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The Mazur–Ulam property for abelian C^*-algebras

by

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Abstract. We prove that every abelian C^*-algebra A has the Mazur–Ulam property, that is, every surjective isometry $T : S(A) \to S(E)$ admits an extension to a surjective real linear isometry from A onto X.

1. Introduction. Let E and F be two normed spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) with unit spheres $S(E)$ and $S(F)$, respectively. A mapping $T : E \to F$ is said to be an isometry if

$$\|Tx - Ty\| = \|x - y\|, \quad \forall x, y \in E.$$

The normed spaces E and F are said to be isometric if there exists a surjective isometric mapping of E to F. In 1932, S. Mazur and S. Ulam [MU] gave the famous Mazur–Ulam theorem.

Theorem 1.1. *Every surjective isometric mapping of a real normed space E to a real normed space F is affine* (*that is, a linear transformation composed with a translation*).

In [M], P. Mankiewicz extended the Mazur–Ulam theorem, showing that an isometric mapping of a connected subset of a real normed space onto an open subset of another real normed space can be extended to an affine mapping of the whole space. In particular, two real normed spaces are linearly isometric if and only if their unit balls are isometric.

On the other hand, there are subsets of normed spaces which are isometric, but not in any sense affinely isometric. Consider the mapping $T : \mathbb{R} \to l_2^\infty$ ($\mathbb{R}^2$ with the max norm) given by

$$T(x) = \begin{cases} (x, x), & x \geq 0, \\ (x, -x), & x < 0. \end{cases}$$

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Then T is an isometry, but is clearly not affine. D. Tingley conjectured that the reason for this is that the range of T does not contain sufficient directions of the space. A very natural set to consider, which one intuitively feels that determines the space, is the unit sphere. In 1987, D. Tingley [Ti] posed the following problem.

TINGLEY'S PROBLEM. *Suppose that $T : S(E) \to S(F)$ is a surjective isometric operator. Does there exist a linear isometric operator $U : E \to F$ such that $U|_{S(E)} = T$ where $S(E)$ and $S(F)$ denote the unit sphere of E and F respectively?*

In [Ti], D. Tingley only proved that $T(-x) = -T(x)$ for any x in $S(E)$, when both E and F are finite-dimensional normed spaces. At early stages, mathematicians studied this problem for surjective isometries between the unit spheres of two real normed spaces of the same type.

A normed space E is said to have the *Mazur–Ulam property* (briefly, *MUP*) provided that for every normed space F, every surjective isometry T between the two unit spheres of E and F is the restriction of a real linear isometry between the two spaces. The Mazur–Ulam property is intrinsically linked to the Tingley's problem. The class of Banach space satisfying the Mazur–Ulam property includes the spaces $c_0(\Gamma, \mathbb{R})$, $\ell_\infty(\Gamma, \mathbb{R})$, $c_0(\Gamma, \mathbb{C})$, $\ell_\infty(\Gamma, \mathbb{C})$ on a discrete set Γ (see [D, L, JMPR, P2]), and the space $C(K, \mathbb{R})$ on a compact Hausdorff space K (see [L]), and the space $L^p(\Omega, \mu)$ on a σ-finite measure space (see [T]). We can see [T1] for more examples of Banach spaces with the Mazur–Ulam property. New achievements prove that the Mazur–Ulam property is satisfied by commutative von Neumann algebras [CP], unital complex C^*-algebras and real von Neumann algebras [MO]. For more details we refer the reader to the surveys [P1, W1]. Tingley's problem for two-dimensional normed space has recently been solved by T. Banakh [B], who proved that any two-dimensional Banach space has the Mazur–Ulam property.

The answer to Tingley's problem is negative when E and F are complex normed spaces: take e.g. $E = F = \mathbb{C}$ and $T(x) = \bar{x}$ for all $x \in \mathbb{C}$ with $|x| = 1$. R. S. Wang [W2] showed that each isometry between the unit spheres of complex normed space $C_0(\Omega)$ and $C_0(K)$ is necessarily the restriction of an isometry between $C_0(\Omega)$ and $C_0(K)$ with one part linear and the other part conjugate linear. That the complex Banach space $C(K)$ has MUP where K is Stonean was proved by María Cueto-Avellaneda and A. M. Peralta [CP], who gave the following theorem.

THEOREM 1.2. *Let $T : S(C(K)) \to S(E)$ be a surjective isometry, where K is a Stonean space and E is a complex Banach space. Then there exist two disjoint clopen subsets K_1 and K_2 of K such that $K = K_1 \cup K_2$ such that if K_1 (respectively, K_2) is non-empty, then there exist a closed subspace E_1*

(*respectively,* E_2) *of* E *and a complex linear* (*respectively, conjugate linear*) *surjective isometry* $T_1 : C(K_1) \to E_1$ (*respectively,* $T_2 : C(K_2) \to E_2$) *such that* $E = E_1 \oplus_\infty E_2$ *and* $T(a) = T_1(\pi_1(a)) + T_2(\pi_2(a))$ *for every* $a \in S(C(K))$, *where* π_j *is the natural projection of* $C(K)$ *onto* $C(K_j)$ *given by* $\pi_j(a) = a|_{K_j}$. *In particular,* T *admits an extension to a surjective real linear isometry from* $C(K)$ *onto* E.

A new tool has been recently added by M. Mori and N. Ozawa [MO] to the wide range of tools developed in the study of extension of surjective isometries. Since for each compact Hausdorff space K, the C^*-algebra $C(K)$, of all complex valued continuous functions on K, is a unital C^*-algebra Theorem 1.2 admits a generalization valid for all compact Hausdorff spaces.

THEOREM 1.3 (see [MO, Theorem 1]). *Let* A *be unital* (*complex*) C^**-algebra or any real von Neumann algebra that is not a type* I_k *factor with* $k = 1, 2$. *Then* A *has the Mazur–Ulam property.*

The purpose of this paper is to show that every abelian C^*-algebra satisfies the Mazur–Ulam property, that is, every surjective isometry between the unit spheres A and $S(E)$ is necessarily the restriction of an isometry between A and E with one part linear and the other part conjugate-linear.

