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A NEW APPROACH TO ROTA–BAXTER COALGEBRAS

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Abstract. Rota–Baxter algebras are a useful tool applied in many branches of mathematics. However, it is difficult to construct examples of Rota–Baxter coalgebras. In this paper, two new approaches to Rota–Baxter coalgebras of weight -1 are introduced, via Hopf coquasigroup theory and Hopf π-algebra theory. In order to do so, the notions of Rota–Baxter linear equation system, Hopf quasicomodule coalgebra, and H_p-Hopf module coalgebra are introduced and discussed. We present numerous new examples of Rota–Baxter coalgebras. Our ideas of constructing such examples may be viewed as a guide to further development.

1. Introduction. Rota–Baxter algebras were introduced in [18] in the context of differential operators on commutative Banach algebras. At present, Rota–Baxter algebras are a useful tool applied in many branches of mathematics, such as combinatorics, Loday type algebras, pre-Lie algebras and pre-Poisson algebras [1, 2, 11], multiple zeta values [7, 8], quantum field theory [4] and other mathematical objects [9, 12, 13, 15, 19, 28, 29]. Rota–Baxter coalgebras [14] are the dual structures to Rota–Baxter algebras. However, construction of Rota–Baxter coalgebras remains an interesting open problem.

We recall that Hopf coquasigroups [10] are generalizations of Hopf algebras [21] that are not required to be coassociative. Hopf quasicomodules and Yetter–Drinfel'd quasicomodules over Hopf coquasigroups were introduced in [9]. Hopf π-algebras [22, 23] are generalizations of Hopf algebras over a family of linear spaces. In this paper, we show that Hopf quasicomodules, Yetter–Drinfel'd quasicomodules and Hopf π-algebras are useful tools in constructing Rota–Baxter coalgebras of weight -1.

The paper is organized as follows.

In Section 1 we recall and investigate some basic definitions and properties of Hopf coquasigroups and Hopf π-algebras.

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In Section 2 we introduce the notion of the Rota–Baxter linear equation system and construct some interesting examples. We show that any solution of the Rota–Baxter linear equation system leads to a Rota–Baxter coalgebra of weight -1.

In Section 3 we present a technique of (i) constructing Rota–Baxter coalgebras of weight -1 by applying Hopf quasicomodule quasicoalgebras and (ii) constructing Hopf quasicomodule quasicoalgebras by means of Yetter–Drinfel'd quasicomodule coalgebras.

In Section 4 we illustrate the construction of Rota–Baxter coalgebras of weight -1 by means of Hopf π-algebras.

2. Preliminaries. Throughout the paper we fix a ground field $\mathbb{F}$ of characteristic 0 and we freely use the Hopf algebra and coalgebra terminology introduced in the monographs [5, 17, 20, 21]. We assume that all linear spaces, maps and tensor products are over $\mathbb{F}$. We use Sweedler's notation to express the coproduct of a coalgebra C as $\Delta(c) = \sum c_1 \otimes c_2$ ([21]). For a left (resp. right) C-comodule coaction $\rho^l : M \to C \otimes M$ (resp. $\rho^r : M \to M \otimes C$) on M, we write $\rho^l(m) = \sum m_{(-1)} \otimes m_0$ (resp. $\rho^r(m) = \sum m_0 \otimes m_{(1)}$) for any $m \in M$. Let U, V, W be linear spaces, and $g : U \to V$ and $f : V \to W$ be linear maps. Then we write simply $fg : U \to W$ for the composite map $f \circ g$ from U to W.

2.1. Algebras, coalgebras, nonassociative bialgebras. Recall from [16] that an *algebra* (A, m) is a linear space A equipped with a map $m : A \otimes A \to A$. A *unital algebra* (A, m, μ) is a linear space A equipped with two maps $m : A \otimes A \to A$ and $\mu : \mathbb{F} \to A$ such that $m(\mathrm{id} \otimes \mu) = \mathrm{id} = m(\mu \otimes \mathrm{id})$. The algebra (A, m, μ) is called *associative* if $m(\mathrm{id} \otimes m) = m(m \otimes \mathrm{id})$.

A *coalgebra* (C, Δ) is a linear space C equipped with a map $\Delta : C \to C \otimes C$. A counital coalgebra (C, Δ, ε) is a linear space C equipped with two maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{F}$ such that $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$. The coalgebra (C, Δ, ε) is called *coassociative* if $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$.

A *nonunital noncounital bialgebra* (B, Δ, m) is an algebra (B, m) and a coalgebra (B, Δ) such that Δ is an algebra homomorphism. A counital bialgebra $(B, \Delta, \varepsilon, m)$ is a counital coalgebra (B, Δ, ε) and an algebra (B, m) such that Δ is an algebra homomorphism and $\varepsilon m = \varepsilon \otimes \varepsilon$.

A *unital bialgebra* (B, Δ, m, μ) is a coalgebra (B, Δ) and a unital algebra (B, m, μ) such that Δ is an algebra homomorphism and preserves μ. A unital counital bialgebra $(B, \Delta, \varepsilon, m, \mu)$ is both a unital bialgebra (B, Δ, m, μ) and a counital bialgebra $(B, \Delta, \varepsilon, m)$ such that $\varepsilon\mu = \mu$.

2.2. Hopf coquasigroups and Hopf quasicomodules. Recall that a *quasigroup* (or "IP loop") is a nonempty set G with a product, an identity e

and with the property that for each $g \in G$ there is $g^{-1} \in G$ such that

$$g^{-1}(gh) = h, \qquad (hg)g^{-1} = h$$

for any $h \in G$. It is easy to see that in any quasigroup G one has unique inverses and

$$(g^{-1})^{-1} = g, \qquad (gh)^{-1} = h^{-1}g^{-1}$$

for any $g, h \in G$.

We first recall from [10] the following notions.

A *Hopf coquasigroup* is a unital associative algebra H equipped with algebra homomorphisms $\Delta : H \to H \otimes H$, $\varepsilon : H \to \mathbb{F}$ forming a possibly non-coassociative but counital coalgebra and armed with an $\mathbb{F}$-map $S : H \to H$ such that

$$\sum S(h_1)h_{21} \otimes h_{22} = 1 \otimes h = \sum h_1 S(h_{21}) \otimes h_{22},$$
$$\sum h_{11} \otimes h_{12}S(h_2) = h \otimes 1 = \sum h_{11} \otimes S(h_{12})h_2,$$

for any $h \in H$.

The following examples were investigated in [9, 10].

Example 2.1. (1) Any Hopf algebra is a Hopf coquasigroup. Conversely, any coassociative Hopf coquasigroup is a Hopf algebra.

(2) Let G be a finite quasigroup. Then $H = \mathbb{F}G$ is a Hopf quasigroup with linear extension of the product and $\Delta(h) = h \otimes h$, $\varepsilon(h) = 1$ and $S(h) = h^{-1}$ on the basis elements $h \in G$. The linear dual vector space $\mathbb{F}G^*$ is a Hopf coquasigroup.

(3) Let A be a Hopf coquasigroup equipped with a linear action of a group G such that $g \cdot (ab) = (g \cdot a)(g \cdot b)$, $g \cdot 1 = g$, $(g \otimes g) \cdot \Delta(a) = \Delta(g \cdot a)$, $\varepsilon(g \cdot a) = \varepsilon(a)$ and $g \cdot (h \cdot a) = (gh) \cdot a$ for all $g, h \in G$ and $a, b \in A$. The cross product algebra $A \rtimes G$ is again a Hopf coquasigroup.

(4) Let H and B be Hopf coquasigroups. Then $H \otimes B$ is a Hopf coquasigroup with canonical tensor multiplication and tensor comultiplication.

