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On the Semadeni derivative of Banach spaces $C(K, X)$

by

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Abstract. The Semadeni derivative of a Banach space X, denoted by $\mathcal{S}(X)$, is the quotient of the space of all weak* sequentially continuous functionals in X^{**} by the canonical copy of X. In a remarkable 1960 paper, Z. Semadeni introduced this concept in order to prove that $C([0,\omega_1])$ is not isomorphic to $C([0,\omega_1]) \oplus C([0,\omega_1])$.

Here we investigate this concept in the context of $C(K, X)$ spaces. In our main result, we prove that if K is a Hausdorff compactum of countable height, then $\mathcal{S}(C(K,X))$ is isometrically isomorphic to $C(K, \mathcal{S}(X))$ for every Banach space X. Additionally, if X is a Banach space with the Mazur property, we explicitly find the derivative of $C([0,\omega_1]^n, X)$ for each $n \geq 1$. Further we obtain an example of a nontrivial Banach space linearly isomorphic to its derivative.

1. Introduction. In his 1960 classical paper, Semadeni [16] gave the first example of a Banach space X that is not isomorphic to its square, i.e., $X \nsim X \oplus X$ solving a problem raised by Banach [2]. The idea was to consider an isomorphic invariant $\mathcal{S}(X)$, the quotient of the space of all weak* sequentially continuous functionals in X^{**} by the canonical copy of X, henceforth called the Semadeni derivative of the Banach space X. Semadeni proved that $\mathcal{S}(C([0,\omega_1])) \sim \mathbb{R}$ and $\mathcal{S}(C([0,\omega_1]) \oplus C([0,\omega_1])) \sim \mathcal{S}(C([0,\omega_1])) \oplus \mathcal{S}(C([0,\omega_1])) \sim \mathbb{R}^2$ and concluded that $C([0,\omega_1]) \nsim C([0,\omega_1]) \oplus C([0,\omega_1])$.

Semadeni's idea was later extended by Kislyakov [12] to classify the Banach spaces $C([0,\alpha])$ for uncountable ordinals, completing the isomorphic classification of spaces $C([0,\alpha])$, initiated by Bessaga and Pełczyński [3].

In this paper, we investigate some properties of the Semadeni derivative for the Banach spaces of the form $C(K, X)$. We are not the first to do this though. We highlight that Galego developed several related tools in the case where K is an ordinal space and X is a Mazur space; see for example [7, 8].

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For scattered compacta K, the intuition about the Semadeni derivative $\mathcal{S}(C(K))$ is that it destroys the points of countable height. In the first part of our research, by using some ideas of Kappeler [11] we prove

THEOREM 1.1. *If K is a scattered Hausdorff compactum of countable height and X is a Banach space, then $\mathcal{S}(C(K,X))$ is linearly isometric to $C(K,\mathcal{S}(X))$.*

We also prove that the Semadeni derivative commutes with c_0-direct sums. More precisely we have

THEOREM 1.2. *If $\{X_i : i \in \Gamma\}$ is a family of Banach spaces, then $\mathcal{S}((\bigoplus_{i\in\Gamma} X_i)_{c_0})$ is linearly isometric to $(\bigoplus_{i\in\Gamma} \mathcal{S}(X_i))_{c_0}$.*

Semadeni was probably the first to notice the exotic geometric properties of the Banach space $C([0,\omega_1])$. For a recent deep investigation of $C([0,\omega_1])$ see [10]. The topological space $[0,\omega_1)^n$, for $n > 1$, shares many common properties with $[0,\omega_1)$, for example, continuous functions $f : [0,\omega_1)^n \to \mathbb{R}$ are eventually constant, that is, there is $\alpha < \omega_1$ and $r \in \mathbb{R}$ such that $f(t) = r$ whenever $t \notin [0,\alpha]^n$. Thus $[0,\omega_1)^n$ is pseudocompact (even more, it is countably compact). So it is to be expected that $C([0,\omega_1^n])$ has interesting exotic properties as well. In the second part of the paper we prove

THEOREM 1.3. *For every integer $n \geq 1$ and every Mazur space X, $\mathcal{S}(C([0,\omega_1]^n, X))$ is isomorphic to $C([0,\omega_1]^{n-1}, X) \oplus \overset{n}{\cdots} \oplus C([0,\omega_1]^{n-1}, X)$.*

In particular, for each $n \geq 1$, it follows that $C([0,\omega_1]^n)$ is not isomorphic to $C([0,\omega_1]^n) \oplus C([0,\omega_1]^n)$ (see Corollary 4.4).

According to a result of Mazurkiewicz and Sierpiński [17, Theorem 8.6.10], whenever $\alpha_1,\dots,\alpha_n$ are countable ordinals, there is a countable ordinal α such that $[0,\alpha_1] \times \cdots \times [0,\alpha_n]$ is homeomorphic to $[0,\alpha]$. There is no similar result for uncountable ordinals. Indeed, suppose that K has a subspace homeomorphic to $[0,\omega_1] \times [0,\omega]$ and let $\infty = (\omega_1,\omega)$. If K was homeomorphic to some ordinal space $[0,\alpha]$, then, since ∞ is the limit of the sequence $((\omega_1,n))_n$, ∞ corresponds to an ordinal $\beta \in [0,\alpha]$ with countable cofinality. But ∞ belongs to the closure of $A = \{(\xi,\omega) : \xi < \omega_1\}$ and no countable subset of A has ∞ as accumulation point. That is a contradiction. Consequently, if $n > 1$, the space $[0,\omega_1]^n$ is not homeomorphic to any ordinal space. Therefore, unless $n = 1$, it is not evident whether previous tools developed for example in [7, 8] can be employed to our analysis.

Further we prove

THEOREM 1.4. *There is a Banach space X such that $\mathcal{S}(X)$ is linearly isomorphic to X.*

The paper is organized as follows. In Section 2 we establish some basic notation. In Section 3 we investigate the Semadeni derivative establishing several auxiliary results and prove Theorems 1.1 and 1.2. In Section 4 we explicitly compute the derivative of $C([0,\omega_1]^n, X)$ and derive some consequences as Theorems 1.3 and 1.4. In Section 5, by applying results obtained in the previous sections, we compute the exact Banach–Mazur distance between the spaces $C([0,\omega^n k], C([0,\omega_1]))$ and $C_0(\mathbb{N}, C([0,\omega_1]))$.

2. Terminology and preliminaries. For a Hausdorff compactum K and a Banach space X, we denote by $C(K,X)$ the Banach space of all continuous functions $f : K \to X$, with the norm $\|f\| = \sup_{t\in K} \|f(t)\|$. When $X = \mathbb{R}$, this space will be simply denoted by $C(K)$.