2. Preliminaries. For any abelian C^*-algebra without identity, the following theorem holds (see [Con, Corollary 2.2]).

THEOREM 2.1. *If* A *is an abelian* C^**-algebra without identity and* Ω *is its maximal ideal space, then the Gelfand transform* $G : A \to C_0(\Omega)$ *is an isometric* $*$*-isomorphism of* A *onto* $C_0(\Omega)$.

So in this paper, instead of talking about C^*-algebras, we consider spaces $C_0(\Omega)$, where Ω is a locally compact Hausdorff space.

The scalar field $\mathbb{K}$ is assumed to be $\mathbb{R}$ or $\mathbb{C}$. A mapping $U : E \to E_1$ between $\mathbb{K}$-linear spaces is *conjugate linear* if

$$U(\alpha x + \beta y) = \bar{\alpha} U(x) + \bar{\beta} U(y)$$

for all x, y in E and all α, β in $\mathbb{C}$.

If Ω is a locally compact Hausdorff space, $C_0(\Omega)$ denotes the Banach space of all continuous functions from Ω into $\mathbb{K}$ which vanish at infinity, i.e. $\{t \in \Omega : |x(t)| \ge \varepsilon\}$ is compact in Ω for all $\varepsilon > 0$. For any f in $S(C_0(\Omega))$, let M_f be the subset of Ω such that

$$M_f = \{t \in \Omega : f(t) \in \mathbb{T}\} = \{t \in \Omega : |f(t)| = 1\} = \{t \in \Omega : |f(t)| \ge 1\}.$$

We always use $\mathbb{T}$ to denote the unit sphere of $\mathbb{C}$, i.e., $\mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\}$.

For functions f and g of Ω into $\mathbb{K}$, we write $f \blacktriangleright g$ if $f(t) = g(t)$ whenever $f(t) \in \mathbb{T}$ and $t \in \Omega$. We say C is a *maximal convex subset* (or *facet*) of $S(C_0(\Omega))$ if it is not properly contained in any other convex subset of $S(C_0(\Omega))$. For each $t_0 \in \Omega$ and each $\lambda \in \mathbb{T}$, we denote

$$A(t_0, \lambda) = \{f \in S(C_0(\Omega)) : f(t_0) = \lambda\}.$$

Then $A(t_0, \lambda)$ is a maximal convex subset of $S(C_0(\Omega))$.

The following theorem was originally established by L. Cheng and Y. Dong [CD], and later rediscovered by R. Tanaka [T2].

THEOREM 2.2 (see [T2, Lemma 3.5]). *Let E and F be Banach spaces. Suppose that $T : S(E) \to S(F)$ is a surjective isometry. Then C is a maximal convex subset of $S(E)$ if and only if $T(C)$ is a maximal convex subset of $S(F)$.*

For normed space E, we denote

$$\begin{aligned}&\operatorname{mac} B(E^*)\\ &\quad = \{\varphi \in \operatorname{ext} B(E^*) : \varphi^{-1}(1) \cap B(E) \text{ is maximal convex subset of } S(E)\}.\end{aligned}$$

Lemma 3.3 in [T3] shows how a suitable application of Eidelheit's separation theorem proves that for every maximal convex subset C of $S(E)$ there exists φ in $S(E^*)$ satisfying $C = \varphi^{-1}(1) \cap B(E)$. Using the Krein–Milman theorem, one can easily prove that every maximal convex subset C of $S(E)$ has the form

$$C = \{x \in B(E) : \varphi(x) = 1\}$$

for some $\varphi \in \operatorname{ext} B(E^*)$.

If $T : S(C_0(\Omega)) \to S(E)$ is a surjective isometry, for any $t_0 \in \Omega$ and $\lambda \in \mathbb{T}$, let

$$\operatorname{supp}(t_0, \lambda) = \{\varphi \in S(E^*) : \varphi(T(f)) = 1, \forall f \in A(t_0, \lambda)\}$$

and

$$M_{Tf} = \{\varphi \in \operatorname{mac} B(E^*) : \varphi(T(f)) \in \mathbb{T}\}.$$

For any $t_0 \in \Omega$ and $\lambda \in \mathbb{T}$, since $A(t_0, \lambda)$ is a maximal convex subset of $S(C_0(\Omega))$, Theorem 2.2 implies that $T(A(t_0, \lambda))$ is a maximal convex subset of $S(E)$. So $\operatorname{supp}(t_0, \lambda) \neq \emptyset$ and $\operatorname{supp}(t_0, \lambda)$ is a convex subset of $B(E^*)$. Because

$$\emptyset \neq \operatorname{ext}(\operatorname{supp}(t_0, \lambda)) \subset \operatorname{ext} B(E^*)$$

we have

$$\emptyset \neq \operatorname{ext}(\operatorname{supp}(t_0, \lambda)) \cap \operatorname{ext} B(E^*) \subset \operatorname{mac} B(E^*).$$

Let us recall some technical results, which will be used later. The following lemma is given by María Cueto-Avellaneda and A. M. Peralta [CP], where Ω is a compact Hausdorff space. In fact, the same statement is true when Ω is a locally compact Hausdorff space.

LEMMA 2.3 (see [CP, Lemma 2.4]). *If Ω is a locally compact Hausdorff space, let $T : S(C_0(\Omega)) \to S(E)$ be a surjective isometry, where E is a complex normed space. Then for each t_0 in Ω and $\lambda \in \mathbb{T}$ we have*

$$\varphi T(f) = -1 \quad \text{for every } f \in A(t_0, -\lambda) \text{ and every } \varphi \in \operatorname{supp}(t_0, \lambda).$$

REMARK 2.4. If $\alpha, \beta \in \mathbb{T}$ satisfy $|1 - \beta| = |1 - \alpha|$ and $|1 + \beta| = |1 + \alpha|$, then $\alpha = \beta$ or $\alpha = \bar{\beta}$.

