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ROTA–BAXTER PAIRED COMODULES AND ROTA–BAXTER PAIRED HOPF MODULES

BY

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Abstract. We introduce the concept of Rota–Baxter paired comodules, which is dual to Rota–Baxter paired modules defined by Zheng et al. (2020). We discuss some properties of Rota–Baxter paired comodules, in particular we give a characterization of generic Rota–Baxter paired comodules, which has an important application for the construction of Rota–Baxter comodules. Moreover, we construct Rota–Baxter paired comodules on Hopf algebras, weak Hopf algebras, weak Hopf modules, dimodules, relative Hopf modules and Rota–Baxter paired comodules. We also introduce the concept of Rota–Baxter paired Hopf modules by combining the notion of Rota–Baxter paired module with Rota–Baxter paired comodule, and prove a structure theorem for generic Rota–Baxter paired Hopf modules.

1. Introduction. A *Rota–Baxter algebra* (first known as a Baxter algebra) is an algebra A with a linear operator P on A that satisfies the Rota–Baxter identity

$$P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy) \quad \text{for all } x, y \in A,$$

where $\lambda \in k$ (the field) is called the *weight* [Bax, R69].

Rota–Baxter algebras originated in the 1960 paper [Bax] of Baxter based on his probability study to understand Spitzer's identity in fluctuation theory. It was not long before the concept attracted attention of many mathematicians, especially Rota, whose fundamental papers around 1970 brought the subject into the areas of algebra and combinatorics. In [A00], a connection with mathematical physics was also established that related a Rota–Baxter algebra of weight 0 to the associative analog of the classical Yang–Baxter equation. Recently, the paper [LMMP] made a detailed study of various structures that may be related to Rota–Baxter operators on BiHom-type algebras.

To study the representations of Rota–Baxter algebras, the authors of [GL] introduced the concept of Rota–Baxter modules related to the ring of Rota–

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Baxter operators. By definition, a *Rota–Baxter module* over a Rota–Baxter algebra (A, P) is a pair (M, T) where M is a (left) A-module and $T : M \to M$ a k-linear operator such that

$$P(a)\cdot T(m) = T(P(a)\cdot m) + T(a\cdot T(m)) + \lambda T(a\cdot m) \quad \text{for all } a \in A, m \in M.$$

Later, Rota–Baxter paired modules were introduced in [ZGZ], without requiring (A, P) to be a Rota–Baxter algebra; they are a natural generalization of Rota–Baxter modules. Many properties of Rota–Baxter modules, and of Rota–Baxter algebras, naturally generalize to Rota–Baxter paired modules. Rota–Baxter paired modules have broader connections and applications, especially to Hopf algebras. We have constructed a large number of Rota–Baxter paired modules from Hopf algebra related structures in [ZGZ].

Representation theory of coalgebras and comodules is extensive and well developed [SW, Sim]. On the basis of comodule theory, we can naturally consider Rota–Baxter operators on comodules. In this paper, we naturally introduce the concept of Rota–Baxter paired comodules, dual to Rota–Baxter paired modules, and give some of their properties. In addition, we give its construction from Hopf algebra related coalgebras and comodules.

A Hopf module on a bialgebra H is also an H-module and an H-comodule, whose action and coaction satisfy a compatibility condition.

As is well known, the structure of Hopf modules has been considered by many scholars. Especially, the relevant structure theorem can describe the integrals of Hopf algebras. Combining Rota–Baxter paired modules and Rota–Baxter paired comodules, we can naturally introduce the concept of Rota–Baxter paired Hopf modules, study Rota–Baxter operators on them, and prove the corresponding structure theorem.

This article is organized as follows. In Section 2, we recall the definition of Rota–Baxter coalgebras, and then introduce the notion of Rota–Baxter paired comodules, which is dual to Rota–Baxter paired modules of [ZGZ]. Moreover, we provide a large number of examples of Rota–Baxter paired comodules. In Section 3, we discuss some properties of Rota–Baxter paired comodules, in particular we obtain in Theorem 3.1 a characterization of generic Rota–Baxter paired comodules, which has an important application for the construction of Rota–Baxter comodules. In Section 4, we construct Rota–Baxter paired comodules on Hopf algebras, weak Hopf algebras, weak Hopf modules, dimodules, relative Hopf modules and Rota–Baxter paired comodules. In particular, we find some Rota–Baxter coalgebras and Rota–Baxter paired comodules by applying (co)integrals in bialgebras, antipod and idempotent elements in (weak) Hopf algebras and R-matrices in quasi-triangular Hopf algebras. In Section 5, we construct pre-Lie comodules from Rota–Baxter paired comodules. In Section 6, we introduce the concept of Rota–Baxter paired Hopf modules by combining the notion of Rota–Baxter paired module with Rota–Baxter paired comodule, and prove a structure theorem for generic Rota–Baxter Hopf modules.

Throughout this paper, we freely use the Hopf algebra and coalgebra terminology introduced in [DNR, R12, Sim, Sw]. We assume that k is a field and, unless otherwise specified, linearity, modules and $\otimes$ are all meant over k. Given a k-coalgebra C, we write its comultiplication $\Delta(c)$ as $c_1 \otimes c_2$ for any $c \in C$; for a left C-comodule M, we denote its coaction by $\rho(m) = m_{(-1)} \otimes m_{(0)}$ for any $m \in M$; for a right C-comodule M, we denote its coaction by $\rho(m) = m_{[0]} \otimes m_{[1]}$ for any $m \in M$; here we omit the summation symbols for convenience.

2. Rota–Baxter coalgebras and Rota–Baxter paired comodules. In this section, we firstly recall the definition of Rota–Baxter coalgebras, and then introduce the notion of Rota–Baxter paired comodules, which is dual to the notion of Rota–Baxter paired modules in [ZGZ]. Moreover, we provide a large number of examples of Rota–Baxter paired comodules.

2.1. Rota–Baxter coalgebras

DEFINITION 2.1 (see [JZ]). Let (C, Δ, ε) be a coalgebra. We call (C, P) a *Rota–Baxter coalgebra of weight* λ if the linear map $P : C \to C$ satisfies

$$(P \otimes P)\Delta = (P \otimes \mathrm{id})\Delta P + (\mathrm{id} \otimes P)\Delta P + \lambda \Delta P,$$

where $\lambda \in k$ and id denotes the identity map.

We refer the reader to [JZ, SW] for further discussion of Rota–Baxter coalgebras and only give the following simple examples which will be revisited later.

EXAMPLE 2.2. (a) Let C be an augmented coalgebra, that is, there exists a coalgebra homomorphism $f : k \to C$. Then it is easy to see that $f(1_k)$ is a group-like element in C. So, (C, P) is a Rota–Baxter coalgebra of weight -1. Here P is given by

$$P : C \to C, \quad c \mapsto \varepsilon(c) f(1_k).$$

Furthermore, if C is a bialgebra with unit u, then C is an augmented coalgebra since $u : k \to C$ is a coalgebra map. So, (C, P) is a Rota–Baxter coalgebra of weight -1, where

$$P : C \to C, \quad c \mapsto \varepsilon(c) 1_C.$$

(b) Let C be a coalgebra and $\xi \in C^*$ (the dual linear space of C). Define

$$P : C \to C, \quad c \mapsto \xi(c_1) c_2.$$

Then (C, P) is a Rota–Baxter coalgebra of weight -1 if and only if $\xi^2 = \xi$, that is, $\xi(c_1)\xi(c_2) = \xi(c)$ for all $c \in C$.

2.2. Rota–Baxter paired comodules

DEFINITION 2.3. Let (C, Δ, ε) be a coalgebra, and M a left C-comodule with coaction ρ. A pair (P, T) of linear maps $P : C \to C$ and $T : M \to M$ is called a *Rota–Baxter paired operator of weight* λ *on* M if

$$(P \otimes T)\rho = (P \otimes \mathrm{id})\rho T + (\mathrm{id} \otimes T)\rho T + \lambda\rho T.$$

The triple (M, P, T) is called a *Rota–Baxter paired* (*left*) C*-comodule of weight* λ. Given a linear map $T : M \to M$, if (M, P, T) is a Rota–Baxter paired C-comodule of weight λ for every linear map $P : C \to C$, then (M, T) is called a *generic Rota–Baxter paired* C*-comodule of weight* λ.

EXAMPLE 2.4. (1) Let C be a coalgebra, regarded as a left C-comodule via its comultiplication Δ. If (C, P) is a Rota–Baxter coalgebra of weight λ, then (C, P, P) is a Rota–Baxter paired C-comodule of weight λ.

In particular, if C is a bialgebra with unit u, then, by Example 2.2, (C, P, P) is a Rota–Baxter paired C-comodule of weight λ, where $P : C \to C$, $c \mapsto \varepsilon(c)1_C$.

(2) Let (M, P, T) be a Rota–Baxter paired C-comodule of weight λ. Then, for any $\mu \in k$, $(M, \mu P, \mu T)$ is a Rota–Baxter paired C-comodule of weight $\lambda\mu$.

(3) Let H be a bialgebra, and (M, P, T) a Rota–Baxter paired H-comodule of weight λ. If $P : H \to H$ is an idempotent bialgebra homomorphism, then $(H \otimes M, \rho, P, T')$ is a Rota–Baxter paired H-comodule of weight λ, where $T' : H \otimes M \to H \otimes M$ and $\rho : H \otimes M \to H \otimes H \otimes M$ are defined by $T'(h \otimes m) = P(h) \otimes T(m)$ and $\rho(h \otimes m) = h_1 m_{(-1)} \otimes h_2 \otimes m_{(0)}$.

In fact, it is easy to prove that $(H \otimes M, \rho)$ is a left H-comodule. Moreover, for any $h \in H$ and $m \in M$, we have

$$\begin{aligned}
&P((h \otimes m)_{(-1)}) \otimes T'((h \otimes m)_{(0)}) \\
&\quad = P(h_1 m_{(-1)}) \otimes P(h_2) \otimes T(m_{(0)}) \\
&\quad = P(h_1)P(m_{(-1)}) \otimes P(h_2) \otimes T(m_{(0)}) \\
&\quad = P(h_1)P(T(m)_{(-1)}) \otimes P(h_2) \otimes T(m)_{(0)} + P(h_1)T(m)_{(-1)} \otimes P(h_2) \\
&\qquad \otimes T(T(m)_{(0)}) + \lambda P(h_1)T(m)_{(-1)} \otimes P(h_2) \otimes T(m)_{(0)}, \\
&P(T'(h \otimes m)_{(-1)}) \otimes T'(h \otimes m)_{(0)} + T'(h \otimes m)_{(-1)} \otimes T'(T'(h \otimes m)_{(0)}) \\
&\qquad + \lambda T'(h \otimes m)_{(-1)} \otimes T'(h \otimes m)_{(0)} \\
&\quad = P(P(h)_1 T(m)_{(-1)}) \otimes P(h)_2 \otimes T(m)_{(0)} + P(h)_1 T(m)_{(-1)} \otimes P(P(h)_2) \\
&\qquad \otimes T(T(m)_{(0)}) + \lambda P(h)_1 T(m)_{(-1)} \otimes P(h)_2 \otimes T(m)_{(0)}
\end{aligned}$$

$$= P(P(h)_1)P(T(m)_{(-1)}) \otimes P(h)_2 \otimes T(m)_{(0)} + P(h)_1T(m)_{(-1)}$$
$$\otimes P(P(h)_2) \otimes T(T(m)_{(0)}) + \lambda P(h)_1T(m)_{(-1)} \otimes P(h)_2 \otimes T(m)_{(0)}$$
$$= P(h_1)P(T(m)_{(-1)}) \otimes P(h_2) \otimes T(m)_{(0)} + P(h_1)T(m)_{(-1)} \otimes P(h_2)$$
$$\otimes T(T(m)_{(0)}) + \lambda P(h_1)T(m)_{(-1)} \otimes P(h_2) \otimes T(m)_{(0)},$$

so, by Definition 2.3, $(H \otimes M, \rho, P, T')$ is a Rota–Baxter paired H-comodule of weight λ.

