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ω-DIAGONALIZABILITY OF F_σ FILTERS

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Abstract. We prove, without any form of determinacy, that F_σ filters are exactly filters ω-diagonalizable by $\mathcal{F}^+$-universal sets. For analytic filters, F_σ filters are exactly P^+(tree)-filters (this is the extension to analytic ideals of a theorem about Borel ideals by Hrušák and Meza-Alcántara). We also show some conditions equivalent to the fact that a filter is a subset of a proper F_σ filter.

1. Preliminaries. We use a standard set-theoretic and topological notation. In particular, the set of natural numbers is identified with the ordinal ω. We denote by $[\omega]^{<\omega}$ ($[\omega]^\omega$, respectively) the family of all finite (infinite, respectively) subsets of ω. We will write $A \subset^\star B$ if $A \setminus B$ is finite.

A *filter on* ω is a non-empty family of subsets of ω upward-closed (i.e. closed under the operation of taking supersets) and closed under taking finite intersections. We denote by $\mathrm{FIN}^\star$ the Fréchet filter (i.e. the family of all cofinite subsets of ω). If not explicitly mentioned otherwise, we assume that a filter is proper ($\neq \mathcal{P}(\omega)$) and contains the Fréchet filter. We can talk about filters on any other countable set by identifying this set with ω via a fixed bijection.

For any family $\mathcal{A} \subset \mathcal{P}(\omega)$ we denote by $\mathcal{A}^\star$ its *dual set* $\{\omega \setminus A \colon A \in \mathcal{A}\}$, and by $\mathcal{A}^c$ its complement $\mathcal{P}(\omega) \setminus \mathcal{A}$. For a filter $\mathcal{F}$ we define the *dual ideal to* $\mathcal{F}$ as $\mathcal{F}^\star$. Moreover, we denote by $\mathcal{F}^+$ the set of all subsets of ω which do not belong to $\mathcal{F}^\star$ (i.e. the *associated coideal of* $\mathcal{F}$).

1.1. Combinatorial properties of a filter $\mathcal{F}$. A filter $\mathcal{F}$ is a *P-filter* if for any sequence $\{F_n \colon n \in \omega\} \subset \mathcal{F}$ there is $G \in \mathcal{F}$ such that $G \subset^\star F_n$ for each n.

We will be interested in trees built from finite subsets of ω. If $s, t \in ([\omega]^{<\omega})^{<\omega}$, we write $s \preceq t$ if $\mathrm{length}(s) \leq \mathrm{length}(t)$ and $t \restriction \mathrm{length}(s) = s$, i.e. $s(i) = t(i)$ for $i = 0, \ldots, \mathrm{length}(s) - 1$. If $s \in ([\omega]^{<\omega})^{<\omega}, a \in [\omega]^{<\omega}$

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and $s = (s(0), \dots, s(k))$ then $s \frown a$ is the sequence $(s(0), \dots, s(k), a)$. We assume that $\emptyset$ is a sequence of length 0 and $\emptyset \preceq t$ for each $t \in ([\omega]^{<\omega})^{<\omega}$.

A set $\mathcal{T} \subset ([\omega]^{<\omega})^{<\omega}$ is a *tree of finite sets* if for each $s \in \mathcal{T}$ and $t \in ([\omega]^{<\omega})^{<\omega}$ with $t \preceq s$, we have $t \in \mathcal{T}$. A *branch* of a tree $\mathcal{T}$ is a function $b\colon \omega \to [\omega]^{<\omega}$ such that $(b(0), \dots, b(k)) \in \mathcal{T}$ for all $k \in \omega$. Sometimes we identify a branch b with $\bigcup_{k\in\omega} b(k)$ (a subset of ω).

We call a tree $\mathcal{T} \subset ([\omega]^{<\omega})^{<\omega}$ an *$\mathcal{X}$-tree of finite sets* for an $\mathcal{X} \subset [\omega]^{\omega}$ if for each $s \in \mathcal{T}$ there is an $X_s \in \mathcal{X}$ such that $s \frown a \in \mathcal{T}$ for all $a \in [X_s]^{<\omega}$.

The next two definitions were introduced by Laflamme [Laf96].

Definition 1.1.