LEMMA 2.5 (see [W2, Lemma 9]). *Suppose $\varphi : \mathbb{T} \to \mathbb{T}$ is an injective map, $\varphi(1) = 1$ and $|\varphi(\alpha) - \varphi(\beta)| \le |\alpha - \beta|$ for all $\alpha, \beta \in \mathbb{T}$. Then φ is an isometry of $\mathbb{T}$ onto $\mathbb{T}$ and either $\varphi(\alpha) = \alpha$ for all $\alpha \in \mathbb{T}$, or $\varphi(\alpha) = \bar{\alpha}$ for all $\alpha \in \mathbb{T}$.*

LEMMA 2.6 (see [CP, Proposition 2.6]). *Suppose Ω is a compact Hausdorff space, E is a complex normed space, and $\lambda \in \mathbb{T}$. Let $T : S(C_0(\Omega)) \to S(E)$ be a surjective isometry. Let $t_0 \in \Omega$ and $\varphi \in \operatorname{supp}(t_0, \lambda)$. Then $\varphi T(h) = 0$ for every $h \in S(C_0(\Omega))$ with $h(t_0) = 0$. Furthermore, $|\varphi T(h)| < 1$ for every $h \in S(C_0(\Omega))$ with $|h(t_0)| < 1$ and every $\varphi \in \operatorname{supp}(t_0, \lambda)$.*

The inequality of the following theorem was first studied by G. G. Ding [Di1]. The results were generalized by X. Fang and J. Wang [FW1] and X. Yang, Z. Hou and X. Fu [YHF], who proved the following theorem.

THEOREM 2.7 (see [YHF, Theorem 2.2]). *Let E and F be normed spaces. Suppose that $T : S(E) \to S(F)$ is a surjective isometry. If*

$$\|T(x) - \lambda T(y)\| \le \|x - \lambda y\|$$

for any $\lambda \in \mathbb{R}^+$ and $x, y \in S(E)$, then T can be extended to a linear isometry on the whole space.

Another technical result of geometric nature, which is applied in this paper, was established by X. N. Fang and J. H. Wang [FW2] and G. G. Ding [Di2].

THEOREM 2.8 (see [FW2, Corollary 2.2]). *Let E and F be normed spaces and let $T : S(E) \to S(F)$ be a surjective isometry. Then, for any x, y in $S(E)$, we have $\|x + y\| = 2$ if and only if $\|T(x) + T(y)\| = 2$.*

3. Main lemmas. To prove the main result of this paper, we need the following lemmas.

LEMMA 3.1. *If A is a facet of $S(C_0(\Omega))$, then there exist $t_0 \in \Omega$ and $\tilde{f} \in S(C_0(\Omega))$ such that $\tilde{f}(t_0) \in \mathbb{T}$ and $A = A(t_0, \tilde{f}(t_0))$.*

Proof. Fix $f_0 \in A$, and for any $f \in A$, let

$$A_f = \{t \in \Omega : f(t) = f_0(t) \in \mathbb{T}\}.$$

The convexity of A implies $A_f \neq \emptyset$.

We first show that $\bigcap_{f\in A} A_f \neq \emptyset$. For any $f_1,\dots,f_n \in A$, since A is convex, it follows that

$$\bar{f} = \frac{f_0 + f_1 + \cdots + f_n}{n+1} \in A.$$

Let $t \in A_{\bar{f}}$. Then $\bar{f}(t) = f_0(t) \in \mathbb{T}$. Because $|f_i(t)| \le 1$ and $f_0(t) \in \mathbb{T}$, it follows that $f_i(t) = f_0(t)$ for $i \in \{1,\dots,n\}$. Hence

$$\emptyset \neq A_{\bar{f}} \subset \bigcap_{i=1}^{n} A_{f_i}.$$

Thus $\bigcap_{f\in A} A_f \neq \emptyset$, because A_f, $f \in A$, are compact subsets of Ω.

We claim that $\bigcap_{f\in A} A_f$ has at most one point. If not, assume that $t_0, t_1 \in \bigcap_{f\in A} A_f$ and $t_0 \neq t_1$. Urysohn's lemma implies that there is a $g \in S(C_0(\Omega))$, $0 \le g \le 1$, such that $g(t_0) = 1$ and $g(t_1) = 0$. For any $f \in A$, it is obvious that $[gf_0, f] \subset S(C_0(\Omega))$. Since A is a maximal convex subset of $S(C_0(\Omega))$, it follows that $gf_0 \in A$. Our choice of g shows that $gf_0(t_1) = 0$, which contradicts $t_1 \in \bigcap_{f\in A} A_f \subset A_{gf_0}$.

Finally, let $\bigcap_{f\in A} A_f = \{t_0\}$ and $\tilde{f} = f_0$. It is obvious that $A \subset A(t_0, \tilde{f}(t_0))$. The opposite inclusion is obvious, since $A(t_0, \tilde{f}(t_0))$ is a convex subset of $S(C_0(\Omega))$ and A is maximal convex subset of $S(C_0(\Omega))$. ■

LEMMA 3.2. *Let $f \in S(C_0(\Omega))$, $\varphi \in \operatorname{mac} B(E^*)$. If $\varphi(T(f)) = 1$, then there exists $t_0 \in \Omega$ such that $f(t_0) \in \mathbb{T}$ and $\varphi \in \operatorname{supp}(t_0, f(t_0))$.*

Proof. Since $\varphi \in \operatorname{mac} B(E^*)$, the set $B = \{x \in S(E) : \varphi(x) = 1\}$ is a facet. Theorem 2.2 implies that $T^{-1}(B)$ is a facet of $S(C_0(\Omega))$. By Lemma 3.1, there exists $t_0 \in \Omega$ such that $f(t_0) \in \mathbb{T}$ and $T^{-1}(B) = A(t_0, f(t_0))$. Thus $\varphi \in \operatorname{supp}(t_0, f(t_0))$. This completes the proof. ■

LEMMA 3.3. *Suppose Ω is a locally compact Hausdorff space. Let $T : S(C_0(\Omega)) \to S(E)$ be a surjective isometry, where E is a complex Banach space. Then the following statements hold:*

(a) *For every $t_1 \neq t_2$ in Ω, and all $\lambda, \mu \in \mathbb{T}$, we have*

$$\mathbb{T} \cdot \operatorname{supp}(t_1, \lambda) \cap \mathbb{T} \cdot \operatorname{supp}(t_2, \mu) = \emptyset.$$

(b) *If $\mu, \nu \in \mathbb{T}$, $\mu \neq \nu$, and $t_0 \in \Omega$, then* $\operatorname{supp}(t_0, \nu) \cap \operatorname{supp}(t_0, \mu) = \emptyset$.

Proof. (a) Since $t_1 \neq t_2$, by Urysohn's lemma there are disjoint open sets W_1, W_2 such that $t_1 \in W_1$, $t_2 \in W_2$, and $0 \le h_1, h_2 \le 1$ in $S(C_0(\Omega))$ such that $h_1(t_1) = 1$, $h_1(t) = 0$ for $t \notin W_1$ and $h_2(t_2) = 1$, $h_2(t) = 0$ for $t \notin W_2$.

