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On nonlinear Rudin–Carleson type theorems

by

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Abstract. We study nonlinear interpolation problems for interpolation and peak-interpolation sets of function algebras. The subject goes back to the classical Rudin–Carleson interpolation theorem. In particular, we prove the following nonlinear version of that theorem: Let $\bar{\mathbb{D}} \subset \mathbb{C}$ be the closed unit disk, $\mathbb{T} \subset \bar{\mathbb{D}}$ the unit circle, $S \subset \mathbb{T}$ a closed subset of Lebesgue measure zero and M a connected complex manifold. Then for every continuous M-valued map f on S there exists a continuous M-valued map g on $\bar{\mathbb{D}}$ holomorphic on its interior such that $g|_S = f$. We also consider similar interpolation problems for continuous maps $f : S \to \bar{M}$, where $\bar{M}$ is a complex manifold with boundary ∂M and interior M. Assuming that $f(S) \cap \partial M \neq \emptyset$, we are looking for holomorphic extensions g of f such that $g(\bar{\mathbb{D}} \setminus S) \subset M$.

1. Formulation of main results

1.1. Let A be a *uniform* algebra on a compact Hausdorff space X, i.e., a closed unital subalgebra of the Banach algebra $C(X)$ of complex continuous functions on X equipped with the norm $\|f\|_{C(X)} := \max_X |f|$ separating points of X. (For the theory of uniform algebras see, e.g., the book [G].)

A compact subset $S \subset X$ is said to be an *interpolation set* for A if the restriction to S maps A onto $C(S)$. The number

$$c_A(S) := \sup_{f \in C(S),\, \|f\|_{C(S)}=1} \inf \{\|F\|_{C(X)} : F \in A,\, F|_S = f\} \tag{1.1}$$

(finite by the Banach open mapping theorem) is called the *interpolation constant* for S.

In this paper we consider interpolation problems for continuous maps of S to complex manifolds. To formulate our results we require several definitions.

The *maximal ideal space* $\mathfrak{M}(A)$ is the set of all nonzero complex homomorphisms of A. It is a compact subset of the closed unit ball of the

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dual space A^* equipped with the weak* topology. The *Gelfand transform* $\hat{} : A \to C(\mathfrak{M}(A))$, $\hat{a}(\varphi) := \varphi(a)$, maps A isometrically onto a uniform subalgebra on $\mathfrak{M}(A)$, and the correspondence $x \mapsto \delta_x$ (the evaluation functional at x) embeds X into $\mathfrak{M}(A)$. Without loss of generality we will identify A with its image under $\hat{}$, and X with its image under the embedding.

A complex manifold M is said to be *Oka* if every holomorphic map $f : K \to M$ from a neighbourhood of a compact convex set $K \subset \mathbb{C}^n$, $n \in \mathbb{N}$, can be approximated uniformly on K by entire maps $\mathbb{C}^n \to M$.

The class of Oka manifolds includes, in particular, complex homogeneous manifolds, complements in $\mathbb{C}^n$, $n > 1$, of complex algebraic subvarieties of codimension ≥ 2 and of compact polynomially convex sets, Hopf manifolds (i.e., nonramified holomorphic quotients of $\mathbb{C}^n \setminus \{0\}$). Also, holomorphic fibre bundles whose bases and fibres are Oka manifolds are Oka manifolds as well. (We refer to the book [F1] and the paper [K] for other examples and basic results of the theory of Oka manifolds.)

In what follows, $C(X,Y)$ stands for the set of continuous maps between topological spaces X and Y. For a uniform algebra A on X and a subspace $M \subset \mathbb{C}^n$, we denote by $A(X,M)$ the set of continuous maps on X taking values in M whose components are in A. For $V \subset C(X,Y)$ and $S \subset X$ the *trace space* $V|_S$ ($\subset C(S,Y)$) consists of restrictions of maps in V to S.

The following result is a particular case of [Br, Thm. 1.4].

THEOREM 1.1. *Let $M \subset \mathbb{C}^n$ be a complex regular submanifold* ([1]) *and an Oka manifold and $S \subset \mathfrak{M}(A)$ be an interpolation set for a uniform algebra A. Then*

$$A(\mathfrak{M}(A), M)|_S = C(\mathfrak{M}(A), M)|_S.$$

In other words, under the above conditions a map $f \in C(S,M)$ extends to a $g \in A(\mathfrak{M}(A), M)$ if and only if it extends to a map in $C(\mathfrak{M}(A), M)$. In the next result, we show that for totally disconnected interpolation sets (such as in the Rudin–Carleson theorem) similar interpolation problems are always solvable in a more general setting.

Let A be a uniform algebra on X. For a family $\mathcal{F} = \{f_1, \dots, f_n\} \subset A$ we denote by $A_{\mathcal{F}} \subset A$ the closed unital subalgebra generated by $f_1, \dots, f_n$. The maximal ideal space $\mathfrak{M}(A_{\mathcal{F}})$ can be naturally identified with the polynomially convex hull of the compact set $\{F(x) := (f_1(x), \dots, f_n(x)) \in \mathbb{C}^n : x \in X\} \subset \mathbb{C}^n$, the *joint spectrum* of $\mathcal{F}$.

Let M be a complex manifold and g be a holomorphic map into M defined on a neighbourhood of $\mathfrak{M}(A_{\mathcal{F}})$. The continuous map $F^*g := g \circ F : \mathfrak{M}(A) \to M$ is said to be *holomorphic*; the set of such maps is denoted

([1]) That is, M is equipped with the induced topology.

by $\mathcal{O}_{\mathcal{F}}(\mathfrak{M}(A), M)$. Note that if $M \subset \mathbb{C}^N$, $N \in \mathbb{N}$, then $\mathcal{O}_{\mathcal{F}}(\mathfrak{M}(A), M) \subset A(\mathfrak{M}(A), M)$ (this is an immediate consequence of the functional calculus in a commutative Banach algebra; see, e.g., [G]).

THEOREM 1.2. *Let $S \subset X$ be a totally disconnected interpolation set for A and let M be a connected complex manifold.*

(a) *If $c_A(S) = 1$, then for every $f \in C(S, M)$ there exists a map $g \in \mathcal{O}_{\mathcal{F}}(\mathfrak{M}(A), M)$, where $|\mathcal{F}| = \dim_{\mathbb{C}} M$, such that $g|_S = f$.*
(b) *Suppose M is an Oka manifold. Let $K \Subset M$ be an open relatively compact subset. There is a subset $L \Subset M$ containing K such that for every $f \in C(S, K)$ there exists a map $g \in \mathcal{O}_{\mathcal{F}}(\mathfrak{M}(A), M)$, where $|\mathcal{F}| = \dim_{\mathbb{C}} M$, such that $g(\mathfrak{M}(A)) \subset L$ and $g|_S = f$.*

Here $|\mathcal{F}|$ stands for the cardinality of $\mathcal{F}$.

