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ON ASYMPTOTIC BASES AND MINIMAL ASYMPTOTIC BASES

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Abstract. Let $\mathbb{N}=\{0,1,2,\dots\}$ and $A\subset\mathbb{N}$. Let $h\ge 2$ and let $r_h(A,n)=\sharp\{(a_1,\dots,a_h)\in A^h : a_1+\dots+a_h=n\}$. The set A is called an asymptotic basis of order h if $r_h(A,n)\ge 1$ for all sufficiently large integers n. An asymptotic basis A of order h is minimal if no proper subset of A is an asymptotic basis of order h. Recently, Chen and Tang resolved a problem of Nathanson on minimal asymptotic bases of order h. In this paper, we generalize this result to g-adic representations.

1. Introduction. Let $\mathbb{N}=\{0,1,2,\dots\}$ and $A\subset\mathbb{N}$. Let $h\ge 2$ and let

$$r_h(A,n)=\sharp\{(a_1,\dots,a_h)\in A^h : a_1+\dots+a_h=n\}.$$

Let W be a nonempty subset of $\mathbb{N}$. The counting function $W(x)$ is defined as the number of positive elements of W not exceeding x. Denote by $\mathcal{F}^*(W)$ the set of all finite, nonempty subsets of W. For any integer $g\ge 2$, let $A_g(W)$ be the set of all numbers of the form $\sum_{f\in F} a_f g^f$ where $F\in\mathcal{F}^*(W)$ and $1\le a_f\le g-1$. The set A is called an *asymptotic basis* of order h if $r_h(A,n)\ge 1$ for all sufficiently large integers n. An asymptotic basis A of order h is *minimal* if no proper subset of A is an asymptotic basis of order h. This means that, for any $a\in A$, the set $E_a=hA\setminus h(A\setminus\{a\})$ is infinite.

In 1955, Stöhr [St55] first introduced the definition of minimal asymptotic basis. In 1956, Härtter [Ha56] gave a nonconstructive proof that there exist uncountably many minimal asymptotic bases of order h. In 1988, Nathanson [Na88] proved that if $W_i=\{n\in\mathbb{N} : n\equiv i \pmod h\}$ $(i=0,\dots,h-1)$, then $\bigcup_{i=0}^{h-1}A_2(W_i)$ is a minimal asymptotic basis of order h. The minimal asymptotic basis is an interesting topic in additive combinatorics. For other related problems, see [Li17], [LT18].

For any partition $\mathbb{N}=W_0\cup\dots\cup W_{h-1}$ it is reasonable to ask whether $\bigcup_{i=0}^{h-1}A_2(W_i)$ is minimal or not. Nathanson proved this is false even for $h=2$. Moreover, he posed the following problem (restated again in [JN89]).

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PROBLEM 1.1. *If $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ is a partition such that $w \in W_r$ implies either $w-1 \in W_r$ or $w+1 \in W_r$, is $A = A_2(W_0) \cup \cdots \cup A_2(W_{h-1})$ a minimal asymptotic basis of order h?*

In 1989, Jia and Nathanson [JN89] extended the example of minimal asymptotic basis by using a power of 2 with a parameter t satisfying $t = \lceil \log(h+1)/\log 2 \rceil$.

THEOREM A. *Let $h \geq 2$ and $t = \lceil \log(h+1)/\log 2 \rceil$. Partition $\mathbb{N}$ into h pairwise disjoint subsets $W_0, \ldots, W_{h-1}$ such that each W_r contains infinitely many intervals of t consecutive integers. Then $A = A_2(W_0) \cup \cdots \cup A_2(W_{h-1})$ is a minimal asymptotic basis of order h.*

In 1996, Jia [Ji96] extended the construction of Theorem A by using powers of any integer g with a similar restriction on t. In 2011, Chen and Chen [CC01] relaxed some restriction in the construction of Jia and Nathanson. They proved that Theorem A only requires each set W_i to contain one interval of t consecutive integers.