(1) A set $\mathcal{Z} = \{z_m : m \in \omega\} \subset [\omega]^{<\omega} \setminus \{\emptyset\}$ is *$\mathcal{A}$-universal* if for each $A \in \mathcal{A}$ there is an $m \in \omega$ such that $z_m \subset A$.
(2) We say that a filter $\mathcal{F}$ is *ω-diagonalizable by $\mathcal{A}$-universal sets* if there exists a sequence $(\mathcal{Z}_N)_{N\in\mathbb{N}}$ of $\mathcal{A}$-universal sets such that for each $F \in \mathcal{F}$ there is $\mathcal{Z}_N = \{z_{N,m}\colon m \in \omega\}$ such that $\forall m\ (z_{N,m} \cap F \neq \emptyset)$.

Remark 1.2. Laflamme's definition of a filter ω-diagonalizable by $\mathcal{A}$-universal sets was slightly different, but it is easy to check that both versions are equivalent.

Definition 1.3. $\mathcal{F}$ is a *P^+(tree)-filter* if every $\mathcal{F}^+$-tree of finite sets has a branch whose union is in $\mathcal{F}^+$.

Example 1.4.

(1) An ultrafilter is a P^+(tree)-filter if and only if it is a P-point.
(2) In forcing theory there is a notion of Canjar filter [Can88]. In [GHMC17] it is shown that every F_σ filter is Canjar, and every Canjar filter is a P^+(tree)-filter, and for Borel filters these three conditions are equivalent (we show in Theorem 2.3 that this is also true for analytic filters).

Proof of (1). Suppose that $\mathcal{G}$ is an ultrafilter. Then $\mathcal{G}$ is non-meager and $\mathcal{G} = \mathcal{G}^+$. The required equivalence follows from Lemma 1.5 below. ∎

Lemma 1.5 ([Laf96, Lem. 1.3]). *A filter $\mathcal{F}$ is a non-meager P-filter if and only if every $\mathcal{F}$-tree of finite sets has a branch whose union is in $\mathcal{F}$.*

Remark 1.6. It is an open question if there exists a non-meager P-filter in ZFC (see e.g. [KMZ15]). The author does not know if there exists a non-meager P^+(tree)-filter in ZFC.

1.2. ω-diagonalizability and F_σ subsets of $\mathcal{P}(\omega)$. By identifying subsets of ω with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the product topology on $\{0,1\}^{\omega}$. It is known that $\mathcal{P}(\omega)$ with this topology is a metrizable compact Polish space without isolated points (it is homeomorphic to the Cantor set).

Lemma 1.7 is technical and simple but it plays a crucial role in all our further considerations about filters of type F_σ.

LEMMA 1.7. *Suppose that $\mathcal{F}$ is ω-diagonalizable by $\mathcal{A}$-universal sets $\mathcal{Z}_N$ ($N < \omega$), where $\mathcal{Z}_N = \{z_{N,n} : n \in \omega\} \subset [\omega]^{<\omega}$ for each N. Then*

$$\mathcal{F} \subset \mathcal{B} \subset (\mathcal{A}^c)^\star,$$

where $\mathcal{B} = \{B : \exists_N \forall_n \ z_{N,n} \cap B \neq \emptyset\}$ is an F_σ subset of $\mathcal{P}(\omega)$. In particular, if $\mathcal{A} = \mathcal{F}^+$ then $\mathcal{F} = \mathcal{B} \in F_\sigma$.

Proof. Since $\mathcal{F}$ is ω-diagonalizable by $\mathcal{Z}_N$'s, it is contained in $\mathcal{B}$. Since $\mathcal{Z}_N$ is $\mathcal{A}$-universal for each N,

$$\mathcal{A} \subset \{B : \forall_N \exists_n \ z_{N,n} \subset B\} = \{B : \forall_N \exists_n \ z_{N,n} \cap B^c = \emptyset\}.$$

Thus

$$\overline{\mathcal{B}} := \{B : \forall_N \exists_n \ z_{N,n} \cap B^c = \emptyset\}^c \subset \mathcal{A}^c,$$

and since

$$\overline{\mathcal{B}} = \{B : \exists_N \forall_n \ z_{N,n} \cap B^c \neq \emptyset\} = \{B^c : \exists_N \forall_n \ z_{N,n} \cap B \neq \emptyset\} = \mathcal{B}^\star,$$

we have $\mathcal{B}^\star \subset \mathcal{A}^c$, which gives $\mathcal{B} \subset (\mathcal{A}^c)^\star$. ∎

REMARK 1.8. Lemma 1.7 is essentially based on the proof of [LR09, Theorem 4]. As an easy corollary we see that for a filter $\mathcal{F}$ which is ω-diagonalizable by $\mathcal{F}$-universal sets we have

$$\mathcal{F} \subset \mathcal{B} \subset \mathcal{F}^+,$$

i.e. $\mathcal{F}$ can be separated by an F_σ set $\mathcal{B}$ from its dual filter. In fact, by [Bou12, Lemma 5.4] it is an equivalence for every filter $\mathcal{F}$.