The set L in part (b) depends on M, K and the interpolation constant $c(S)$ only. If $M = \mathbb{C}$ and $K = \mathbb{D}$, then L is the disk $\{z \in \mathbb{C} : |z| < c(S)\}$. Similarly, in the nonlinear case, L can be thought of as an analog of dilation of K by scale factor $c(S)$. In particular, the manifold M must allow such nonlinear 'dilations' by arbitrarily large scale factors. This is where the hypothesis that M is Oka is used.

In turn, part (a) shows that if $c(S) = 1$ and $K \subset M$ is connected, then one can take $L = K$. Note that there is a converse to part (a) of the theorem. Specifically, if the conclusion of part (a) holds for $M = \mathbb{D}$ and every $f \in C(S, \mathbb{D})$, then $c(S) = 1$; see (1.1).

EXAMPLE 1.3. (1) Let $A(\mathbb{D}^n) \subset C(\bar{\mathbb{D}}^n)$ be the uniform algebra of continuous functions on the closure $\bar{\mathbb{D}}^n$ of the open unit polydisk $\mathbb{D}^n \subset \mathbb{C}^n$ holomorphic on $\mathbb{D}^n$. Let $\mathbb{T}^n \subset \bar{\mathbb{D}}^n$ be the boundary torus. It is proved in [RS] that every compact subset $S \subset \mathbb{T}^n$ of zero 1-dimensional Hausdorff measure is an interpolation set for $A(\mathbb{D}^n)$ with the interpolation constant 1. Since such an S is totally disconnected, Theorem 1.2 implies the following extension of the Rudin–Carleson theorem (see [R], [S]):

Let M be a connected complex manifold. For every $f \in C(S, M)$ there exists a $g \in C(\bar{\mathbb{D}}^n, M)$ holomorphic on $\mathbb{D}^n$ such that $g|_S = f$.

(For instance, this implies that every continuous complex-valued function f on a subset S of the unit circle $\mathbb{T}$ of Lebesgue measure zero extends to a disk algebra element mapping $\bar{\mathbb{D}}$ into an arbitrarily small connected neighbourhood of $f(S)$.)

(2) Let Z be a connected complex manifold such that the algebra $H^\infty(Z)$ of bounded holomorphic functions on Z separates points. A sequence $S = \{s_n\}_{n\in\mathbb{N}} \subset Z$ is called *interpolating* for $H^\infty(Z)$ if $H^\infty(Z)|_S$ coincides with the Banach space of bounded complex-valued functions on S equipped with the

supremum norm. The interpolation constant $c(S)$ is defined similarly to (1.1). Note that interpolation constants $c(S)$ can be arbitrarily large (see, e.g., the Carleson theorem on interpolating sequences in $H^\infty(\mathbb{D})$ [Ga, Ch. VII, Thm. 1.1]).

Let X be the closure of Z in the maximal ideal space $\mathfrak{M}(H^\infty(Z))$. We identify $H^\infty(Z)$ with its image in $C(\mathfrak{M}(H^\infty(Z)))$ under the Gelfand transform. Then the closure $\bar{S} \subset X$ of S is an interpolation set for $H^\infty(Z)$ with interpolation constant $c(S)$. Moreover, $\bar{S}$ is homeomorphic to the Stone–Čech compactification of $\mathbb{N}$, and hence is totally disconnected. Now, Theorem 1.2 implies that S is also an interpolating sequence for bounded holomorphic maps into connected Oka manifolds:

Let M be a connected Oka manifold and $K \subset M$ be a compact subset. There exists a compact subset $L \subset M$ depending on M, K and $c(S)$ only such that every map $f : S \to K$ can be extended to a holomorphic map $g : Z \to M$ with image in L.

1.2. In this part we consider nonlinear interpolation problems for peak-interpolation sets. (For instance, interpolation sets in the Rudin–Carleson theorem and in [RS] are peak-interpolation.)

Recall that a compact subset $S \subset X$ is said to be a *peak-interpolation* set for a uniform algebra A on X if every non-identically-zero function $f \in C(S)$ extends to a $g \in A$ that satisfies

$$|g(x)| < \max_S |f| \qquad \forall x \in S^c := X \setminus S. \tag{1.2}$$

Equivalently, every $f \in C(S, \bar{\mathbb{D}})$ extends to a $g \in A$ that satisfies $g(S^c) \subset \mathbb{D}$. Thus, for peak-interpolation sets it is natural to consider interpolation problems for maps into complex manifolds with boundaries.

Remark 1.4. A set can be an interpolation set and have interpolation constant 1 but not be a peak-interpolation set. For instance, each point in $\mathbb{D}$ is an interpolation set with interpolation constant 1 for the algebra $A(\mathbb{D})$ but it is not a peak-interpolation set due to the maximum modulus principle.

In what follows, a *Banach manifold* is a manifold modelled on Banach spaces. A subset S of a complex Banach space B is said to be *a topological Banach submanifold with boundary* if S is the image of a topological Banach manifold with boundary Y under a continuous injection $i : Y \to B$. (In particular, the submanifold topology on S $(:= i(Y))$ need not be the relative topology induced from B.) If, in addition, the interior of Y has the structure of a complex Banach manifold and the map $i : Y \to B$ is holomorphic on the interior, then $S = i(Y) \subset B$ is said to be a *complex Banach submanifold with continuous boundary.*

For a uniform algebra A on X we denote by $A(X,S)$ the set of maps $f \in C(X,S)$ such that $\varphi(f) \in A$ for every $\varphi \in B^*$. We are interested in submanifolds subject to the following definition: A complex Banach submanifold $\bar{M} \subset B$ with continuous boundary ∂M and interior M is said to be *universal* if for every compact Hausdorff space X, a uniform algebra $A \subset C(X)$ and a peak-interpolation set $S \subset X$ for A the following holds:

Every $f \in C(S, \bar{M})$ extends to a map $g \in A(X, \bar{M})$ such that $g(S^c) \subset M$.

The class of universal manifolds has the following properties.

PROPOSITION 1.5.

(1) *A direct product of universal manifolds is a universal manifold.*
(2) *The set of universal submanifolds of a complex Banach space B is invariant with respect to the action of the group of invertible complex affine transformations of B.*
(3) *If $M \subset \mathbb{C}^n$ is a universal submanifold and $F : \mathbb{C}^n \to \mathbb{C}^m$ is a holomorphic embedding, then $F(M)$ is a universal submanifold of $\mathbb{C}^m$.*
(4) *Every paracompact universal manifold is contractible.*

It is known that a closed ball of a complex Banach space B is universal (see [S]). Our next result generalizes this fact.

Let $p_M : B \to [0,\infty)$ be the Minkowski functional of an open absorbing subset M of a complex Banach space B, i.e.,

$$p_M(v) := \inf_{tv \in M,\, t>0} \frac{1}{t}. \tag{1.3}$$

Then p_M is homogeneous, i.e., $p_M(rv) = r p_M(v)$ for all $r \in \mathbb{R}_+$, $v \in B$.