If $\{X_i : i \in \Gamma\}$ is a family of Banach spaces, then $\big(\bigoplus_{i\in\Gamma} X_i\big)_{c_0}$ denotes its c_0-direct sum, that is, the space consisting of all $(x_i)_{i\in\Gamma} \in \prod_{i\in\Gamma} X_i$ such that $\{i \in \Gamma : \|x_i\| \geq \epsilon\}$ is finite for every $\epsilon > 0$, endowed with the norm $\|(x_i)_{i\in\Gamma}\| = \sup_{i\in\Gamma} \|x_i\|$. We use the notation $X_1 \oplus \cdots \oplus X_n$ when $\Gamma = \{1,\ldots,n\}$. When $X_i = X$ for each $i \in \Gamma$, these spaces will be denoted by $c_0(\Gamma, X)$ and X^n respectively.

A topological space K is said to be *scattered* if every nonempty subset $L \subset K$ has an isolated point in L. For a scattered space, there is always an ordinal number γ such that the γ-Cantor–Bendixson derivative $K^{(\gamma)}$ is empty [17, Definition 8.5.1]. The least such ordinal is called the *height* of K and will be denoted by $\mathfrak{H}(K)$.

Ordinal numbers, as topological spaces, will always be endowed with the usual order topology. For such spaces we use the classical interval notation. As usual, ω represents the first infinite ordinal and ω_1 the first uncountable ordinal.

For a Tikhonov space L, βL denotes its Stone–Čech compactification.

We say that Banach spaces X and Y are *isomorphic* if there is a bijective bounded linear operator $T : X \to Y$. If an operator $T : X \to Y$ is an isomorphism onto its image, we will say that T is a *linear embedding* of X into Y. In any of these cases, the number $\|T\|\,\|T^{-1}\|$ will be called the *distortion* of T. We will often write $X \sim Y$ to indicate that there is an isomorphism $T : X \to Y$ and we write $X \overset{\lambda}{\sim} Y$ to emphasize that $\|T\|\,\|T^{-1}\| \leq \lambda$. For linear embeddings, in turn, we write $X \hookrightarrow Y$ and $X \overset{\lambda}{\hookrightarrow} Y$.

For every Banach space X, X^* denotes its topological dual and X^{**} its bidual. By abuse of notation, X will always be identified with its canonical copy in X^{**}.

If Y is a closed subspace of X, the quotient map of X onto X/Y will be indicated by $x \mapsto [x]$. The following elementary fact on quotient spaces will play an important role in this article.

THEOREM 2.1. *For Banach spaces X and Y let $T : X \to Y$ be a bounded linear operator. Then the map $\widehat{T} : X/\ker(T) \to Y$ given by $\widehat{T}([x]) = T(x)$*

is a bounded linear operator with $\|\widehat{T}\| = \|T\|$. *If additionally* T *is surjective, then* $\widehat{T}$ *is an isomorphism.*

We will also need the following vector-valued Tietze extension theorem. The proof is an adaptation of an answer of Bill Johnson to a question from MathOverflow [14].

THEOREM 2.2. *Let* K *be a Hausdorff compactum and let* X *be a Banach space. If* A *is a closed subset of* K, *then every continuous function* $f : A \to X$ *can be extended to a continuous function* $F : K \to X$.

Proof. Suppose first that $X = \ell_\infty(\Gamma)$ for some set Γ. We identify isometrically $\ell_\infty(\Gamma)$ with $C(\beta\Gamma)$ and consider a continuous function $f : A \to C(\beta\Gamma)$. By the usual scalar Tietze extension theorem, we can extend the function $g : A \times \beta\Gamma \to \mathbb{R}$, $g(t, s) = f(t)(s)$, to a continuous function $G : K \times \beta\Gamma \to \mathbb{R}$. We are done in this case by taking $F : K \to C(\beta\Gamma)$, $F(t)(s) = G(t, s)$.

For the general case, let $f : A \to X$ be a continuous function where X is an arbitrary Banach space. According to [20], there is a homeomorphism $\varphi : X \to c_0(\kappa)$ where κ denotes the density character of X. Let $\iota : c_0(\kappa) \to \ell_\infty(\kappa)$ be the natural embedding. By applying the previous case, we can extend the composition $g = \iota \circ \varphi \circ f : A \to \ell_\infty(\kappa)$ to a continuous function $G : K \to \ell_\infty(\kappa)$. Since, by [6, Proposition 12], there is a Lipschitz retraction $P : \ell_\infty(\kappa) \to c_0(\kappa)$, we are done by taking $F = \varphi^{-1} \circ P \circ G$. ∎

Any other standard terminology of Banach space theory and set-theoretic topology follows [17].

3. The Semadeni derivative. Given a Banach space X, a functional $x^{**} \in X^{**}$ is called *weak* sequentially continuous* if $\langle x^{**}, x_n^*\rangle \to 0$ whenever $(x_n^*)_n$ is a weak* null sequence in X^*. Following Semadeni [16] we consider

$$X^{\mathfrak{s}} = \{x^{**} \in X^{**} : x^{**} \text{ is a weak}^* \text{ sequentially continuous functional}\}.$$

It is evident that $X^{\mathfrak{s}}$ is a closed subspace of X^{**} containing the canonical copy of X. The *Semadeni derivative* of a Banach space X (introduced in [16]) is the quotient $\mathcal{S}(X) = X^{\mathfrak{s}}/X$.

For a large class of Banach spaces X, $\mathcal{S}(X) = \{\vec{0}\}$. For example, if X is separable, then any norm bounded subset of X^* endowed with the weak* topology is metrizable. In this case, if $x^{**} \in X^{\mathfrak{s}}$, then x^{**} is weak* continuous on the norm bounded subsets of X^* and hence, by the Banach–Dieudonné theorem, $x^{**} \in X$. Therefore, $\mathcal{S}(X) = \{\vec{0}\}$. The Banach spaces X such that $\mathcal{S}(X) = \{\vec{0}\}$, namely, where every weak* sequentially continuous functional in X^{**} is weak* continuous, are called *Mazur spaces*. This class includes several important subclasses, like WCG spaces [1]. We recommend [11, 13] for more examples.

In [16], Semadeni used the fact that his derivative is an isomorphic invariant. In the first result of this section, with a slight modification of [12, Lemma 1.4], we show that it also preserves distortion.

THEOREM 3.1. *For all Banach spaces X and Y, if $X \overset{\lambda}{\hookrightarrow} Y$, then $\mathcal{S}(X) \overset{2\lambda}{\hookrightarrow} \mathcal{S}(Y)$. If $X \overset{\lambda}{\sim} Y$, then $\mathcal{S}(X) \overset{\lambda}{\sim} \mathcal{S}(Y)$.*

Proof. Let $T : X \to Y$ be a linear embedding. Since $T^{**}(X^{\mathfrak{s}}) \subset Y^{\mathfrak{s}}$ [11, Proposition 2.2(1)], the map $Q : X^{\mathfrak{s}} \to \mathcal{S}(Y)$ given by $Q(x^{**}) = [T^{**}(x^{**})]$ is a well defined bounded linear operator with $\|Q\| \le \|T\|$. It is also clear, since $T^{**}(X) \subset Y$, that $X \subset \ker(Q)$. On the other hand, if $Q(x^{**}) = \vec{0}$, then $T^{**}(x^{**}) = y$ for some $y \in Y$. Since T is an embedding, x^{**} is weak* continuous, whence $x^{**} = x$. We deduce that $\ker(Q) = X$. By Theorem 2.1, there is an injective operator $\widehat{Q} : \mathcal{S}(X) \to \mathcal{S}(Y)$ with $\|\widehat{Q}\| = \|Q\| \le \|T\|$. In addition, if T is surjective, with similar arguments we may obtain the inverse operator $\widehat{Q}^{-1}$ with $\|\widehat{Q}^{-1}\| \le \|T^{-1}\|$. In this case $\widehat{Q}$ is an isomorphism with distortion $\|\widehat{Q}\|\,\|\widehat{Q}^{-1}\| \le \|T\|\,\|T^{-1}\|$.