1.3. The game $G(\mathcal{A}, [\omega]^{<\omega}, \mathcal{Z})$. For any pair of families $\mathcal{A}, \mathcal{Z} \subset \mathcal{P}(\omega)$ define an infinite game $G(\mathcal{A}, [\omega]^{<\omega}, \mathcal{Z})$ as follows: in the nth move, **I** plays an element X_n from $\mathcal{A}$, and then **II** plays any finite set $a_n \in [X_n]^{<\omega}$. **II** wins when $\bigcup\{a_n : n \in \omega\} \in \mathcal{Z}$. Otherwise **I** wins.

This game was investigated by Laflamme [Laf96] for several variants of $\mathcal{A}$ and $\mathcal{Z}$; in [LL02] there is a table with characterization of winning strategies for all combinations of $\mathcal{A}, \mathcal{Z} \in \{\mathrm{FIN}^\star, \mathcal{F}, \mathcal{F}^+\}$. We slightly generalize some of these results in Appendix A.

Note that a strategy for **I** in $G(\mathcal{A}, [\omega]^{<\omega}, \mathcal{Z})$ is an $\mathcal{A}$-tree $\mathcal{T}$ of finite sets. $\mathcal{T}$ is winning if and only if there is no branch of $\mathcal{T}$ in $\mathcal{Z}$ (cf. Def. 1.3).

2. F_σ filters. An ideal $\mathcal{I}$ is an F_σ *ideal* (*analytic ideal*, respectively) if $\mathcal{I}$ is an F_σ subset of $\mathcal{P}(\omega)$ (an analytic subset of $\mathcal{P}(\omega)$, respectively).

A map $\phi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure* if $\phi(\emptyset) = 0$, ϕ is monotone (i.e. $\phi(A) \le \phi(B)$ whenever $A \subset B$) and ϕ is subadditive (i.e. $\phi(A \cup B) \le \phi(A) + \phi(B)$). We will also assume that $\phi(\mathbb{N}) > 0$.

A submeasure ϕ is *lower semicontinuous* (lsc, for short) if for all $A \subset \mathbb{N}$ we have $\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0, 1, \ldots, n-1\})$.

For a lsc submeasure ϕ we define $\mathrm{FIN}(\phi) = \{A \subset \mathbb{N} : \phi(A) < \infty\}$. Mazur [Maz91] proved that $\mathcal{F} \in F_\sigma$ if and only if there exists a lsc submeasure ϕ such that $\mathcal{F}^\star = \mathrm{FIN}(\phi)$.

Examples of filters of type F_σ are the Fréchet filter $\mathrm{FIN}^\star$ and filters dual to summable ideals (these are filters defined by a submeasure $\phi(A) = \sum_{i\in A} s_i$, where $\sum_{i\in\omega} s_i$ is a positive divergent series).

THEOREM 2.1. *Fix a filter $\mathcal{F}$. The following conditions are equivalent:*

(1) *$\mathcal{F} \in F_\sigma$;*
(2) *$\mathcal{F}$ is ω-diagonalizable by a family of $\mathcal{F}^+$-universal sets.*

Proof. "(1) → (2)". If $\mathcal{F} \in F_\sigma$ then by Mazur's characterization there exists a lsc submeasure ϕ such that $\mathcal{F}^\star = \mathrm{FIN}(\phi)$. Consider the family of sets $\mathcal{Z}_N = \{A \subset [\omega]^{<\omega} : \phi(A) \geq N\}$. These sets are $\mathcal{F}^+$-universal and for each $B \in \mathcal{F}$ with $\phi(\omega \setminus B) < M$, $B \cap A \neq \emptyset$ for any $A \in \mathcal{Z}_N$ $(N \geq M)$.