PROPOSITION 1.6. *Suppose M satisfies the condition*:

(1.4) *if $v \in \bar{M}$ (the closure of M) then the entire segment $[0,v)$ lies in M.*

Then p_M is a continuous function.

Conversely, if $p : B \to [0,\infty)$ is a continuous homogeneous function, then $M := \{v \in B : p(v) < 1\}$ is an open absorbing set satisfying (1.4) *and $p = p_M$.*

Note that if M satisfies (1.4), then $M = \{v \in B : p_M(v) < 1\}$ and $\partial M = \{v \in B : p_M(v) = 1\}$ (the boundary of M). Moreover, $\bar{M}$ is a complex Banach manifold with boundary modelled on B (in fact, $\bar{M} \setminus p_M^{-1}(0)$ is homeomorphic to $(0,1] \times \partial M$).

For instance, an open convex neighbourhood of $0 \in B$ satisfies (1.4). In this case the function p_M is subadditive (i.e., $p_M(v+v') \le p_M(v) + p_M(v')$). Also, every star body $M \subset \mathbb{C}^n$ containing 0 satisfies (1.4). If, in addition, such an M is bounded and 0 is an interior point of its kernel, then p_M is a Lipschitz function; see, e.g., [T].

Theorem 1.7. *Suppose $M \subset B$ is an open absorbing subset satisfying* (1.4). *Then $\bar{M}$ is a universal manifold.*

Remark 1.8. Since p_M is a homogeneous function, the theorem can be restated as follows:

Given a uniform algebra A on X and a peak-interpolation set $S \subset X$ for A, every $f \in C(S, \bar{M})$ such that $p_M \circ f \not\equiv 0$ has an extension $g \in A(X, \bar{M})$ that satisfies

$$p_M(g(x)) < \max_{y \in S} p_M(f(y)) \quad \forall x \in S^c.$$

For a subset $K \subset B$ we denote by $\mathrm{co}(K)$ the convex hull of K. Also, by $[K]_\varepsilon \subset B$ we denote the open ε-neighbourhood of K:

$$[K]_\varepsilon := \Big\{ v \in B : \inf_{v' \in K} \|v - v'\|_B < \varepsilon \Big\}.$$

Corollary 1.9. *Let A be a uniform algebra on X and $S \subset X$ be a peak-interpolation set for A. Let $f \in C(S, B)$. For every $\varepsilon > 0$ there is a g_ε in $A(X, [\mathrm{co}(f(S))]_\varepsilon)$ such that $g_\varepsilon|S = f$. Moreover, if $B = \mathbb{C}^n$ and $\mathrm{co}(f(S))$ has a nonempty interior $(\mathrm{co}(f(S)))^\circ$, then there is a $g \in A(X, \mathrm{co}(f(S)))$ extending f such that $g(S^c) \subset (\mathrm{co}(f(S)))^\circ$.*

For the last statement, note that since $f(S) \subset \mathbb{C}^n$ is compact, $\mathrm{co}(f(S))$ is compact as well by the Carathéodory theorem, and $(\mathrm{co}(f(S)))^\circ \neq \emptyset$ provided that $f(S)$ contains $2n+1$ linearly independent vectors over $\mathbb{R}$.

Remark 1.10. Inspired by Theorem 1.2, one can consider an analogous interpolation problem for totally disconnected peak-interpolation sets.

Problem 1.11. Let $\bar{M}$ be a domain with boundary in a complex manifold N. For what $\bar{M}$ does the following hold?

(∗) For every uniform algebra A on X, a totally disconnected peak-interpolation set $S \subset X$ for A and a map $f \in C(S, \bar{M})$ there are a subset $\mathcal{F} \subset A$ with $|\mathcal{F}| = \dim_{\mathbb{C}} N$ and a map $g \in \mathcal{O}_{\mathcal{F}}(\mathfrak{M}(A), N)$ such that $g|_S = f$ and $g(S^c) \subset M$.

Theorem 1.2(a) asserts that every $\bar{M}$ is *near-optimal*, meaning that for every open neighbourhood $O \subset N$ of $\bar{M}$ there exists a map $g \in \mathcal{O}_{\mathcal{F}}(\mathfrak{M}(A), O)$ with $g|_S = f$. We conjecture that for $\bar{M}$ with a 'nice' boundary (e.g., for strongly pseudoconvex domains $\bar{M} \subset \mathbb{C}^n$) such near-optimal g can be deformed to obtain one satisfying condition (∗).

Clearly, if $\bar{M}_i \subset N_i$, $i = 1, 2$, satisfy (∗), then $\bar{M}_1 \times \bar{M}_2 \subset N_1 \times N_2$ satisfies (∗) as well. Also, universal submanifolds $\bar{M} \subset \mathbb{C}^n$ satisfy (∗). The following result gives an example of nonuniversal $\bar{M}$ satisfying (∗) (cf. Proposition 1.5(4)).

THEOREM 1.12. *Let $\bar{M}$ be a connected Riemann surface with boundary embedded in a Riemann surface N such that the inclusion $\bar{M} \hookrightarrow N$ is a homotopy equivalence. Then $\bar{M}$ satisfies condition* $(*)$.

Note that every connected Riemann surface with boundary $\bar{M}$ can be embedded in its double W. Then there is an open neighbourhood $N \subset W$ of $\bar{M}$ that satisfies the hypothesis of the theorem.

2. Proofs of Theorems 1.2 and 1.12

2.1. Proof of Theorem 1.2. (a) Due to the main theorem of [FS] there is a finite locally biholomorphic surjective map $h : \mathbb{D}^n \to M$, where $n = \dim_{\mathbb{C}} M$. Since $f(S) \subset M$ is compact and h is locally biholomorphic, there exist a finite open cover $(U_i)_{1\le i\le k}$ of $f(S)$ and holomorphic maps $\tilde{h}_i : U_i \to \mathbb{D}^n$ such that $h \circ \tilde{h}_i = \mathrm{id}_{U_i}$, $1 \le i \le k$. Consider the finite open cover $\mathfrak{V} = (f^{-1}(U_i))_{1\le i\le k}$ of S. Since S is compact and totally disconnected, its covering dimension is zero (for basic results of dimension theory, see, e.g., [N]). In particular, there is a refinement $(W_s)_{1\le s\le m}$ of $\mathfrak{V}$ by clopen pairwise disjoint subsets. Let $\tau : \{1, \dots, m\} \to \{1, \dots, k\}$ be the refinement map, i.e., $W_s \subset f^{-1}(U_{\tau(s)})$, $1 \le s \le m$. Let us define a map $\tilde{f} : S \to \mathbb{D}^n$ by

$$\tilde{f}(x) := \tilde{h}_{\tau(s)}(f(x)), \quad x \in W_s,\ 1 \le s \le m. \tag{2.1}$$

Then $\tilde{f} \in C(S, \mathbb{D}^n)$ and $h \circ \tilde{f} = f$. Since S is an interpolation set with $c_A(S) = 1$, we can extend coordinates of $\tilde{f}$ to get a continuous map $\tilde{g} : \mathfrak{M}(A) \to \mathbb{D}^n$ with coordinates in A such that $\tilde{g}|_S = \tilde{f}$. Let $\mathcal{F}$ be the family of coordinates of $\tilde{g}$. Then $g := h \circ \tilde{g} \in \mathcal{O}_{\mathcal{F}}(\mathfrak{M}(A), M)$ is the required map interpolating f on S.