If T is not surjective, suppose that there is $[x^{**}] \in \mathcal{S}(X)$ such that $\|[x^{**}]\| \ge 3\|T^{-1}\|/2$ and $\|\widehat{Q}([x^{**}])\| < 3/4$. Pick $y \in Y$ with $\|T^{**}(x^{**}) - y\| < 3/4$. For each $x \in X$,

$$\begin{aligned}\frac{3}{2} \le \frac{1}{\|T^{-1}\|}\|x^{**} - x\| &\le \|T^{**}(x^{**}) - T^{**}(x)\| \\ &\le \|T^{**}(x^{**}) - y\| + \|y - T^{**}(x)\|.\end{aligned}$$

Hence, $\inf_{x\in X} \|y - T(x)\| \ge 3/4$. By the Hahn–Banach theorem, there is y^* in the unit sphere of Y^* with $\langle y^*, y\rangle \ge 3/4$ and $T^*(y^*) = \vec{0}$. It follows that

$$\tfrac{3}{4} \le |\langle y^*, y\rangle| = |\langle T^{**}(x^{**}) - y, y^*\rangle| \le \|T^{**}(x^{**}) - y\| < \tfrac{3}{4},$$

a contradiction. We deduce that for each $[x^{**}] \in \mathcal{S}(X)$,

$$\frac{1}{2\|T^{-1}\|}\|[x^{**}]\| \le \|\widehat{Q}([x^{**}])\| \le \|T\|\,\|[x^{**}]\|.$$

Therefore, $\widehat{Q}$ is a linear embedding with $\|\widehat{Q}\|\,\|\widehat{Q}^{-1}\| \le 2\|T\|\,\|T^{-1}\|$. ■

Before proving the main results of this section, we need to establish a number of auxiliary results.

LEMMA 3.2. *Let K be a zero-dimensional Hausdorff compactum and let X be a Banach space. Then for each $f \in C(K,X)$ there is a net $(f_j)_{j\in\Gamma}$ in $C(K,X)$, of functions with finite image, such that* $\lim_{j\to\infty} \|f_j - f\| = 0$.

Proof. Let Γ be the set of all partitions of K into a finite number of pairwise disjoint nonempty clopen subsets of K, directed by following relation: $i \le j$ if and only if j is finer than i, that is, every element of i is the union of elements of j. Let $f \in C(K,X)$ be arbitrary. For each nonempty clopen set $U \subset K$ we fix $t_U \in U$ and, for each $j \in \Gamma$, we define $f_j : K \to X$ by

$$f_j(t) = \sum_{U \in j} f(t_U) \cdot \chi_U(t),$$

where, for each U, χ_U denotes the characteristic function of U. It is clear that $(f_j)_{j\in\Gamma}$ is a net in $C(K, X)$. Given $\epsilon > 0$, for each $t \in K$, we fix a clopen set V_t with $t \in V_t \subset f^{-1}(B_{\epsilon/2}(f(t)))$ $(B_{\epsilon/2}(f(t)) = \{x \in X : \|x - f(t)\| < \epsilon/2\})$. Let $t_1, \ldots, t_n$ be such that $K = V_{t_1} \cup \cdots \cup V_{t_n}$ (we assume that n is the least possible integer for which such a collection exists). We define $U_1 = V_{t_1}$ and, for each $1 < k \le n$, $U_k = V_{t_k} \setminus \bigcup_{s<k} V_{t_s}$. It follows that $j_0 = \{U_1, \ldots, U_n\} \in \Gamma$ and it is straightforward checking that $\|f - f_j\| < \epsilon$ whenever $j \ge j_0$. ∎

When K is a scattered compactum, $C(K)^*$ can be isometrically identified with $\ell_1(K)$ [17, Theorem 19.7.6]. With a similar reasoning but applying Singer's representation theorem [18, p. 192], it follows that for every Banach space X, $C(K, X)^*$ can be isometrically identified with $\ell_1(K, X^*)$, and hence $C(K, X)^{**}$ can be isometrically identified with $\ell_\infty(K, X^{**})$. This identification will be widely used from now on.

THEOREM 3.3. *Let K be a scattered Hausdorff compactum and X be a Banach space. Then*

$$C(K, X^{\mathfrak{s}}) \subset C(K, X)^{\mathfrak{s}} \subset \ell_\infty(K, X^{\mathfrak{s}}).$$

Proof. We first prove that $C(K, X^{\mathfrak{s}}) \subset C(K, X)^{\mathfrak{s}}$. Indeed, let U be a nonempty clopen subset of K, $x^{**} \in X^{\mathfrak{s}}$ be arbitrary and consider $F = x^{**} \cdot \chi_U \in C(K, X^{\mathfrak{s}})$. Let $(g_n)_n$ be a weak* null sequence in $\ell_1(K, X^*)$. For each $n \in \mathbb{N}$, if $g_n = \sum_{t\in K} x_t^* \cdot \delta_t$, then $\sum_{t\in K} \|x_t^*\| < \infty$. Hence, $G_n = \sum_{t\in U} x_t^*$ is an element of X^*. For an arbitrary $x \in X$, consider the function $f_x = x \cdot \chi_U \in C(K, X)$. Since $(g_n)_n$ is weak* null, we have

$$\lim_{n\to\infty} \langle G_n, x\rangle = \lim_{n\to\infty} \langle g_n, f_x\rangle = 0.$$

We deduce that $(G_n)_n$ is a weak* null sequence in X^*. Hence, since $x^{**} \in X^{\mathfrak{s}}$, we have

$$\lim_{n\to\infty} \langle F, g_n\rangle = \lim_{n\to\infty} \langle x^{**}, G_n\rangle = 0.$$

Therefore, $C(K, X)^{\mathfrak{s}}$ contains the set $\mathcal{F}(K, X^{\mathfrak{s}})$ of all continuous functions $F : K \to X^{\mathfrak{s}}$ with finite image. Since, by Lemma 3.2, $\mathcal{F}(K, X^{\mathfrak{s}})$ is a dense subset of $C(K, X^{\mathfrak{s}})$, we conclude that $C(K, X^{\mathfrak{s}}) \subset C(K, X)^{\mathfrak{s}}$.