The following notions and examples were investigated in [9].

Definition 2.2. Let H be a Hopf coquasigroup and M a linear space with a linear map $\rho^r : M \to M \otimes H$ (resp. $\rho^l : M \to H \otimes M$) such that $(\mathrm{id} \otimes \varepsilon)\rho^r = \mathrm{id}$ (resp. $(\varepsilon \otimes \mathrm{id})\rho^l = \mathrm{id}$). Then M is called a right (resp. left) *H-quasicomodule* if

$$\sum m_{00} \otimes m_{0(1)}S(m_{(1)}) = m \otimes 1 = \sum m_{00} \otimes S(m_{0(1)})m_{(1)},$$

resp.

$$\sum (S(m_{(-1)}))m_{0(-1)} \otimes m_{00} = 1 \otimes m = \sum m_{(-1)}S(m_{0(-1)}) \otimes m_{00}),$$

for any $h \in H$ and $m \in M$. Furthermore, let M be an associative and unital right (resp. left) H-module with action φ (usually denoted by $\cdot$). Then M is called a right (resp. left) *H-Hopf quasicomodule* if

$$\rho^r(m\cdot h)=\sum m_0\cdot h_1\otimes m_{(1)}h_2, \tag{1.1}$$

resp.

$$\rho^l(h\cdot m)=\sum h_1m_{(-1)}\otimes h_2\cdot m_0,$$

for any $h\in H$ and $m\in M$.

In what follows, H is a Hopf coquasigroup.

EXAMPLE 2.3. (1) Let M be a linear space. Then $M\otimes H$ is a right H-Hopf quasicomodule by setting

$$(m\otimes h)\cdot g=m\otimes hg,\qquad \rho^r_{M\otimes H}(m\otimes h)=\sum(m\otimes h_1)\otimes h_2,$$

for any $g,h\in H$ and $m\in M$. In particular H can be viewed as a right H-Hopf quasicomodule.

(2) Let M be a linear space. Then $H\otimes M$ is a left H-Hopf quasicomodule by setting

$$g\cdot(h\otimes m)=gh\otimes m,\qquad \rho^r_{H\otimes M}(h\otimes m)=\sum h_1\otimes h_2\otimes m,$$

for any $g,h\in H$ and $m\in M$. In particular, H can be viewed as a left H-Hopf quasicomodule.

LEMMA 2.4. *Let H be a Hopf coquasigroup and M a right (resp. left) H-Hopf quasicomodule, and $Q_R(m)=\sum m_0\cdot S(m_{(1)})$ (resp. $Q_L(m)=\sum S(m_{(-1)})\cdot m_0$). Then*

$$Q_R(m\cdot h)=Q_R(m)\varepsilon(h),\qquad (Q_R\otimes \mathrm{id})\rho^r(m\cdot h)=\sum Q_R(m_0)\otimes m_{(1)}h, \tag{1.2}$$

resp.

$$Q_L(h\cdot m)=\varepsilon(h)Q_L(m),\qquad (\mathrm{id}\otimes Q_L)\rho^l(h\cdot m)=\sum hm_{(-1)}\otimes Q_L(m_0),$$

for any $h\in H$ and $m\in M$.

Proof. Since M has the usual associative module structure, the proof is straightforward. ∎

EXAMPLE 2.5. (1) If $(M,\cdot,\rho^r)$ is a right H-Hopf quasicomodule, then $(M,\cdot,\phi^r)$ is a right H-Hopf quasicomodule with

$$\phi^r(m)=\sum Q_R(m_0)\cdot m_{(1)1}\otimes m_{(1)2}\quad \text{for any } m\in M.$$

(2) If $(M,\cdot,\rho^l)$ is a left H-Hopf quasicomodule, then $(M,\cdot,\phi^l)$ is a left H-Hopf quasicomodule with

$$\phi^l(m)=\sum m_{(-1)1}\otimes m_{(-1)2}\cdot Q_L(m_0)\quad \text{for any } m\in M.$$

The following notions were introduced in [3, 9].

Let H be a Hopf coquasigroup and M an associative and unital left H-module with action $\cdot$ and a left H-quasicomodule with coaction ρ^l. Then M is called a left *H-Yetter–Drinfel'd quasicomodule* (briefly, H-$\mathcal{YD}$-quasicomodule) if

$$\sum h_1 m_{(-1)} \otimes h_2 \cdot m_0 = \sum (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_0, \tag{1.3}$$

$$\sum h_1 \cdot m \otimes h_{21} \otimes h_{22} = \sum h_{11} \cdot m \otimes h_{12} \otimes h_2, \tag{1.4}$$

$$\sum h_1 \otimes h_{21} \cdot m \otimes h_{22} = \sum h_{11} \otimes h_{12} \cdot m \otimes h_2, \tag{1.5}$$

for any $h \in H$ and $m \in M$. The equality (1.3) is equivalent to

$$\sum (h \cdot m)_{(-1)} \otimes (h \cdot m)_0 = \sum h_{11} m_{(-1)} S(h_2) \otimes h_{12} \cdot m_0.$$

Recall from [9, Definition 3.8] that a *coquasitriangular Hopf coquasigroup* B is a Hopf coquasigroup endowed with a linear map $\sigma : B \otimes B \to \mathbb{F}$ such that σ is convolution invertible in $\mathrm{Hom}_{\mathbb{F}}(B \otimes B, \mathbb{F})$ and the following conditions are satisfied:

$$\sigma(xy, z) = \sum \sigma(y, z_1) \sigma(x, z_2), \tag{QCT1}$$

$$\sigma(x, yz) = \sum \sigma(x_1, y) \sigma(x_2, z), \tag{QCT2}$$

$$\sum \sigma(x_1, y_1) y_2 x_2 = \sum x_1 y_1 \sigma(x_2, y_2), \tag{QCT3}$$

$$\sum x_1 \otimes x_{22} \sigma(x_{21}, y) = \sum x_{11} \otimes x_2 \sigma(x_{12}, y), \tag{QCT4}$$

$$\sum \sigma(x_1, y) x_{21} \otimes x_{22} = \sum \sigma(x_{11}, y) x_{12} \otimes x_2, \tag{QCT5}$$

for any $x, y, z \in B$.

Similarly, recall from [9, Definition 3.6] that a Hopf coquasigroup is defined to be *quasitriangular* if there exists an invertible element $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ such that

$$(\Delta \otimes \mathrm{id})(R) = R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}, \tag{QQT1}$$

$$(\mathrm{id} \otimes \Delta)(R) = R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}, \tag{QQT2}$$

$$R \Delta(h) = \Delta^{\mathrm{cop}}(h) R, \tag{QQT3}$$

for any $h \in H$.

EXAMPLE 2.6. (1) Let (B, σ) be a coquasitriangular Hopf coquasigroup and M a left B-quasicomodule. Then M is a left B-$\mathcal{YD}$-quasicomodule with the left B-module structure

$$x \cdot m = \sigma(x, m_{(-1)}) \otimes m_0 \quad \text{for any } x \in B \text{ and } m \in M.$$

(2) Let (H,R) be a quasitriangular Hopf coquasigroup and M an associative and unital left H-module which satisfy

$$\sum h_1\cdot m\otimes h_{21}\otimes h_{22}=\sum h_{11}\cdot m\otimes h_{12}\otimes h_2,$$
$$\sum h_1\otimes h_{21}\cdot m\otimes h_{22}=\sum h_{11}\otimes h_{12}\cdot m\otimes h_2,$$

for any $h\in H$ and $m\in M$. Then M is a left H-$\mathcal{YD}$-quasicomodule with the left H-quasicomodule structure $\rho^l(m)=\sum R^{(2)}\otimes R^{(1)}\cdot m$ for any $m\in M$.