"(2) → (1)" is an immediate consequence of Lemma 1.7. ∎

REMARK 2.2. Theorem 2.1 is "dual" to the characterization proved by Bouziad [Bou12, Lemma 5.4]: for any filter $\mathcal{F}$ the following conditions are equivalent:

(1) $\mathcal{F}$ is F_σ-separated from its dual ideal;
(2) $\mathcal{F}$ is ω-diagonalizable by a family of $\mathcal{F}$-universal sets.

Lemma 1.7 can also be used to give a short proof of the following theorem which was originally proved by Hrušák and Meza-Alcántara for Borel filters ([HMA11, Thm. 2.6], see also [MA09, Thm. 3.2.11]).

THEOREM 2.3. *An analytic filter $\mathcal{F}$ is F_σ if and only if $\mathcal{F}$ is a P^+(tree)-filter.*

Proof. Consider the game $G := G(\mathcal{F}^+, [\omega]^{<\omega}, \mathcal{F}^+)$. By [Laf96, Thm. 2.20] (or Proposition A.1):

(1) **I** has a winning strategy in G if and only if $\mathcal{F}$ is not a P^+(tree)-filter;
(2) **II** has a winning strategy in G if and only if $\mathcal{F}$ is ω-diagonalizable by $\mathcal{F}^+$-universal sets.

For $\mathcal{F}$ analytic the family $\mathcal{F}^+$ is coanalytic, and by Proposition A.2 the game G is determined, so either $\mathcal{F}$ is not a P^+(tree)-filter, or $\mathcal{F}$ is ω-diagonalizable by $\mathcal{F}^+$-universal sets. By Theorem 2.1 the second condition is equivalent to $\mathcal{F} \in F_\sigma$. ∎

3. Extendability to F_σ filters

LEMMA 3.1. *A filter $\mathcal{F}$ is a subset of some F_σ filter if and only if there is a filter $\mathcal{G}$ such that $\mathcal{F}$ is ω-diagonalizable by a family of $\mathcal{G}^+$-universal sets.*

Proof. "$\rightarrow$". Suppose that $\mathcal{F} \subset \mathcal{G} \in F_\sigma$. By Theorem 2.1, $\mathcal{G}$ is ω-diagonalizable by a family of $\mathcal{G}^+$-universal sets. It follows from the definition of ω-diagonalizability that $\mathcal{F}$ is also ω-diagonalizable by the same family of sets.

"$\leftarrow$". By Lemma 1.7 there is an F_σ set $\mathcal{B}$ such that

$$\mathcal{F} \subset \mathcal{B} \subset \mathcal{G}.$$

Let $\mathcal{B} = \bigcup_{n\in\omega} B_n$, where each B_n is closed.

From the definition of $\mathcal{B}$ in the formulation of Lemma 1.7 it follows that if $B \in \mathcal{B}$ then $B' \in \mathcal{B}$ for each $B' \supset B$, so $\mathcal{B}$ is hereditary. Thus by [Maz91, Prop. 1.1] we may assume that all B_n's are hereditary, closed, and $B_n \subset B_{n+1}$ for each n.

Observe that a function $\underline{\cap}\colon \mathcal{P}(\omega) \times \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ defined by $\underline{\cap}(x, y) = x \cap y$ is continuous. Then $B_n^m := B_n \underline{\cap} \cdots \underline{\cap} B_n$ (m times) is closed for each n and m. Moreover, since each B_n is a subset of a filter $\mathcal{G}$, we have $B_n^m \subset \mathcal{G}$. Then $\bigcup_{n\in\omega} B_n^n \subset \mathcal{G}$ is a proper F_σ filter. ■

THEOREM 3.2. *For an analytic filter $\mathcal{F}$ the following conditions are equivalent:*

(1) *$\mathcal{F} \subset \mathcal{G}$ for some F_σ filter $\mathcal{G}$;*
(2) *there exists a filter $\mathcal{G}$ such that $\mathcal{F}$ is ω-diagonalizable by a family of $\mathcal{G}^+$-universal sets;*
(3) *there exists a filter $\mathcal{G}$ such that every $\mathcal{G}^+$-tree of finite sets has a branch whose union is in $\mathcal{F}^+$;*
(4) *$\mathcal{F} \subset \mathcal{G}$ for some $P^+(tree)$-filter $\mathcal{G}$.*

Proof. "(1) $\leftrightarrow$ (2)". This is Lemma 3.1.