For the second inclusion let $f \in C(K, X)^{\mathfrak{s}}$ be arbitrary. Then f is an element of $\ell_\infty(K, X^{**})$. Given $t \in K$, let $(x_n^*)_n$ be a weak* null sequence in X^*. Then $(\delta_t \cdot x_n^*)_n$ is a weak* null sequence in $\ell_1(K, X^*)$ and we have

$$\lim_{n\to\infty} \langle f, \delta_t \cdot x_n^*\rangle = \lim_{n\to\infty} \langle f(t), x_n^*\rangle = 0.$$

Thus, $f(t) \in X^{\mathfrak{s}}$. We deduce that $C(K, X)^{\mathfrak{s}} \subset \ell_\infty(K, X^{\mathfrak{s}})$. ∎

REMARK 3.4. When X is a Mazur space, Theorem 3.3 gives $C(K,X) \subset C(K,X)^{\mathfrak{s}} \subset \ell_\infty(K,X)$.

PROPOSITION 3.5. *Let K be a scattered Hausdorff compactum and X be a Banach space. For each $f \in C(K,X)^{\mathfrak{s}}$ and for each closed subset $V \subset K$, $f{\restriction}_V \in C(V,X)^{\mathfrak{s}}$.*

Proof. Let $f \in C(K,X)^{\mathfrak{s}}$ be arbitrary, $V \subset K$ be a nonempty closed subset and $(g_n)_n$ be weak* null sequence in $\ell_1(V,X^*)$. For each n we extend g_n to an element $\overline{g}_n \in \ell_1(K,X^*)$ with same support. It is evident that $(\overline{g}_n)_n$ is a weak* null sequence in $\ell_1(K,X^*)$. Then, since $f \in C(K,X)^{\mathfrak{s}}$,

$$\lim_{n\to\infty} \langle f{\restriction}_V, g_n\rangle = \lim_{n\to\infty} \langle f, \overline{g}_n\rangle = 0.$$

We deduce that $f{\restriction}_V \in C(V,X)^{\mathfrak{s}}$. ∎

The following crucial lemma is a slight modification of [11, Theorem 4.1].

LEMMA 3.6. *If K is a scattered Hausdorff compactum of countable height and X is a Banach space, then $C(K,X)^{\mathfrak{s}} = C(K,X^{\mathfrak{s}})$.*

Proof. From Theorem 3.3 we have $C(K,X^{\mathfrak{s}}) \subset C(K,X)^{\mathfrak{s}}$. To establish the reverse inclusion, we proceed by transfinite induction through the formula

$$P(\alpha) : \forall K \ (K \text{ is a compactum with } \mathfrak{H}(K) = \alpha+1 \Rightarrow C(K,X)^{\mathfrak{s}} \subset C(K,X^{\mathfrak{s}})).$$

If $\alpha = 0$, then $P(0)$ is true by Theorem 3.3. Let $1 \le \alpha < \omega_1$ be arbitrary and suppose that $P(\beta)$ holds for every $\beta < \alpha$. For an arbitrary compactum K of height $\alpha+1$ we pick an element $f \in C(K,X)^{\mathfrak{s}}$. According to Theorem 3.3, f is a bounded function from K to $X^{\mathfrak{s}}$. We will be done by proving that f is continuous in K.

If $t \in K \setminus K^{(\alpha)}$, there is a clopen subset V of K such that $t \in V \subset K \setminus K^{(\alpha)}$. Since, by Proposition 3.5, the restriction $f{\restriction}_V$ is an element of $C(V,X)^{\mathfrak{s}}$ and $V^{(\alpha)} = \emptyset$, the induction hypothesis applies and we deduce that f is continuous at t. It remains to prove the continuity of f at all points of $K^{(\alpha)}$. Since $K^{(\alpha)}$ is finite, there is no loss of generality in assuming that $K^{(\alpha)} = \{\infty\}$. If $\alpha = \beta+1$, then $K^{(\beta)}$ is the Aleksandrov one-point compactification of an infinite discrete set Γ, i.e., $K^{(\beta)} = \Gamma \cup \{\infty\}$. If $f_\beta := f{\restriction}_{K^{(\beta)}}$ is not continuous at ∞, there is $\epsilon > 0$ such that, for each finite set $j \subset \Gamma$, we may fix a point $t_j \in \Gamma \setminus j$ such that $\|f_\beta(t_j) - f_\beta(\infty)\| \ge \epsilon$. Let $(t_{j_n})_n$ be any sequence of distinct such points. For each $n \in \mathbb{N}$, let x_n^* in the unit sphere of X^* be such that

$$\langle x_n^*, f_\beta(t_n) - f_\beta(\infty)\rangle \ge \epsilon/2.$$

It follows that $(g_n)_n$, $g_n = (\delta_{t_{j_n}} - \delta_\infty) \cdot x_n^*$, is a weak* null sequence in $\ell_1(K^{(\beta)}, X^*)$. By Proposition 3.5 we have $f_\beta \in C(K^{(\beta)}, X)^{\mathfrak{s}}$. Then

$$\lim_{n\to\infty} \langle f_\beta, g_n\rangle = \lim_{n\to\infty} \langle x_n^*, f_\beta(t_{j_n}) - f_\beta(\infty)\rangle = 0,$$

which is a contradiction.

Let $\epsilon > 0$ be arbitrary. Since, by the previous argument, $f\restriction_{K^{(\beta)}}$ is continuous at ∞, there is $\{t_1, \ldots, t_n\} \subset \Gamma$ such that $\|f(t) - f(\infty)\| < \epsilon/2$ for each $t \in K^{(\beta)} \setminus \{t_1, \ldots, t_n\}$. Since f is continuous at every point of Γ, each point $t \in \Gamma \setminus \{t_1, \ldots, t_n\}$ admits a clopen neighborhood $V_t \subset K \setminus \{t_1, \ldots, t_n, \infty\}$ so that $f(V_t) \subset B_{\epsilon/2}(f(t))$. If $V = \bigcup_{t\in\Gamma\setminus\{t_1,\ldots,t_n\}} V_t$, then $(K \setminus V)^{(\alpha)} = \emptyset$ and, by Proposition 3.5, $f\restriction_{K\setminus V} \in C(K \setminus V, X)^{\mathfrak{s}}$. By the induction hypothesis there is an open set $U \subset K$ such that $\infty \in U$ and $\|f(t) - f(\infty)\| < \epsilon$ for each $t \in U \cap (K \setminus V) = U \setminus V$. Hence $f(U) \subset B_\epsilon(\infty)$. We deduce that f is continuous at ∞.

If α is a limit ordinal, then $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$. Let $\epsilon > 0$ be arbitrary. We assert that there is $\beta < \alpha$ such that $\|f(t) - f(\infty)\| < \epsilon/2$ for each $t \in K^{(\beta)}$. Indeed, otherwise we fix any bijection $\varphi : \omega \to \alpha$ and construct recursively sequences $(\xi_n)_n$ and $(t_n)_n$ in the following way. We fix $\xi_0 = \varphi(0)$ and $t_0 \in K^{(\xi_0)} \setminus f^{-1}(B_{\epsilon/2}(f(\infty)))$, and for each $n < \omega$ we let $\xi_{n+1} = \varphi(\min\{k < \omega : t_n \notin K^{(\varphi(k))}\})$ and fix $t_{n+1} \in K^{(\xi_{n+1})} \setminus f^{-1}(B_{\epsilon/2}(f(\infty)))$. It is readily seen that $(\xi_n)_n$ is cofinal in α and $(t_n)_n$ has pairwise distinct terms. Moreover, by construction, $(t_n)_n$ cannot accumulate at any other point than ∞. Therefore, $t_n \to \infty$. For each n we fix x_n^* in the unit sphere X^* such that

$$\langle x_n^*, f(t_n) - f(\infty)\rangle \geq \epsilon/3.$$

It is clear that $(g_n)_n$, given by $g_n = (\delta_{t_n} - \delta_\infty) \cdot x_n^*$ is a weak* null sequence in $\ell_1(K, X^*)$. Then, since $f \in C(K, X)^{\mathfrak{s}}$, we have

$$\lim_{n\to\infty} \langle f, g_n\rangle = \lim_{n\to\infty} \langle x_n^*, f(t_n) - f(\infty)\rangle = 0,$$

a contradiction.