(4) Let M be a left C-comodule with coaction ρ, and V a vector space. Then $M \otimes V$ has a left C-comodule structure, whose comodule structure map is given by $\rho \otimes \mathrm{id}$. So, if (M, P, T) is a Rota–Baxter paired C-comodule of weight λ, we easily see that $(M \otimes V, P, T \otimes \mathrm{id})$ is a Rota–Baxter paired C-comodule of weight λ.

In particular, if (C, P) is a Rota–Baxter coalgebra of weight λ, then $(C \otimes V, P, P \otimes \mathrm{id})$ is a Rota–Baxter paired C-comodule of weight λ.

Furthermore, if (M, T) is a generic Rota–Baxter paired C-comodule of weight λ, then $(M \otimes V, T \otimes \mathrm{id})$ is also a generic Rota–Baxter paired C-comodule of weight λ.

(5) Let M be a left C-comodule, and $T : M \to M$ an idempotent epimorphism in $\mathrm{End}(M)$. Then, (M, id, T) is a Rota–Baxter paired C-comodule of weight -1.

Based on Definition 2.3, we can prove the result as follows:

Since T is epic, for any $m \in M$ there exists $n \in M$ such that $m = T(n)$. Then

$$(\mathrm{id} \otimes T)\rho(m) = (\mathrm{id} \otimes T)\rho(T(n)) = (\mathrm{id} \otimes T)\rho(T^2(n)) = (\mathrm{id} \otimes T)\rho(T(m)),$$

i.e., (M, id, T) is a Rota–Baxter paired C-comodule of weight -1.

A *Rota–Baxter paired C-subcomodule* N of a Rota–Baxter paired C-comodule (M, P, T) is a C-subcomodule of M such that $T(N) \subseteq N$. A *Rota–Baxter paired comodule map* $f : (M, P, T) \to (M', P', T')$ of the same weight λ is a C-comodule map such that $fT = T'f$.

Proposition 2.5. *Let $f : (M, P, T) \to (M', P', T')$ be a Rota–Baxter paired comodule map of weight λ. Then the following conclusions hold:*

(a) $\mathrm{Ker}\, f$ *is a Rota–Baxter paired C-subcomodule of M.*
(b) *If K is a Rota–Baxter paired C-subcomodule of M, then $f(K)$ is a Rota–Baxter paired C-subcomodule of M'.*

In particular, if T is C-colinear, then $T(M)$ is a Rota–Baxter paired C-subcomodule of M.

(c) *If L is a Rota–Baxter paired C-subcomodule of M', then $f^{-1}(L)$ is a Rota–Baxter paired C-subcomodule of M.*

Proof. (a) Since f is a C-comodule map, $\operatorname{Ker} f$ is a C-subcomodule of M'. In addition, for any $x \in \operatorname{Ker} f$, $fT(x) = T'f(x) = 0$, so $T(\operatorname{Ker} f) \subseteq \operatorname{Ker} f$. Hence $\operatorname{Ker} f$ is a Rota–Baxter paired C-subcomodule of M.

(b) It is obvious that $f(K)$ is a subcomodule of M', so we have only to verify that $T'(f(K)) \subseteq f(K)$. Since $T(K) \subseteq K$, and $fT = T'f$, we have $T'f(K) = fT(K) \subseteq f(K)$.

(c) We consider the composition πf of comodule maps π and f, where $\pi : M' \to M'/L$ is the projection. By (a), $\operatorname{Ker} f$ is a C-subcomodule of M, so $\operatorname{Ker}(\pi f) = f^{-1}(L)$ and is a subcomodule of M. In addition, $fT(f^{-1}(L)) = T'f(f^{-1}(L)) = T'(L) \subseteq L$, so $T(f^{-1}(L)) \subseteq f^{-1}(L)$. ∎

3. Some properties of Rota–Baxter paired comodules. Recall that a linear operator $T : M \to M$ is called *quasi-idempotent* [ZGZ] of weight λ if $T^2 = -\lambda T$. We have the following characterization of generic Rota–Baxter paired comodules, which has important application for the construction of Rota–Baxter comodules.

THEOREM 3.1. *Let C be a coalgebra, and M a left C-comodule. If there exists a C-colinear map $T : M \to M$, then the following are equivalent:*

(a) *(M, T) is a generic Rota–Baxter paired C-comodule of weight λ.*
(b) *There is a linear operator $P : C \to C$ such that (M, P, T) is a Rota–Baxter paired C-comodule of weight λ.*
(c) *T is quasi-idempotent of weight λ.*

Proof. Under the C-colinearity condition of T, for any linear operator $P : A \to A$ and $m \in M$, we have

$$\begin{aligned}
P(m_{(-1)}) \otimes T(m_{(0)}) &= P(T(m)_{(-1)}) \otimes T(m)_{(0)} \\
&\quad + T(m)_{(-1)} \otimes T(T(m)_{(0)}) + \lambda T(m)_{(-1)} \otimes T(m)_{(0)} \\
\iff P(m_{(-1)}) \otimes T(m_{(0)}) &= P(m_{(-1)}) \otimes T(m_{(0)}) + m_{(-1)} \otimes T^2(m_{(0)}) \\
&\quad + \lambda m_{(-1)} \otimes T(m_{(0)}) \\
\iff 0 &= m_{(-1)} \otimes T^2(m_{(0)}) + \lambda m_{(-1)} \otimes T(m_{(0)}).
\end{aligned}$$

If (a) holds, then applying $\varepsilon \otimes \mathrm{id}$ to both sides of the above equality, we get $T^2 = -\lambda T$. Conversely, if $T^2 = -\lambda T$, it is obvious that (a) holds.

The equivalence (b)$\Leftrightarrow$(c) can be proved in a similar way. ∎

PROPOSITION 3.2. *Let M be a left C-comodule. There exists a left C-comodule map $T : M \to M$ such that (M, T) is a generic Rota–Baxter C-comodule of weight -1 if and only if there is a C-comodule direct sum decomposition $M = M_1 \oplus M_2$ such that $T : M \to M_1 \subseteq M$ is the projection of M onto M_1: $T(m_1 + m_2) = m_1$ for $m_1 \in M_1$ and $m_2 \in M_2$.*

Proof. Suppose M has a direct sum decomposition $M = M_1 \oplus M_2$ of C-comodules, where M_1 and M_2 are subcomodules of M. Then the projection T of M onto M_1 is idempotent, since for $m = m_1 + m_2 \in M$ with $m_1 \in M_1$ and $m_2 \in M_2$, we have $T^2(m) = T^2(m_1 + m_2) = T(m_1) = m_1 = T(m)$.

Furthermore, we have

$$\begin{aligned}(\mathrm{id} \otimes T)\rho(m) &= (\mathrm{id} \otimes T)\rho(m_1 + m_2)\\ &= m_{1(-1)} \otimes T(m_{1(0)} + 0) + m_{2(-1)} \otimes T(0 + m_{2(0)})\\ &= m_{1(-1)} \otimes m_{1(0)} = \rho(m_1) = \rho T(m),\end{aligned}$$

so T is a left C-comodule map. Again by Theorem 3.1, the pair (M, T) is a generic Rota–Baxter paired C-comodule of weight -1.

Conversely, if (M, T) is a generic Rota–Baxter paired C-comodule of weight -1 and T a left C-comodule map, then by Theorem 3.1, T is idempotent.

Let $M_1 = T(M)$ and $M_2 = (\mathrm{id} - T)(M)$. Because T is a left C-comodule map, both M_1 and M_2 are subcomodules of M. Also, for any $m \in M$, $m = T(m) + (\mathrm{id} - T)(m)$, so $M = M_1 + M_2$. Furthermore, if $n \in M_1 \cap M_2$, then $n = T(x) = (\mathrm{id} - T)(y)$ for some $x, y \in M$. Thus $n = T(x) = T^2(x) = T(\mathrm{id} - T)(y) = (T - T^2)(y) = 0$. Therefore $M = M_1 \oplus M_2$.

Finally, since $m = T(m) + (\mathrm{id} - T)(m)$ is the decomposition of $m \in M$ as $m = m_1 + m_2$ with $m_1 \in M_1$ and $m_2 \in M_2$, we see that T is the projection of M onto M_1. ∎

PROPOSITION 3.3. *Let M be a C-comodule and $P : C \to C$, $T : M \to M$ linear maps. (M, P, T) is a Rota–Baxter paired C-comodule of weight $\lambda \neq 0$ if and only if there is a map $f : M \to C \otimes M$ such that*

$$(P \otimes T)\rho = fT, \quad (\overline{P} \otimes \overline{T})\rho = -f\overline{T},$$

where $\overline{P} = -P - \lambda\,\mathrm{id}$ and $\overline{T} = -T - \lambda\,\mathrm{id}$.

Proof. Let (M, P, T) be a Rota–Baxter paired C-comodule of weight λ. Then

$$(P \otimes T)\rho = (P \otimes \mathrm{id})\rho T + (\mathrm{id} \otimes T)\rho T + \lambda\rho T.$$

Let $f = (P \otimes \mathrm{id})\rho + (\mathrm{id} \otimes T)\rho + \lambda\rho$. Then the above equation yields

$$(P \otimes T)\rho = fT,$$

so we obtain

$$(\overline{P} \otimes \overline{T})\rho = -f\overline{T}.$$

To prove the converse, assume that there exists a map $f : M \to C \otimes M$ such that $(P \otimes T)\rho = fT$ and $(\overline{P} \otimes \overline{T})\rho = -f\overline{T}$. Then

$$\begin{aligned}-\lambda f &= f\overline{T} + fT = (P \otimes T)\rho - (\overline{P} \otimes \overline{T})\rho\\ &= (P \otimes T)\rho - ((-\lambda\,\mathrm{id} - P) \otimes (-\lambda\,\mathrm{id} - T))\rho\end{aligned}$$

$$= (P \otimes T)\rho - (\lambda^2 \operatorname{id} \otimes \operatorname{id} + \lambda \operatorname{id} \otimes T + \lambda P \otimes \operatorname{id} + P \otimes T)\rho$$
$$= -\lambda(\lambda \operatorname{id} \otimes \operatorname{id} + \operatorname{id} \otimes T + P \otimes \operatorname{id})\rho,$$

so

$$f = (P \otimes \operatorname{id})\rho + (\operatorname{id} \otimes T)\rho + \lambda\rho.$$

Furthermore, $(P \otimes T)\rho = ((P \otimes \operatorname{id})\rho + (\operatorname{id} \otimes T)\rho + \lambda\rho)T$. Hence (M, P, T) is a Rota–Baxter paired C-comodule of weight $\lambda \neq 0$. ■

By the above definition of $\overline{T}$ and $\overline{P}$ in Proposition 3.3, there are also the following relationships.