"(2) $\leftrightarrow$ (3)". Consider the game $G := G(\mathcal{G}^+, [\omega]^{<\omega}, \mathcal{F}^+)$. The equivalence "(2) $\leftrightarrow$ (3)" is a consequence of Proposition A.1 and the fact that the game considered is determined (see Proposition A.2).

"(1) $\rightarrow$ (4)". This implication follows from Theorem 2.3.

"(4) $\rightarrow$ (3)". If $\mathcal{F} \subset \mathcal{G}$ then $\mathcal{G}^+ \subset \mathcal{F}^+$, so if a $\mathcal{G}^+$-tree of finite sets has a branch whose union is in $\mathcal{G}^+$ then it is also in $\mathcal{F}^+$. ■

4. Remarks about extendability to F_σ filters. This section is a comment on the problem of characterizing those analytic filters on natural numbers which can be extended to proper F_σ filters. It does not contain any original results, just collects the existing results in one place.

DEFINITION 4.1 ([FMRS07]). A filter $\mathcal{F}$ of subsets of the naturals has the *finite Bolzano–Weierstrass property* (is FinBW, for short) if for every sequence $(\mathcal{P}_n)$ of finite partitions of ω such that each $\mathcal{P}_n$ is refined by $\mathcal{P}_{n+1}$, there exist a decreasing sequence (A_n) and a set $Z \in \mathcal{F}^+$ such that $Z \subset^\star A_n \in \mathcal{P}_n$ for each n.

Remark 4.2. Observe that if $\mathcal{F}$ is a P^+(tree)-filter then $\mathcal{F}$ is FinBW. It follows that every F_σ filter is FinBW.

Example 4.3. The filter on the rationals dual to the set

$$\mathrm{CONV} := \{A \subset \mathbb{Q} \colon A \text{ has finitely many cluster points}\}$$

is an example of a filter without the finite Bolzano–Weierstrass property.

The following characterizations are known in the literature:

Proposition 4.4.

(1) (CH): *any F_σ filter can be extended to a maximal P-filter* (see [Zap09, Claim 2.2], [FMRS07, Lem. 4.4])*;*
(2) (ZFC): *any analytic filter which can be extended to a maximal P-filter can be extended to an F_σ filter* [Zap09, Claim 3.2]*;*
(3) (ZFC+LC): *any universally Baire filter which can be extended to a maximal P-filter can be extended to an F_σ filter* [Zap09, Claim 3.2]*;*
(4) (ZFC): *an analytic P-filter can be extended to an F_σ filter iff it is* FinBW [FMRS07, Thm. 4.2].

Remark 4.5. By Theorem 3.2 we get an alternative proof of Proposition 4.4(2). Indeed, this is a consequence of Theorem 3.2 and the fact that every maximal P-filter is a P^+(tree)-filter (see Example 1.4).

Recall some classical partial orderings on the family of ideals on ω:

Definition 4.6. *Katětov–Blass order*: $\mathcal{F} \leq_{\mathrm{KB}} \mathcal{G}$ if there is a finite-to-one $f \colon \omega \to \omega$ such that $A \in \mathcal{F} \Rightarrow f^{-1}[A] \in \mathcal{G}$;

Katětov order: $\mathcal{F} \leq_{\mathrm{K}} \mathcal{G}$ if there is an $f \colon \omega \to \omega$ such that $A \in \mathcal{F} \Rightarrow f^{-1}[A] \in \mathcal{G}$.

Proposition 4.7 ([Kwe18, Lemma 2.5]). *The following are equivalent for any filter $\mathcal{F}$:*

(1) *$\mathcal{F} \subset \mathcal{G}$ for some F_σ filter $\mathcal{G}$;*
(2) *$\mathcal{F} \leq_{\mathrm{KB}} \mathcal{G}$ for some F_σ filter $\mathcal{G}$;*
(3) *$\mathcal{F} \leq_{\mathrm{K}} \mathcal{G}$ for some F_σ filter $\mathcal{G}$.*

Recall the open question of Hrušák (see also [UA19, Question 7.8]):

Problem 4.8 ([Hru11, Q. 5.16]). *Let $\mathcal{F}$ be a Borel filter. Are the following conditions equivalent:*

(i) $\mathrm{CONV}^\star \not\leq_{\mathrm{K}} \mathcal{F}$*;*
(ii) *$\mathcal{F}$ can be extended to a proper F_σ filter?*

The original question can be reformulated using the following characterization.