2.3. Hopf π-algebras. The following notions were introduced in [22–26].

DEFINITION 2.7. Let π be a group. A *Hopf π-algebra* is a family $H=\{H_p\}_{p\in\pi}$ of $\mathbb{F}$-spaces endowed with the following data:

1. A family $m=\{m_{p,q}:H_p\otimes H_q\to H_{pq}\}_{p,q\in\pi}$ of $\mathbb{F}$-maps (the multiplication) and an $\mathbb{F}$-map $\mu:\mathbb{F}\to H_e$ (the unit) such that
$$h(gf)=(hg)f,\quad 1h=h=h1,$$
for any $p,q,r\in\pi$, $h\in H_p$, $g\in H_q$, and $f\in H_r$.
2. The triple $(H_p,\Delta_p,\varepsilon_p)$ is a coassociative counital coalgebra, for any $p\in\pi$.
3. $\{\Delta_{p,q}\}$ and ε are algebra maps, i.e.
$$\sum(hg)_{(1,pq)}\otimes(hg)_{(2,pq)}=\sum h_{(1,p)}g_{(1,q)}\otimes h_{(2,p)}g_{(2,q)},$$
$$\Delta_e(1)=1\otimes1,\qquad \varepsilon_p(hf)=\varepsilon_p(h)\varepsilon_p(f),\qquad \varepsilon_e(1)=1_{\mathbb{F}},$$
for any $p,q\in\pi$, $h,f\in H_p$, and $g\in H_q$.
4. A family $S=\{S_p:H_p\to H_{p^{-1}}\}_{p\in\pi}$ of $\mathbb{F}$-maps (the antipode) such that
$$\sum S_p(h_{(1,p)})h_{(2,p)}=\varepsilon_p(h)1=\sum h_{(1,p)}S_p(h_{(2,p)})$$
for any $p\in\pi$ and $h\in H_p$.

Note that $(H_e,m_{e,e},1_e,\Delta_{e,e},\varepsilon,S_e)$ is the usual Hopf algebra.

Let H be a Hopf π-algebra. Then the following equalities hold:

1. $S_{pq}(hg)=S_q(g)S_p(h)$ for any $p,q\in\pi$, $h\in H_p$ and $g\in H_q$.
2. $S_e(1)=1$.
3. We have
$$\sum S_p(h)_{(1,p^{-1})}\otimes S_p(h)_{(2,p^{-1})}=\sum S_p(h_{(2,p)})\otimes S_p(h_{(1,p)})$$
for any $p\in\pi$ and $h\in H_p$.
4. We have

$$\varepsilon_{p^{-1}}S_p(h)=\varepsilon_p(h)\quad\text{for any } p\in\pi \text{ and } h\in H_p. \tag{1.6}$$

EXAMPLE 2.8. (1) Recall from [22, 23] that a Hopf group-coalgebra $H = \{H_p\}_{p\in\pi}$ over a group π is a group-coalgebra $H = \{H_p, \Delta = \{\Delta_{p,q}\}\}_{p,q\in\pi}$ and a family of algebras $\{H_p, m_p, \eta_p\}_{p\in\pi}$ endowed with an antipode $S = \{S_p : H_p \to H_{p^{-1}}\}_{p\in\pi}$ such that

$$\begin{cases} \sum h_{(1,p)} \otimes \Delta_{q,r}(h_{(2,qr)}) = \Delta_{p,q}(h_{(1,pq)}) \otimes h_{(2,r)}, \quad \forall h \in H_{pqr}, \\ \Delta_{p,q}(hg) = \Delta_{p,q}(h)\Delta_{p,q}(g), \quad \Delta_{p,q}(1_{pq}) = 1_p \otimes 1_q, \quad \forall h, g \in H_{pq}, \\ \varepsilon(hg) = \varepsilon(h)\varepsilon(g), \quad \varepsilon(1_e) = 1, \quad \forall h, g \in H_e, \\ \sum h_{(1,p)}\varepsilon(h_{(2,e)}) = h = \sum \varepsilon(h_{(1,e)})h_{(2,p)}, \quad \forall h \in H_p, \\ \sum S_{p^{-1}}(h_{(1,p^{-1})})h_{(2,p)} = \varepsilon(h)1_p = \sum h_{(1,p)}S_{p^{-1}}(h_{(2,p^{-1})}), \quad \forall h \in H_e. \end{cases}$$

Let $H = \{H_p, \Delta = \{\Delta_{p,q}\}\}_{p,q\in\pi}$ be a finite-type Hopf group-coalgebra. Then $H^* = \{H_p^*, \Delta^* = \{\Delta_{p,q}^*\}\}_{p,q\in\pi}$ is a Hopf π-algebra.

(2) The reader is referred to [22–26] for various examples of Hopf π-algebras.

3. Rota–Baxter linear equation system and Rota–Baxter coalgebras. Let $\lambda \in \mathbb{F}$. Recall from [14] that a pair (C, Q) is called a *Rota–Baxter coalgebra* of weight λ if C is a coassociative coalgebra and $Q : C \to C$ is a linear endomorphism of C satisfying

$$\sum Q(c_1) \otimes Q(c_2) = \sum Q(c)_1 \otimes Q(Q(c)_2) + Q(Q(c)_1) \otimes Q(c)_2 + \lambda Q(c)_1 \otimes Q(c)_2$$

for any $c \in C$.

The map Q is then called a *Rota–Baxter operator*. If $Q^2 = Q$, then (C, Q) is called *idempotent*.

Now we give the definition of the Rota–Baxter linear equation system:

DEFINITION 3.1. Let C be a coassociative coalgebra and $Q : C \to C$ a linear endomorphism of C. If the equations

$$\text{(RBE1)} \qquad \begin{aligned} \sum Q(c)_1 \otimes Q(c)_2 &= \sum Q(Q(c)_1) \otimes Q(c)_2, \\ \sum Q(c_1) \otimes Q(c_2) &= \sum Q(c)_1 \otimes Q(Q(c)_2), \end{aligned}$$

or

$$\text{(RBE2)} \qquad \begin{aligned} \sum Q(c)_1 \otimes Q(c)_2 &= \sum Q(c)_1 \otimes Q(Q(c)_2), \\ \sum Q(c_1) \otimes Q(c_2) &= \sum Q(Q(c)_1) \otimes Q(c)_2, \end{aligned}$$

are satisfied for any $c \in C$, then Q is said to be a solution of the *Rota–Baxter linear equation system*.

Lemma 3.2. *If Q is a solution of the Rota–Baxter linear equation system, then (C,Q) is a Rota–Baxter coalgebra of weight -1.*

Proof. Obvious. ∎

Example 3.3. Let C be a coassociative coalgebra.

(1) If there is an idempotent linear map $L : C \to C$ satisfying $L(c_1)\otimes c_2 = L(c)_1 \otimes L(c)_2$ for any $c \in C$, then L is a solution of (RBE1). If there is an idempotent linear map $R : C \to C$ satisfying $c_1 \otimes R(c_2) = R(c)_1 \otimes R(c)_2$ for any $c \in C$, then R is a solution of (RBE2).

(2) If $Q : C \to C$ is an idempotent coalgebra homomorphism, then Q is a solution of both (RBE1) and (RBE2). In particular, the identity map id_C of C is a solution of both (RBE1) and (RBE2).

Example 3.4. Let H be a Hopf coquasigroup. The natural map $Q_\varepsilon : H \to H$, $h \mapsto \varepsilon(h)1$, is a solution of both (RBE1) and (RBE2).