PROPOSITION 3.4. *Let (M, P, T, ρ) be a Rota–Baxter paired C-comodule of weight λ. Then, for any $m \in M$, we have*

$$P(m_{(-1)}) \otimes \overline{T}(m_{(0)}) = T(m)_{(-1)} \otimes \overline{T}(T(m)_{(0)}) + P(\overline{T}(m)_{(-1)}) \otimes \overline{T}(m)_{(0)},$$
$$\overline{P}(m_{(-1)}) \otimes T(m_{(0)}) = \overline{P}(T(m)_{(-1)}) \otimes T(m)_{(0)} + \overline{T}(m)_{(-1)} \otimes T(\overline{T}(m)_{(0)}).$$

Proof. By using the compatible condition of Rota–Baxter paired comodules, we can directly verify both equations. ■

The following result shows how close it is for an idempotent Rota–Baxter operator of comodule to have weight -1.

PROPOSITION 3.5. *Let (M, P, T) be a Rota–Baxter paired C-comodule of weight λ.*

(a) *If T is idempotent, then $(1 + \lambda)T(m)_{(-1)} \otimes T(T(m)_{(0)}) = 0$ for any $m \in M$.*
(b) *If P and T are idempotent, then $(1 + \lambda)P(T(m)_{(-1)}) \otimes T(m)_{(0)} = 0$ for any $m \in M$.*
(c) *If P and T are idempotent, then $(1 + \lambda)(P(T(m)_{(-1)}) \otimes T(m)_{(0)} - \lambda T(m)_{(-1)} \otimes T(m)_{(0)}) = 0$ for any $m \in M$.*

Proof. (a) Since $T^2 = T$, for any $m \in M$ we obtain

$$\begin{aligned} P(m_{(-1)}) \otimes T(m_{(0)}) &= P(m_{(-1)}) \otimes T(T(m_{(0)})) \\ &= P(T(m)_{(-1)}) \otimes T(T(m)_{(0)}) + T(m)_{(-1)} \otimes T^2(T(m)_{(0)}) \\ &\quad + \lambda T(m)_{(-1)} \otimes T(T(m)_{(0)}) \\ &= P(T(m)_{(-1)}) \otimes T(T(m)_{(0)}) + T(m)_{(-1)} \otimes T(T(m)_{(0)}) \\ &\quad + \lambda T(m)_{(-1)} \otimes T(T(m)_{(0)}), \end{aligned}$$

that is, $(P \otimes T)\rho = (P \otimes T)\rho T + (\operatorname{id} \otimes T)\rho T + \lambda(\operatorname{id} \otimes T)\rho T$.

Based on the above equality, we have

$$(P \otimes T)\rho T = (P \otimes T)\rho T^2 + (\operatorname{id} \otimes T)\rho T^2 + \lambda(\operatorname{id} \otimes T)\rho T^2.$$

Then by $T^2 = T$, we can get

$$(\operatorname{id} \otimes T)\rho T + \lambda(\operatorname{id} \otimes T)\rho T = 0.$$

(b) Since $P^2 = P$, for any $m \in M$ we obtain

$$\begin{aligned} P(m_{(-1)}) \otimes T(m_{(0)}) &= P(P(m_{(-1)})) \otimes T(m_{(0)}) \\ &= P^2(T(m)_{(-1)}) \otimes T(m)_{(0)} + P(T(m)_{(-1)}) \otimes T(T(m)_{(0)}) \\ &\quad + \lambda P(T(m)_{(-1)}) \otimes T(m)_{(0)} \\ &= P(T(m)_{(-1)}) \otimes T(m)_{(0)} + P(T(m)_{(-1)}) \otimes T(T(m)_{(0)}) \\ &\quad + \lambda P(T(m)_{(-1)}) \otimes T(m)_{(0)}, \end{aligned}$$

that is, $(P \otimes T)\rho = (P \otimes \mathrm{id})\rho T + (P \otimes T)\rho T + \lambda(P \otimes \mathrm{id})\rho T$.

Applying $T^2 = T$ to the above equality, we obtain $(1+\lambda)P(T(m)_{(-1)}) \otimes T(m)_{(0)} = 0$ for any $m \in M$.

(c) Apply (a) and (b). ∎

COROLLARY 3.6. *Any idempotent Rota–Baxter operator of a comodule is of weight* -1.

Proof. Apply Proposition 3.5. ∎

4. Constructions of Rota–Baxter paired comodules. In this section, we construct Rota–Baxter paired comodules on comodules, weak Hopf algebras, weak Hopf modules, dimodules, relative Hopf modules and Rota–Baxter paired comodules.

Firstly, we construct Rota–Baxter paired comodules from comodules.

4.1. A construction on comodules

PROPOSITION 4.1. *Let C be a coalgebra, and M a left C-comodule. Define two maps $T : M \to M$ and $P : C \to C$ by $T(m) = \chi(m_{(-1)})m_{(0)}$ and $P(c) = \chi(c_1)c_2$, for any $m \in M$, $c \in C$, $\chi \in C^*$. Then (M, P, T) is a Rota–Baxter paired C-comodule of weight -1 if χ is idempotent under the convolution product.*

Proof. For any $m \in M$, we have

$$\begin{aligned} P(m_{(-1)}) \otimes T(m_{(0)}) &= \chi(m_{(-1)1})m_{(-1)2} \otimes \chi(m_{(0)(-1)})m_{(0)(0)} \\ &= \chi(m_{(-1)1})m_{(-1)2} \otimes \chi(m_{(-1)3})m_{(0)}. \end{aligned}$$

Moreover, for any $m \in M$, $\rho(T(m)) = \rho(\chi(m_{-1})m_0) = \chi(m_{-1})m_{(0)(-1)} \otimes m_{(0)(0)} = \chi(m_{(-1)1})m_{(-1)2} \otimes m_{(0)}$, so

$$\begin{aligned} &P(T(m)_{(-1)}) \otimes T(m)_{(0)} + T(m)_{(-1)} \otimes T(T(m)_{(0)}) - T(m)_{(-1)} \otimes T(m)_{(0)} \\ &= \chi(m_{(-1)1})P(m_{(-1)2}) \otimes m_{(0)} + \chi(m_{(-1)1})m_{(-1)2} \otimes T(m_{(0)}) \\ &\quad - \chi(m_{(-1)1})m_{(-1)2} \otimes m_{(0)} \\ &= \chi(m_{(-1)1})\chi(m_{(-1)2})m_{(-1)3} \otimes m_{(0)} + \chi(m_{(-1)1})m_{(-1)2} \\ &\quad \otimes \chi(m_{(0)(-1)})m_{(0)(0)} - \chi(m_{(-1)1})m_{(-1)2} \otimes m_{(0)} \end{aligned}$$

$$= \chi(m_{(-1)1})\chi(m_{(-1)2})m_{(-1)3} \otimes m_{(0)} + \chi(m_{(-1)1})m_{(-1)2} \otimes \chi(m_{(-1)3})m_{(0)}$$
$$- \chi(m_{(-1)1})m_{(-1)2} \otimes m_{(0)}$$
$$= \chi^2(m_{(-1)1})m_{(-1)2} \otimes m_{(0)} + \chi(m_{(-1)1})m_{(-1)2} \otimes \chi(m_{(-1)3})m_{(0)}$$
$$- \chi(m_{(-1)1})m_{(-1)2} \otimes m_{(0)}$$
$$= \chi(m_{(-1)1})m_{(-1)2} \otimes m_{(0)} + \chi(m_{(-1)1})m_{(-1)2} \otimes \chi(m_{(-1)3})m_{(0)}$$
$$- \chi(m_{(-1)1})m_{(-1)2} \otimes m_{(0)}$$
$$= \chi(m_{(-1)1})m_{(-1)2} \otimes \chi(m_{(-1)3})m_{(0)}.$$

Hence $P(m_{(-1)}) \otimes T(m_{(0)}) = P(T(m)_{(-1)}) \otimes T(m)_{(0)} + T(m)_{(-1)} \otimes T(T(m)_{(0)}) - T(m)_{(-1)} \otimes T(m)_{(0)}$ and (M, P, T) is a Rota–Baxter paired H-comodule of weight -1. ∎

Let H be a bialgebra. If there exists $\varsigma \in H^*$ such that $f\varsigma = \varepsilon_{H^*}(f)\varsigma$ for any $f \in H^*$, then we call ς a *left cointegral* of H^*. Furthermore, if $\varepsilon_{H^*}(\varsigma) = 1$, we call H a *cosemisimple bialgebra*, and easily see $\varsigma^2 = \varsigma$, that is, ς is idempotent.

As a consequence of Proposition 4.1 and Example 2.2(b) we obtain

Corollary 4.2. *Let H be a cosemisimple bialgebra with cointegral ς, and M a left H-comodule. Define $T : M \to M$ and $P : H \to H$ by $T(m) = \varsigma(m_{(-1)})m_{(0)}$ and $P(h) = \varsigma(h_1)h_2$, for any $m \in M$ and $h \in H$.*

(a) *(M, P, T) is a Rota–Baxter paired H-comodule of weight -1.*
(b) *(H, P) is a Rota–Baxter coalgebra of weight -1.*

4.2. A construction on weak Hopf algebras

Definition 4.3 (see [BNS]). Let H be both an algebra and a coalgebra. Then H is called a *weak bialgebra* if it satisfies the following conditions:

(1) $\Delta(xy) = \Delta(x)\Delta(y)$ for all $x, y \in H$,
(2) $\varepsilon(xyz) = \varepsilon(xy_1)\varepsilon(y_2z) = \varepsilon(xy_2)\varepsilon(y_1z)$ for any $x, y, z \in H$,
(3) $\Delta^2(1_H) = (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) = 1_1 \otimes 1_2 1'_1 \otimes 1'_2$
$= (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H) = 1_1 \otimes 1'_1 1_2 \otimes 1'_2,$
where $\Delta(1_H) = 1_1 \otimes 1_2 = 1'_1 \otimes 1'_2$ and $\Delta^2 = (\Delta \otimes \mathrm{id}_H) \circ \Delta$.

Moreover, if there exists a linear map $S : H \to H$, called an *antipode*, satisfying the following axioms for all $h \in H$:

$$h_1 S(h_2) = \varepsilon(1_1 h)1_2, \quad S(h_1)h_2 = \varepsilon(h 1_2)1_1, \quad S(h_1)h_2 S(h_3) = S(h),$$

then the weak bialgebra H is called a *weak Hopf algebra*.