If $\mathbb{T} \cdot \operatorname{supp}(t_1, \lambda) \cap \mathbb{T} \cdot \operatorname{supp}(t_2, \mu) \neq \emptyset$, there exist $\varphi \in \operatorname{supp}(t_1, \lambda)$ and $\alpha \in \mathbb{T}$ such that $\alpha\varphi \in \operatorname{supp}(t_2, \mu)$. Thus $\varphi T(\lambda h_1) = 1$ and $\alpha\varphi T(\mu h_2) = 1$.

By Lemma 2.3 we obtain

$$|1-\overline{\alpha}| = |\varphi T(\lambda h_1) - \varphi T(\mu h_2)| \le \|\lambda h_1 - \mu h_2\| = 1, \tag{1}$$
$$|1+\overline{\alpha}| = |\varphi T(\lambda h_1) - \varphi T(-\mu h_2)| \le \|\lambda h_1 + \mu h_2\| = 1. \tag{2}$$

(1) and (2) imply that $\alpha = 0$, which contradicts $\alpha \in \mathbb{T}$.

(b) The proof of (b), coming from [CP], is given for completeness. Let $\varphi \in \operatorname{supp}(t_0,\nu) \cap \operatorname{supp}(t_0,\mu)$ with $\mu \neq \nu$, and $h_0 \in A(t_0,1)$. Since $\mu h_0 \in A(t_0,\mu)$ and $\nu h_0 \in A(t_0,\nu)$, we get

$$2 = \varphi f(\nu h_0) + \varphi f(\mu h_0) \le \|f(\nu h_0) + f(\mu h_0)\| \le 2,$$

and by Theorem 2.8 we have $2 = \|\nu h_0 + \mu h_0\| = |\mu + \nu|$, which holds if and only if $\mu = \nu$. ■

LEMMA 3.4. *If $f, g \in S(C_0(\Omega))$, then $f \blacktriangleright g$ if and only if for all $\varphi \in \operatorname{mac} B(E^*)$ with $\varphi T(f) \in \mathbb{T}$, we have $\varphi T(f) = \varphi T(g)$.*

Proof. Suppose $f \blacktriangleright g$. Let $\varphi \in \operatorname{mac} B(E^*)$ with $\varphi T(f) = \alpha \in \mathbb{T}$. Then $\frac{1}{\alpha}\varphi(T(f)) = 1$. By Lemma 3.1, there exists $t_0 \in \Omega$ such that $f(t_0) \in \mathbb{T}$ and $\frac{1}{\alpha}\varphi \in \operatorname{supp}(t_0, f(t_0))$. Since $f \blacktriangleright g$, it follows that $g(t_0) = f(t_0)$. Then $\frac{1}{\alpha}\varphi(T(g)) = 1$. Thus $\varphi T(f) = \alpha = \varphi T(g)$.

Conversely, assume that $t_0 \in \Omega$ and $f(t_0) \in \mathbb{T}$. Then there exists

$$\varphi \in \operatorname{supp}(t_0, f(t_0)) \cap \operatorname{mac} B(E^*)$$

such that $\varphi T(f) = 1$. Since $\varphi T(g) = \varphi T(f) = 1$, Lemmas 3.2 and 3.3 imply that $g(t_0) = f(t_0)$, so $f \blacktriangleright g$. ■

REMARK 3.5. Let $\varphi \in \operatorname{mac} B(E^*)$. Then $\varphi^{-1}(1) \cap B(E)$ is a facet. By Theorem 2.2 we see that $T^{-1}(\varphi^{-1}(1) \cap B(E))$ is a facet in $C_0(\Omega)$, so there exist $t_0 \in \Omega$ and $\alpha \in \mathbb{T}$ such that $\varphi \in \operatorname{supp}(t_0,\alpha)$. If $\beta \in \mathbb{T}$, it is obvious that $\beta\varphi \in \operatorname{mac} B(E^*)$. So there exist $t_1 \in \Omega$ and $\gamma \in \mathbb{T}$ such that $\beta\varphi \in \operatorname{supp}(t_1,\gamma)$. Lemma 3.3 implies that $t_0 = t_1$ and γ is uniquely determined by β. Let $m(\beta) = \gamma$; then m is a function of $\mathbb{T}$ to $\mathbb{T}$.

LEMMA 3.6. *Let $m : \mathbb{T} \to \mathbb{T}$ be as in Remark 3.5. Then m is a surjective isometry of $\mathbb{T}$ to $\mathbb{T}$.*

Proof. Lemma 3.3 implies that m is injective.

Let $\beta_1, \beta_2 \in \mathbb{T}$ and $f \in A(t_0,\alpha)$. By the definition of m, we obtain

$$\beta_1\varphi\biggl(T\biggl(\frac{m(\beta_1)}{\alpha}f\biggr)\biggr) = 1 \quad \text{and} \quad \beta_2\varphi\biggl(T\biggl(\frac{m(\beta_2)}{\alpha}f\biggr)\biggr) = 1.$$

So

$$|\beta_1-\beta_2| = \left|\varphi\Big(T\Big(\frac{m(\beta_1)}{\alpha}f\Big)\Big)-\varphi\Big(T\Big(\frac{m(\beta_1)}{\alpha}f\Big)\Big)\right| \le \left\|T\Big(\frac{m(\beta_1)}{\alpha}f\Big)-T\Big(\frac{m(\beta_2)}{\alpha}f\Big)\right\| = |m(\beta_1)-m(\beta_2)|. \tag{3}$$

It is obvious that $m(1)=\alpha$ and $m(-1)=-\alpha$. By (3), for any $\beta\in\mathbb{T}$, we obtain

$$|m(1)-m(\beta)| = |\alpha-m(\beta)| \ge |1-\beta|,$$
$$|m(-1)-m(\beta)| = |\alpha+m(\beta)| \ge |1+\beta|.$$