Proposition 3.5. *Let C be an ordinary bialgebra and H a Hopf algebra with an antipode S [21]. Suppose there are bialgebra homomorphisms $i : C \to H$ and $\pi : H \to C$ such that $i\pi = \mathrm{id}_H$. Then:*

(a) *The linear map $Q_L : C \to C$ defined by $Q_L(c) = \sum \pi Si(c_1)c_2$ is a solution of* (RBE1).
(b) *The linear map $Q_R : C \to C$ defined by $Q_R(c) = \sum c_1\pi Si(c_2)$ is a solution of* (RBE2).
(c) *(C, Q_L) and (C, Q_R) are idempotent Rota–Baxter idempotent Rota–Baxter coalgebras of weight -1.*

Proof. For any $c \in C$, we have

$$
\begin{aligned}
\sum Q_L(Q_L(c)_1) \otimes Q_L(c)_2 &= \sum Q_L((\pi Si(c_1)c_2)_1) \otimes (\pi Si(c_1)c_2)_2 \\
&= \sum Q_L(\pi Si(c_1)_1 c_{21}) \otimes \pi Si(c_1)_2 c_{22} \\
&= \sum Q_L(\pi Si(c_{12})c_{21}) \otimes \pi Si(c_{11})c_{22} \\
&= \sum \pi Si((\pi Si(c_{12})c_{21})_1)(\pi Si(c_{12})c_{21})_2 \otimes \pi Si(c_{11})c_{22} \\
&= \sum \pi Si(\pi Si(c_{122})c_{211})(\pi Si(c_{121})c_{212}) \otimes \pi Si(c_{11})c_{22} \\
&= \sum \pi Si(c_{211})\pi S^2 i(c_{122})\pi Si(c_{121})c_{212} \otimes \pi Si(c_{11})c_{22} \\
&= \sum \pi S(i(c_{121})Si(c_{122})i(c_{211}))c_{212} \otimes \pi Si(c_{11})c_{22} \\
&= \sum \pi Si(c_{12})c_{21} \otimes \pi Si(c_{11})c_{22} \\
&= \sum (\pi Si(c_1)c_2)_1 \otimes (\pi Si(c_1)c_2)_2 = \sum Q_L(c)_1 \otimes Q_L(c)_2
\end{aligned}
$$

and

$$\begin{aligned}\sum Q_L(c)_1\otimes Q_L(Q_L(c)_2) &= \sum(\pi Si(c_1)c_2)_1\otimes Q_L((\pi Si(c_1)c_2)_2)\\ &= \sum \pi Si(c_{12})c_{21}\otimes Q_L(\pi Si(c_{11})c_{22})\\ &= \sum \pi Si(c_{12})c_{21}\otimes \pi Si((\pi Si(c_{11})c_{22})_1)(\pi Si(c_{11})c_{22})_2\\ &= \sum \pi Si(c_{12})c_{21}\otimes \pi Si(\pi Si(c_{112})c_{221})(\pi Si(c_{111})c_{222})\\ &= \sum \pi Si(c_{12})c_{21}\otimes \pi Si(c_{221})\pi Si\pi Si(c_{112})\pi Si(c_{111})c_{222}\\ &= \sum \pi Si(c_{12})c_{21}\otimes \pi Si(c_{221})\varepsilon(i(c_{11}))c_{222}\\ &= \sum \pi Si(c_{11})c_{12}\otimes \pi Si(c_{21})c_{22} = \sum Q_L(c_1)\otimes Q_L(c_2).\end{aligned}$$

Consequently, Q_L is a solution of (RBE1) and (a) follows. The statement (b) follows in a similar way.

(c) Note that

$$\begin{aligned}Q_L^2(c) &= \sum Q_L(\pi Si(c_1)c_2)\\ &= \pi Si((\pi Si)(c_1)c_2)_1(\pi Si(c_1)c_2)_2\\ &= \sum \pi Si(\pi Si(c_{12})c_{21})(\pi Si(c_{11})c_{22})\\ &= \sum \pi Si(c_{21})\pi S^2 i(c_{12})\pi Si(c_{11})c_{22}\\ &= \sum \pi Si(c_1)c_2 = Q_L(c).\end{aligned}$$

It follows that (C, Q_L) is a Rota–Baxter coalgebra with weight -1. Since the proof for (C, Q_R) is analogous, the statement (c) is proved. ∎

EXAMPLE 3.6. Let H be a Hopf algebra and C a braided Hopf algebra in the category of H-$\mathcal{YD}$-modules. Let $C\star H$ be the Hopf algebra defined in [17] by setting

$$\begin{cases}(c\star h)(d\star g) = \sum c(h_1\cdot d)\star h_2 g,\\ 1_{C\star H} = 1_C\otimes 1_H,\quad \varepsilon_{C\star H} = \varepsilon_C\otimes\varepsilon_H,\\ \Delta_{C\star Q}(c\star h) = \sum(c_1\star c_{2(-1)}h_1)\otimes(c_{2(0)}\star h_2),\\ S_{C\star H}(c\star h) = (1\star S_H(c_{(-1)}h))(S(c_0)\star 1)\end{cases}$$

for any $c, d\in C$ and $h, g\in H$. Let π and i be the bialgebra homomorphisms $\pi: H\to C\star H$, $h\mapsto 1\star h$, and $i: C\star H\to H$, $c\star h\mapsto \varepsilon(c)h$. Since obviously $i\pi = \mathrm{id}_H$, by Proposition 3.5 the maps

$$\begin{aligned}&Q_L: C\star H\to C\star H, && Q_L(c\star h) = \sum(1\star S(c_{(-1)}h_1))(c_0\star h_2),\\ &Q_R: C\star H\to C\star H, && Q_R(c\star h) = c\star 1\varepsilon(h),\end{aligned}$$

with $c\in C$ and $h\in H$, define solutions of (RBE1) and (RBE2), respectively.

EXAMPLE 3.7. Let (H, R) be a quasitriangular Hopf algebra and V a finite-dimensional left H-module. We view the algebra $\mathrm{End}_{\mathbb{F}}(V)$ as a left H-module, with the structure

$$(h \bullet f)(v) = \sum h_1 \bullet (f(S(h_2) \cdot v))$$

for any $h \in H, v \in V$ and $f \in \mathrm{End}_{\mathbb{F}}(V)$. By [27], we have Radford's biproduct $C = U(\mathrm{End}_{\mathbb{F}}(V)^{(-)}) \star H$ with the following Hopf structures:

$$\begin{aligned}
&(f \star h)(g \star l) = \sum f(h_1 \bullet g_0) \star h_2 l, \\
&\Delta(f \star h) = \sum (f \star h_1) \otimes (1 \star h_2) + (1 \star R^{(2)} h_1) \otimes (R^{(1)} \bullet f \star h_2), \\
&S(f \star h) = \sum (S(R^{(2)}{}_2 h_2) \bullet S(R^{(1)} \bullet f) \star S(R^{(2)}{}_1 h_1)),
\end{aligned}$$

for any $f, g \in \mathrm{End}_{\mathbb{F}}(V)$ and $h, l \in H$. Then according to Proposition 3.5, we have two solutions of the Rota–Baxter linear equation system:

$$\begin{aligned}
&Q_L : C \to C, \quad Q_L(f \star h) = \sum (1 \star S(R^{(2)} h_1))(R^{(1)} \bullet f \star h_2), \\
&Q_R : C \to C, \quad Q_R(f \star h) = \varepsilon(h)[f \star 1],
\end{aligned}$$

for any $f \in \mathrm{End}_{\mathbb{F}}(V)$ and $h \in H$.