For any weak bialgebra H, define maps $\sqcap^L, \sqcap^R : H \to H$ by

$$\sqcap^L(h) = \varepsilon(1_1 h)1_2, \quad \sqcap^R(h) = \varepsilon(h 1_2)1_1.$$

Denote by H^L the image of $\sqcap^L$ and by H^R the image of $\sqcap^R$, where H^L and H^R are respectively called the *target algebra* and the *source algebra* of the weak bialgebra H.

By [BNS], if H is a weak Hopf algebra with antipode S, then for any $h, g \in H$, we have the following conclusions:

$$\sqcap^L \circ \sqcap^L = \sqcap^L, \quad \sqcap^R \circ \sqcap^R = \sqcap^R; \tag{4.1}$$

$$\sqcap^L(h_1) \otimes h_2 = S(1_1) \otimes 1_2 h, \quad h_1 \otimes \sqcap^R(h_2) = h1_1 \otimes S(1_2); \tag{4.2}$$

$$\sqcap^L(1_1) \otimes 1_2 = S(1_1) \otimes 1_2, \quad 1_1 \otimes \sqcap^R(1_2) = 1_1 \otimes S(1_2); \tag{4.3}$$

$$\sqcap^L(\sqcap^L(h)g) = \sqcap^L(h)\sqcap^L(g), \quad \sqcap^R(h\sqcap^R(g)) = \sqcap^R(h)\sqcap^R(g). \tag{4.4}$$

Note that $\sqcap^L(1_1) \otimes 1_2$ and $1_1 \otimes \sqcap^R(1_2)$ are separable idempotents of H^L and H^R respectively by [BNS, Proposition 2.11]. So, by (4.2), we have

$$h\sqcap^L(1_1) \otimes 1_2 = \sqcap^L(1_1) \otimes 1_2 h, \quad h1_1 \otimes \sqcap^R(1_2) = 1_1 \otimes \sqcap^R(1_2)h. \tag{4.5}$$

Again according to [WCZ], for any $x \in H^L$ and $y \in H^R$,

$$\Delta(1) = 1_1 \otimes 1_2 \in H^R \otimes H^L, \quad xy = yx; \tag{4.6}$$

$$\Delta(x) = 1_1 x \otimes 1_2, \quad \Delta(y) = 1_1 \otimes y1_2; \tag{4.7}$$

$$xS(1_1) \otimes 1_2 = S(1_1) \otimes 1_2 x, \quad y1_1 \otimes S(1_2) = 1_1 \otimes S(1_2)y. \tag{4.8}$$

Again by (4.6) and (4.7), we have

$$\Delta(xy) = 1_1 x \otimes 1_2 y \in H^L H^R \otimes H^L H^R. \tag{4.9}$$

According to (4.9), we know that $H^L H^R$ is a subcoalgebra of H.

PROPOSITION 4.4. *Let H be a weak Hopf algebra with antipode S.*

(a) $(H^L H^R, \sqcap^L, \sqcap^L)$ *is a Rota–Baxter paired H-comodule of weight -1, whose comodule structure map is given by the comultiplication Δ of H.*
(b) $(H^L H^R, \sqcap^L)$ *is a Rota–Baxter coalgebra of weight -1.*

Proof. By (4.9), $H^L H^R$ is a left H-comodule via the comultiplication Δ of H. Moreover, for any $x \in H^L, y \in H^R$, we have

$$\begin{aligned}(\sqcap^L \otimes \sqcap^L)\Delta(xy) &\overset{(4.9)}{=} (\sqcap^L \otimes \sqcap^L)(1_1 x \otimes 1_2 y) \\ &= \sqcap^L(1_1 x) \otimes \sqcap^L(1_2 y) \overset{(4.6)}{=} \sqcap^L(x1_1) \otimes \sqcap^L(1_2 y) \\ &\overset{(4.4)}{=} x\sqcap^L(1_1) \otimes 1_2 \sqcap^L(y) \overset{(4.5)}{=} x\sqcap^L(y)\sqcap^L(1_1) \otimes 1_2,\end{aligned}$$

$$
\begin{aligned}
(\sqcap^L \otimes \mathrm{id})\Delta \sqcap^L(xy) &= (\sqcap^L \otimes \mathrm{id})\Delta(x \sqcap^L(y)) \\
&= (\sqcap^L \otimes \mathrm{id})(x_1 \sqcap^L(y)_1 \otimes x_2 \sqcap^L(y)_2) \\
&\overset{(4.7)}{=} (\sqcap^L \otimes \mathrm{id})(1_1 x 1_{1'} \sqcap^L(y) \otimes 1_2 1_{2'}) \\
&= (\sqcap^L \otimes \mathrm{id})(1_1 1_{1'} x \sqcap^L(y) \otimes 1_2 1_{2'}) \\
&= (\sqcap^L \otimes \mathrm{id})(1_1 x \sqcap^L(y) \otimes 1_2) \\
&= \sqcap^L(1_1 x \sqcap^L(y)) \otimes 1_2 \\
&\overset{(4.4)}{=} x \sqcap^L(y) \sqcap^L(1_1) \otimes 1_2, \\
(\mathrm{id} \otimes \sqcap^L)\Delta \sqcap^L(xy) &= (\mathrm{id} \otimes \sqcap^L)\Delta(x \sqcap^L(y)) = (\mathrm{id} \otimes \sqcap^L)(1_1 x \sqcap^L(y) \otimes 1_2) \\
&= 1_1 x \sqcap^L(y) \otimes \sqcap^L(1_2) \overset{(4.1)}{=} 1_1 x \sqcap^L(y) \otimes 1_2 = \Delta \sqcap^L(xy).
\end{aligned}
$$

Hence, we obtain

$$(\sqcap^L \otimes \mathrm{id})\Delta \sqcap^L(xy) + (\mathrm{id} \otimes \sqcap^L)\Delta \sqcap^L(xy) - \Delta \sqcap^L(xy) = (\sqcap^L \otimes \sqcap^L)\Delta(xy),$$

that is, $(H^L H^R, \sqcap^L, \sqcap^L)$ is a Rota–Baxter paired H-comodule of weight -1 and $(H^L H^R, \sqcap^L)$ is a Rota–Baxter coalgebra of weight -1. ■

In a similar way, we obtain

Remark. Let H be a weak Hopf algebra with antipode S.

(1) $(H^L H^R, \sqcap^R, \sqcap^R)$ is a Rota–Baxter paired H-comodule of weight -1, and so $(H^L H^R, \sqcap^R)$ is a Rota–Baxter coalgebra of weight -1.
(2) $(H, \sqcap^L, \sqcap^L)$ is a Rota–Baxter paired H-comodule of weight -1, and so $(H, \sqcap^L)$ is a Rota–Baxter coalgebra of weight -1.
(3) $(H, \sqcap^R, \sqcap^R)$ is a Rota–Baxter paired H-comodule of weight -1, and so $(H, \sqcap^R)$ is a Rota–Baxter coalgebra of weight -1.

4.3. A construction on weak Hopf modules. In this subsection, we assume that H is a weak Hopf algebra with antipode S. Then S is both an antimultiplication map and an anticomultiplication map, that is, for any $h, g \in H$,

$$S(hg) = S(g)S(h), \quad S(1) = 1, \quad \Delta S(h) = S(h_2) \otimes S(h_1), \quad \varepsilon S(h) = \varepsilon(h),$$

and we have

$$h_1 \otimes \sqcap^L(h_2) = 1_1 h \otimes 1_2, \quad \sqcap^R(h_1) \otimes h_2 = 1_1 \otimes h 1_2. \tag{4.10}$$

Definition 4.5. Suppose that H is a weak Hopf algebra with antipode S. A *weak right H-Hopf module* is defined to be a triple $(M, \cdot, \rho)$, where $(M, \cdot)$ is a right H-module and (M, ρ) a right H-comodule, such that

$$\rho(m \cdot h) = m_{[0]} \cdot h_1 \otimes m_{[1]} h_2$$

for any $m \in M$ and $h \in H$.

Define a map $T : M \to M$ by

$$T(m) = m_{[0]} \cdot S(m_{[1]}) \quad \text{for any } m \in M.$$

Then, according to [ZGZ, Proposition 3.8], T is idempotent. Moreover,

$$\rho T(m) = T(m) \cdot 1_1 \otimes 1_2. \tag{4.11}$$

Indeed, for any $m \in M$, we have

$$\begin{aligned}
\rho T(m) &= m_{[0][0]} \cdot S(m_{[1]})_1 \otimes m_{[0][1]} S(m_{[1]})_2 \\
&= m_{[0]} \cdot S(m_{[1]2})_1 \otimes m_{[1]1} S(m_{[1]2})_2 \\
&= m_{[0]} \cdot S(m_{[1]3}) \otimes m_{[1]1} S(m_{[1]2}) = m_{[0]} \cdot S(m_{[1]2}) \otimes \sqcap^L(m_{[1]1}) \\
&\overset{(4.2)}{=} m_{[0]} \cdot S(1_2 m_{[1]}) \otimes S(1_1) = m_{[0]} \cdot S(m_{[1]}) S(1_2) \otimes S(1_1) \\
&= T(m) \cdot S(1_2) \otimes S(1_1) = T(m) \cdot 1_1 \otimes 1_2.
\end{aligned}$$

A weak Hopf algebra H is said to be *quantum commutative* if $h_1 g \sqcap^R(h_2) = hg$ for any $h, g \in H$. By [BFS, Proposition 4.1], H is quantum commutative if and only if $H^R \subseteq Z(H)$ (the center of H).

PROPOSITION 4.6. *Let H be a quantum commutative weak Hopf algebra, and M a weak right H-Hopf module. Then $(M, \sqcap^L, T)$ is a Rota–Baxter paired H-comodule of weight -1.*

Proof. According to (4.10), for any $m \in M$, we have

$$\begin{aligned}
(T \otimes \sqcap^L)\rho(m) &= T(m_{[0]}) \otimes \sqcap^L(m_{[1]}) = m_{[0][0]} \cdot S(m_{[0][1]}) \otimes \sqcap^L(m_{[1]}) \\
&= m_{[0]} \cdot S(m_{[1]1}) \otimes \sqcap^L(m_{[1]2}) \overset{(4.10)}{=} m_{[0]} \cdot S(1_1 m_{[1]}) \otimes 1_2 \\
&= m_{[0]} \cdot S(m_{[1]}) S(1_1) \otimes 1_2 = T(m) \cdot S(1_1) \otimes 1_2, \\
(\mathrm{id} \otimes \sqcap^L)\rho T(m) &\overset{(4.11)}{=} T(m) \cdot 1_1 \otimes \sqcap^L(1_2) \overset{(4.1)}{=} T(m) \cdot 1_1 \otimes 1_2 = \rho T(m), \\
(T \otimes \mathrm{id})\rho T(m) &= T(T(m) \cdot 1_1) \otimes 1_2 = T(m)_{[0]} \cdot 1_1 S(T(m)_{[1]} 1_2) \otimes 1_3 \\
&= T(m)_{[0]} \cdot 1_1 S(1_2) S(T(m)_{[1]}) \otimes 1_3 \\
&= T(m)_{[0]} \cdot \sqcap^L(1_1) S(T(m)_{[1]}) \otimes 1_2 \\
&\overset{(4.3)}{=} T(m)_{[0]} \cdot S(1_1) S(T(m)_{[1]}) \otimes 1_2 \\
&= T(m)_{[0]} \cdot S(T(m)_{[1]} 1_1) \otimes 1_2 \\
&= T(m)_{[0]} \cdot S(1_1 T(m)_{[1]}) \otimes 1_2 \quad (H^R \subseteq Z(H)) \\
&= T(m)_{[0]} \cdot S(T(m)_{[1]}) S(1_1) \otimes 1_2 \\
&= T(m) \cdot S(1_1) \otimes 1_2,
\end{aligned}$$

so we obtain

$$(T \otimes \sqcap^L)\rho(m) = (\mathrm{id} \otimes \sqcap^L)\rho T(m) + (T \otimes \mathrm{id})\rho T(m) - \rho T(m).$$

Hence $(M, T, \sqcap^L)$ is a Rota–Baxter paired H-comodule of weight -1. ∎

Remark. Let H be a weak Hopf algebra. Then H is a weak right H-Hopf module whose action and coaction are given by the multiplication and comultiplication of H. If H is quantum commutative, then, by the above proposition and $h_1S(h_2) = \sqcap^L(h)$ for $h \in H$, we know that $(H, \sqcap^L, \sqcap^L)$ is a Rota–Baxter paired H-comodule of weight -1.