But

$$4=|1-\beta|^2+|1+\beta|^2 \le |\alpha-m(\beta)|^2+|\alpha+m(\beta)|^2=4,$$

which implies that $|1+\beta|=|\alpha+m(\beta)|$ and $|1-\beta|=|\alpha-m(\beta)|$. Remark 2.4 implies $m(\beta)=\alpha\beta$ or $m(\beta)=\alpha\bar\beta$. Since $m(\bar\beta)=\alpha\bar\beta$ or $m(\bar\beta)=\alpha\beta$, the injectivity of m implies that

$$\{m(\beta),m(\bar\beta)\}=\{\alpha\beta,\alpha\bar\beta\}.$$

So m is surjective. By Lemma 2.5, m is a surjective isometry of $\mathbb{T}$ to $\mathbb{T}$. ∎

LEMMA 3.7. *Let $f,g\in S(C_0(\Omega))$. Then $M_f\cap M_g=\emptyset$ if and only if $M_{T(f)}\cap M_{T(g)}=\emptyset$.*

Proof. Let $f,g\in S(C_0(\Omega))$ with $M_f\cap M_g=\emptyset$. Since Ω is locally compact and the sets M_f,M_g are compact, there exist disjoint open subsets U_1 and U_2 of Ω with compact closures such that $M_f\subseteq U_1, M_g\subseteq U_2$. By Urysohn's lemma, there are $0\le h_1,h_2\le 1$ in $S(C_0(\Omega))$ such that $h_1(M_f)=1=h_2(M_g)$ and $h_1(U_1^c)=0=h_2(U_2^c)$. It is easy to see that $f\blacktriangleright h_1f\blacktriangleright f$, $g\blacktriangleright h_2g\blacktriangleright g$ and $M_f=M_{h_1f}$, $M_g=M_{h_2g}$. Since U_1 and U_2 are disjoint, we obtain $h_1f+h_2g\in S(C_0(\Omega))$ and $h_1f,h_2g\blacktriangleright(h_1f+h_2g)$.

If $M_{Tf}\cap M_{Tg}\ne\emptyset$, let $\varphi_0\in M_{Tf}\cap M_{Tg}$. Then $\varphi_0(T(f)),\varphi_0(T(g))\in\mathbb{T}$. Since $h_1f,h_2g\blacktriangleright(h_1f+h_2g)$, Lemma 3.4 implies that

$$\varphi_0(Tf)=\varphi_0(T(h_1f))=\varphi_0(T(h_1f+h_2g))=\varphi_0(T(h_2g))=\varphi_0(Tg).$$

Lemmas 3.2 and 3.3 imply that there exists $t_0\in\Omega$ such that

$$h_1(t_0)f(t_0)=h_2(t_0)g(t_0),$$

thus $t_0\in M_f\cap M_g$, which contradicts $M_f\cap M_g=\emptyset$.

Conversely, suppose $f,g\in S(C_0(\Omega))$ with $M_{Tf}\cap M_{Tg}=\emptyset$, but $M_f\cap M_g\ne\emptyset$. Let $t_0\in M_f\cap M_g$, $\alpha=f(t_0)$, $\beta=g(t_0)$ and $\varphi\in\operatorname{supp}(t_0,\alpha)\cap\operatorname{mac}B(E^*)$, Remark 3.5 and Lemma 3.6 imply that $m^{-1}(\beta)\varphi\in\operatorname{supp}(t_0,\beta)$. So $\varphi\in M_{Tf}\cap M_{Tg}$, which contradicts $M_{Tf}\cap M_{Tg}=\emptyset$. ∎

LEMMA 3.8. *Let $f, g \in S(C_0(\Omega))$. Then $M_f \subseteq M_g$ if and only if $M_{T(f)} \subseteq M_{T(g)}$.*

Proof. Suppose $M_{T(f)} \subseteq M_{T(g)}$ and there exists $t_0 \in M_f \setminus M_g$. Then there exists $h \in S(C_0(\Omega))$ such that $0 \le h \le 1$, $h(t_0) = 1$ and $h(M_g) = 0$. Let $\widetilde{f} = hf$. Then $\widetilde{f} \in S(C_0(\Omega))$ and $M_{\widetilde{f}} \cap M_g = \emptyset$. By Lemma 3.7, we have $M_{T\widetilde{f}} \cap M_{Tg} = \emptyset$. But obviously $\widetilde{f} \blacktriangleright f$, and Lemma 3.4 implies that $M_{T\widetilde{f}} \subseteq M_{Tf} \subseteq M_{Tg}$, which contradicts $M_{T\widetilde{f}} \cap M_{Tg} = \emptyset$.

Conversely, suppose $M_f \subseteq M_g$ and there exists $\varphi_0 \in M_{Tf} \setminus M_{Tg}$. We can assume that $\varphi_0 Tf = 1$. By Lemma 3.2, there exists $t_0 \in \Omega$ such that $f(t_0) \in \mathbb{T}$ and $\varphi_0 \in \operatorname{supp}(t_0, f(t_0))$.

Since $M_f \subseteq M_g$, $g(t_0) \in \mathbb{T}$. Let $\varphi = \varphi_0$. By Lemma 3.6, we obtain

$$m^{-1}(g(t_0))\varphi_0 \in \operatorname{supp}(t_0, g(t_0)).$$

But it is obvious that $m^{-1}(g(t_0))\varphi_0 \in \operatorname{mac} B(E^*)$, so $m^{-1}(g(t_0))\varphi_0 \in M_{Tg}$ and $\varphi_0 \in M_{Tg}$, which contradicts $\varphi_0 \in M_{Tf} \setminus M_{Tg}$. ∎

LEMMA 3.9. *For any $t \in \Omega$,*

$$\phi_T(t) = \bigcap \{M_{T(f)} : f \in S(C_0(\Omega)) \text{ and } f(t) \in \mathbb{T}\} \neq \emptyset.$$