(b) According to [F2, Thm. 1.1] there is a surjective holomorphic map $h : \mathbb{C}^n \to M$, where $n = \dim_{\mathbb{C}} M$, such that for every $x \in M$ there are an open neighbourhood $U_x \Subset M$ of x and a holomorphic map $\tilde{h}_x : U_x \to \mathbb{C}^n$ such that $h \circ \tilde{h}_x = \mathrm{id}_{U_x}$. Let $(U_{x_i})_{1\le i\le k}$ be a finite open cover of the compact set $\bar{K}$ (the closure of K). Then $V := \bigcup_{i=1}^k \tilde{h}_{x_i}(U_{x_i})$ and $\tilde{K} := V \cap h^{-1}(K)$ are open relatively compact subsets of $\mathbb{C}^n$. Let $D_{\tilde{K}} \Subset \mathbb{C}^n$ be the minimal open polydisk centred at 0 containing $\tilde{K}$ and let $c_S(A)D_{\tilde{K}}$ be the dilation of $D_{\tilde{K}}$ with scalar factor $c_S(A)$. We define

$$L := h(c_S(A)D_{\tilde{K}}). \tag{2.2}$$

Let $f \in C(S, K)$. Then as in the proof of part (a) of the theorem we construct a map $\tilde{f} \in C(S, D_{\tilde{K}})$ such that $f = h \circ \tilde{f}$. By the definition of an interpolation set, there is a map $\tilde{g} \in C(\mathfrak{M}(A), c_S(A)D_{\tilde{K}})$ with coordinates in A such that $\tilde{g}|_S = \tilde{f}$. Let $\mathcal{F}$ be the family of coordinates of $\tilde{g}$. Then $g := h \circ \tilde{g} \in \mathcal{O}_{\mathcal{F}}(\mathfrak{M}(A), M)$ satisfies $g(\mathfrak{M}(A)) \subset L$ and $g|_S = f$, as required. ∎

2.2. Proof of Theorem 1.12. Let $r : N_u \to N$ be the universal covering of M. Then $\bar{M}_u := r^{-1}(\bar{M})$ is the universal covering of $\bar{M}$, and $\partial M_u := r^{-1}(\partial M)$ and $M_u := r^{-1}(M)$ are the boundary and the interior of $\bar{M}_u$. By our assumption, M_u and N_u are biholomorphic to $\mathbb{D}$. Thus, without loss of generality, we assume that N_u coincides with $\mathbb{D}$. Then M_u is a simply connected domain in $\mathbb{D}$ and (as $\bar{M}_u$ is a manifold with boundary) ∂M_u is homeomorphic to a one-dimensional manifold and there is a neighbourhood of ∂M_u in $\bar{M}_u$ homeomorphic to $\partial M_u \times (0, 1]$. In particular, each open arc in ∂M_u is a *free boundary arc* (see [P, Sec. 3.1]). Hence, a biholomorphic map $h : M_u \to \mathbb{D}$ extends to an injective continuous map $\bar{M}_u \to \bar{\mathbb{D}}$ (see [P, Thm. 3.1]).

Let A be a uniform algebra on X and $S \subset X$ be a totally disconnected peak-interpolation set for A. Let $f \in C(S, \bar{M})$ be such that $f(S) \cap \partial M \neq \emptyset$. Since r is locally biholomorphic, as in the proof of Theorem 1.2 we can construct a map $\tilde{f} \in C(S, \bar{M}_u)$ such that $f = r \circ \tilde{f}$. Consider the map $h \circ \tilde{f} \in C(S, \bar{\mathbb{D}})$. By our hypothesis, $(h \circ \tilde{f})(S) \cap \mathbb{T} \neq \emptyset$. Then by the definition of a peak-interpolation set, there exists $g' \in A$ with $g'|_S = h \circ \tilde{f}$ and $g'(S^c) \subset \mathbb{D}$. In turn, $\tilde{g} := h^{-1} \circ g'$ maps S^c in M_u and coincides with $\tilde{f}$ on S.

Let us prove that $\tilde{g} \in A$.

In fact, let $K := g'(X)$. Then K is a compact subset of $L \cup \mathbb{D}$, where $L := (h \circ \tilde{f})(S)$. Suppose that the open set $\mathbb{T} \setminus L \subset \mathbb{T}$ is non-void. Let $z \in \mathbb{T} \setminus L$. Then there is an open disk D centred at z such that $D \cap K = \emptyset$. In particular, for a point $z' \in D \setminus \bar{\mathbb{D}}$ sufficiently close to z, the function $p(w) := \frac{1}{w - z'}$, $w \in \bar{\mathbb{D}}$, lies in $A(\mathbb{D})$ and satisfies

$$|p(z)| > \max_{w \in K} |p(w)|.$$

This implies that the polynomially convex hull $\hat{K}$ of K does not contain points from $\mathbb{T} \setminus L$. Thus $\hat{K}$ is a compact subset of $L \cup \mathbb{D}$ as well. Similarly, if $\mathbb{T} \setminus L = \emptyset$, then $\hat{K} = L \cup \mathbb{D} = \bar{\mathbb{D}}$. Further, the function h^{-1} is continuous on $L \cup \mathbb{D}$ and holomorphic on $\mathbb{D}$. Thus by the Mergelyan theorem [M], $h^{-1}|_{\hat{K}}$ can be uniformly approximated by holomorphic polynomials. Hence, $\tilde{g} := h^{-1} \circ g'$ can be uniformly approximated on X by holomorphic polynomials in g', i.e., it lies in A, as claimed.

Now, $\tilde{g}(X)$ is a compact subset of $\bar{M}_u \subset \mathbb{D}$ $(=: N_u)$. Hence, $\tilde{g}(\mathfrak{M}(A))$ is a compact subset of $\mathbb{D}$. In particular, the map $g := r \circ \tilde{g} : \mathfrak{M}(A) \to N$ lies in $\mathcal{O}_{\mathcal{F}}(\mathfrak{M}(A), M)$, where $\mathcal{F} := \{\tilde{g}\}$, and $g|_S = f$, $g(S^c) \subset M$. ∎

3. Proofs of Propositions 1.5 and 1.6

3.1. Proof of Proposition 1.5. Parts (1) and (2) follow directly from the definition of a universal manifold. (In part (1) we also use the fact that if in a direct product of topological Banach manifolds one of the factors has a boundary, the product manifold also has a boundary; see the Appendix for details.)

