Kér2] and [Gam2], $\vartheta_\alpha(\kappa_r(Q))$ is invertible.

A question appears: does there exist an operator Q_r such that $T = \kappa_r(Q_r)$? It is possible to prove that there exists $Q_{1r} \in \{T\}'$ such that $\widehat{\gamma}_T(Q_{1r}) = \kappa_r^{-1}$ and $\sigma(Q_{1r})$ is a proper subarc of $\mathbb{T}$. But the estimate obtained by the author is $\|Q_{1r}^n\| \leq Cn(\log n)^2$ for sufficiently large $n \in \mathbb{N}$, which does not allow one to define $\kappa_r(Q_{1r})$.

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