THEOREM B. *Let $h \geq 2$ and t be the least integer with $t > \log h/\log 2$. Let $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ be a partition such that each set W_i is infinite and contains t consecutive integers. Then $A = A_2(W_0) \cup \cdots \cup A_2(W_{h-1})$ is a minimal asymptotic basis of order h.*

Recently, Yong-Gao Chen and the first author [CT18] proved the following result:

THEOREM C. *Let h and t be integers with $2 \leq t \leq \log h/\log 2$. Then there exists a partition $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ such that each set W_r is a union of infinitely many intervals of at least t consecutive integers and $A = A_2(W_0) \cup \cdots \cup A_2(W_{h-1})$ is not a minimal asymptotic basis of order h.*

In [CT18], the authors showed that if $h > 2^t$, then $A_2(W_0)$ is an asymptotic basis of order h; if $h = 2^t$, then $A \setminus \{4\}$ is an asymptotic basis of order h.

It is natural to pose the following g-adic version of Problem 1.1:

PROBLEM 1.2. *Let $g \geq 2$ be an integer. If $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ is a partition such that $w \in W_r$ implies either $w-1 \in W_r$ or $w+1 \in W_r$, is $A = A_g(W_0) \cup \cdots \cup A_g(W_{h-1})$ a minimal asymptotic basis of order h?*

Similar to the proof of Theorem B, Ling and Tang [LT15] remarked that Theorem B can be extended to all $g \geq 2$ as follows.

THEOREM D. *Let $h \geq 2$ and t be the least integer with $t > \max\left\{1, \frac{\log h}{\log g}\right\}$, let $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ be a partition such that each set W_i is infinite and contains t consecutive integers. Then $A = A_g(W_0) \cup \cdots \cup A_g(W_{h-1})$ is a minimal asymptotic basis of order h.*

In this paper, we solve Problem 1.2.

THEOREM 1.3. *Let $g \geq 2$, h and t be integers with $2 \leq t \leq \log h/\log g$. Then there exists a partition $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ such that each set W_r is a union of infinitely many intervals of at least t consecutive integers and $A = A_g(W_0) \cup \cdots \cup A_g(W_{h-1})$ is not a minimal asymptotic basis of order h.*

Combining Theorem D and Theorem 1.3 (Theorem 1.3 is a negative result), we find that the answer to Problem 1.2 is affirmative for $2 \leq h < g^2$ and negative for $h \geq g^2$.

For $h > g^t(g-1)$, the following stronger result is obtained.

THEOREM 1.4. *Let $g \geq 2$, h and t be nonnegative integers with $h > g^t(g-1)$. Then there exists a partition $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ such that each set W_r contains infinitely many intervals of at least t consecutive integers and $n \in hA_g(W_0)$ for all $n \geq h$.*

2. Proof of Theorem 1.3. We need the following lemma:

LEMMA 2.1 ([Le93, Lemma 1]). *Let $g \geq 2$ be an integer.*

(a) *If W_1 and W_2 are disjoint subsets of $\mathbb{N}$, then $A_g(W_1) \cap A_g(W_2) = \emptyset$.*
(b) *If $W \subseteq \mathbb{N}$ and $W(x) = \theta x + O(1)$ for some $\theta \in (0,1]$, then there exist positive constants c_1 and c_2 such that*

$$c_1 x^\theta < A_g(W)(x) < c_2 x^\theta$$

for all x sufficiently large.
(c) *Let $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$, where $W_i \neq \emptyset$ for $i = 0, \ldots, h-1$. Then $A = A_g(W_0) \cup \cdots \cup A_g(W_{h-1})$ is an asymptotic basis of order h.*

For $a < b$, let $[a,b]$ denote the set of all integers in the interval $[a,b]$. Let $\{m_i\}_{i=1}^{\infty}$ be a sequence of integers with $m_1 > g^{h+2}$ and $m_{i+1} - m_i > g^{h+2}$ $(i \geq 1)$. Let

$$W_0 = [0, m_1] \cup \bigcup_{i=1}^{\infty} [m_i + t + 1, m_{i+1}]$$

and

$$W_j = \bigcup_{\substack{i=1 \\ i \equiv j \ (\mathrm{mod}\, h-1)}}^{\infty} [m_i + 1, m_i + t], \quad j = 1, \ldots, h-1.$$

Write

$$A = A_g(W_0) \cup \cdots \cup A_g(W_{h-1}).$$

By Lemma 2.1, we know that A is an asymptotic basis of order h. It is clear that $g^2 \in A_g(W_0)$. Now we prove that $E_{g^2} = hA \setminus h(A \setminus \{g^2\})$ is a finite set. Thus A is not a minimal asymptotic basis of order h.