Thus, we may fix $\beta < \alpha$ according to our assertion. Since f is continuous in $K \setminus \{\infty\}$, for every $t \in K^{(\beta)}$ there exists an open neighborhood of t, $V_t \subset K \setminus \{\infty\}$, such that $\|f(s) - f(t)\| < \epsilon/2$ for each $s \in V_t$. Then, if $s \in V_t$ one obtains $\|f(s) - f(\infty)\| < \epsilon$. Let $V = \bigcup_{t\in K^{(\beta)}} V_t$. Then $K \setminus V$ is compact and $(K \setminus V)^{(\beta)} \subset (K \setminus V) \cap K^{(\beta)} = \{\infty\}$, hence $(K \setminus V)^{(\alpha)} = \emptyset$. By the induction hypothesis there is an open neighborhood of ∞, $U \subset K$, such that $\|f(t) - f(\infty)\| < \epsilon$ for each $t \in U \cap (K \setminus V) = U \setminus V$. Thus $f(U) \subset B_\epsilon(f(\infty))$. We deduce that f is continuous at ∞. ∎

The following result, which can also be deduced from [11, Theorem 4.1 and Proposition 5.1], is an immediate consequence of Lemma 3.6.

COROLLARY 3.7. *If K is a Hausdorff compactum with countable height and X is a Mazur space, then $C(K, X)$ is a Mazur space.*

COROLLARY 3.8. *Let K be a scattered Hausdorff compactum and let X be a Banach space. If $f \in C(K, X)^{\mathfrak{s}}$, then f is a bounded function from K to $X^{\mathfrak{s}}$, continuous at every point of countable height.*

Proof. Let $f \in C(K,X)^{\mathfrak{s}}$ be arbitrary. According to Proposition 3.3, f is a bounded function from K to $X^{\mathfrak{s}}$. If $t_0 \in K$ is a point of countable height, let V be a clopen subset of K such that $t \in V \subset K \setminus K^{(\omega_1)}$. By Proposition 3.5, $f\restriction_V$ is an element of $C(V,X)^{\mathfrak{s}}$. Hence, by Theorem 3.6, $f\restriction_V$ is continuous. We deduce that f is continuous at t_0. ∎

LEMMA 3.9. *Let $X_1, \dots, X_n$ be Banach spaces and $Y_k \subset X_k$ be a closed subspace for every $k = 1, \dots, n$. Let $X = X_1 \oplus \cdots \oplus X_n$ and $Y = Y_1 \oplus \cdots \oplus Y_n$. For every $f \in X$,*

$$\inf_{g\in Y} \max_{1\le k\le n} \|f(k) - g(k)\| = \max_{1\le k\le n} \inf_{x\in Y_k} \|f(k) - x\|.$$

Proof. For each $1 \le k \le n$ and $g \in Y$ we have

$$\inf_{x\in Y_k} \|f(k) - x\| \le \|f(k) - g(k)\| \le \max_{1\le k\le n} \|f(k) - g(k)\|.$$

Therefore,

$$\max_{1\le k\le n} \inf_{x\in Y_k} \|f(k) - x\| \le \inf_{g\in Y} \max_{1\le k\le n} \|f(k) - g(k)\|.$$

For the opposite inequality, let $\ell = \max_{1\le k\le n} \inf_{x\in Y_k} \|f(k)-x\|$ and pick $\epsilon > 0$ arbitrary. For each $1 \le k \le n$ we fix $x_k \in Y_k$ so that $\|f(k)-x_k\| < \ell+\epsilon$. Then

$$\inf_{g\in Y} \max_{1\le k\le n} \|f(k) - g(k)\| \le \max_{1\le k\le n} \|f(k) - x_k\| < \ell + \epsilon.$$

Since ϵ is arbitrary we deduce

$$\inf_{g\in Y} \max_{1\le k\le n} \|f(k) - g(k)\| \le \max_{1\le k\le n} \inf_{x\in Y_k} \|f(k) - x\|. \ ∎$$

We are now in a position to prove our first main result.

Proof of Theorem 1.1. From Lemma 3.6, we may define a bounded linear map $T : C(K,X)^{\mathfrak{s}} \to C(K, \mathcal{S}(X))$ by $T(f)(t) = [f(t)]$. It is readily seen that $T(f) = 0$ if and only if $f(t) \in X$ for every $t \in K$. Hence $\ker(T) = C(K,X)$ and, by Theorem 2.1, the induced operator $\widehat{T} : \mathcal{S}(C(K,X)) \to C(K, \mathcal{S}(X))$ satisfies $\|\widehat{T}([f])\| \le \|[f]\|$ for each $f \in C(K,X)^{\mathfrak{s}}$.

On the other hand, if $f \in C(K,X)^{\mathfrak{s}}$, then $f \in C(K, X^{\mathfrak{s}})$ by Lemma 3.6, and we may fix a net $(f_j)_{j\in\Gamma}$ as in Lemma 3.2 such that $\lim_{j\to\infty} \|f_j - f\| = 0$. For each $j \in \Gamma$, if $f_j(t) = \sum_{U\in j} f(t_U) \cdot \chi_U$, applying Lemma 3.9 we obtain

$$\begin{aligned}\|\widehat{T}([f_j])\| = \sup_{t\in K} \|[f_j(t)]\| &= \sup_{t\in K} \inf_{x\in X} \|f_j(t) - x\| = \max_{U\in j} \inf_{x\in X} \|f(t_U) - x\| \\ &= \inf_{U\in j,\, x_U\in X} \max_{U\in j} \|f(t_U) - x_U\| \\ &\ge \inf_{g\in C(K,X)} \sup_{t\in K} \|f_j(t) - g(t)\| = \|[f_j]\|.\end{aligned}$$

By taking the limit in both sides of the previous relation we obtain $\|\widehat{T}([f])\| \ge \|[f]\|$. We deduce that $\widehat{T}$ is an isometry onto its image.

To see that $\widehat{T}$ is surjective, note that if $g = [x^{**}] \cdot \chi_U$, where U is a clopen subset of K and $x^{**} \in X^{\mathfrak{s}}$, then $T(x^{**} \cdot \chi_U) = g$. Therefore, the image of T contains the set $\mathcal{F}(K, \mathcal{S}(X))$ of all continuous functions with finite image. We are done since $\widehat{T}$ is a linear embedding and $\mathcal{F}(K, \mathcal{S}(X))$ is a dense subset of $C(K, \mathcal{S}(X))$. ∎

For the second main result we need a lemma that was inspired by [13, Theorem 4.2].