4.4. A construction on dimodules. In this subsection, we construct Rota–Baxter paired comodules on dimodules.

Definition 4.7 (see [C94]). Assume that H is a bialgebra. A k-module M which is both a left H-module and a right H-comodule is called a *left-right H-dimodule* if for any $h \in H$ and $m \in M$,

$$\rho(h \cdot m) = h \cdot m_{[0]} \otimes m_{[1]},$$

where ρ is the right H-comodule structure map of M.

Proposition 4.8. *Let H be a bialgebra with an idempotent element e, and M a left-right H-dimodule. If $T : M \to M$ is defined by $T(m) = e \cdot m$, then (M, T) is a generic Rota–Baxter paired H-comodule of weight* -1.

Proof. For any $m \in M$, we have

$$T^2(m) = T(e \cdot m) = e^2 \cdot m = e \cdot m = T(m),$$

that is, T is idempotent. Since M is a left-right H-dimodule, for any $m \in M$ we have

$$(T \otimes \mathrm{id})\rho(m) = T(m_{[0]}) \otimes m_{[1]} = e \cdot m_{[0]} \otimes m_{[1]} = \rho(e \cdot m) = \rho T(m),$$

that is, T is a comodule map. Thus by Theorem 3.1, (M, T) is a generic Rota–Baxter paired H-comodule of weight -1. ∎

Remark. (1) Let G be a finite group. Then $H = (kG)^* = \mathrm{Hom}_k(kG, k)$ is a Hopf algebra with dual basis $\{p_g \mid p_g(h) = \delta_{gh}\}$. According to [CF], the p_g are orthogonal idempotents, for any $g \in G$. Thus, by the above proposition, for any left-right H-dimodule M, (M, T_g) is a generic Rota–Baxter paired H-comodule of weight -1 for every $g \in G$, where $T_g(m) = p_g \cdot m$ for $m \in M$.

(2) Let H be a bialgebra. If there is an element $x \in H$ such that $hx = \varepsilon(h)x$ for any $h \in H$, then we call x a *left integral* of H.

Suppose that H is a finite-dimensional semisimple Hopf algebra. Then by [Sw, Theorem 5.1.8], there exists a nonzero left integral e such that $\varepsilon(e) = 1$. It is obvious that $e^2 = e$. Hence the following conclusions hold.

(i) If $(H, \mathcal{R})$ is a quasitriangular Hopf algebra, then, by [ZGZ, Example 3.12], (H, ρ) is a left-right H-dimodule, whose action is given by its multiplication and coaction $\rho : H \to H \otimes H$ given by $\rho(h) = h\mathcal{R}_i \otimes \mathcal{R}_j$. So, by Proposition 4.8, (H, T) is a generic Rota–Baxter paired H-comodule of weight -1, where $T : H \to H$ is given by $T(h) = e \cdot h$.

(ii) Let M be a left-right H-dimodule. Define $T(m) = e \cdot m$ for $m \in M$. Then, by Proposition 4.8, (M, T) is a generic Rota–Baxter paired H-comodule of weight -1.

Note that T is a left H-module map: for any $h \in H$ and $m \in M$,

$$T(h \cdot m) = \varepsilon(h)e \cdot m = he \cdot m = h \cdot T(m).$$

Again T is idempotent by Proposition 4.8, thus (M, T) is a generic Rota–Baxter paired H-module of weight -1 by [ZGZ, Theorem 2.4]. Hence (M, T) is a generic Rota–Baxter paired left-right H-dimodule of weight -1.

In particular, if $(H, \mathcal{R})$ is a quasitriangular Hopf algebra, then, according to the above conclusions, (H, T) is a generic Rota–Baxter paired left-right H-dimodule of weight -1.

4.5. A construction on relative Hopf modules. In this subsection, we construct Rota–Baxter paired comodules on relative Hopf modules.

Definition 4.9 (see [Z97]). Let H be a bialgebra and C a right H-module coalgebra. A *relative* $[C, H]$*-Hopf module* M is a right C-comodule which is also a right H-module such that the following compatibility condition holds: for all $m \in M$ and $h \in H$,

$$\rho(m \cdot h) = m_{[0]} \cdot h_1 \otimes m_{[1]} \cdot h_2.$$

Proposition 4.10. *Let H be a Hopf algebra with an antipode S, and M a relative $[C, H]$-Hopf module. If there is a right H-module coalgebra map $\phi : C \to H$, define $E_C : C \to C$ and $E_M : M \to M$ by*

$$E_C(c) = c_1 \cdot S\phi(c_2), \qquad E_M(m) = m_{[0]} \cdot S\phi(m_{[1]}),$$

for any $c \in C$ and $m \in M$. Then (M, E_C, E_M) is a Rota–Baxter paired right C-comodule of weight -1. Here H is a right H-module coalgebra whose action is given by the multiplication of H.

Proof. By the definition of Rota–Baxter paired comodule, it is sufficient to prove that

$$(E_M \otimes E_C)\rho(m) = (\mathrm{id} \otimes E_C)\rho E_M + (E_M \otimes \mathrm{id})\rho E_M - \rho E_M$$

for any $m \in M$.

To see this, we note that, for any $m \in M$ and $c \in C$, we have

$$\begin{aligned}
\rho E_M(m) &= m_{[0][0]} \cdot (S\phi(m_{[1]}))_1 \otimes m_{[0][1]} \cdot (S\phi(m_{[1]}))_2 \\
&= m_{[0]} \cdot (S\phi(m_{[1]2}))_1 \otimes m_{[1]1} \cdot (S\phi(m_{[1]2}))_2 \\
&= m_{[0]} \cdot S\phi(m_{[1]3}) \otimes m_{[1]1} \cdot S\phi(m_{[1]2}) \\
&= m_{[0]} \cdot S\phi(m_{[1]2}) \otimes E_C(m_{[1]1}),
\end{aligned}$$

$$\begin{aligned}E_C^2(c) &= E_C(c_1\cdot S\phi(c_2)) = (c_1\cdot S\phi(c_2))_1\cdot S\phi((c_1\cdot S\phi(c_2))_2)\\ &= c_1\cdot (S\phi(c_4)S\phi(c_2\cdot S\phi(c_3)))\\ &= c_1\cdot (S\phi(c_4)S(S\phi(c_3))S\phi(c_2))\\ &= c_1\cdot S(\phi(c_2)S\phi(c_3)\phi(c_4)) = c_1\cdot S\phi(c_2) = E_C(c).\end{aligned}$$

The above equations yield

$$\begin{aligned}&(\mathrm{id}\otimes E_C)\rho E_M(m) + (E_M\otimes \mathrm{id})\rho E_M(m) - \rho E_M(m)\\ &\quad = (\mathrm{id}\otimes E_C)(m_{[0]}\cdot S\phi(m_{[1]2})\otimes E_C(m_{[1]1})) + (E_M\otimes \mathrm{id})(m_{[0]}\cdot S\phi(m_{[1]2})\\ &\qquad \otimes E_C(m_{[1]1})) - m_{[0]}\cdot S\phi(m_{[1]2})\otimes E_C(m_{[1]1})\\ &\quad = m_{[0]}\cdot S\phi(m_{[1]2})\otimes E_C^2(m_{[1]1}) + E_M(m_{[0]}\cdot S\phi(m_{[1]2}))\otimes E_C(m_{[1]1})\\ &\qquad - m_{[0]}\cdot S\phi(m_{[1]2})\otimes E_C(m_{[1]1})\\ &\quad = E_M(m_{[0]}\cdot S\phi(m_{[1]2}))\otimes E_C(m_{[1]1}).\end{aligned}$$

Finally, we prove that $E_M(m_{[0]}\cdot S\phi(m_{[1]2}))\otimes E_C(m_{[1]1}) = (E_M\otimes E_C)\rho(m)$. Indeed,

$$\begin{aligned}&E_M(m_{[0]}\cdot S\phi(m_{[1]2}))\otimes E_C(m_{[1]1})\\ &\quad = (m_{[0]}\cdot S\phi(m_{[1]2}))_{[0]}\cdot S\phi((m_{[0]}\cdot S\phi(m_{[1]2}))_{[1]})\otimes E_C(m_{[1]1})\\ &\quad = (m_{[0][0]}\cdot (S\phi(m_{[1]2}))_1)\cdot S\phi(m_{[0][1]}\cdot (S\phi(m_{[1]2}))_2)\otimes E_C(m_{[1]1})\\ &\quad = m_{[0]}\cdot (S\phi(m_{[1]4})S(S\phi(m_{[1]3}))S\phi(m_{[1]1}))\otimes E_C(m_{[1]2})\\ &\quad = m_{[0]}\cdot (S(S\phi(m_{[1]3})\phi(m_{[1]4}))S\phi(m_{[1]1}))\otimes E_C(m_{[1]2})\\ &\quad = m_{[0]}\cdot S\phi(m_{[1]1})\otimes E_C(m_{[1]2})\\ &\quad = m_{[0][0]}\cdot S\phi(m_{[0][1]})\otimes E_C(m_{[1]})\\ &\quad = E_M(m_{[0]})\otimes E_C(m_{[1]}).\end{aligned}$$

So (M, E_C, E_M) is a Rota–Baxter paired right C-comodule of weight -1. ■

Remark. Let H be a Hopf algebra and M a relative $[C, H]$-Hopf module. Then it is easy to show that $H\otimes M$ is a relative $[C, H]$-Hopf module via

$$\rho : H\otimes M \to H\otimes M\otimes C, \qquad \rho(h\otimes m) = h_1\otimes m_{[0]}\otimes m_{[1]}\cdot h_2,$$

$$\cdot : H\otimes M\otimes H \to H\otimes M, \qquad (h\otimes m)\cdot g = hg\otimes m.$$

Hence, by Proposition 4.10, $(H\otimes M, E_C, E_{H\otimes M})$ is a right Rota–Baxter paired C-comodule of weight -1 if there exists a right H-module coalgebra map $\phi : C\to H$, where

$$E_C(c) = c_1\cdot S\phi(c_2), \qquad E_{H\otimes M}(h\otimes m) = h_1 S\phi(m_{[1]}\cdot h_2)\otimes m_{[0]}.$$

4.6. A construction on Rota–Baxter paired comodules

PROPOSITION 4.11. *Let (M, P, T) be a Rota–Baxter paired C-comodule of weight λ. Define*

$$\overline{P} = -P - \lambda\,\mathrm{id}, \quad \overline{T} = -T - \lambda\,\mathrm{id}\,.$$

Then $(M, \overline{P}, \overline{T})$ is also a Rota–Baxter paired C-comodule of weight λ.