EXAMPLE 3.8. Let (B, σ) be a coquasitriangular Hopf algebra with a bijective antipode S and V a finite-dimensional left B-comodule. View the algebra $\mathrm{End}_{\mathbb{F}}(V)$ as a left B-comodule with the structure

$$\rho(a)(v) := \sum a(v_0)_{(-1)} S^{-1}(v_{(-1)}) \otimes f(v_0)_0$$

for any $v \in V$ and $a \in \mathrm{End}_{\mathbb{F}}(V)$. By [27], we have Radford's biproduct $K = U(\mathrm{End}_{\mathbb{F}}(V)^{(-)}) \star B$ with the following Hopf structures:

$$\begin{aligned}
&(a \star x)(b \star y) = \sum \sigma(x_1, b_{(-1)}) a b_0 \star x_2 y, \\
&\tilde{\Delta}(a \star x) = \sum (a \star x_1) \otimes (1 \star x_2) + (1 \star a_{(-1)} x_1) \otimes (a_0 \star x_2), \\
&\tilde{S}(a \star x) = \sum (1 \star S(a_{(-1)} x))(S(a_0) \star 1),
\end{aligned}$$

for any $a, b \in \mathrm{End}_{\mathbb{F}}(V)$ and $x, y \in B$. In this case, by a straightforward computation, we show that the maps

$$\begin{aligned}
&Q_L : K \to K, \quad Q_L(a \star x) = \sum (1 \star S(a_{(-1)} x_1))(a_0 \star x_2), \\
&Q_R : K \to K, \quad Q_R(a \star h) = \varepsilon(h)[a \star 1],
\end{aligned}$$

with $a \in \mathrm{End}_{\mathbb{F}}(V)$ and $x \in B$, define solutions of the Rota–Baxter linear equation system.

PROPOSITION 3.9. *Let C be a coassociative coalgebra. If P and Q are solutions of* (RBE1) (*resp.* (RBE2)) *such that $PQ = QP$, then PQ is also a solution of* (RBE1) (*resp.* (RBE2)).

Proof. For (RBE1), for any $c \in C$, we have

$$\begin{aligned}\sum PQ(c)_1 \otimes PQ(c)_2 &= \sum P(PQ(c)_1) \otimes PQ(c)_2\\ &= \sum P(QP(c)_1) \otimes QP(c)_2\\ &= \sum PQ(QP(c)_1) \otimes QP(c)_2\\ &= \sum PQ(PQ(c)_1) \otimes PQ(c)_2,\end{aligned}$$

and

$$\begin{aligned}\sum PQ(c_1) \otimes PQ(c_2) &= \sum P(Q(c)_1) \otimes P(Q(Q(c)_2))\\ &= \sum P(Q(c)_1) \otimes PQ(Q(c)_2)\\ &= \sum P(Q(c)_1) \otimes QP(Q(c)_2)\\ &= \sum PQ(c)_1 \otimes QP(PQ(c)_2)\\ &= \sum PQ(c)_1 \otimes PQ(PQ(c)_2).\end{aligned}$$

For (RBE2), for any $c \in C$, we have

$$\begin{aligned}\sum PQ(c)_1 \otimes PQ(c)_2 &= \sum PQ(c)_1 \otimes P(PQ(c)_2)\\ &= \sum QP(c)_1 \otimes P(QP(c)_2)\\ &= \sum QP(C)_1 \otimes PQ(QP(c)_2)\\ &= \sum PQ(c)_1 \otimes PQ(PQ(c)_2),\end{aligned}$$

and

$$\begin{aligned}\sum PQ(c_1) \otimes PQ(c_2) &= \sum PQ(Q(c)_1) \otimes P(Q(c)_2)\\ &= \sum QP(Q(c)_1) \otimes P(Q(c)_2)\\ &= \sum QP(PQ(c)_1) \otimes PQ(c)_2\\ &= \sum PQ(PQ(c)_1) \otimes PQ(c)_2. \blacksquare\end{aligned}$$

4. A new approach to Rota–Baxter linear equation systems by Hopf coquasigroup theory. In this section we present some methods of constructing Rota–Baxter coalgebras of weight -1 from Hopf quasicomodule coalgebras. Moreover, we construct new Hopf quasicomodule coalgebras by applying Yetter–Drinfel'd quasicomodule coalgebras.

We start with the following definition of Hopf quasicomodule coquasialgebra.

Definition 4.1. Let H be a Hopf coquasigroup. A right (resp. left) *H-Hopf quasicomodule coalgebra* is a coassociative coalgebra M which is a right (resp. left) H-Hopf quasicomodule such that

$$\sum (m\cdot h)_1 \otimes (m\cdot h)_2 = \sum m_1\cdot h_1 \otimes m_2\cdot h_2, \tag{1.1}$$

resp.

$$\sum (h\cdot m)_1 \otimes (h\cdot m)_2 = \sum h_1\cdot m_1 \otimes h_2\cdot m_2, \tag{1.2}$$

for any $h \in H$ and $m \in M$. Furthermore, M is called a right (resp. left) *H-Hopf quasicomodule coquasialgebra* if

$$\begin{aligned} \sum m_{01}\cdot S(m_{(1)}) \otimes Q_R(m_{02}) &= \sum Q_R(m)_1 \otimes Q_R(m)_2,\\ \sum Q_R(m_{01}) \otimes m_{02}\cdot S(m_{(1)}) &= \sum Q_R(m_1)\otimes Q_R(m_2), \end{aligned} \tag{1.3}$$

resp.

$$\begin{aligned} \sum Q_L(m)_1 \otimes Q_L(m)_2 &= \sum Q_L(m_{01}) \otimes S(m_{(-1)})\cdot m_{02},\\ \sum Q_L(m_1)\otimes Q_L(m_2) &= \sum S(m_{(-1)})\cdot m_{01} \otimes Q_L(m_{02}) \end{aligned} \tag{1.4}$$

for any $m \in M$.

Example 4.2. According to Lemma 2.4 and Example 1.3, we have $Q_R(m\otimes h) = m\otimes 1\varepsilon(h)$ and $Q_L(m\otimes h) = 1\otimes m\varepsilon(h)$ for any $m \in M$ and $h \in H$. If M is a coassociative coalgebra, then $M\otimes H$ (resp. $H\otimes M$) is a right (resp. left) H-Hopf quasicomodule coquasialgebra with canonical comultiplication. In particular, H can be viewed as a right and a left H-Hopf quasicomodule coquasialgebra since in this case we have $Q_R = Q_L = \varepsilon$.

By applying the definitions of Q_R and Q_L and the equation (1.2), we obtain

Lemma 4.3. *Let H be a Hopf coquasigroup and (M,ρ^r) (resp. (M,ρ^l)) be a right (resp. left) H-Hopf quasicomodule coalgebra. Then $Q_R^2 = Q_R$ and $Q_L^2 = Q_L$.*

Now we are able to prove the main result of this section:

Theorem 4.4. *If H is a Hopf coquasigroup and M a right (resp. left) H-Hopf quasicomodule coquasialgebra, then Q_R (resp. Q_L) is a solution of* (RBE2) *(resp.* (RBE1)*).*

Proof. For any $m \in M$, we have

$$\begin{aligned} Q_R(Q_R(m)_1)\otimes Q_R(m)_2 &= \sum Q_R((m_0\cdot S(m_{(1)})_1))\otimes (m_0\cdot S(m_{(1)}))_2\\ &\overset{(1.3)}{=} \sum Q_R(m_{01}\cdot S(m_{(1)2}))\otimes (m_{02}\cdot S(m_{(1)1}))\\ &\overset{(1.2)}{=} \sum Q_R(m_{01})\otimes m_{02}\cdot S(m_{(1)})\\ &\overset{(1.3)}{=} Q_R(m_1)\otimes Q_R(m_2) \end{aligned}$$

and

$$\begin{aligned}Q_R(m)_1\otimes Q_R(Q_R(m)_2) &= \sum(m_0\cdot S(m_{(1)}))_1\otimes Q_R((m_0\cdot S(m_{(1)}))_2)\\ &\overset{(1.3)}{=} \sum(m_{01}\cdot S(m_{(1)2})\otimes Q_R(m_{02}\cdot S(m_{(1)1}))\\ &\overset{(1.2)}{=} \sum m_{01}\cdot S(m_{(1)})\otimes Q_R(m_{02})\\ &\overset{(1.3)}{=} Q_R(m)_1\otimes Q_R(m)_2.\end{aligned}$$