PROPOSITION 4.9 ([MA09]). *For any filter $\mathcal{F}$ the following conditions are equivalent:*

(i) $\mathrm{CONV}^\star \not\leq_{\mathrm{K}} \mathcal{F}$;
(ii) $\mathcal{F}$ *is* FinBW.

By Proposition 4.4(4), if $\mathcal{J}$ is an analytic P-ideal then the answer to Hrušák's question is affirmative. (Added in proof (August 2021): Recently Adam Kwela reported that he found an example of a Borel FinBW filter which is not extendable to any F_σ filter.)

Appendix A. Game $G(\mathcal{G}^+, [\omega]^{<\omega}, \mathcal{F}^+)$ and analytic filters. By the straightforward modification of the proof of Theorem 2.20 from [Laf96] we get Proposition A.1. We give the proof here because the corresponding arguments are "scattered" throughout [Laf96].

PROPOSITION A.1. *Fix a filter $\mathcal{F}$ and any $\mathcal{A} \subset \mathcal{P}(\omega)$, and consider the game $G := G(\mathcal{A}, [\omega]^{<\omega}, \mathcal{F}^+)$.*

(1) **I** *has no winning strategy in G iff every $\mathcal{A}$-tree of finite sets has a branch whose union is in $\mathcal{F}^+$;*
(2) **II** *has a winning strategy in G iff $\mathcal{F}$ is ω-diagonalizable by a family of $\mathcal{A}$-universal sets.*

Proof of (1). Strategy for **I** is nothing other than an $\mathcal{A}$-tree of finite sets, so **I** has no winning strategy iff any $\mathcal{A}$-tree of finite sets has a branch in $\mathcal{F}^+$. ∎

Proof of (2). First fix $\mathcal{A}$-universal sets $\langle X_n \colon n \in \omega\rangle$ ω-diagonalizing $\mathcal{F}$. At stage k, after **I** produced a set $Y_k \in \mathcal{A}$, **II** responds with $s_k \in X_k \cap [Y_k]^{<\omega}$. At the end of the play, **II** has produced $S = \bigcup_{k\in\omega} s_k$ which contains members of each X_n, and therefore its intersection with any $F \in \mathcal{F}$ is non-empty. Thus $S \in \mathcal{F}^+$.

Now let \$ be a winning strategy for **II** and define a tree $\mathcal{T} \subset ([\omega]^{<\omega})^{<\omega}$ such that the successors of each node $\overline{s} \in \mathcal{T}$ form an $\mathcal{A}$-universal set and $\langle X_{\overline{s}} \colon \overline{s} \in \mathcal{T}\rangle$ ω-diagonalize $\mathcal{F}$.

Define $X_\emptyset := \{\$(Y) \colon Y \in \mathcal{A}\}$. Note that it is an $\mathcal{A}$-universal set. For each $s \in X_\emptyset$ choose $X^s_\emptyset \in \mathcal{A}$ such that $\$(X^s_\emptyset) = s$.

In general, given $X^t_{\overline{s}} \in \mathcal{A}$, for $\overline{s} = \langle s_0, s_1, \dots, s_i\rangle \in ([\omega]^{<\omega})^{<\omega}$ define

$$X_{\overline{s}\frown t} := \{\$(X^{s_0}_\emptyset, X^{s_1}_{\langle s_0\rangle}, X^{s_2}_{\langle s_0,s_1\rangle}, \dots, X^t_{\overline{s}}, Y) \colon Y \in \mathcal{A}\},$$

and note that it is an $\mathcal{A}$-universal set. For each $u \in X_{\overline{s}\frown t}$ choose $X^u_{\overline{s}\frown t} \in \mathcal{A}$ such that

$$\$(X^{s_0}_\emptyset, X^{s_1}_{\langle s_0\rangle}, \dots, X^t_{\overline{s}}, X^u_{\overline{s}\frown t}) = u.$$