Proof. Fix $t_0 \in \Omega$. We need to show that the family

$$\{M_{T(f)} : f \in S(C_0(\Omega)) \text{ and } f(t_0) \in \mathbb{T}\}$$

of subsets of $\operatorname{ext} B(E^*)$ has the finite intersection property. For any $f_1, \dots, f_n$ in $S(C_0(\Omega))$ with $f_i(t_0) \in \mathbb{T}$ $(i = 1, \dots, n)$, define

$$f_0(t) = \frac{1}{n} \sum_{i=1}^{n} |f_i(t)|, \qquad \forall t \in \Omega.$$

Then $\|f_0\| \le 1$ and $f_0(t_0) = 1$, so $f_0 \in S(C_0(\Omega))$. Since $|f_i(t)| \le 1$ $(i = 1, \dots, n)$, we have $|f_i(t)| = 1$ $(i = 1, \dots, n)$ whenever $|f_0(t)| = 1$. Therefore

$$M_{f_0} \subseteq M_{f_i} \quad (i = 1, \dots, n).$$

It follows from Lemma 3.8 that

$$M_{T(f_0)} \subseteq M_{T(f_i)} \quad (i = 1, \dots, n),$$

thus $M_{T(f_0)} \subseteq \bigcap_{i=1}^{n} M_{T(f_i)}$. Now $M_{T(f_0)} \neq \emptyset$ implies $\bigcap_{i=1}^{n} M_{T(f_i)} \neq \emptyset$. Hence the family

$$\{M_{T(f)} : f \in S(C_0(\Omega)) \text{ and } f(t_0) \in \mathbb{T}\}$$

has the finite intersection property, which shows that $\phi_T(t_0) \neq \emptyset$. ∎

Let Φ be s set-valued function from a set A to the subsets of a set B. A *selection* of Ψ of Φ is a mapping $\Psi : A \to B$ such that $\Psi(a) \in \Phi(a)$ for all a in A.

Lemma 3.10. *For any t in Ω, the set-valued function ϕ_T admits a selection ψ_T such that $\psi_T(t)(Tf) = 1$ whenever $f \in S(C_0(\Omega))$ and $f(t) = 1$.*

Proof. For any t in Ω, let f_0 in $S(C_0(\Omega))$ with $f_0(t) = 1$. Choose $x^* \in \phi_T(t)$ with

$$x^*(Tf_0) = 1.$$

For any $f \in S(C_0(\Omega))$ with $f(t) = 1$, let $h = \frac{1}{2}(f_0 + f)$. Then $\|h\| = 1$ and $h(t) = 1$. Obviously, $h \blacktriangleright f_0, f$. By Lemma 3.4 we obtain

$$x^*(T(h)) = x^*(T(f_0)) = x^*(T(f)).$$

It follows that $\psi_T(t) = x^*$ is well defined and $\psi_T(t)(Tf) = 1$ whenever $f \in S(C_0(\Omega))$ and $f(t) = 1$. ∎

Remark 3.11. According to Lemma 3.10, for any t in Ω, we have $\psi_T(t) \in \operatorname{supp}(t, 1)$.

Lemma 3.12. *Let ψ_T be as in Lemma 3.10. For any $t_0 \in \Omega$ and $\alpha \in \mathbb{T}$, if $g \in S(C_0(\Omega))$ with $g(t_0) = \alpha \in \mathbb{T}$, then $\psi_T(t_0)(Tg) = \alpha$ or $\psi_T(t_0)(Tg) = \bar{\alpha}$.*

Proof. Let $f = \bar{\alpha}g$. Since $g(t_0) = \alpha f(t_0) = \alpha$ and $\psi_T(t_0) \in M_{Tg}$, it follows that $\psi_T(t_0)(Tg) \in \mathbb{T}$. Let $\psi_T(t_0)(Tg) = \beta \in \mathbb{T}$. We have

$$\|f - g\| = \|f - \alpha f\| = |1 - \alpha|\,\|f\| = |1 - \alpha|. \tag{4}$$

Moreover,

$$\begin{aligned} \|f - g\| &= \|Tf - Tg\| \ge \|\psi_T(t_0)(Tf - Tg)\| \\ &= \|\psi_T(t_0)(Tf) - \psi_{T_0}(t)(Tg)\| = |1 - \beta|. \end{aligned} \tag{5}$$

Then (4) and (5) show that $|1 - \alpha| \ge |1 - \beta|$.

Since $-f(t_0) = -1$, by Lemma 2.3, $\psi_T(t_0)T(-f) = -1$. Further,

$$\begin{aligned} |1 + \beta| &= \|\psi_T(t_0)(Tf) + \psi_T(t_0)(Tg)\| = \|\psi_T(t_0)(Tg) - \psi_T(t_0)(T(-f))\| \\ &= \|\psi_T(t_0)(Tg - T(-f))\| \\ &\le \|(Tg - T(-f))\| = \|g + f\| \\ &= \|\alpha g + g\| = |1 + \alpha|\,\|g\| = |1 + \alpha|. \end{aligned}$$

But

$$4 = |1 - \beta|^2 + |1 + \beta|^2 \le |1 - \alpha|^2 + |1 + \alpha|^2 = 4,$$

which implies that $|1 + \beta| = |1 + \alpha|$ and $|1 - \beta| = |1 - \alpha|$. By Remark 2.4, $\beta = \alpha$ or $\beta = \bar{\alpha}$. So $\psi_T(t_0)(Tg) = \alpha$ or $\psi_T(t_0)(Tg) = \bar{\alpha}$. ∎

Lemma 3.13. *Let ψ_T be as in Lemma 3.10. For any $t_0 \in \Omega$, if for some $f_0 \in S(C_0(\Omega))$ with $f_0(t_0) = \alpha \in \mathbb{T} \setminus \{\pm 1\}$, $\psi_T(t_0)(Tf_0) = \alpha$ (respectively, $\psi_T(t_0)(Tf_0) = \bar{\alpha}$), then $\psi_T(t_0)(Tf) = \alpha$ (respectively, $\psi_T(t_0)(Tf) = \bar{\alpha}$) for any $f \in S(C_0(\Omega))$ with $f(t_0) = \alpha$.*

Proof. Fix $f \in S(C_0(\Omega))$ with $f(t_0) = \alpha$. Let $n \in \mathbb{N}$ and

$$f_i = \frac{i}{n} f + \left(1 - \frac{i}{n}\right) f_0.$$

Then $f_i \in S(C_0(\Omega))$ and $f_i(t_0) = \alpha$, for $i = 0, 1, \ldots, n$.

Since

$$\begin{aligned} \frac{2}{n} \geq \frac{1}{n}\|f - f_0\| &= \|f_i - f_{i+1}\| = \|Tf_i - Tf_{i+1}\| \\ &\geq \|\psi_T(t_0)Tf_i - \psi_T(t_0)Tf_{i+1}\| \end{aligned} \tag{6}$$

for $i = 0, 1, \ldots, n-1$. We can choose n large enough so that $\frac{2}{n} < |\alpha - \bar{\alpha}|$.