Lemma 3.10. *Let $\{X_i : i \in \Gamma\}$ be a nonempty family of Banach spaces. Then*

$$\Big(\bigoplus_{i\in\Gamma} X_i\Big)^{\mathfrak{s}}_{c_0} = \Big(\bigoplus_{i\in\Gamma} X_i^{\mathfrak{s}}\Big)_{c_0}.$$

Proof. Let $F = (x_i^{**})_{i\in\Gamma} \in (\bigoplus_{i\in\Gamma} X_i)^{\mathfrak{s}}_{c_0}$ be arbitrary. For every $j \in \Gamma$, if $(x_n^*)_n$ is a weak* null sequence in X_j^*, then $(\delta_j \cdot x_n^*)_n$ is a weak* null sequence in $(\bigoplus_{i\in\Gamma} X_i^*)_{\ell_1}$. Hence

$$\lim_{n\to\infty} \langle x_j^{**}, x_n^*\rangle = \lim_{n\to\infty} \langle F, \delta_j \cdot x_n^*\rangle = 0.$$

We deduce $(\bigoplus_{i\in\Gamma} X_i)^{\mathfrak{s}}_{c_0} \subset (\bigoplus_{i\in\Gamma} X_i^{\mathfrak{s}})_{c_0}$.

On the other hand, let $F = (x_i^{**})_{i\in\Gamma} \in (\bigoplus_{i\in\Gamma} X_i^{\mathfrak{s}})_{c_0}$ and let $(g_n)_n$, $g_n = (x^*_{(i,n)})_{i\in\Gamma}$, be a weak* null sequence in the unit ball of $(\bigoplus_{i\in\Gamma} X_i^*)_{\ell_1}$. Given $\epsilon > 0$, let $A = \{i \in \Gamma : \|x_i^{**}\| \geq \epsilon/2\}$. For each $j \in \Gamma$, $(x^*_{(j,n)})_n$ is a weak* null sequence in X_j^*. Since A is a finite set, there is $N \in \mathbb{N}$ such that $|\langle x_j^{**}, x^*_{(j,n)}\rangle| < \epsilon/(2(|A|+1))$ for each $n \geq N$ and each $j \in A$. Then for each $n \geq N$,

$$\begin{aligned} |\langle F, g_n\rangle| &\leq \sum_{i\in\Gamma} |\langle x_i^{**}, x^*_{(i,n)}\rangle| = \sum_{j\in A} |\langle x_j^{**}, x^*_{(j,n)}\rangle| + \sum_{i\in\Gamma\setminus A} |\langle x_i^{**}, x^*_{(i,n)}\rangle| \\ &< \sum_{j\in A} |\langle x_j^{**}, x^*_{(j,n)}\rangle| + \frac{\epsilon}{2} \sum_{i\in\Gamma\setminus A} \|x^*_{(i,n)}\| < \frac{\epsilon}{2(|A|+1)}|A| + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Hence $\lim_{n\to\infty} \langle F, g_n\rangle = 0$. We deduce $(\bigoplus_{i\in\Gamma} X_i^{\mathfrak{s}})_{c_0} \subset (\bigoplus_{i\in\Gamma} X_i)^{\mathfrak{s}}_{c_0}$. ∎

Proof of Theorem 1.2. The proof follows a similar argument to Theorem 1.1. From Lemma 3.10 we may define a bounded linear operator $T : (\bigoplus_{i\in\Gamma} X_i)^{\mathfrak{s}}_{c_0} \to (\bigoplus_{i\in\Gamma} \mathcal{S}(X_i))_{c_0}$ by $((x_i^{**})_{i\in\Gamma})(t) = ([x_i^{**}])_{i\in\Gamma}$. It is readily seen that T is surjective and $\ker(T) = (\bigoplus_{i\in\Gamma} X_i)_{c_0}$. Then, according to Theorem 2.1, the induced operator $\widehat{T} : \mathcal{S}((\bigoplus_{i\in\Gamma} X_i)_{c_0}) \to (\bigoplus_{i\in\Gamma} \mathcal{S}(X_i))_{c_0}$ is an isomorphism with $\|\widehat{T}\| = \|T\| \leq 1$.

To check that $\widehat{T}$ is an isometry, let

$$f = (x_i^{**})_{i\in\Gamma} \in \Big(\bigoplus_{i\in\Gamma} X_i\Big)^{\mathfrak{s}}_{c_0} = \Big(\bigoplus_{i\in\Gamma} X_i^{\mathfrak{s}}\Big)_{c_0}$$

be arbitrary. If $\mathcal{P}$ is the collection of all finite subsets of Γ ordered by inclusion, then the net $(f_A)_{A\in\mathcal{P}}$, given by $f_A=\sum_{i\in A}x_i^{**}\cdot\chi_{\{i\}}$, converges in norm to f. By applying Lemma 3.9, for each $A\in\mathcal{P}$, we have

$$\begin{aligned}\|\widehat{T}([f_A])\| &= \max_{i\in A}\|[x_i^{**}]\| = \max_{i\in A}\inf_{x\in X_i}\|x_i^{**}-x\| \\ &= \inf_{i\in A,\, x_i\in X_i}\max_{i\in A}\|x_i^{**}-x_i\| \ge \inf_{g\in(\bigoplus_{i\in\Gamma}X_i)_{c_0}}\|f_A-g\| = \|[f_A]\|.\end{aligned}$$

Therefore, $\|\widehat{T}([f])\|\ge\|[f]\|$. We deduce that $\widehat{T}$ is an isometry. ∎

4. The derivative of $C([0,\omega_1]^n,X)$. In this section, X denotes an arbitrary Mazur space, $L=[0,\omega_1)$, $K=[0,\omega_1]$ and $n\ge 1$ denotes a natural number. The space K^n is endowed with the usual product topology and the symbol ∞ stands for the corner point $(\omega_1,\overset{n}{\ldots},\omega_1)\in K^n$. For each $A\subset\{1,\dots,n\}$, we define the function $\varphi_A:K^n\to K^n$ by setting, for each $x=(x_1,\dots,x_n)\in K^n$, $\varphi_A(x)=(y_1,\dots,y_n)$ where

$$y_j=\begin{cases}x_j & \text{if } j\in A,\\ \omega_1 & \text{otherwise.}\end{cases}$$

For each $0\le k\le n$, we let $\mathcal{F}_k=\{A\subset\{1,\dots,n\}:|A|=k\}$ and define

$$\Omega_k=\bigcup_{A\in\mathcal{F}_k}\varphi_A(L^n).$$

It is clear that Ω_k is homeomorphic to the topological sum of $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ copies of L^k, in symbols,

$$\Omega_k\approx\bigoplus_{j=1}^{\binom{n}{k}}L^k.$$

Since L^k is pseudocompact for every k, the collection $\{\Omega_k:0\le k\le n\}$ constitutes a partition of K^n into pairwise disjoint pseudocompact subsets. Furthermore, by [9, Theorem 1], we have

$$\beta\Omega_k\approx\bigoplus_{j=1}^{\binom{n}{k}}\beta(L^k)\approx\bigoplus_{j=1}^{\binom{n}{k}}(\beta L)^k\approx\bigoplus_{j=1}^{\binom{n}{k}}K^k.$$