Let $n > m_2$. We will show that

$$n \notin E_{g^2} = hA \setminus h(A \setminus \{g^2\}).$$

This is equivalent to proving that $n \in h(A \setminus \{g^2\})$.

Let the g-adic expansion of n be

$$n = \sum_{f \in F_n} a_f g^f, \quad 1 \le a_f \le g-1.$$

It is clear that $F_n \subseteq \mathbb{N} = W_0 \cup W_1 \cup \cdots \cup W_{h-1}$. We consider the following three cases:

CASE 1: $F_n \cap W_0 = \emptyset$. Then $F_n \subseteq W_1 \cup \cdots \cup W_{h-1}$.

SUBCASE 1.1: $|F_n| \ge h$. Then F_n has a partition $F_n = L_1 \cup \cdots \cup L_h$, where $L_i \ne \emptyset$ $(1 \le i \le h)$ and for every L_i there exists a W_j $(j \ge 1)$ with $L_i \subseteq W_j$. Let

$$n_i = \sum_{l \in L_i} a_l g^l, \quad 1 \le i \le h.$$

Then $n_i \in A \setminus \{g^2\}$ and $n = n_1 + \cdots + n_h$. Hence $n \in h(A \setminus \{g^2\})$.

SUBCASE 1.2: $1 \le |F_n| \le h-1$. Write

$$F_n = \{f_0, \ldots, f_{l-1}\}, \quad f_0 > \cdots > f_{l-1}.$$

Then $1 \le l \le h-1$. Note that

$$n = \begin{cases} \sum\limits_{j=1}^{l-1} a_{f_j} g^{f_j} + (g-1) \sum\limits_{j=1}^{h-l} g^{f_0 - j} + g^{f_0-(h-l)} & \text{if } a_{f_0} = 1, \\ \sum\limits_{j=1}^{l-1} a_{f_j} g^{f_j} + (a_{f_0}-1) g^{f_0} & \\ \qquad + (g-1) \sum\limits_{j=1}^{h-l-1} g^{f_0-j} + g^{f_0-(h-l-1)} & \text{if } a_{f_0} > 1; \end{cases}$$

moreover, $f_0 \ge m_1 + 1 > g^{h+2} > h+2$, thus $f_0 - (h-l) > l+2 \ge 3$ and $f_0 - (h-l-1) > l+3 \ge 4$. Hence $n \in h(A \setminus \{g^2\})$.

CASE 2: $F_n \cap W_0 \ne \emptyset$ and $F_n \cap W_0 \ne \{2\}$.

SUBCASE 2.1: $|F_n \setminus W_0| \ge h-1$. Then $F_n \setminus W_0$ has a partition

$$F_n \setminus W_0 = L_1 \cup \cdots \cup L_{h-1},$$

where $L_i \ne \emptyset$ $(1 \le i \le h-1)$ and for every L_i there exists a W_j $(j \ge 1)$ with $L_i \subseteq W_j$. Let $L_0 = F_n \cap W_0$ and

$$n_i = \sum_{l \in L_i} a_l g^l, \quad 0 \le i \le h-1.$$

Then $n_i \in A \setminus \{g^2\}$ and $n = n_0 + \cdots + n_{h-1}$. Hence $n \in h(A \setminus \{g^2\})$.

SUBCASE 2.2: $1 \le |F_n \setminus W_0| \le h-2$. Write

$$F_n \setminus W_0 = \{f_0, \dots, f_{l-1}\}, \quad f_0 > \cdots > f_{l-1}.$$

Then $1 \le l \le h-2$. Note that

$$n = \begin{cases} \sum_{f \in F_n \cap W_0} a_f g^f + \sum_{j=1}^{l-1} a_{f_j} g^{f_j} \\ \quad + (g-1) \sum_{j=1}^{h-l-1} g^{f_0 - j} + g^{f_0-(h-l-1)} & \text{if } a_{f_0} = 1, \\ \sum_{f \in F_n \cap W_0} a_f g^f + \sum_{j=1}^{l-1} a_{f_j} g^{f_j} + (a_{f_0}-1) g^{f_0} \\ \quad + (g-1) \sum_{j=1}^{h-l-2} g^{f_0 - j} + g^{f_0-(h-l-2)} & \text{if } a_{f_0} > 1; \end{cases}$$

moreover, $f_0 \ge m_1 + 1 > g^{h+2} > h+2$, thus $f_0 - (h-l-1) > l+3 \ge 4$ and $f_0 - (h-l-2) > l+4 \ge 5$. Hence $n \in h(A \setminus \{g^2\})$.