(3) Since $M \subset \mathbb{C}^n$ is universal and $F : \mathbb{C}^n \to \mathbb{C}^m$ is a holomorphic embedding, in order to prove that $F(M)$ is universal it suffices to check that if $g \in A(X, \bar{M})$, then $F \circ g \in A(X, F(\bar{M}))$. In fact, let $\widehat{g(X)} \Subset \mathbb{C}^n$ be the polynomially convex hull of the compact set $g(X) \subset \mathbb{C}^n$. Then the coordinates of F are uniformly approximated on a neighbourhood of $\widehat{g(X)}$ by holomorphic polynomials: the partial sums of their power series. Since A is a uniform algebra, this implies the required statement.

(4) Let $\bar{\mathbb{B}}^n \subset \mathbb{R}^n$ be the closed unit Euclidean ball and $\mathbb{S}^{n-1} \subset \bar{\mathbb{B}}^n$ be the unit sphere. Since $\mathbb{S}^{n-1}$ is a peak-interpolation set for the algebra $C(\bar{\mathbb{B}}^n)$, by the definition of a universal manifold every $f \in C(\mathbb{S}^{n-1}, \bar{M})$ extends to a $g \in C(\bar{\mathbb{B}}^n, \bar{M})$. This shows that all homotopy groups of $\bar{M}$ are trivial. In turn, since $\bar{M}$ is a paracompact Banach manifold, the latter implies that M is contractible; see [Pa, Corollary after Thm. 15]. ∎

3.2. Proof of Proposition 1.6. First, we prove that under condition (1.4) the Minkowski functional $p_M : B \to [0, \infty)$ is continuous, i.e., for every $v \in B$ and every $\{v_n\}_{n\in\mathbb{N}} \subset B$ converging to v,

$$\lim_{n\to\infty} p_M(v_n) = p_M(v).$$

To this end, for $v \in B$ we set

$$\overrightarrow{0v} := \{tv \in B : t \in \mathbb{R}_+\}.$$

Due to (1.4) for $v \neq 0$ there is some $t(v) \in (0, \infty]$ such that

$$[0, t(v)v) \subset M, \quad \overrightarrow{0v} \setminus [0, t(v)v] \not\subset \bar{M} \quad \text{and} \quad t(v)v \in \partial M \quad \text{if } t(v) < \infty.$$

In particular, $p_M(v) = \frac{1}{t(v)}$.

Let $\{v_n\}_{n\in\mathbb{N}} \subset B \setminus \{0\}$ be a sequence converging to v. If $v = 0$, then since M is open, $\lim_{n\to\infty} t(v_n) = \infty$. Thus,

$$\lim_{n\to\infty} p_M(v_n) = \lim_{n\to\infty} \frac{1}{t(v_n)} = 0 = p_M(0).$$

If $v \neq 0$, then $tv \in M$ for every $t \in [0, t(v))$. Since $\{tv_n\}_{n\in\mathbb{N}}$ converges to tv and M is open, there is some $n(t) \in \mathbb{N}$ such that $tv_n \in M$ for all $n \geq n(t)$. This implies that

$$\overline{\lim_{n\to\infty}}\, p_M(v_n) = \overline{\lim_{n\to\infty}} \frac{1}{t(v_n)} \leq \inf_{t\in[0,t(v))} \frac{1}{t} = p_M(v). \tag{3.1}$$

If $p_M(v) = 0$, the latter implies that $\lim_{n\to\infty} p_M(v_n) = p_M(v)$. For otherwise, $t(v) < \infty$. Hence, $tv \in \bar{M}^c$ for every $t > t(v)$. Since $\{tv_n\}_{n\in\mathbb{N}}$ converges to tv and $\bar{M}^c$ is open, there is some $n(t) \in \mathbb{N}$ such that $tv_n \in \bar{M}^c$ for all $n \geq n(t)$. This implies that

$$p_M(v) = \sup_{t>t(v)} \frac{1}{t} \leq \underline{\lim_{n\to\infty}} \frac{1}{t(v_n)} = \underline{\lim_{n\to\infty}}\, p_M(v_n). \tag{3.2}$$

From (3.1), (3.2) we deduce that $\lim_{n\to\infty} p_M(v_n) = p_M(v)$. Thus, p_M is a continuous function.

Now, assume that $p : B \to [0,\infty)$ is a continuous homogeneous function. Let $M := \{v \in B : p(v) < 1\}$. Then M is an open set containing 0 such that if $v \in M$, then $tv \in M$ for all $t \in [0,1]$, i.e., M is absorbing. Further, if $v \in \bar{M}$, then continuity and homogeneity of p imply that $p(v) \le 1$ and $tv \in M$ for all $t \in [0,1)$. Hence, M satisfies condition (1.4).

Finally, for $v \neq 0$,

$$p_M(v) := \inf_{tv\in M,\, t>0} \frac{1}{t} = \inf_{p(tv)<1,\, t>0} \frac{1}{t} = \frac{1}{1/p(v)} = p(v),$$

as required. ∎

4. Proofs of Theorem 1.7 and Corollary 1.9

4.1. This part contains some results used in the proof of Theorem 1.7.

Let $\theta : [0,1] \to [0,\pi/4]$ be a continuous function positive on $(0,1)$ and equal to zero at $\{0,1\}$ and let

$$\Omega := \{z = re^{i\theta} \in \mathbb{C} : 0 < \theta < \theta(r),\ r \in (0,1)\}. \tag{4.1}$$

Let A be a uniform algebra on X and let $S \subset X$ be a peak-interpolation set for A.

Lemma 4.1. *Given $\varepsilon \in (0,1)$ and a compact set $E \subset S^c$ there is a function $h_\varepsilon \in A$ such that*

$$h_\varepsilon(X) \subset \bar{\Omega}, \quad h_\varepsilon|_S = 1, \quad |h_\varepsilon(x)| < 1 \quad \forall x \in S^c, \quad |h_\varepsilon(x)| \le \varepsilon \quad \forall x \in E.$$

Proof. Since Ω is a simply connected domain whose boundary is the Jordan curve

$$\gamma(t) := \begin{cases} 2t, & 0 \le t \le 1/2, \\ (2-2t)e^{i\theta(2-2t)}, & 1/2 \le t \le 1, \end{cases}$$

by the Carathéodory theorem (see, e.g., [P]) there is a conformal map $\mathbb{D} \to \Omega$ that extends to a homeomorphism $G : \bar{\mathbb{D}} \to \bar{\Omega}$. Let $z_0 := G^{-1}(0)$, $z_1 := G^{-1}(1) \in \mathbb{T}$. Let $\chi \in A$ be such that $\chi|_S = 1$ and $|\chi(x)| < 1$ for all $x \in S^c$ (existing by the definition of a peak-interpolation set). Then there exists $r \in (0,1)$ such that $\chi(E) \subset \mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. Consider the set of Möbius transformations of $\bar{\mathbb{D}}$:

$$g_a(z) := \frac{z - az_0}{1 - az_0^{-1}z}, \qquad z \in \bar{\mathbb{D}},\ -1 < a < 1.$$