Therefore, Q_R is a solution of (RBE2). By Lemma 2.2, we have

$$\begin{aligned}\sum Q_R(m_1)\otimes Q_R(m_2) = {} & \sum Q_R(m)_1\otimes Q_R(Q_R(m)_2)\\ & + Q_R(Q_R(m)_1)\otimes Q_R(m)_2 - Q_R(m)_1\otimes Q_R(m)_2.\end{aligned}$$

Note that a similar computation for Q_L with (1.4) gives rise to the following equalities:

$$\begin{aligned}Q_L(m)_1\otimes Q_L(m)_2 &= Q_L(Q_L(m)_1)\otimes Q_L(m)_2,\\ Q_L(m_1)\otimes Q_L(m_2) &= Q_L(m)_1\otimes Q_L(Q_L(m)_2).\end{aligned}$$

This shows that Q_L is a solution of (RBE1) and so

$$\begin{aligned}\sum Q_L(m_1)\otimes Q_L(m_2) = {} & \sum Q_L(m)_1\otimes Q_L(Q_L(m)_2)\\ & + Q_L(Q_L(m)_1)\otimes Q_L(m)_2 - Q_L(m)_1\otimes Q_L(m)_2.\end{aligned}$$

Consequently, (M, Q_R) (resp. (M, Q_L)) is an idempotent Rota–Baxter coalgebra of weight -1. ■

Definition 4.5. Let H be a Hopf coquasigroup. A left *H-$\mathcal{YD}$-quasicomodule coalgebra* is a left H-$\mathcal{YD}$-quasicomodule (M, φ, ρ^l) which is a coalgebra such that

$$\begin{aligned}\sum m_{(-1)}\otimes m_{01}\otimes m_{02} &= \sum m_{1(-1)}m_{2(-1)}\otimes m_{10}\otimes m_{20},\\ \sum(h\cdot m)_1\otimes(h\cdot m)_2 &= \sum h_1\cdot m_1\otimes h_2\cdot m_2,\end{aligned}$$

for any $h\in H$ and $m\in M$.

Definition 4.6. Let H be a Hopf coquasigroup and M a left H-$\mathcal{YD}$-quasicomodule coalgebra. The *smash coproduct coalgebra* $M\rtimes H$ is an $\mathbb{F}$-space $M\otimes H$ with

$$\Delta(m\rtimes h) = \sum m_1\rtimes m_{2(-1)}h_1\otimes m_{20}\rtimes h_2$$

for any $h\in H$ and $m\in M$.

Proposition 4.7. *Let H be a Hopf coquasigroup and M a left H-$\mathcal{YD}$-quasicomodule coalgebra. The coalgebra $M\rtimes H$ is a right H-Hopf quasicomodule coalgebra with*

$$(m\rtimes h)\cdot g = m\rtimes hg \quad \text{and} \quad \rho^r(m\rtimes h) = \sum m\rtimes h_1\otimes h_2$$

for any $h, g \in H$ and $m \in M$. Moreover $M \rtimes H$ is a right H-Hopf quasicomodule coquasialgebra with $Q_R : M \rtimes H \to M \rtimes H$ defined by

$$Q_R(m \rtimes h) = m \rtimes \varepsilon(h)1_H \quad \text{for any } h \in H \text{ and } m \in M.$$

Proof. In view of Example 2.3(1), $M \otimes H$ is a right H-Hopf quasicomodule and we have

$$\begin{aligned}\sum((m \rtimes h) \cdot g)_1 \otimes ((m \rtimes h) \cdot g)_2 &= \sum(m \rtimes hg)_1 \otimes (m \rtimes hg)_2 \\ &= \sum m_1 \rtimes m_{2(-1)}(hg)_1 \otimes m_{20} \rtimes (hg)_2 \\ &= \sum m_1 \rtimes m_{2(-1)}h_1g_1 \otimes m_{20} \rtimes h_2g_2 \\ &= \sum(m_1 \rtimes m_{2(-1)}h_1) \cdot g_1 \otimes (m_{20} \rtimes h_2) \cdot g_2 \\ &= \sum(m \rtimes h)_1 \cdot g_1 \otimes (m \rtimes h)_2 \cdot g_2.\end{aligned}$$

Now, we verify (1.3) for Q_R:

$$\begin{aligned}&\sum(m \rtimes h_1)_1 \cdot S(h_2) \otimes Q_R((m \rtimes h_1)_2) \\ &= \sum(m_1 \rtimes m_{2(-1)}h_{11}) \cdot S(h_2) \otimes Q_R(m_{20} \rtimes h_{12}) \\ &= \sum(m_1 \rtimes m_{2(-1)}h_{11}) \cdot S(h_2) \otimes m_{20} \rtimes 1\varepsilon(h_{12}) \\ &= \sum m_1 \rtimes m_{2(-1)} \otimes m_{20} \rtimes 1\varepsilon(h) \\ &= \sum(m \rtimes 1)_1 \otimes (m \rtimes 1)_2\varepsilon(h) \\ &= \sum Q_R(m \rtimes h)_1 \otimes Q_R(m \rtimes h)_2\end{aligned}$$

and

$$\begin{aligned}&\sum Q_R((m \rtimes h)_{01}) \otimes (m \rtimes h)_{02} \cdot S((m \rtimes h)_{(1)}) \\ &= \sum Q_R((m \rtimes h_1)_1) \otimes (m \rtimes h_1)_2 \cdot S(h_2) \\ &= \sum Q_R(m_1 \rtimes m_{2(-1)}h_{11}) \otimes (m_{20} \rtimes h_{12}) \cdot S(h_2) \\ &= \sum m_1 \rtimes 1 \otimes m_{20} \rtimes h_{12}S(h_2)\varepsilon(m_{2(-1)}h_{11}) \\ &= \sum m_1 \rtimes 1 \otimes m_2 \rtimes 1\varepsilon(h) \\ &= \sum Q_R(m_1 \rtimes m_{2(-1)}h_1) \otimes Q_R(m_{20} \rtimes h_2) \\ &= \sum Q_R((m \rtimes 1)_1) \otimes Q_R((m \rtimes h)_2).\end{aligned}$$

This shows that $M \rtimes H$ is a right H-Hopf quasicomodule coquasialgebra. ∎

PROPOSITION 4.8. *Let H be a Hopf coquasigroup and M a left H-$\mathcal{YD}$-quasicomodule coalgebra. The smash coproduct coalgebra $M \rtimes H$ with*

$$g \cdot (m \rtimes h) = \sum g_1 \cdot m \rtimes g_2h, \quad \rho^l(m \rtimes h) = \sum m_{(-1)}h_1 \otimes m_0 \rtimes h_2$$

is a left H-Hopf quasicomodule coalgebra.