SUBCASE 2.3: $F_n \setminus W_0 = \emptyset$. That is, $F_n \subseteq W_0$. Write

$$F_n = \{f_0, \dots, f_{k-1}\}, \quad f_0 > \cdots > f_{k-1}.$$

Since

$$n > m_2 > g^{h+2} > (g-1)(1 + g + g^2 + \cdots + g^{h+1}),$$

we have $f_0 \ge h+2$.

If $k \ge 3$, then

$$n = \begin{cases} \sum_{j=1}^{k-1} a_{f_j} g^{f_j} + (g-1) \sum_{j=1}^{h-k} g^{f_0-j} + g^{f_0-(h-k)} & \text{if } a_{f_0} = 1, \\ \sum_{j=1}^{k-1} a_{f_j} g^{f_j} + (a_{f_0}-1) g^{f_0} + (g-1) \sum_{j=1}^{h-k-1} g^{f_0-j} \\ \quad + g^{f_0-(h-k-1)} & \text{if } a_{f_0} > 1. \end{cases}$$

Hence $n \in h(A \setminus \{g^2\})$.

If $k = 2$, then $n = a_{f_0} g^{f_0} + a_{f_1} g^{f_1}$.

If $a_{f_1} > 1$, or $a_{f_1} = 1, f_1 \ne 2$, then

$$n = \begin{cases} a_{f_1} g^{f_1} + (g-1) \sum_{j=1}^{h-2} g^{f_0-j} + g^{f_0-(h-2)} & \text{if } a_{f_0} = 1, \\ a_{f_1} g^{f_1} + (a_{f_0}-1) g^{f_0} + (g-1) \sum_{j=1}^{h-3} g^{f_0-j} + g^{f_0-(h-3)} & \text{if } a_{f_0} > 1. \end{cases}$$

Hence $n \in h(A \setminus \{g^2\})$.

If $a_{f_1} = 1$, $f_1 = 2$, then

$$n = \begin{cases} (g-1)g + g + (g-1)\sum_{j=1}^{h-3} g^{f_0-j} + g^{f_0-(h-3)} & \text{if } a_{f_0} = 1, \\ (g-1)g + g + (a_{f_0}-1)g^{f_0} \\ \qquad + (g-1)\sum_{j=1}^{h-4} g^{f_0-j} + g^{f_0-(h-4)} & \text{if } a_{f_0} > 1. \end{cases}$$

Hence $n \in h(A \setminus \{g^2\})$.

If $k = 1$, then

$$n = \begin{cases} (g-1)g^{f_0-1} + \cdots + (g-1)g^{f_0-(h-1)} + g^{f_0-(h-1)} & \text{if } a_{f_0} = 1, \\ (a_{f_0}-1)g^{f_0} + (g-1)\sum_{j=1}^{h-2} g^{f_0-j} + g^{f_0-(h-2)} & \text{if } a_{f_0} > 1. \end{cases}$$

Hence $n \in h(A \setminus \{g^2\})$.

Case 3: $F_n \cap W_0 = \{2\}$. As $n > m_2$, we have $F_n \setminus W_0 \neq \emptyset$. If $f \in F_n \setminus W_0$, then $f > m_1 > g^{h+2}$. Let

$$n = a_2 g^2 + \sum_{f \in F_n \setminus \{2\}} a_f g^f. \tag{2.1}$$

Subcase 3.1: $a_2 > 1$ and $|F_n \setminus \{2\}| \geq h-1$. Then $F_n \setminus \{2\}$ has a partition

$$F_n \setminus \{2\} = L_1 \cup \cdots \cup L_{h-1},$$

where $L_i \neq \emptyset$ $(1 \leq i \leq h-1)$ and for every L_i there exists a W_j $(j \geq 1)$ with $L_i \subseteq W_j$. Let $n_0 = a_2 g^2$ and

$$n_i = \sum_{l \in L_i} a_l g^l, \quad 1 \leq i \leq h-1.$$

Then $n_i \in A \setminus \{g^2\}$ and $n = n_0 + \cdots + n_{h-1}$. Hence $n \in h(A \setminus \{g^2\})$.