Now suppose that the sets $X_{\overline{s}}$ for $\overline{s} \in \mathcal{T}$ do not ω-diagonalize $\mathcal{F}$, i.e. there is an $F \in \mathcal{F}$ such that for each $\overline{s} \in \mathcal{T}$ there is an $a_{\overline{s}} \in X_{\overline{s}}$ with $a_{\overline{s}} \cap F = \emptyset$. The sequence $\langle a_k \colon k \in \omega\rangle$ given by

$$a_k = \begin{cases} a_{\emptyset} & \text{for } k = 0,\\ a_{\langle a_0, a_1, \ldots, a_{k-1}\rangle} & \text{for } k > 0,\end{cases}$$

is a branch of $\mathcal{T}$, so it is an outcome of a play of G. But $\bigcup_{k\in\omega} a_k \cap F = \emptyset$, which contradicts \$ being a winning strategy for **II**. ∎

Using a trick similar to that used by Kwela and Sabok [KS15, proof of Thm. 1.6] (a more detailed argument is in [Kwe14, Lem. 3.1.5, Lem. 3.1.6]) one can show coanalytic determinacy of our game. (In fact, from their proof it follows that the game $G(\mathcal{F}, [\omega]^{<\omega}, \mathcal{Z})$ is determined for any filter $\mathcal{F}$ and $\mathcal{Z} \subset \mathcal{P}(\omega)$ analytic.)

PROPOSITION A.2. *Suppose that $\mathcal{Z} \subset \mathcal{P}(\omega)$ is coanalytic and upward-closed. Then the game $G(\mathcal{A}, [\omega]^{<\omega}, \mathcal{Z})$ is determined for each $\mathcal{A} \subset \mathcal{P}(\omega)$.*

Proof. Write $G := G(\mathcal{A}, [\omega]^{<\omega}, \mathcal{Z})$. Let $\mathcal{S} = \mathcal{P}(\omega) \setminus \mathcal{Z}$. Clearly, $\mathcal{S}$ is analytic and downward-closed. There is a closed set $\mathcal{S}' \subset \mathcal{P}(\omega) \times \omega^\omega$ such that $\mathcal{S}$ is the projection of $\mathcal{S}'$. The unfolded game $G' := G'(\mathcal{A}, [\omega]^{<\omega}, \mathcal{S}')$ is played by **I** and **II** as follows: at stage $k \in \omega$,

- **I** chooses a pair $\langle X_k, m_k\rangle$ with $X_k \in \mathcal{A}$ and $m_k \in k \cup \{-1\}$, and then
- **II** responds with a set $a_k \in [X_k]^{<\omega}$.

After ω-many steps **I** wins if $\langle \bigcup_{k\in\omega} a_k, \overline{m}\rangle \in \mathcal{S}'$, where $\overline{m}$ is the sequence of those m_k's which are not equal to -1. Otherwise **II** wins.

Since $\mathcal{S}'$ is closed, the game G' is determined. To finish the proof it is enough to prove two claims:

(1) If **I** has a winning strategy in G', then **I** also has a winning strategy in G.
(2) If **II** has a winning strategy in G', then **II** also has a winning strategy in G.

(1): Let \$′ be a winning strategy for **I** in G'. For a sequence $\langle a_0, a_1, \ldots, a_k\rangle$ of sets played by **II** in G (or G') and

$$\$'(a_0, a_1, \ldots, a_k) = \langle X, m\rangle \in \mathcal{A} \times (k \cup \{-1\})$$

let $\$(a_0, a_1, \ldots, a_k) = X$. Then \$ is a strategy for **I** in G. We will show that it is also a winning strategy for **I**.

Suppose that a_k, $k \in \omega$, is a sequence of moves of **II** in G and $X_k = \$(a_0, a_1, \ldots, a_k)$ is a sequence of **I**'s responses. Since \$′ is a winning strategy for **I** in G', $\langle \bigcup_{k\in\omega} a_k, \overline{m}\rangle \in \mathcal{S}'$. Thus $\bigcup_{k\in\omega} a_k \in \mathrm{proj}_1(\mathcal{S}') = \mathcal{S} = \mathcal{P}(\omega) \setminus \mathcal{Z}$, and so **I** wins in G.

(2): Let $\$'$ be a winning strategy for **II** in G'. We define a strategy for **II** in G by

$$\$(X_0, X_1, \dots, X_k) = \bigcup\{\$'(\langle X_0, m_0\rangle, \langle X_1, m_1\rangle, \dots, \langle X_k, m_k\rangle)\colon m_i \in i \cup \{-1\} \text{ for all } i \leq k\}.$$

We will show that it is also a winning strategy for **II** in G.