Proof. We have only to verify that

$$(\overline{P} \otimes \overline{T})\rho = (\overline{P} \otimes \mathrm{id})\rho\overline{T} + (\mathrm{id} \otimes \overline{T})\rho\overline{T} + \lambda\rho\overline{T}.$$

Indeed, for any $m \in M$, we have

$$\begin{aligned}
(\overline{P} \otimes \overline{T})\rho(m) &= (\overline{P} \otimes \overline{T})(m_{(-1)} \otimes m_{(0)}) = \overline{P}(m_{(-1)}) \otimes \overline{T}((m_{(0)}))\\
&= (-P(m_{(-1)}) - \lambda m_{(-1)}) \otimes (-T(m_{(0)}) - \lambda m_{(0)})\\
&= P(m_{(-1)}) \otimes T(m_{(0)}) + \lambda P(m_{(-1)}) \otimes m_{(0)}\\
&\quad + \lambda m_{(-1)} \otimes T(m_{(0)}) + \lambda^2 m_{(-1)} \otimes m_{(0)}\\
&= P(T(m)_{(-1)}) \otimes T(m)_{(0)} + T(m)_{(-1)} \otimes T(T(m)_{(0)})\\
&\quad + \lambda T(m)_{(-1)} \otimes T(m)_{(0)} + \lambda P(m_{(-1)}) \otimes m_{(0)}\\
&\quad + \lambda m_{(-1)} \otimes T(m_{(0)}) + \lambda^2 m_{(-1)} \otimes m_{(0)},
\end{aligned}$$

$$\begin{aligned}
&((\overline{P} \otimes \mathrm{id})\rho\overline{T} + (\mathrm{id} \otimes \overline{T})\rho\overline{T} + \lambda\rho\overline{T})(m)\\
&= (\overline{P} \otimes \mathrm{id} + \mathrm{id} \otimes \overline{T} + \lambda)\rho(-T(m) - \lambda m)\\
&= -(\overline{P} \otimes \mathrm{id})(T(m)_{(-1)} \otimes T(m)_{(0)}) - \lambda(\overline{P} \otimes \mathrm{id})(m_{(-1)} \otimes m_{(0)})\\
&\quad - (\mathrm{id} \otimes \overline{T})(T(m)_{(-1)} \otimes T(m)_{(0)}) - \lambda(\mathrm{id} \otimes \overline{T})(m_{(-1)} \otimes m_{(0)})\\
&\quad - \lambda T(m)_{(-1)} \otimes T(m)_{(0)} - \lambda^2 m_{(-1)} \otimes m_{(0)}\\
&= -\overline{P}(T(m)_{(-1)}) \otimes T(m)_{(0)} - \lambda\overline{P}(m_{(-1)}) \otimes m_{(0)}\\
&\quad - T(m)_{(-1)} \otimes \overline{T}(T(m)_{(0)}) - \lambda m_{(-1)} \otimes \overline{T}(m_{(0)})\\
&\quad - \lambda T(m)_{(-1)} \otimes T(m)_{(0)} - \lambda^2 m_{(-1)} \otimes m_{(0)}\\
&= (P(T(m)_{(-1)}) + \lambda T(m)_{(-1)}) \otimes T(m)_{(0)} - \lambda(-P(m_{(-1)}) - \lambda m_{(-1)})\\
&\quad \otimes m_{(0)} - T(m)_{(-1)} \otimes (-T(T(m)_{(0)}) - \lambda T(m)_{(0)}) - \lambda m_{(-1)}\\
&\quad \otimes (-T(m_{(0)}) - \lambda m_{(0)}) - \lambda T(m)_{(-1)} \otimes T(m)_{(0)} - \lambda^2 m_{(-1)} \otimes m_{(0)}\\
&= P(T(m)_{(-1)}) \otimes T(m)_{(0)} + \lambda T(m)_{(-1)} \otimes T(m)_{(0)} + \lambda P(m_{(-1)}) \otimes m_{(0)}\\
&\quad + T(m)_{(-1)} \otimes T(T(m)_{(0)}) + \lambda m_{(-1)} \otimes T(m_{(0)}) + \lambda^2 m_{(-1)} \otimes m_{(0)}
\end{aligned}$$

as desired. ■

PROPOSITION 4.12. *Let (C, P) be a Rota–Baxter coalgebra of weight λ, and (M, P, T) a Rota–Baxter paired comodule of weight λ. Define another* *comultiplication* Δ' *on* C *by*

$$\Delta' = (\mathrm{id} \otimes P)\Delta + (P \otimes \mathrm{id})\Delta + \lambda\Delta,$$

and another operation of M *by*

$$\rho' = (P \otimes \mathrm{id})\rho + (\mathrm{id} \otimes T)\rho + \lambda\rho.$$

Then the following conclusions hold.

(a) (C, Δ', P) *is also a (noncounitary) Rota–Baxter coalgebra of weight* λ.
(b) $\rho' T = (P \otimes T)\rho$.
(c) (M, ρ') *is a noncounitary* (C, Δ')*-comodule.*
(d) (M, P, T) *is a Rota–Baxter paired* (C, Δ')*-comodule of weight* λ*, whose comodule structure map is given by* ρ'.

Proof. (a) Since (C, Δ, P) is a Rota–Baxter coalgebra of weight λ, we have

$$(P \otimes P)\Delta = (P \otimes \mathrm{id})\Delta P + (\mathrm{id} \otimes P)\Delta P + \lambda\Delta P.$$

Hence we get $\Delta' P = (P \otimes P)\Delta$. Moreover, for any $c \in C$, we have

$$\begin{aligned}
(P \otimes P)\Delta'(c) &= (P \otimes P)((P \otimes \mathrm{id})\Delta + (\mathrm{id} \otimes P)\Delta + \lambda\Delta)(c) \\
&= (P \otimes P)(P(c_1) \otimes c_2 + c_1 \otimes P(c_2) + \lambda c_1 \otimes c_2) \\
&= P^2(c_1) \otimes P(c_2) + P(c_1) \otimes P^2(c_2) + \lambda P(c_1) \otimes P(c_2) \\
&= (P \otimes \mathrm{id} + \mathrm{id} \otimes P + \lambda)(P(c_1) \otimes P(c_2)) \\
&= (P \otimes \mathrm{id} + \mathrm{id} \otimes P + \lambda)(P \otimes P)\Delta(c) \\
&= ((P \otimes \mathrm{id})\Delta' P + (\mathrm{id} \otimes P)\Delta' P + \lambda\Delta' P)(c).
\end{aligned}$$

(b) We check directly that $\rho' T = (P \otimes \mathrm{id})\rho T + (\mathrm{id} \otimes T)\rho T + \lambda\rho T = (P \otimes T)\rho$.

(c) We only need to prove that $(\mathrm{id} \otimes \rho')\rho' = (\Delta' \otimes \mathrm{id})\rho'$: from $\Delta' P = (P \otimes P)\Delta$ and (b), for any $m \in M$, we have

$$\begin{aligned}
&(\mathrm{id} \otimes \rho')\rho'(m) \\
&\quad = P(m_{(-1)}) \otimes \rho'(m_{(0)}) + m_{(-1)} \otimes \rho'(T(m_{(0)})) + \lambda m_{(-1)} \otimes \rho'(m_{(0)}) \\
&\quad = P(m_{(-1)}) \otimes P(m_{(0)(-1)}) \otimes m_{(0)(0)} + P(m_{(-1)}) \otimes m_{(0)(-1)} \otimes T(m_{(0)(0)}) \\
&\qquad + P(m_{(-1)}) \otimes \lambda m_{(0)(-1)} \otimes m_{(0)(0)} + m_{(-1)} \otimes \rho'(T(m_{(0)})) \\
&\qquad + \lambda m_{(-1)} \otimes P(m_{(0)(-1)}) \otimes m_{(0)(0)} + \lambda m_{(-1)} \otimes m_{(0)(-1)} \otimes T(m_{(0)(0)}) \\
&\qquad + \lambda m_{(-1)} \otimes \lambda m_{(0)(-1)} \otimes m_{(0)(0)}
\end{aligned}$$

$$
\begin{aligned}
&= P(m_{(-1)1}) \otimes P(m_{(-1)2}) \otimes m_{(0)} + P(m_{(-1)1}) \otimes m_{(-1)2} \otimes T(m_{(0)}) \\
&\quad \underbrace{+ P(m_{(-1)1}) \otimes \lambda m_{(-1)2} \otimes m_{(0)}} + m_{(-1)} \otimes \rho'(T(m_{(0)})) \\
&\quad \underbrace{+ \lambda m_{(-1)1} \otimes P(m_{(-1)2}) \otimes m_{(0)}} + \lambda m_{(-1)1} \otimes m_{(-1)2} \otimes T(m_{(0)}) \\
&\quad + \underbrace{\lambda m_{(-1)1} \otimes \lambda m_{(-1)2} \otimes m_{(0)}} \\
&= P(m_{(-1)1}) \otimes P(m_{(-1)2}) \otimes m_{(0)} + P(m_{(-1)1}) \otimes m_{(-1)2} \otimes T(m_{(0)}) \\
&\quad + \underbrace{\Delta'(m_{(-1)}) \otimes \lambda m_{(0)}} + m_{(-1)} \otimes \rho'(T(m_{(0)})) \\
&\quad + \lambda m_{(-1)1} \otimes m_{(-1)2} \otimes T(m_{(0)}) \\
&= P(m_{(-1)1}) \otimes P(m_{(-1)2}) \otimes m_{(0)} + P(m_{(-1)1}) \otimes m_{(-1)2} \otimes T(m_{(0)}) \\
&\quad + \Delta'(m_{(-1)}) \otimes \lambda m_{(0)} + m_{(-1)} \otimes (P \otimes T)\rho(m_{(0)}) \\
&\quad + \lambda m_{(-1)1} \otimes m_{(-1)2} \otimes T(m_{(0)}) \\
&= P(m_{(-1)1}) \otimes P(m_{(-1)2}) \otimes m_{(0)} \underbrace{+ P(m_{(-1)1}) \otimes m_{(-1)2} \otimes T(m_{(0)})} \\
&\quad + \Delta'(m_{(-1)}) \otimes \lambda m_{(0)} + \underbrace{m_{(-1)1} \otimes P(m_{(-1)2}) \otimes T(m_{(0)})} \\
&\quad + \underbrace{\lambda m_{(-1)1} \otimes m_{(-1)2} \otimes T(m_{(0)})} \\
&= P(m_{(-1)1}) \otimes P(m_{(-1)2}) \otimes m_{(0)} + \underbrace{\Delta'(m_{(-1)}) \otimes T(m_{(0)})} \\
&\quad + \Delta'(m_{(-1)}) \otimes \lambda m_{(0)} \\
&= \Delta' P(m_{(-1)}) \otimes m_{(0)} + \Delta'(m_{(-1)}) \otimes T(m_{(0)}) + \Delta'(m_{(-1)}) \otimes \lambda m_{(0)} \\
&= (\Delta' \otimes \mathrm{id})((P \otimes \mathrm{id})\rho(m) + (\mathrm{id} \otimes T)\rho(m) + \lambda\rho(m)) \\
&= (\Delta' \otimes \mathrm{id})\rho'(m).
\end{aligned}
$$