Proof. It is obvious that $M \rtimes H$ is a left H-module. For any $m \rtimes h \in M \rtimes H$, we have

$$\begin{aligned}
&\sum S((m \rtimes h)_{(-1)})(m \rtimes h)_{0(-1)} \otimes (m \rtimes h)_{00} \\
&\qquad = \sum S(m_{(-1)}h_1)(m_0 \rtimes h_2)_{(-1)} \otimes (m_0 \rtimes h_2)_0 \\
&\qquad = \sum S(m_{(-1)}h_1)(m_{0(-1)}h_{21}) \otimes (m_{00} \rtimes h_{22}) \\
&\qquad = \sum S(h_1)S(m_{(-1)})m_{0(-1)}h_{21} \otimes m_{00} \rtimes h_{22} \\
&\qquad = \sum S(h_1)h_{21} \otimes m \rtimes h_{22} = 1 \otimes m \rtimes h.
\end{aligned}$$

Similarly, we obtain $\sum (m \rtimes h)_{(-1)} S((m \rtimes h)_{0(-1)}) \otimes (m \rtimes h)_{00} = 1 \otimes m \rtimes h$. Consequently, $M \rtimes H$ is a right H-quasicomodule.

Now we verify the compatibility condition:

$$\begin{aligned}
\rho^l(g \cdot (m \rtimes h)) &= \sum (g_1 \cdot m \rtimes g_2h)_{(-1)} \otimes (g_1 \cdot m \rtimes g_2h)_0 \\
&= \sum (g_1 \cdot m)_{(-1)}(g_2h)_1 \otimes (g_1 \cdot m)_0 \rtimes (g_2h)_2 \\
&= \sum (g_1 \cdot m)_{(-1)}g_{21}h_1 \otimes (g_1 \cdot m)_0 \rtimes g_{22}h_2 \\
&\overset{(1.4)}{=} (g_{11} \cdot m)_{(-1)}g_{12}h_1 \otimes (g_{11} \cdot m)_0 \rtimes g_2h_2 \\
&\overset{(1.3)}{=} \sum g_{11}m_{(-1)}h_1 \otimes g_{12} \cdot m_0 \rtimes g_2h_2 \\
&\overset{(1.5)}{=} \sum g_1(m_{(-1)}h_1) \otimes g_{21} \cdot m_0 \rtimes g_{22}h_2 \\
&= \sum g_1(m_{(-1)}h_1) \otimes g_2 \cdot (m_0 \rtimes h_2) \\
&= \sum g_1(m \rtimes h)_{(-1)} \otimes g_2 \cdot (m \rtimes h)_0 = \Delta(g)\rho^l(m \rtimes h).
\end{aligned}$$

Consequently, $M \rtimes H$ is a left H-Hopf quasicomodule and therefore it remains to show that $M \rtimes H$ is a left H-Hopf quasicomodule coalgebra. This is a consequence of the following equalities:

$$\begin{aligned}
\Delta(g \cdot (m \rtimes h)) &= \sum \Delta(g_1 \cdot m \rtimes g_2h) \\
&= \sum (g_1 \cdot m)_1 \rtimes (g_1 \cdot m)_{2(-1)}(g_2h)_1 \otimes (g_1 \cdot m)_{20} \rtimes (g_2h)_2 \\
&= \sum (g_1 \cdot m)_1 \rtimes (g_1 \cdot m)_{2(-1)}g_{21}h_1 \otimes (g_1 \cdot m)_{20} \rtimes g_{22}h_2 \\
&= \sum g_{11} \cdot m_1 \rtimes (g_{12} \cdot m_2)_{(-1)}g_{21}h_1 \otimes (g_{12} \cdot m_2)_0 \rtimes g_{22}h_2 \\
&\overset{(1.4)}{=} \sum g_1 \cdot m_1 \rtimes (g_{21} \cdot m_2)_{(-1)}g_{221}h_1 \otimes (g_{21} \cdot m_2)_0 \rtimes g_{222}h_2 \\
&\overset{(1.4)}{=} \sum g_1 \cdot m_1 \rtimes (g_{211} \cdot m_2)_{(-1)}g_{212}h_1 \otimes (g_{211} \cdot m_2)_0 \rtimes g_{22}h_2 \\
&\overset{(1.3)}{=} \sum g_1 \cdot m_1 \rtimes g_{211}m_{2(-1)}h_1 \otimes g_{212} \cdot m_{20} \rtimes g_{22}h_2 \\
&\overset{(1.5)}{=} \sum g_1 \cdot m_1 \rtimes g_{21}m_{2(-1)}h_1 \otimes g_{221} \cdot m_{20} \rtimes g_{222}h_2 \\
&\overset{(1.4)}{=} \sum g_{11} \cdot m_1 \rtimes g_{12}m_{2(-1)}h_1 \otimes g_{21} \cdot m_{20} \rtimes g_{22}h_2
\end{aligned}$$

$$= \sum g_1 \cdot (m_1 \rtimes m_{2(-1)} h_1) \otimes g_2 \cdot (m_{20} \rtimes h_2)$$
$$= \sum g_1 \cdot (m \rtimes h)_1 \otimes g_2 \cdot (m \rtimes h)_2. \ \blacksquare$$

EXAMPLE 4.9. Let H be a (co)quasitriangular Hopf coquasigroup and M a left H-$\mathcal{YD}$-quasicomodule coalgebra defined in Example 2.6. Then $M \rtimes H$ is a left H-Hopf quasicomodule coalgebra and a right H-Hopf quasicomodule coquasialgebra with $Q_R : M \rtimes H \to M \rtimes H$, $Q_R(m \rtimes h) = m \rtimes 1\varepsilon(h)$, for any $m \in M$ and $h \in H$.

5. A new approach to Rota–Baxter linear equation systems by Hopf π-algebra theory. In this section, we introduce the notion of H_p-Hopf modules and show that they are useful tools in constructing Rota–Baxter coalgebras of weight -1.

DEFINITION 5.1. Let H be a Hopf π-algebra and p an idempotent element of π. A right *H_p-Hopf module* (resp. left *H_p-Hopf module*) is defined to be an $\mathbb{F}$-space M endowed with the following data (i)–(iii):

(i) we have

$$(m \cdot h) \cdot g = m \cdot (hg) \quad (\text{resp. } h \cdot (g \cdot m) = (hg) \cdot m),$$

for any $m \in M$ and $h, g \in H_p$,

(ii) there is an $\mathbb{F}$-map $\rho_p^r : M \to M \otimes H_p$ (resp. $\rho_p^l : M \to H_p \otimes M$) such that

$$(\mathrm{id} \otimes \Delta_p)\rho_p^r = (\rho_p^r \otimes \mathrm{id})\rho_p^r \quad (\text{resp. } (\Delta_p \otimes \mathrm{id})\rho_p^l = (\mathrm{id} \otimes \rho_p^l)\rho_p^l) \tag{1.1}$$

for any $m \in M$ and we denote

$$\rho_p^r(m) = m_{(0)} \otimes m_{(1,p)} \quad (\text{resp. } \rho_p^l(m) = m_{(-1,p)} \otimes m_{(0)}),$$

(iii) the compatibility condition holds:

$$(m \cdot h)_{(0)} \otimes (m \cdot h)_{(1,p)} = m_{(0)} \cdot h_{(1,p)} \otimes m_{(1,p)} h_{(2,p)}, \tag{1.2}$$

resp.

$$(m \cdot h)_{(-1,p)} \otimes (m \cdot h)_{(0)}) = m_{(-1,p)} h_{(1,p)} \otimes m_{(0)} \cdot h_{(2,p)},$$

for all $h \in H_p$ and $m \in M$.