Then $g_a(z_0) = z_0$ for all a and $\lim_{a\to-1} g_a(\mathbb{D}_r) = \{z_0\}$ (convergence in the Hausdorff metric). In particular, there exists $a_\varepsilon \in (-1,0)$ such that $G \circ g_{a_\varepsilon}$ maps $\mathbb{D}_r$ into $\Omega \cap \mathbb{D}_\varepsilon$. This map sends $g_{a_\varepsilon}^{-1}(z_1)$ to 1. Consider the function $\chi_\varepsilon := g_{a_\varepsilon}^{-1}(z_1)\chi \in A$. Since $|g_{a_\varepsilon}^{-1}(z_1)| = 1$, we have $\chi_\varepsilon|_S = g_{a_\varepsilon}^{-1}(z_1)$,

$\chi_\varepsilon(S^c) \subset \mathbb{D}$ and $\chi_\varepsilon(E) \subset \mathbb{D}_r$. We set $h_\varepsilon := G \circ g_{a_\varepsilon} \circ \chi_\varepsilon$. Since $G \circ g_{a_\varepsilon} \in A(\mathbb{D})$, it is a uniform limit of a sequence of holomorphic polynomials. Hence, $h_\varepsilon \in A$ and it has the required properties. ∎

LEMMA 4.2. *Let $\Psi : [0,1] \to \mathbb{R}_+$ be a continuous strictly increasing function equal to* 0 *at* 0. *There exists a sequence $\{\psi_i\}_{i\in\mathbb{N}} \subset C([0,1])$ of functions positive on $(0,1)$ and equal to zero on $\{0,1\}$ such that*

$$\sum_{i=1}^{k} \psi_i(r_i) \le \Psi\Big(\sum_{i=1}^{k} \frac{r_i}{2^i}\Big), \qquad r_i \in [0,1],\ 1 \le i \le k,\ k \in \mathbb{N};$$

here equality holds if and only if all r_i are zero.

Proof. We set $\psi_1(r_1) := (1-r_1)\Psi(r_1/2)$ $(\le \Psi(r_1/2))$, $r_1 \in [0,1]$. Suppose the required ψ_i are already defined for all $i \le k-1$. Then we define

$$\psi_k(r_k) := (1 - r_k) \cdot \min_{r_1,\dots,r_{k-1}\in[0,1]} \Big\{\Psi\Big(\sum_{i=1}^{k} \frac{r_i}{2^i}\Big) - \sum_{i=1}^{k-1} \psi_i(r_i)\Big\}, \qquad r_k \in [0,1].$$

By the induction hypothesis, ψ_k is continuous equal to 0 at $\{0,1\}$ and (since Ψ is strictly increasing) for $r_k \notin \{0,1\}$,

$$\psi_k(r_k) > (1 - r_k) \cdot \min_{r_1,\dots,r_{k-1}\in[0,1]} \Big\{\Psi\Big(\sum_{i=1}^{k-1} \frac{r_i}{2^i}\Big) - \sum_{i=1}^{k-1} \psi_i(r_i)\Big\} = 0.$$

That is, ψ_k is positive on $(0,1)$.

Next, if one of r_i, $1 \le i \le k$, is not zero, then by the induction hypothesis,

$$\psi_k(r_k) \le (1 - r_k)\Big(\Psi\Big(\sum_{i=1}^{k} \frac{r_i}{2^i}\Big) - \sum_{i=1}^{k-1} \psi_i(r_i)\Big) < \Psi\Big(\sum_{i=1}^{k} \frac{r_i}{2^i}\Big) - \sum_{i=1}^{k-1} \psi_i(r_i).$$

This proves the required statement. ∎

4.2. Proof of Theorem 1.7. Let $f \in C(S, \bar{M})$, $f(S) \cap \partial M \ne \emptyset$. According to [S] there is a map $g \in A(X, B)$ such that $g|_S = f$. Consider the compact set $K := g(X) \subset B$. By Mazur's theorem (see, e.g., [Co, Ch. VI, 4.8]), the closure of the convex balanced hull of K,

$$\hat{K} := \mathrm{cl}\Big\{\sum_{i=1}^{n} c_i v_i : v_i \in K,\ c_i \in \mathbb{D},\ \sum_{i=1}^{n} |c_i| = 1,\ n \in \mathbb{N}\Big\},$$

is compact. Let $V \subset B$ be the subspace generated by vectors in $\hat{K}$ equipped with the norm $\|\cdot\|_B$ and let $M|_V := M \cap V$. Then $M|_V$ is an open subset of V and its Minkowski functional (defined on V) coincides with $p_M|_V$, i.e., it is continuous. Consider the modulus of continuity of $p_M|_{\hat{K}}$,

$$\omega_{p_M|_{\hat{K}}}(t) := \sup\{|p_M(v) - p_M(v')| : v, v' \in \hat{K},\ \|v - v'\|_B \le t\}, \qquad t \ge 0.$$

Since K is compact and convex, $\omega_{p_M|_{\hat{K}}} : [0, \operatorname{diam} \hat{K}] \to [0, \infty)$ is a nondecreasing continuous subadditive function equal to zero at 0. We set

$$\omega(t) := t + \omega_{p_M|_{\hat{K}}}(t), \quad t \in [0, \operatorname{diam} \hat{K}]. \tag{4.2}$$

Then ω is a strictly increasing continuous subadditive function. Moreover, since $p_M|_{\hat{K}}$ attains value 1 on K and equals 0 at 0, the range of ω contains the interval $[0,1]$. Thus the inverse $\omega^{-1} : [0,T] \to [0, \operatorname{diam} \hat{K}]$, $T := \omega(\operatorname{diam} \hat{K}) \geq 1$, of ω is an increasing continuous function and equals 0 at 0.

We define the required extension $h \in A(X, \bar{M})$ of f by the formula

$$h(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} h_k(x) g(x), \quad x \in X,$$

for some $h_k \in A$, $|h_k| \leq 1$. Here h_k maps X into the closure of the domain

$$\Omega_k := \{z = re^{i\theta} \in \mathbb{C} : 0 < \theta < \theta_k(r),\, r \in (0,1)\},$$

where $\theta_k : [0,1] \to [0, \pi/4]$ is a continuous function positive on $(0,1)$ and equal to zero on $\{0,1\}$.

To this end, we set

$$\mathfrak{m} := \max_{x \in X} \|g(x)\|_B, \quad \mathfrak{m}' := \max_{x \in X} p_M(g(x))$$

and choose continuous functions $\tilde{\theta}_k : [0,1] \to \mathbb{R}_+$ positive on $(0,1)$ and equal to 0 on $\{0,1\}$ such that for every $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \tilde{\theta}_k(r_k) \leq \frac{1}{2} \omega^{-1}\left(\sum_{k=1}^{n} \frac{r_k}{2^k}\right), \quad r_k \in [0,1],\, k \in \mathbb{N}. \tag{4.3}$$

This is possible due to Lemma 4.2.