By Lemma 3.12 and (6), we obtain

$$\psi_T(t_0)(Tf) = \psi_T(t_0)(Tf_n) = \psi_T(t_0)(Tf_{n-1}) = \cdots = \psi_T(t_0)(Tf_0).$$

This completes the proof. ∎

Lemma 3.14. *Let ψ_T be as in Lemma 3.10. For any $t_0 \in \Omega$, if for some $f_0 \in S(C_0(\Omega))$ with $\psi_T(t_0)(Tf_0) = f_0(t_0) \in \mathbb{T} \setminus \{\pm 1\}$ (respectively, $\psi_T(t_0)(Tf_0) = \overline{f_0(t_0)} \in \mathbb{T} \setminus \{\pm 1\}$), then $\psi_T(t_0)(Tf) = f(t_0)$ (respectively, $\psi_T(t_0)(Tf) = \overline{f(t_0)}$) for any $f \in S(C_0(\Omega))$ with $f(t_0) \in \mathbb{T}$.*

Proof. The proof of the two cases is similar. We consider only the case $\psi_T(t_0)(Tf_0) = f_0(t_0)$.

Let $\alpha = f_0(t_0)$, $\beta = f(t_0)$ and $g = \bar{\beta}\alpha f$. By Lemma 3.13, we obtain

$$\psi_T(t_0)(Tf_0) = f_0(t_0) = \alpha = g(t_0) = \psi_T(t_0)(Tg).$$

If $\psi_T(t_0)(Tf) = \bar{\beta}$, then

$$\|f - g\| = \|f - \bar{\beta}\alpha f\| = |1 - \bar{\beta}\alpha|\,\|f\| = |\alpha - \beta|$$

and

$$\|f - g\| = \|Tf - Tg\| \geq \|\psi_T(t_0)(Tf - Tg)\| = |\alpha - \bar{\beta}|.$$

So we obtain

$$|\alpha - \beta| \geq |\alpha - \bar{\beta}|.$$

Lemma 2.3 implies $\psi_T(t_0)(T(-g)) = -\psi_T(t_0)(T(g))$, so

$$\begin{aligned} |\alpha + \bar{\beta}| &= \|\psi_T(t_0)(Tf) + \psi_T(t_0)(Tg)\| \\ &= \|\psi_T(t_0)(Tf) - \psi_T(t_0)(T(-g))\| \\ &\leq \|T(f) - T(-g)\| = \|f + g\| = |\alpha + \beta|. \end{aligned} \tag{7}$$

But

$$4 = |\alpha - \bar{\beta}|^2 + |\alpha + \bar{\beta}|^2 \leq |\alpha - \beta|^2 + |\alpha + \beta|^2 = 4.$$

Hence $|\alpha - \beta| = |\alpha - \bar{\beta}|$ and $|\alpha + \beta| = |\alpha + \bar{\beta}|$. Any of the previous identities holds if and only if

$$2 + 2\Re e(\alpha\bar{\beta}) = |\alpha|^2 + |\beta|^2 + 2\Re e(\alpha\bar{\beta}) = |\alpha + \beta|^2 = |\alpha + \bar{\beta}|^2 = 2 + 2\Re e(\alpha\beta),$$

i.e.

$$\Re e(\alpha)\Re e(\beta)+\Im m(\alpha)\Im m(\beta)=\Re e(\alpha)\Re e(\beta)-\Im m(\alpha)\Im m(\beta),$$

which is impossible since $\alpha,\beta\notin\mathbb{R}$. So $\psi_T(t_0)(Tf)=\beta$. ∎

LEMMA 3.15. *Let $t\in\Omega$. Then either $\psi_T(t)(Tf)=f(t)$ for any $f\in S(C_0(\Omega))$ or $\psi_T(t)(Tf)=\overline{f(t)}$ for any $f\in S(C_0(\Omega))$.*

Proof. Fix $t_0\in\Omega$. Let $u\in S(C_0(\Omega))$ with $u(t_0)=i$. By Lemma 3.12, we have either $\psi_T(t_0)(Tu)=i$ or $\psi_T(t_0)(Tu)=\overline{i}$. Assume, without loss of generality, that $\psi_T(t_0)(Tu)=i$.

Let $f_0\in S(C_0(\Omega))$. Lemma 2.6 implies that $\psi_T(t_0)(Tf_0)=0$ when $f_0(t_0)=0$. So we can assume that $f_0(t_0)\neq 0$. For any $\varepsilon\in(0,|f_0(t_0)|)$,

$$U_\varepsilon=\{t\in\Omega:\big||f_0(t)|-|f_0(t_0)|\big|<\varepsilon\}$$

is an open subset of Ω with $t_0\in U_\varepsilon$ and the closure of U_ε is contained in the compact subset $\{t\in\Omega:|f_0(t)|\geq|f_0(t_0)|-\varepsilon\}$ of Ω. Take $h_\varepsilon\in S(C_0(\Omega))$ such that $0\leq h_\varepsilon\leq 1$, $h_\varepsilon(t_0)=1$ and $h_\varepsilon(U_\varepsilon^c)=0$. Define

$$g_\varepsilon(t)=f_0(t)+h_\varepsilon(t)(1-|f_0(t)|)\frac{f_0(t)}{|f_0(t)|}$$

for all $t\in\Omega$. It is easy to check that $g_\varepsilon\in S(C_0(\Omega))$ and $g_\varepsilon(t_0)=\frac{f_0(t_0)}{|f_0(t_0)|}\in\mathbb{T}$. So

$$\|g_\varepsilon-f_0\|\leq 1-|f_0(t_0)|+\varepsilon. \tag{8}$$

By Lemma 3.14, we have $\psi_T(t_0)(Tg_\varepsilon)=\frac{f_0(t_0)}{|f_0(t_0)|}$. Then

$$\begin{aligned}\left|\frac{f_0(t_0)}{|f_0(t_0)|}-\psi_T(t_0)(Tf_0)\right|&\leq|\psi_T(t_0)(Tg_\varepsilon)-\psi_T(t_0)(Tf_0)|\\&\leq\|Tg_\varepsilon-Tf_0\|=\|g_\varepsilon-f_0\|\\&\leq 1-|f_0(t_0)|+\varepsilon.\end{aligned} \tag{9}$$