DEFINITION 5.2. Let H be a Hopf π-algebra. A right (resp. left) *H_p-Hopf module coalgebra* is defined to be a right (resp. left) H_p-Hopf module (M, φ, ρ^r) (resp. (M, ψ, ρ^l)) with comultiplication $\delta : M \to M \otimes M$ ($\delta(m) = m_1 \otimes m_2$) such that

$$\begin{aligned} &m_{(0)1} \otimes m_{(0)2} \otimes m_{(1,p)} = m_1 \otimes m_{2(0)} \otimes m_{2(1,p)}, \\ &\delta(m \cdot h) = m_1 \cdot h_{(1,p)} \otimes m_2 \cdot h_{(2,p)}, \end{aligned} \tag{1.3}$$

resp.

$$m_{(-1,p)} \otimes m_{(0)1} \otimes m_{(0)2} = m_{1(-1,p)} \otimes m_{1(0)} \otimes m_2,$$
$$\delta(h \cdot m) = h_{(1,p)} \cdot m_1 \otimes h_{(2,p)} \cdot m_2,$$

for any $h \in H_p$ and $m \in M$.

EXAMPLE 5.3. Let H be a Hopf π-algebra and p an idempotent element of π. Then H_p is an H_p-Hopf module with multiplication $m_{p,p}$ and comultiplication Δ_p.

LEMMA 5.4. *Let H be a Hopf π-algebra and M a right (resp. left) H_p-Hopf-module. For the map*

$$Q_R^p : M \to M, \quad m \mapsto \sum m_{(0)} \cdot S_p(m_{(1,p)}),$$

resp.

$$Q_L^p : M \to M, \quad m \mapsto \sum S_p(m_{(-1,p)}) \cdot m_{(0)},$$

the following equalities hold:

$$\sum Q_R^p(m \cdot h) = Q_R^p(m)\varepsilon_p(h), \quad Q_R^p Q_R^p(m) = Q_R^p(m), \tag{1.4}$$

resp.

$$\sum Q_L^p(h \cdot m) = Q_L^p(m)\varepsilon_p(h), \quad Q_L^p Q_L^p(m) = Q_L^p(m),$$

for any $m \in M$ and $h \in H_p$.

Proof. For all $m \in M$ and $h \in H_p$, we have

$$\begin{aligned} Q_R^p(m \cdot h) &= \sum (m \cdot h)_{(0)} \cdot S_p((m \cdot h)_{(1,p)}) \\ &\overset{(1.2)}{=} \sum (m_{(0)} \cdot h_{(1,p)}) \cdot S_p(m_{(1,p)} h_{(2,p)}) \\ &= \sum (m_{(0)} \cdot (h_{(1,p)} S_p(h_{(2,p)}))) \cdot S_p(m_{(1,p)}) \\ &= \sum m_{(0,e)} \cdot S_p(m_{(1,p)})\varepsilon_p(h) = Q_R^p(m)\varepsilon_p(h), \end{aligned}$$

and

$$\begin{aligned} Q_R^p(Q_R^p(m)) &= Q_R^p(m_{(0)} \cdot S_p(m_{(1,p)})) \\ &= Q_R^p(m_{(0)})\varepsilon_p S_p(m_{(1,p)}) = Q_R^p(m_{(0)})\varepsilon_p(m_{(1,p)}) = Q_R^p(m). \end{aligned}$$

By a similar calculation we obtain the result for Q_L^p. ∎

Now we are able to prove the following main theorem of this section:

THEOREM 5.5. *If H is a Hopf π-algebra and M is a right (resp. left) H_p-Hopf module coalgebra, then (M, Q_R^p) (resp. (M, Q_L^p)) is an idempotent Rota–Baxter coalgebra of weight -1.*

Proof. For all $p \in \pi$ and $m, n \in M$, we have

$$
\begin{aligned}
&Q_R^p(Q_R^p(m)_1) \otimes Q_R^p(m)_2 \\
&\qquad = Q_R^p((m_{(0)} \cdot S_p(m_{(1,p)}))_1) \otimes (m_{(0)} \cdot S_p(m_{(1,p)}))_2 \\
&\qquad \overset{(1.2)}{=} Q_R^p(m_{(0)1} \cdot S_p(m_{(1,p)(2,p)})) \otimes (m_{(0)2} \cdot S_p(m_{(1,p)(1,p)})) \\
&\qquad \overset{(1.4)}{=} Q_R^p(m_{(0)1}) \otimes m_{(0)2} \cdot S_p(m_{(1,p)(1,p)})\varepsilon_p(m_{(1,p)(2,p)}) \\
&\qquad \overset{(1.6)}{=} Q_R^p(m_{(0)1}) \otimes m_{(0)2} \cdot S_p(m_{(1,p)}) \\
&\qquad = Q_R^p(m_1) \otimes m_{2(0)} \cdot S_p(m_{(2(1,p))}) = Q_R^p(m_1) \otimes Q_R^p(m_2)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_R^p(m)_1 \otimes Q_R^p(m)_2 &= (m_{(0)} \cdot S_p(m_{(1,p)}))_1 \otimes (m_{(0)} \cdot S_p(m_{(1,p)}))_2 \\
&\overset{(1.2)}{=} m_{(0)1} \cdot S_p(m_{(1,p)(2,p)}) \otimes m_{(0)2} \cdot S_p(m_{(1,p)(1,p)}) \\
&\overset{(1.3)}{=} m_{(0)1} \cdot S_p(m_{(1,p)}) \otimes m_{(0)2(0)} \cdot S_p(m_{(0)2(1,p)}) \\
&= m_{(0)1} \cdot S_p(m_{(1,p)}) \otimes Q_R^p(m_{(0)2}) \\
&= m_{(0)1} \cdot S_p(m_{(1,p)(2,p)}) \otimes Q_R^p(m_{(0)2})\varepsilon_p S_p(m_{(1,p)(1,p)}) \\
&\overset{(1.4)}{=} m_{(0)1} \cdot S_p(m_{(1,p)(2,p)}) \otimes Q_R^p(m_{(0)2} \cdot S_p(m_{(1,p)(1,p)})) \\
&\overset{(1.2)}{=} (m_{(0)} \cdot S(m_{(1,p)}))_1 \otimes Q_R^p((m_{(0)} \cdot S_p(m_{(1,p)}))_2) \\
&= Q_R^p(m)_1 \otimes Q_R^p(Q_R^p(m)_2).
\end{aligned}
$$

Consequently, Q_R^p is a solution of (RBE2). Since the proof for Q_L^p is analogous, the proof is complete. ∎

EXAMPLE 5.6. Let H be a Hopf π-algebra and p an idempotent element of π. Then H_p is an idempotent Rota–Baxter coalgebra of weight -1 with $Q_R^p(h) = h_{(1,p)}S_p(h_{(2,p)})$ or $Q_L^p(h) = S_p(h_{(1,p)})h_{(2,p)}$.

EXAMPLE 5.7. Let C and H be Hopf π-algebras and p an idempotent element of π. Suppose there are maps $i_p : H_p \to C_p$ and $\pi_p : C_p \to H_p$ such that $\pi_p i_p = \mathrm{id}_{H_p}$ and

$$
\begin{aligned}
&i_p(hg) = i_p(h)i_p(g), \qquad \pi(cb) = \pi(c)\pi(b), \\
&i_p(h)_{(1,p)} \otimes i_p(h)_{(2,p)} = i_p(h_{(1,p)}) \otimes i_p(h_{(2,p)}), \\
&\pi_p(c)_{(1,p)} \otimes \pi_p(c)_{(2,p)} = \pi_p(c_{(1,p)}) \otimes \pi_p(c_{(2,p)}),
\end{aligned}
$$

for any $h, g \in H_p$ and $c, b \in C$. Define linear maps $Q_L^p, Q_R^p : C_p \to C_p$ by $Q_L^p(c) = \sum(i_p S_p \pi_p)(c_{(1,p)})c_{(2,p)}$ and $Q_R^p(c) = \sum c_{(1,p)}(i_p S_p \pi_p)(c_{(2,p)})$. Then (C_p, Q_L^p) and (C_p, Q_R^p) are Rota–Baxter coalgebras of weight -1.

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