LEMMA 4.1. *Let $f:K^n\to X$ be a function such that $f\restriction_{\Omega_k}$ is continuous for each $0\le k\le n$. Then $f\in C(K^n,X)^{\mathfrak{s}}$.*

Proof. Let $(g_m)_m$, $g_m=(x^*_{(t,m)})_{t\in K^n}$, be a weak* null sequence in $\ell_1(K^n,X^*)$. For each $m\in\mathbb{N}$, since g_m has countable support, there is $\alpha_m<\omega_1$ such that

$$\{t\in K^n: x^*_{(t,m)}\ne\vec{0}\}\subset\bigcup_{0\le j\le n}\bigcup_{A\in\mathcal{F}_j}\varphi_A([0,\alpha_m]^n).$$

Let $\alpha = \sup_{m\in\mathbb{N}} \alpha_m$ and consider the set

$$V = \bigcup_{0\le j\le n} \bigcup_{A\in\mathcal{F}_j} \varphi_A([0,\alpha]^n).$$

Since V is closed in K^n and $f{\restriction}_V$ is continuous, we can apply Theorem 2.2 and extend $f{\restriction}_V$ to a continuous function $F : K^n \to X$. For each $m \in \mathbb{N}$, since V contains the support g_m, we have

$$\begin{aligned}\langle f, g_m\rangle &= \sum_{t\in V} \langle f(t), x^*_{(t,m)}\rangle = \sum_{t\in V} \langle x^*_{(t,m)}, f(t)\rangle \\ &= \sum_{t\in V} \langle x^*_{(t,m)}, F(x)\rangle = \sum_{t\in K^n} \langle x^*_{(t,m)}, F(x)\rangle = \langle g_m, F\rangle.\end{aligned}$$

Hence, $\lim_{m\to\infty}\langle f, g_m\rangle = \lim_{m\to\infty}\langle g_m, F\rangle = 0$, so $f \in C(K^n, X)^{\mathfrak{s}}$. ∎

Theorem 4.2. *For every natural number $n \ge 1$,*

$$\mathcal{S}(C(K^n, X)) \overset{2}{\sim} C(\beta\Omega_{n-1}, X) \oplus \cdots \oplus C(\beta\Omega_1, X) \oplus C(\beta\Omega_0, X).$$

Proof. If $f \in C(K^n, X)^{\mathfrak{s}}$, by Remark 3.4, $f \in \ell_\infty(K, X)$. Furthermore, if $0 \le k \le n$, by Proposition 3.5 and Corollary 3.8, $f{\restriction}_{\Omega_k}$ is a continuous function. Since Ω_k is pseudocompact, $f(\Omega_k)$ is a compact subset of X [19, Theorem 2.3]. By the extension property of the Stone–Čech compactification, $f{\restriction}_{\Omega_k}$ can be uniquely extended to a continuous function $R_k(f) : \beta\Omega_k \to X$. We deduce that, for each $0 \le k \le n$, the formula $f \mapsto R_k(f)$ defines a bounded linear operator $R_k : C(K^n, X)^{\mathfrak{s}} \to C(\beta\Omega_k, X)$ with $\|R_k\| \le 1$. Since $\beta\Omega_n = K^n$, we have $R_n(C(K^n, X)^{\mathfrak{s}}) \subset C(K^n, X) \subset C(K^n, X)^{\mathfrak{s}}$. Hence, denoting $Y = C(\beta\Omega_{n-1}, X) \oplus \cdots \oplus C(\beta\Omega_1, X) \oplus C(\beta\Omega_0, X)$, the formula

$$T(f) = \big(R_{n-1}(f - R_n(f)), \ldots, R_1(f - R_n(f)), R_0(f - R_n(f))\big)$$

defines a bounded linear operator $T : C(K^n, X)^{\mathfrak{s}} \to Y$ with $\|T\| \le 2$.

It is readily seen that $T(f) = \vec{0}$ if and only if $R_n(f) = f$, that is, $\ker(T) = C(K^n, X)$.

For each $g = (g_{n-1}, \ldots, g_1, g_0) \in Y$ consider the function $f : K^n \to X$ given by

$$f(t) = \begin{cases} g_k(t) & \text{if } t \in \Omega_k \text{ for some } k < n,\\ 0 & \text{if } t \in \Omega_n.\end{cases}$$

From Proposition 4.1 we know that $f \in C(K^n, X)^{\mathfrak{s}}$. It is clear that $\|f\| = \|g\|$ and, since $f(t) = 0$ for each $t \in L^n$,

$$\begin{aligned}T(f) &= \big(R_{n-1}(f - R_n(f)), \ldots, R_1(f - R_n(f)), R_0(f - R_n(f))\big)\\ &= \big(R_{n-1}(f), \ldots, R_1(f), R_0(f)\big) = (g_{n-1}, \ldots, g_1, g_0) = g.\end{aligned}$$

We deduce that for each $g \in Y$ there is $f \in C(K^n, X)^{\mathfrak{s}}$ such that $T(f) = g$

and $\|f\| = \|T(f)\| \le 2\|f\|$. Therefore, the induced operator

$$\widehat{T} : C(K^n, X)^{\mathfrak{s}}/C(K^n, X) \to Y$$

given by Theorem 2.1 is an isomorphism with distortion $\|\widehat{T}\|\,\|\widehat{T}^{-1}\| \le 2$. ■

For the next proof it is worth recalling that since K is an infinite scattered Hausdorff compactum, K has a nontrivial convergent sequence, which leads to a complemented subspace of $C(K)$ isomorphic to c_0. Then, for every Banach space X, $C(K, X)$ has a complemented subspace isomorphic to $C_0(\mathbb{N}, X) = (\bigoplus_{j\in\mathbb{N}} X)_{c_0}$.

Proof of Theorem 1.3. For each $0 \le k \le n$, since $\beta\Omega_k \approx \bigoplus_{j=1}^{\binom{n}{k}} K^k$, from Theorem 4.2 we have

$$\mathcal{S}(C(K^n, X)) \sim \Big(\bigoplus_{k=0}^{n-1} \bigoplus_{j=1}^{\binom{n}{k}} C(K^k, X)\Big)_{c_0}.$$

For each $1 \le k \le n-1$, since $C(K^k, X) \sim C(K, C(K^{k-1}, X))$ has a complemented subspace isomorphic to $(\bigoplus_{j\in\mathbb{N}} C(K^{k-1}, X))_{c_0}$, we have

$$\begin{aligned} &C(K^k, X) \oplus C(K^{k-1}, X)^{\binom{n}{k-1}} \\ &\qquad \sim Y \oplus \Big(\bigoplus_{j\in\mathbb{N}} C(K^{k-1}, X)\Big)_{c_0} \oplus C(K^{k-1}, X)^{\binom{n}{k-1}} \\ &\qquad \sim Y \oplus \Big(\bigoplus_{j\in\mathbb{N}} C(K^{k-1}, X)\Big)_{c_0} \sim C(K^k, X). \end{aligned}$$

We deduce $\mathcal{S}(C(K^n, X)) \sim C(K^{n-1}, X)^n$. ■

The next result was inspired by [15, Corollary 5.3]. It also relates to [4, Corollary 1.2 and Remark 4.9].