(d) By (b) and (c), it is sufficient to prove that

$$
\begin{aligned}
&(P \otimes T)\rho'(m) \\
&\quad = (P \otimes T)(P(m_{(-1)}) \otimes m_{(0)} + m_{(-1)} \otimes T(m_{(0)}) + \lambda m_{(-1)} \otimes m_{(0)}) \\
&\quad = P^2(m_{(-1)}) \otimes T(m_{(0)}) + P(m_{(-1)}) \otimes T^2(m_{(0)}) + \lambda P(m_{(-1)}) \otimes T(m_{(0)}) \\
&\quad = (P \otimes \mathrm{id} + \mathrm{id} \otimes T + \lambda)(P \otimes T)\rho(m) \\
&\quad = (P \otimes \mathrm{id} + \mathrm{id} \otimes T + \lambda)\rho' T(m) \\
&\quad = (P \otimes \mathrm{id})\rho' T(m) + (\mathrm{id} \otimes T)\rho' T(m) + \lambda \rho' T(m)
\end{aligned}
$$

for any $m \in M$, so (d) holds. ∎

COROLLARY 4.13. *Let (M, P, T) be a Rota–Baxter paired C-comodule of weight λ. Then $(M, \overline{P}, \overline{T})$ is also a Rota–Baxter paired (C, Δ')-comodule of weight λ. Here the coaction ρ' of M and the comultiplication Δ' of C are defined in Proposition 4.12, and $\overline{P}, \overline{T}$ are defined in Proposition 4.11.*

Proof. Apply Propositions 4.11 and 4.12. ∎

5. From Rota–Baxter paired comodules to pre-Lie comodules. In this section, we mainly construct pre-Lie comodules from Rota–Baxter paired comodules.

DEFINITION 5.1. A *pre-Lie coalgebra* is a pair (C, Δ) consisting of a linear space C and a linear map $\Delta : C \to C \otimes C$ satisfying

$$\Delta_C - \Phi_{(12)}\Delta_C = 0,$$

where $\Delta_C = (\Delta \otimes \mathrm{id})\Delta - (\mathrm{id} \otimes \Delta)\Delta$ and $\Phi_{(12)}(c_1 \otimes c_2 \otimes c_3) = c_2 \otimes c_1 \otimes c_3$.

DEFINITION 5.2. Let (C, Δ) be a pre-Lie coalgebra. A *left C-pre-Lie comodule* (M, ρ) is a space M together with a map $\rho : M \to C \otimes M$ such that

$$\rho_M - (\tau \otimes \mathrm{id})\rho_M = 0,$$

where $\rho_M = (\mathrm{id} \otimes \rho)\rho - (\Delta \otimes \mathrm{id})\rho$, and $\tau(c \otimes d) = d \otimes c$ for any $c, d \in C$.

LEMMA 5.3 (see [ML]). *Let (C, Q) be a Rota–Baxter coalgebra of weight -1. Define an operation $\widetilde{\Delta}$ on C by*

$$\widetilde{\Delta}(c) = Q(c_1) \otimes c_2 - Q(c_2) \otimes c_1 - c_1 \otimes c_2.$$

Then $\widetilde{C} = (C, \widetilde{\Delta})$ is a pre-Lie coalgebra.

PROPOSITION 5.4. *Let (C, Q) be a Rota–Baxter coalgebra of weight -1, and (M, Q, T) a Rota–Baxter paired C-comodule of weight -1. Define a map $\widetilde{\rho} : M \to C \otimes M$ by*

$$\widetilde{\rho}(m) = Q(m_{(-1)}) \otimes m_{(0)} + m_{(-1)} \otimes T(m_{(0)}) - m_{(-1)} \otimes m_{(0)}.$$

Then $(M, \widetilde{\rho})$ is a left $\widetilde{C}$-pre-Lie comodule, where $\widetilde{C}$ is defined as in Lemma 5.3.

Proof. By Lemma 5.3, we know $\widetilde{C} = (C, \widetilde{\Delta})$ is a pre-Lie coalgebra, so we only need to prove that $\widetilde{\rho}_M - (\tau \otimes \mathrm{id})\widetilde{\rho}_M = 0$.

As a matter of fact, for any $m \in M$, we have

$$\begin{aligned}
\widetilde{\rho}_M(m) &= (\mathrm{id} \otimes \widetilde{\rho})\widetilde{\rho}(m) - (\widetilde{\Delta} \otimes \mathrm{id})\widetilde{\rho}(m) \\
&= (\mathrm{id} \otimes \widetilde{\rho})(Q(m_{(-1)}) \otimes m_{(0)} + m_{(-1)} \otimes T(m_{(0)}) - m_{(-1)} \otimes m_{(0)}) \\
&\quad - (\widetilde{\Delta} \otimes \mathrm{id})(Q(m_{(-1)}) \otimes m_{(0)} + m_{(-1)} \otimes T(m_{(0)}) - m_{(-1)} \otimes m_{(0)}) \\
&= Q(m_{(-1)}) \otimes \widetilde{\rho}(m_{(0)}) + m_{(-1)} \otimes \widetilde{\rho}(T(m_{(0)})) - m_{(-1)} \otimes \widetilde{\rho}(m_{(0)}) \\
&\quad - \widetilde{\Delta}(Q(m_{(-1)})) \otimes m_{(0)} - \widetilde{\Delta}(m_{(-1)}) \otimes T(m_{(0)}) \\
&\quad + \widetilde{\Delta}(m_{(-1)}) \otimes m_{(0)}
\end{aligned}$$

$$
\begin{aligned}
&= Q(m_{(-1)}) \otimes Q(m_{(0)(-1)}) \otimes m_{(0)(0)} + Q(m_{(-1)}) \otimes m_{(0)(-1)} \\
&\quad \otimes T(m_{(0)(0)}) - Q(m_{(-1)}) \otimes m_{(0)(-1)} \otimes m_{(0)(0)} \\
&\quad + m_{(-1)} \otimes Q(T(m_{(0)})_{(-1)}) \otimes T(m_{(0)})_{(0)} + m_{(-1)} \otimes T(m_{(0)})_{(-1)} \\
&\quad \otimes T(T(m_{(0)})_{(0)}) - m_{(-1)} \otimes T(m_{(0)})_{(-1)} \otimes T(m_{(0)})_{(0)} \\
&\quad - m_{(-1)} \otimes Q(m_{(0)(-1)}) \otimes m_{(0)(0)} - m_{(-1)} \otimes m_{(0)(-1)} \otimes T(m_{(0)(0)}) \\
&\quad + m_{(-1)} \otimes m_{(0)(-1)} \otimes m_{(0)(0)} - Q(Q(m_{(-1)})_1) \otimes Q(m_{(-1)})_2 \otimes m_{(0)} \\
&\quad + Q(Q(m_{(-1)})_2) \otimes Q(m_{(-1)})_1 \otimes m_{(0)} + Q(m_{(-1)})_1 \otimes Q(m_{(-1)})_2 \\
&\quad \otimes m_{(0)} - Q(m_{(-1)1}) \otimes m_{(-1)2} \otimes T(m_{(0)}) + Q(m_{(-1)2}) \otimes m_{(-1)1} \\
&\quad \otimes T(m_{(0)}) + m_{(-1)1} \otimes m_{(-1)2} \otimes T(m_{(0)}) + Q(m_{(-1)1}) \otimes m_{(-1)2} \\
&\quad \otimes m_{(0)} - Q(m_{(-1)2}) \otimes m_{(-1)1} \otimes m_{(0)} - m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)} \\
&= Q(m_{(-1)1}) \otimes Q(m_{(-1)2}) \otimes m_{(0)} + Q(m_{(-1)1}) \otimes m_{(-1)2} \otimes T(m_{(0)}) \\
&\quad - Q(m_{(-1)1}) \otimes m_{(-1)2} \otimes m_{(0)} + m_{(-1)} \otimes Q(T(m_{(0)})_{(-1)}) \\
&\quad \otimes T(m_{(0)})_{(0)} + m_{(-1)} \otimes T(m_{(0)})_{(-1)} \otimes T(T(m_{(0)})_{(0)}) - m_{(-1)} \\
&\quad \otimes T(m_{(0)})_{(-1)} \otimes T(m_{(0)})_{(0)} - m_{(-1)} \otimes Q(m_{(0)(-1)}) \otimes m_{(0)(0)} \\
&\quad - m_{(-1)1} \otimes m_{(-1)2} \otimes T(m_{(0)}) + m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)} \\
&\quad - Q(Q(m_{(-1)})_1) \otimes Q(m_{(-1)})_2 \otimes m_{(0)} + Q(Q(m_{(-1)})_2) \\
&\quad \otimes Q(m_{(-1)})_1 \otimes m_{(0)} + Q(m_{(-1)})_1 \otimes Q(m_{(-1)})_2 \otimes m_{(0)} \\
&\quad - Q(m_{(-1)1}) \otimes m_{(-1)2} \otimes T(m_{(0)}) + Q(m_{(-1)2}) \otimes m_{(-1)1} \\
&\quad \otimes T(m_{(0)}) + m_{(-1)1} \otimes m_{(-1)2} \otimes T(m_{(0)}) + Q(m_{(-1)1}) \otimes m_{(-1)2} \\
&\quad \otimes m_{(0)} - Q(m_{(-1)2}) \otimes m_{(-1)1} \otimes m_{(0)} - m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)} \\
&= Q(m_{(-1)1}) \otimes Q(m_{(-1)2}) \otimes m_{(0)} + m_{(-1)} \otimes Q(T(m_{(0)})_{(-1)}) \\
&\quad \otimes T(m_{(0)})_{(0)} + m_{(-1)} \otimes T(m_{(0)})_{(-1)} \otimes T(T(m_{(0)})_{(0)}) - m_{(-1)} \\
&\quad \otimes T(m_{(0)})_{(-1)} \otimes T(m_{(0)})_{(0)} - m_{(-1)} \otimes Q(m_{(0)(-1)}) \otimes m_{(0)(0)} \\
&\quad - Q(Q(m_{(-1)})_1) \otimes Q(m_{(-1)})_2 \otimes m_{(0)} + Q(Q(m_{(-1)})_2) \\
&\quad \otimes Q(m_{(-1)})_1 \otimes m_{(0)} + Q(m_{(-1)})_1 \otimes Q(m_{(-1)})_2 \otimes m_{(0)} \\
&\quad + Q(m_{(0)(-1)}) \otimes m_{(-1)} \otimes T(m_{(0)(0)}) - Q(m_{(0)(-1)}) \otimes m_{(-1)} \otimes m_{(0)(0)}.
\end{aligned}
$$

From the above equalities and (C, Q) being a Rota–Baxter coalgebra of weight -1, we easily prove that $\widetilde{\rho}_M(m) = (\tau \otimes \mathrm{id})\widetilde{\rho}_M(m)$ for any $m \in M$. Hence $(M, \widetilde{\rho})$ is a left $\widetilde{C}$-pre-Lie comodule. ∎

Lemma 5.5 (see [ML]). *Let (C, Δ, Q) be a Rota–Baxter coalgebra of weight 0. Define an operation $\widetilde{\Delta}$ on C by*

$$\widetilde{\Delta}(c) = Q(c_1) \otimes c_2 - Q(c_2) \otimes c_1.$$

Then $\widetilde{C} = (C, \widetilde{\Delta})$ is a pre-Lie coalgebra.