Next, we define

$$\theta_k(r_k) := \min\left\{\frac{2^k}{\mathfrak{m}}\, \tilde{\theta}_k(1 - r_k), \frac{\pi}{4}\right\}, \quad r_k \in [0,1],\, k \in \mathbb{N}. \tag{4.4}$$

Also, we define a sequence $\{\varepsilon_n\}_{n \in \mathbb{Z}_+}$ of positive numbers converging to 0 by the formulas

$$\varepsilon_0 := 1,\; \varepsilon_{n+1} := \min\left\{\varepsilon_n, \frac{2^n}{\mathfrak{m}} \omega^{-1}\left(\frac{\varepsilon_n}{\max\{\mathfrak{m}'/\mathfrak{m}, 1\}}\right), \frac{1}{2^{n+2}}\right\},\; n \geq 0. \tag{4.5}$$

Fix a proper open neighbourhood $U \subset X$ of S. Let

$$U_n := \{x \in U : p_M(g(x)) < 1 + \varepsilon_n\}, \quad n \in \mathbb{N}. \tag{4.6}$$

Then $U_n \subseteq U$ is an open neighbourhood of S. We choose $h_n \in A$ with image in $\bar{\Omega}_n$ such that $h_n(S) = 1$, $|h_n|_{S^c}| < 1$ and $|h_n(x)| \leq \varepsilon_n$ for all $x \in U_n^c (\neq \emptyset)$ (see Lemma 4.1). Note that $U_1 \supseteq U_2 \supseteq \cdots$ because the sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ is nonincreasing. We set for convenience $U_0 := X$, $h_0 := 0$, $\theta_0 := 0$.

Since the maps

$$h(x)=\sum_{k=1}^{\infty}\frac{1}{2^k}h_k(x)g(x)\quad\text{and}\quad h'(x):=\sum_{k=1}^{\infty}\frac{1}{2^k}|h_k(x)|g(x),\qquad x\in X,$$

map X into $\hat{K}$, we have

$$\begin{aligned}(4.7)\qquad \|h(x)-h'(x)\|_B&=\left\|\sum_{k=1}^{\infty}\frac{1}{2^k}|h_k(x)|(e^{i\operatorname{Arg}(h_k(x))}-1)g(x)\right\|_B\\ &\le\sum_{k=1}^{\infty}\frac{\mathfrak{m}}{2^k}|h_k(x)|\,2\sin\left(\frac{\operatorname{Arg}(h_k(x))}{2}\right)\le\sum_{k=1}^{\infty}\frac{\mathfrak{m}}{2^k}|h_k(x)|\theta_k(|h_k(x)|)\\ &\le\sum_{k=1}^{\infty}\frac{\mathfrak{m}}{2^k}\theta_k(|h_k(x)|)\le\operatorname{diam}\hat{K},\qquad x\in X.\end{aligned}$$

Suppose $x\in U_n\setminus U_{n+1}$, $n\in\mathbb{Z}_+$. Then (4.7), the subadditivity of $\omega_{p_M|_{\hat{K}}}$ and (4.5) imply

$$\begin{aligned}(4.8)\quad p_M(h(x))&\le p_M(h'(x))+\omega_{p_M|_{\hat{K}}}(\|h(x)-h'(x)\|_B)\\ &\le\left(\sum_{k=0}^{\infty}\frac{1}{2^k}|h_k(x)|\right)p_M(g(x))+\omega_{p_M|_{\hat{K}}}\left(\sum_{k=0}^{\infty}\frac{\mathfrak{m}}{2^k}|h_k(x)|\theta_k(|h_k(x)|)\right)\\ &\le\left[\left(\sum_{k=0}^{n}\frac{1}{2^k}|h_k(x)|\right)(1+\varepsilon_n)+\omega_{p_M|_{\hat{K}}}\left(\sum_{k=0}^{n}\frac{\mathfrak{m}}{2^k}|h_k(x)|\theta_k(|h_k(x)|)\right)\right]\\ &+\left[\left(\sum_{k=n+1}^{\infty}\frac{1}{2^k}|h_k(x)|\right)p_M(g(x))+\omega_{p_M|_{\hat{K}}}\left(\sum_{k=n+1}^{\infty}\frac{\mathfrak{m}}{2^k}|h_k(x)|\theta_k(|h_k(x)|)\right)\right]\\ &\le\left[\left(\sum_{k=0}^{n}\frac{1}{2^k}|h_k(x)|\right)(1+\varepsilon_n)+\omega\left(\sum_{k=0}^{n}\frac{\mathfrak{m}}{2^k}\theta_k(|h_k(x)|)\right)\right]\\ &+\left[\frac{\mathfrak{m}'\varepsilon_{n+1}}{2^n}+\omega_{p_M|_{\hat{K}}}\left(\frac{\mathfrak{m}\,\varepsilon_{n+1}}{2^n}\right)\right].\end{aligned}$$

Note that due to (4.5),

$$(4.9)\qquad \frac{\mathfrak{m}'\varepsilon_{n+1}}{2^n}+\omega_{p_M|_{\hat{K}}}\left(\frac{\mathfrak{m}\,\varepsilon_{n+1}}{2^n}\right)\le\max\left\{\frac{\mathfrak{m}'}{\mathfrak{m}},1\right\}\cdot\omega\left(\frac{\mathfrak{m}\,\varepsilon_{n+1}}{2^n}\right)\le\varepsilon_n.$$

Also, for $n\ge 1$, since $\varepsilon_n\le\frac{1}{2^{n+1}}$ and $|h_k|\le 1$, we have

$$\begin{aligned}(1-\varepsilon_n)-\left(\sum_{k=1}^{n}\frac{1}{2^k}|h_k(x)|\right)(1+\varepsilon_n)&\ge\sum_{k=1}^{n}\frac{1-|h_k(x)|}{2^k}+\frac{1}{2^n}-\varepsilon_n\left(2-\frac{1}{2^n}\right)\\ &>\sum_{k=1}^{n}\frac{1-|h_k(x)|}{2^k}.\end{aligned}$$

This and (4.3), (4.4) imply, for $n \geq 1$,

$$\Big(\sum_{k=1}^{n}\frac{1}{2^k}|h_k(x)|\Big)(1+\varepsilon_n)+\omega\Big(\sum_{k=1}^{n}\frac{\mathfrak{m}}{2^k}\theta_k(|h_k(x)|)\Big) < 1-\varepsilon_n+\omega\Big(\sum_{k=1}^{n}\frac{\mathfrak{m}}{2^k}\theta_k(|h_k(x)|)\Big)-\sum_{k=1}^{n}\frac{1-|h_k(x)|}{2^k}\leq 1-\varepsilon_n. \tag{4.10}$$

Thus, applying estimates (4.9) and (4.10) to (4.8) we deduce that for all $x \in U_n \setminus U_{n+1}$, $n \in \mathbb{Z}_+$, we have $p_M(h(x)) < 1$, i.e., $h(x) \in M$ in this case.