Suppose to the contrary that there exists a sequence $\langle X_i\rangle_{i\in\omega}$ of elements of $\mathcal{A}$ such that $\bigcup_{k\in\omega} \$(X_0, X_1, \dots, X_k) \in S$. Since S is a projection of S', there exists a sequence $m := \langle m_i\rangle_{i\in\omega} \in \omega^\omega$ such that

$$\Big\langle \bigcup_{k\in\omega} \$(X_0, X_1, \dots, X_k), m\Big\rangle \in S'.$$

By Lemma A.3, without loss of generality, we can assume that

$$\text{(A)} \qquad \langle A, m\rangle \in S' \quad \text{for all } A \subset \bigcup_{k\in\omega} \$(X_0, X_1, \dots, X_k).$$

Let $m' := \langle m'_i\rangle_{i\in\omega}$ be a sequence of integers such that

$$m'_i \in i \cup \{-1\} \quad \text{for all } i, \quad \overline{m'} = m,$$

i.e. m' is constructed from m by inserting blocks of consecutive -1's, so that it is possible to meet the condition $m'_i < i$. Then consider the game G' with the sequence of **I**'s moves $\langle X_k, m'_k\rangle$, $k = 0, 1, \dots$. For each k let $a_k = \$'(\langle X_0, m'_0\rangle, \langle X_1, m'_1\rangle, \dots, \langle X_k, m'_k\rangle)$ be **II**'s response. Since $\$'$ is a winning strategy for **II**,

$$\Big\langle \bigcup_{k\in\omega} a_k, \overline{m'}\Big\rangle \notin S',$$

and since $\overline{m'} = m$,

$$\text{(B)} \qquad \Big\langle \bigcup_{k\in\omega} a_k, m\Big\rangle \notin S'.$$

From the definition of $\$$ it follows that $a_k \subset \$(X_0, X_1, \dots, X_k)$ for each k, so

$$\text{(C)} \qquad \bigcup_{k\in\omega} a_k \subset \bigcup_{k\in\omega} \$(X_0, X_1, \dots, X_k).$$

Now (A) combined with (B) and (C) give us the desired contradiction. ■

Lemma A.3 (folklore). *Suppose that $S \subset \mathcal{P}(\omega)$ is downward-closed (i.e. closed under taking subsets), Y is a metric space and $S' \subset \mathcal{P}(\omega) \times Y$ is closed with* $\operatorname{proj}_1 S' = S$. *Then there exists a closed $S'' \subset \mathcal{P}(\omega) \times Y$ such that* $\operatorname{proj}_1 S'' = S$ *and for each $y \in Y$,*

$$\{s \in \mathcal{P}(\omega)\colon \langle s, y\rangle \in S''\}$$

is downward-closed.

Proof. For each $y \in Y$ let $S'_y := \{s\colon \langle s, y\rangle \in S'\}$ and $S''_y := \{z\colon \exists_{s\in S'_y}\ z \subset s\}$. Define $S'' := \{\langle z, y\rangle\colon z \in S''_y\}$. Then $\{s \in \mathcal{P}(\omega)\colon \langle s, y\rangle \in S''\} = S''_y$ is downward-closed. Since S is downward-closed, and $S'' \supset S$, and each horizontal section of S'' is downward-closed, it follows that $\mathrm{proj}_1(S'') = S$. We have to show that S'' is a closed subset of $\mathcal{P}(\omega) \times Y$.

Suppose that $\langle z_n, y_n\rangle \in S''$ for each $n \in \omega$ and $\langle z_n, y_n\rangle \to \langle z, y\rangle$. For each n there exists $s_n \in S'_{y_n}$ such that $z_n \subset s_n$. Since $\mathcal{P}(\omega)$ is compact, we may assume that $\langle s_n, y_n\rangle \to \langle s, y\rangle$. Since $\langle s_n, y_n\rangle \in S'$ $(n \in \omega)$ and S' is closed, we see that $\langle s, y\rangle \in S' \subset S''$. To finish the proof it is enough to check that $z \subset s$.

Fix $m \in z$. Then there exists $N \in \omega$ such that $m \in z_n \subset s_n$ for all $n > N$. Since $s_n \to s$, it follows that $m \in s$. ∎

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