By applying Lemma 5.5, we can prove the following proposition in a similar way to Proposition 5.4.

PROPOSITION 5.6. *Let* (C, Q) *be a Rota–Baxter coalgebra of weight* 0, *and* (M, Q, T) *a Rota–Baxter paired* C*-comodule of weight* 0. *Define a map* $\widetilde{\rho} : M \to C \otimes M$ *by*

$$\widetilde{\rho}(m) = Q(m_{(-1)}) \otimes m_{(0)} + m_{(-1)} \otimes T(m_{(0)})$$

for any $m \in M$. *Then* $(M, \widetilde{\rho})$ *is a left* $\widetilde{C}$*-pre-Lie comodule, where* $\widetilde{C}$ *is defined as in Lemma* 5.5.

6. Rota–Baxter paired Hopf modules. In this section, we combine Rota–Baxter paired modules and Rota–Baxter paired comodules, and introduce the concept of Rota–Baxter paired Hopf modules, and prove the structure theorem for Rota–Baxter paired Hopf modules.

DEFINITION 6.1. Let H be a bialgebra, and M a left H-Hopf module. A triple (M, P, T) is called a *Rota–Baxter paired left* H*-Hopf module of weight* λ if (M, P, T) is both a Rota–Baxter paired left H-module of weight λ, and a Rota–Baxter paired left H-comodule of weight λ.

We call (M, T) a *generic Rota–Baxter paired left* H*-Hopf module of weight* λ if (M, T) is both a generic Rota–Baxter paired left H-module of weight λ and a generic Rota–Baxter paired left H-comodule of weight λ.

A *Rota–Baxter* H*-Hopf submodule* N of a Rota–Baxter paired H-Hopf-module (M, P, T) is an H-Hopf submodule of M such that $T(N) \subseteq N$. Then (N, P, T) is a Rota–Baxter paired H-Hopf module.

Let (M, P, T) and (M', P', T') be Rota–Baxter paired H-Hopf modules of the same weight λ. A *Rota–Baxter* H*-Hopf module map* $f : (M, P, T) \to (M', P', T')$ of weight λ is a Hopf module map such that $f \circ T = T' \circ f$.

EXAMPLE 6.2. (1) Let H be a bialgebra. Then H is an augmented coalgebra (there is a coalgebra map $u : k \to H$) and an augmented algebra (there is an algebra map $\varepsilon : H \to k$).

Define a map $P : H \to H$ given by $P(h) = \varepsilon(h)1_H$. Then, by Example 2.2, (H, P) is a Rota–Baxter coalgebra of weight -1, and a Rota–Baxter algebra of weight -1 by [ZGZ, Example 2.1]. So, (H, P, P) is a Rota–Baxter paired H-comodule of weight -1 and a Rota–Baxter paired H-module of weight -1.

It is obvious that H is a right H-Hopf module via its multiplication and its comultiplication. Hence (H, P, P) is a Rota–Baxter paired H-Hopf module of weight -1.

(2) Let H be a bialgebra, and (H, P, P) a Rota–Baxter bialgebra of weight (λ, λ) given in [ML], that is, (H, P) is both a Rota–Baxter algebra of weight λ, and a Rota–Baxter coalgebra of weight λ. Then (H, P, P) is

both a Rota–Baxter paired H-module of weight λ, whose action is given by the multiplication of H, and a Rota–Baxter paired H-comodule of weight λ, whose coaction is given by the comultiplication of H. So, (H, P, P) is a Rota–Baxter paired H-Hopf module of weight λ.

(3) Let H be a quantum commutative weak Hopf algebra and M a weak H-Hopf module. Then, by [ZGZ, Remark 3.17], $(M, \sqcap^L, T)$ is a Rota–Baxter paired H-module of weight -1, where T is given in Proposition 4.6.

Again by Proposition 4.6, $(M, \sqcap^L, T)$ is also a Rota–Baxter paired H-comodule of weight -1. So, $(M, \sqcap^L, T)$ is a Rota–Baxter paired H-Hopf module of weight -1.

In particular, $(H, \sqcap^L, \sqcap^L)$ is a Rota–Baxter paired H-Hopf module of weight -1 for every quantum commutative weak Hopf algebra H.

Combining Theorem 3.1 and [ZGZ, Theorem 2.4], we get the following result.

PROPOSITION 6.3. *Let H be a bialgebra and M a left H-Hopf module. Suppose that there is a Hopf module map T from M to M. Then the following are equivalent:*

(1) *(M, T) is a generic Rota–Baxter paired H-Hopf module of weight λ.*
(2) *There is a linear operator $P : H \to H$ such that (M, P, T) is a Rota–Baxter paired H-Hopf module of weight λ.*
(3) *T is quasi-idempotent of weight λ.*

EXAMPLE 6.4. Let H be a bialgebra and C a coalgebra. Then $H \otimes C$ is a left H-Hopf module via $h \cdot (g \otimes c) = hg \otimes c$ and $\rho(h \otimes c) = h_1 \otimes h_2 \otimes c$ for any $h, g \in H$ and $c \in C$. Define a map $T : H \otimes C \to H \otimes C$ by $T(h \otimes c) = h \otimes \varepsilon(c)e$, where $e \in C$ satisfies $\varepsilon(e) = 1$. It is not difficult to prove that $T^2 = T$ and T is a Hopf module map, so by Proposition 6.3, $(H \otimes C, T)$ is a generic Rota–Baxter paired H-Hopf module of weight -1.

Now we prove the structure theorem for generic Rota–Baxter paired Hopf modules.

THEOREM 6.5. *Let H be a Hopf algebra, (M, T) a generic Rota–Baxter paired H-Hopf module of weight λ and T a Hopf module map. Then there is an isomorphism*

$$(M, T) \cong (H \otimes M^{\mathrm{co}\,H}, T')$$

of generic Rota–Baxter left H-Hopf modules of weight λ, where T' is defined by

$$T'(h \otimes m) = h \otimes T(m), \qquad h \in H,\ m \in M^{\mathrm{co}\,H},$$

and $M^{\mathrm{co}\,H} = \{m \in M \mid \rho(m) = 1 \otimes m\}$, and $H \otimes M^{\mathrm{co}\,H}$ is a left H-Hopf module via $h \cdot (g \otimes m) = hg \otimes m$ and $\rho(h \otimes m) = h_1 \otimes h_2 \otimes m$ for any $h, g \in H$ and $m \in M^{\mathrm{co}\,H}$.

Proof. Since T is a left H-comodule map, we easily see that $T(m) \in M^{\mathrm{co}\,H}$ for $m \in M^{\mathrm{co}\,H}$. Hence T' is well defined.

According to [Sw], it is obvious that $H \otimes M^{\mathrm{co}\,H}$ is a left H-Hopf module.

First, by applying Proposition 6.3, we prove that $(H \otimes M^{\mathrm{co}\,H}, T')$ is a generic Rota–Baxter left H-Hopf module of weight λ.

As a matter of fact, for any $h \in H$ and $m \in M^{\mathrm{co}\,H}$, we have

$$T'^2(h \otimes m) = T'(h \otimes T(m)) = h \otimes T^2(m) = -\lambda h \otimes T(m) = -\lambda T'(h \otimes m),$$

so T' is quasi-idempotent of weight λ. Again, for any $h, g \in H$ and $m \in M^{\mathrm{co}\,H}$, we get

$$\begin{aligned} T'(h \cdot (g \otimes m)) &= T'(hg \otimes m) = hg \otimes T(m) = h \cdot T'(g \otimes m), \\ T'(h \otimes m)_{(-1)} \otimes T'(h \otimes m)_{(0)} &= (h \otimes T(m))_{(-1)} \otimes (h \otimes T(m))_{(0)} \\ &= h_1 \otimes (h_2 \otimes T(m)) = h_1 \otimes T'(h_2 \otimes m) \\ &= (h \otimes m)_{(-1)} \otimes T'((h \otimes m)_{(0)}), \end{aligned}$$

so T' is a left H-Hopf module map. Hence $(H \otimes M^{\mathrm{co}\,H}, T')$ is a generic Rota–Baxter left H-Hopf module of weight λ by Proposition 6.3.

According to [Sw, Theorem 4.1.1], we have an H-Hopf module isomorphism

$$\alpha : H \otimes M^{\mathrm{co}\,H} \to M, \quad h \otimes m \mapsto h \cdot m,$$

with inverse

$$\beta : M \to H \otimes M^{\mathrm{co}\,H}, \quad m \mapsto m_{(-1)} \otimes E_M(m_{(0)}),$$

where $E_M(m)$ is given by $S(m_{(-1)}) \cdot m_{(0)}$ for $m \in M$.

Moreover, for any $m \in M$, we obtain

$$\begin{aligned} T' \circ \beta(m) &= T'(m_{(-1)} \otimes E_M(m_{(0)})) = m_{(-1)} \otimes T(E_M(m_{(0)})) \\ &= m_{(-1)} \otimes T(S(m_{(0)(-1)}) \cdot m_{(0)(0)}) \\ &= m_{(-1)} \otimes S(m_{(0)(-1)}) \cdot T(m_{(0)(0)}) \\ &= m_{(-1)} \otimes S(T(m_{(0)})_{(-1)}) \cdot T(m_{(0)})_{(0)} = m_{(-1)} \otimes E_M(T(m_{(0)})) \\ &= T(m)_{(-1)} \otimes E_M(T(m)_{(0)}) = \beta \circ T(m), \end{aligned}$$

so $T' \circ \beta = \beta \circ T$. In a similar way, we prove that $\alpha \circ T' = T \circ \alpha$. Hence $(M, T) \cong (H \otimes M^{\mathrm{co}\,H}, T')$ as generic Rota–Baxter left H-Hopf modules of weight λ. ■

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