Define

$$k_\varepsilon(t)=\begin{cases}h_\varepsilon(t)\frac{f_0(t)}{|f_0(t)|}, & f_0(t)\neq 0,\\ 0, & f_0(t)=0,\end{cases}$$

for all $t\in\Omega$. It is easy to check that $k_\varepsilon\in S(C_0(\Omega))$, $k_\varepsilon(t_0)=\frac{f_0(t_0)}{|f_0(t_0)|}\in\mathbb{T}$ and

$$\|k_\varepsilon+f_0\|\leq 1+|f_0(t_0)|+\varepsilon. \tag{10}$$

Indeed, for any $t\in\Omega$, if $t\in U_\varepsilon$, then

$$\begin{aligned}|k_\varepsilon(t)+f_0(t)|&=\left|h_\varepsilon(t)\frac{f_0(t)}{|f_0(t)|}+f_0(t)\right|\leq\left|h_\varepsilon(t)\frac{f_0(t)}{|f_0(t)|}\right|+|f_0(t)|\\&\leq 1+|f_0(t_0)|+\varepsilon.\end{aligned}$$

If $t \in U_\varepsilon^c$, then

$$|k_\varepsilon(t) + f_0(t)| = |f_0(t)| \le 1 \le 1 + |f_0(t_0)| + \varepsilon.$$

Hence

$$\|k_\varepsilon + f_0\| \le 1 + |f_0(t_0)| + \varepsilon.$$

By Lemma 2.3, we obtain $\psi_T(t_0)(T(-k_\varepsilon)) = -\frac{f_0(t_0)}{|f_0(t_0)|}$. Then

$$\begin{aligned} (11)\qquad \left|\frac{f_0(t_0)}{|f_0(t_0)|} + \psi_T(t_0)(Tf_0)\right| &\le |\psi_T(t_0)(T(-k_\varepsilon)) - \psi_T(t_0)(Tf_0)| \\ &\le \|T(-k_\varepsilon) - Tf_0\| = \|k_\varepsilon + f_0\| \\ &\le 1 + |f_0(t_0)| + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (9) and (11) give $\psi_T(t_0)(Tf_0) = f_0(t_0)$. ∎

4. Main Theorem

Main Theorem 4.1. *Suppose Ω is a locally compact Hausdorff space, and $T : S(C_0(\Omega)) \to S(E)$ is a surjective isometry, where E is a complex Banach space. Then there exists an isometry U of $C_0(\Omega)$ onto E and two disjoint subsets A, B of Ω such that:*

(a) $A \cup B = \Omega$;
(b) $U|_{C_0(A)}$ *is linear and* $U|_{C_0(B)}$ *is conjugate linear* (*where* $C_0(A) = \{f \in C_0(\Omega) : f|_{A^c} = 0\}$ *and* $C_0(B) = \{f \in C_0(\Omega) : f|_{B^c} = 0\}$);
(c) $U|_{S(C_0(\Omega))} = T$.

Proof. Set

$$A = \{t \in \Omega : \psi_T(t)(Tf) = f(t), \forall f \in S(C_0(\Omega))\},$$
$$B = \{t \in \Omega : \psi_T(t)(Tf) = \overline{f(t)}, \forall f \in S(C_0(\Omega))\}.$$

It is obvious that A and B are well defined and disjoint. Lemma 3.15 implies that $A \cup B = \Omega$. Let $A_1 = \psi_T(A)$ and $B_1 = \psi_T(B)$. It follows from Lemma 3.7 and the definition of ϕ_T that A_1 and B_1 are disjoint subsets of $\operatorname{ext} B(E^*)$.

Now, define $U : C_0(\Omega) \to E$ as follows:

$$U(f) = \begin{cases} \|f\|T(f/\|f\|) & \text{if } \|f\| \neq 0, \\ 0 & \text{if } \|f\| = 0. \end{cases}$$

It is obvious that $U|_{S(C_0(\Omega))} = T$. For any $f, g \in C_0(\Omega)$, by Lemma 3.15 and the definition of U, we have

$$\begin{aligned} (12)\qquad \|U(f) - U(g)\| &\ge \sup_{t\in\Omega} |\psi_T(t)(U(f) - U(g))| \\ &= \sup_{t\in\Omega} |f(t) - g(t)| = \|f - g\|. \end{aligned}$$

By Theorem 2.7, U is an isometry of $C_0(\Omega)$ onto E. Also, we have

$$\psi_T(t)U(f) = f(t)\chi_A(t) + \overline{f(t)}\chi_B(t)$$

for all $f \in C_0(\Omega)$ and all $t \in \Omega$ (where χ_D is the characteristic function of D). If $f, g \in C_0(A)$ and $k_1, k_2 \in \mathbb{C}$, then

$$\begin{aligned}\psi_T(t)U(k_1 f + k_2 g) &= (k_1 f(t) + k_2 g(t))\chi_A(t) + (\overline{k_1 f(t)} + \overline{k_2 g(t)})\chi_B(t) \\ &= (k_1 f(t) + k_2 g(t))\chi_A(t) = k_1\psi_T(t)U(f) + k_2\psi_T(t)U(g) \\ &= \psi_T(t)(k_1 U(f) + k_2 U(g)).\end{aligned}$$

Note that the weak*-closed absolutely convex hull of $\{\psi_T(t) : t \in \Omega\}$ is $B(E^*)$, i.e., $\overline{\mathrm{aco}}^{w^*}\{\psi_T(t) : t \in \Omega\} = B(E^*)$. Indeed, for any $x \in S(E)$, let $f = T^{-1}(x)$. Then $\|f\| = 1$, hence there is a $t_0 \in \Omega$ with $|f(t_0)| = 1$. Then $\psi_T(t_0)(x) = \psi_T(t_0)(Tf) \in \{f(t_0), \overline{f(t_0)}\}$. It follows that $\{\psi_T(t) : t \in \Omega\}$ is 1-norming for E, hence the bipolar theorem shows that $\overline{\mathrm{aco}}^{w^*}\{\psi_T(t) : t \in \Omega\}$ $= B(E^*)$. We have

$$U(k_1 f + k_2 g) = k_1 U(f) + k_2 U(g).$$

So $U|_{C_0(A)}$ is linear. Similarly, $U|_{C_0(B)}$ is conjugate-linear on $C_0(B)$. ■

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