THEOREM 4.3. *Let M_1 and M_2 be metric compacta. For each $m, n \in \mathbb{N}$, if $C(M_1 \times K^n) \sim C(M_2 \times K^m)$, then $m = n$ and $C(M_1) \sim C(M_2)$.*

Proof. Since $C(M_1)$ and $C(M_2)$ are separable, they are Mazur spaces. If $m \ge n$, taking the nth derivative and applying Theorems 1.3 and 3.1 we obtain

$$\begin{aligned} C(M_1)^{n!} &\sim \mathcal{S}^n(C(K^n, C(M_1))) \sim \mathcal{S}^n(C(K^m, C(M_2))) \\ &\sim C(K^{m-n}, C(M_2))^{\frac{m!}{(m-n+1)!}}. \end{aligned}$$

Then $C(K^{m-n}, C(M_2))$ is separable, whence $m = n$ and $C(M_1)^{n!} \sim C(M_2)^{n!}$. If M_1 is finite, then $C(M_1)^{n!} \sim (\mathbb{R}^p)^{n!}$ for some $p \in \mathbb{N}$. Then M_2 is also finite, has the same cardinality as M_1 and we have $C(M_1) \sim C(M_2)$. If M_1 and M_2 are infinite, then $C(M_1) \sim C(M_1)^{n!}$ and $C(M_2) \sim C(M_2)^{n!}$ [17, Theorems 21.5.10 and 21.5.11]. Hence $C(M_1) \sim C(M_2)$. ■

By noticing that $C([0,\omega_1]^n) \sim C(\{0\}\times[0,\omega_1]^n)$ and $C([0,\omega_1]^n)\oplus C([0,\omega_1]^n) \sim C(\{0,1\}\times[0,\omega_1]^n)$, Theorem 4.3 gives the following generalization of [16].

Corollary 4.4. *For each $n\in\mathbb{N}$, $C([0,\omega_1]^n) \nsim C([0,\omega_1]^n)\oplus C([0,\omega_1]^n)$.*

For the next proof we will use the following elementary fact: for every $\Gamma\subset\mathbb{N}$,

$$C(K^n,c_0) \overset{1}{\sim} C\Big(K^n,\Big(\bigoplus_{j\in\Gamma} c_0\Big)_{c_0}\Big) \overset{1}{\sim} \Big(\bigoplus_{j\in\Gamma} C(K^n,c_0)\Big)_{c_0}.$$

Proof of Theorem 1.4. We prove that $X=(\bigoplus_{n=0}^{\infty} C([0,\omega_1]^n,c_0))_{c_0}$ has the desired property.

For each $k\le n$, since $\beta\Omega_k \approx \bigoplus_{j=1}^{\binom{n}{k}} K^k$, we have

$$C(\beta\Omega_k,c_0) \overset{1}{\sim} \Big(\bigoplus_{j=1}^{\binom{n}{k}} C([0,\omega_1]^k,c_0)\Big)_{c_0} \overset{1}{\sim} C([0,\omega_1]^k,c_0).$$

Applying Theorem 4.2 we deduce, for each $n\ge 1$,

$$\mathcal{S}(C([0,\omega_1]^n,c_0)) \overset{2}{\sim} c_0\oplus C([0,\omega_1],c_0)\oplus\cdots\oplus C([0,\omega_1]^{n-1},c_0).$$

Then, by using Theorem 1.2 and the fact that $(\bigoplus_{j=1}^{\infty} C([0,\omega_1]^n,c_0))_{c_0} \overset{1}{\sim} C([0,\omega_1]^n,c_0)$ for each n, it follows that

$$\begin{aligned}\mathcal{S}(X) &\overset{1}{\sim} \Big(\bigoplus_{n=0}^{\infty}\mathcal{S}(C([0,\omega_1]^n,c_0))\Big)_{c_0} \overset{2}{\sim} \Big(\bigoplus_{n=1}^{\infty}\Big(\bigoplus_{0\le k<n} C([0,\omega_1]^k,c_0)\Big)_{c_0}\Big)_{c_0}\\ &\overset{1}{\sim} \Big(\bigoplus_{n=0}^{\infty}\Big(\bigoplus_{k=0}^{\infty} C([0,\omega_1]^n,c_0)\Big)_{c_0}\Big)_{c_0} \overset{1}{\sim} \Big(\bigoplus_{n=0}^{\infty} C([0,\omega_1]^n,c_0)\Big)_{c_0} = X.\end{aligned}$$

We conclude that $\mathcal{S}(X) \overset{2}{\sim} X$. ∎

5. Further applications. We recall that the Banach–Mazur distance between Banach spaces X and Y is defined (as $+\infty$ if $X\nsim Y$) as the infimum of the distortions $\|T\|\,\|T^{-1}\|$ taken over all isomorphisms $T:X\to Y$.

Theorem 5.1. *For each $1\le k,n<\omega$, the Banach–Mazur distance between $C([0,\omega^n k],C([0,\omega_1]))$ and $C_0(\mathbb{N},C([0,\omega_1]))$ is exactly $2n+1$.*

Proof. Let ℓ be the required Banach–Mazur distance. According to [5, Theorem 1.3.], for $1\le k,n<\omega$, there exists an isomorphism $T:C([0,\omega^n k]) \to c_0$ with distortion $\|T\|\,\|T^{-1}\|=2n+1$. This operator can be lifted to an isomorphism from $C([0,\omega^n k],C([0,\omega_1]))$ to $C_0(\mathbb{N},C([0,\omega_1]))$ with the same distortion [4, Proposition 3.7]. Therefore, $\ell\le 2n+1$. On the other hand, if $T : C([0,\omega^n k], C([0,\omega_1])) \to C_0(\mathbb{N}, C([0,\omega_1])$ is an isomorphism with $\|T\|\,\|T^{-1}\| = \lambda$, combining Theorems 1.1, 1.2 and 3.1 we obtain

$$C([0,\omega^n k]) \overset{1}{\sim} \mathcal{S}(C([0,\omega^n k], C([0,\omega_1]))) \overset{\lambda}{\sim} \mathcal{S}\Big(\Big(\bigoplus_{k\in\mathbb{N}} C([0,\omega_1])\Big)_{c_0}\Big)$$
$$\overset{1}{\sim} \Big(\bigoplus_{k\in\mathbb{N}} \mathcal{S}(C([0,\omega_1]))\Big)_{c_0} \overset{1}{\sim} c_0.$$

Then, by [5, Theorem 1.2], we have $\lambda \geq 2n+1$, so $\ell \geq 2n+1$. ∎

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