Further, if $x \in \bigcap_{n\in\mathbb{Z}_+} U_n$, then $p_M(g(x)) \leq 1$. Hence, as in (4.8), using continuity of ω and ω^{-1} and (4.3), (4.4) we obtain

$$\begin{aligned} p_M(h(x)) &\leq \Big(\sum_{k=1}^{\infty}\frac{1}{2^k}|h_k(x)|\Big)+\omega\Big(\sum_{k=1}^{\infty}\frac{\mathfrak{m}}{2^k}\theta_k(|h_k(x)|)\Big)\\ &= -\Big(\sum_{k=1}^{\infty}\frac{1-|h_k(x)|}{2^k}\Big)+1+\omega\Big(\sum_{k=1}^{\infty}\frac{\mathfrak{m}}{2^k}\theta_k(|h_k(x)|)\Big)\\ &= 1+\lim_{n\to\infty}\Big(\omega\Big(\sum_{k=1}^{n}\frac{\mathfrak{m}}{2^k}\theta_k(|h_k(x)|)\Big)-\sum_{k=1}^{n}\frac{1-|h_k(x)|}{2^k}\Big)\\ &\leq 1+\omega\Big(\frac{1}{2}\omega^{-1}\Big(\sum_{k=1}^{\infty}\frac{1-|h_k(x)|}{2^k}\Big)\Big)-\sum_{k=1}^{\infty}\frac{1-|h_k(x)|}{2^k}\leq 1. \end{aligned} \tag{4.11}$$

Since $\frac{1}{2}\omega^{-1}(t) < \omega^{-1}(t)$ for $t > 0$, equality holds in (4.11) if and only if

$$\sum_{k=1}^{\infty}\frac{1-|h_k(x)|}{2^k}=0.$$

This implies that $|h_k(x)| = 1$ for all $k \in \mathbb{N}$. In turn, according to our construction (see Lemma 4.1), the latter implies that $x \in S$. Thus, $p_M(h(x)) < 1$ for $x \in \big(\bigcap_{n\in\mathbb{Z}_+} U_n\big) \setminus S$, i.e., $h(x) \in M$ for such x as well.

Therefore h maps X into $\bar{M}$, $h|_S = f$ and $h(S^c) \subset M$, as required. ∎

4.3. Proof of Corollary 1.9. The first statement follows directly from Theorem 1.7 (see also Remark 1.8) as $[\mathrm{co}(f(S))]_\varepsilon \subset B$ is a bounded open convex set containing $f(S)$. The second one is a consequence of Theorem 1.7 as well as of the fact that $(\mathrm{co}(f(S)))^\circ \subset \mathbb{C}^n$ is a bounded open convex set and $f(S) \cap \partial(\mathrm{co}(f(S))) \neq \emptyset$. ∎

5. Appendix. The editors advised the author to add some basics of the theory of Banach topological manifolds. This section contains the corresponding material.

Let M be a topological manifold modelled on a (real) Banach space X with boundary ∂M and interior $\mathring{M}$. Thus each point in $\mathring{M}$ has an open neighbourhood homeomorphic to the open ball B_X of X and each point in ∂M has an open neighbourhood homeomorphic to a (relatively) open neighbourhood of $0 \in X$ in the set $H_+ = \{x \in X : f(x) \geq 0\}$ for some $f \in X^* \setminus \{0\}$ and this homeomorphism maps points of $\mathring{M}$ into the interior of H_+ and points of ∂M into $H := \{x \in X : f(x) = 0\}$.

It is readily seen that any two closed subspaces of X of codimension 1 are linearly homeomorphic. Indeed, if $H_1 \neq H_2$ are such subspaces, then $H_1 \cap H_2$ is a closed subspace of codimension 1 in each H_i and so it is complemented there. Hence, H_i are isomorphic as Banach spaces to the direct product of Banach spaces $(H_1 \cap H_2) \times \mathbb{R}$.

Thus without loss of generality we may assume that in the definition of M the Banach space X is of the form $X = Y \times \mathbb{R}$ and that a point of ∂M has an open neighbourhood homeomorphic to $B_Y \times [0, 1) \subset X$ and this homeomorphism maps the set of points of ∂M onto $B_Y \times \{0\}$.

Now, assume that M_i are topological manifolds with boundaries modelled on Banach spaces $X_i = Y_i \times \mathbb{R}$, $i = 1, 2$. Then the direct product $M_1 \times M_2$ is the union of $\mathring{M}_1 \times \mathring{M}_2$ and the set $\mathit{\Gamma} := (M_1 \times \partial M_2) \cup (\partial M_1 \times M_2)$. Let us show that $M_1 \times M_2$ is a topological manifold with boundary $\mathit{\Gamma}$ modelled on $X_1 \times X_2$.

Clearly, each point of $\mathring{M}_1 \times \mathring{M}_2$ has an open neighbourhood homeomorphic to $B_{X_1 \times X_2} := B_{X_1} \times B_{X_2}$. Now if $m = (m_1, m_2) \in \partial M_1 \times \partial M_2$, then it has an open neighbourhood homeomorphic to $(B_{Y_1} \times [0, 1)) \times (B_{Y_2} \times [0, 1))$ and this homeomorphism maps the set of points of $\mathit{\Gamma}$ onto

$$\big((B_{Y_1} \times \{0\}) \times (B_{Y_2} \times [0, 1))\big) \cup \big((B_{Y_1} \times [0, 1)) \times (B_{Y_2} \times \{0\})\big).$$

The previous sets are homeomorphic to

$$B_{Y_1 \times Y_2} \times ([0, 1) \times [0, 1)) \quad \text{and} \quad B_{Y_1 \times Y_2} \times \big(([0, 1) \times \{0\}) \cup (\{0\} \times [0, 1))\big),$$

respectively. Further, $[0, 1) \times [0, 1)$ is homeomorphic to $(-1, 1) \times [0, 1)$ and this homeomorphism maps $([0, 1) \times \{0\}) \cup (\{0\} \times [0, 1))$ onto $(-1, 1)$. In turn, the set $B_{Y_1 \times Y_2} \times (-1, 1)$ is linearly homeomorphic to the open unit ball B_Z of a closed subspace $Z \subset X_1 \times X_2$ of codimension 1. Thus, combining these homeomorphisms we see that m has an open neighbourhood homeomorphic to an open neighbourhood of $Z \times [0, 1) \subset M_1 \times M_2$ under the homeomorphism that maps the set of points of $\mathit{\Gamma}$ onto $B_Z \times \{0\}$.

The cases of $m = (m_1, m_2) \in \partial M_1 \times M_2$ or $m = (m_1, m_2) \in M_1 \times \partial M_2$ are easier and can be treated similarly.

This gives the required result.

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