Subcase 3.2: $a_2 > 1$ and $1 \leq |F_n \setminus \{2\}| \leq h-2$. Write

$$F_n \setminus \{2\} = \{f_0, \ldots, f_{l-1}\}, \qquad f_0 > \cdots > f_{l-1}.$$

Then $1 \leq l \leq h-2$. Note that

$$n = \begin{cases} a_2 g^2 + \sum_{j=1}^{l-1} a_{f_j} g^{f_j} + (g-1)\sum_{j=1}^{h-l-1} g^{f_0-j} + g^{f_0-(h-l-1)} & \text{if } a_{f_0} = 1, \\ a_2 g^2 + \sum_{j=1}^{l-1} a_{f_j} g^{f_j} + (a_{f_0}-1)g^{f_0} \\ \qquad + (g-1)\sum_{j=1}^{h-l-2} g^{f_0-j} + g^{f_0-(h-l-2)}, & \text{if } a_{f_0} > 1; \end{cases}$$

moreover, $f_0 \geq m_1 + 1 > g^{h+2} > h+2$, thus $f_0 - (h-l-1) > l+3 \geq 4$ and $f_0 - (h-l-2) > l+4 \geq 5$. Hence $n \in h(A \setminus \{g^2\})$.

SUBCASE 3.3: $a_2 = 1$ and $|F_n \setminus \{2\}| \geq h-2$. Then $F_n \setminus \{2\}$ has a partition

$$F_n \setminus \{2\} = L_1 \cup \cdots \cup L_{h-2},$$

where $L_i \neq \emptyset$ $(1 \leq i \leq h-2)$ and for every L_i there exists a W_j $(j \geq 1)$ with $L_i \subseteq W_j$. Let $n_0 = (g-1)g$ and $n_1 = g$ and

$$n_{i+1} = \sum_{l \in L_i} a_l g^l, \quad 1 \leq i \leq h-2.$$

Then $n_i \in A \setminus \{g^2\}$ and $n = n_0 + \cdots + n_{h-1}$. Hence $n \in h(A \setminus \{g^2\})$.

SUBCASE 3.4: $a_2 = 1$ and $1 \leq |F_n \setminus \{2\}| \leq h-3$. Write

$$F_n \setminus \{2\} = \{f_0, \ldots, f_{l-1}\}, \quad f_0 > \cdots > f_{l-1}.$$

Then $1 \leq l \leq h-3$. We have $f_0 \geq m_1 + 1 > g^{h+2} > h+2$. Hence $f_0 - (h-l-2) > l+4 \geq 5$ and $f_0 - (h-l-3) > l+5 \geq 6$. Further,

$$n = \begin{cases} (g-1)g + g + \sum_{j=1}^{l-1} a_{f_j} g^{f_j} \\ \quad + (g-1) \sum_{j=1}^{h-l-2} g^{f_0-j} + g^{f_0-(h-l-2)} & \text{if } a_{f_0} = 1, \\ (g-1)g + g + \sum_{j=1}^{l-1} a_{f_j} g^{f_j} + (a_{f_0} - 1) g^{f_0} \\ \quad + (g-1) \sum_{j=1}^{h-l-3} g^{f_0-j} + g^{f_0-(h-l-3)} & \text{if } a_{f_0} > 1. \end{cases}$$

Hence $n \in h(A \setminus \{g^2\})$.

This completes the proof of Theorem 1.3.

3. Proof of Theorem 1.4. Fix an integer m such that $g^m > g^{t+2}h$. Put

$$W_0 = \{mk + s : k \in \mathbb{N},\ s \in [0, m-t-1]\},$$
$$W_i = \{mk + s : k \equiv i \pmod{h-1},\ s \in [m-t, m-1]\}, \quad 1 \leq i \leq h-1.$$

Let

$$A = A_g(W_0) \cup \cdots \cup A_g(W_{h-1}).$$

We will use induction on n to prove that every integer $n \geq h$ is in $hA_g(W_0)$. This implies that A is not a minimal asymptotic basis of order h.

If $n = h$, then by $0 \in W_0$ $(k = 0)$ we have $n = hg^0 \in hA_g(W_0)$. For $i = 1, \ldots, g(g-1)$, we have $i+1 \in A_g(W_0)$, so $n = h+i = (h-1)g^0 + (i+1) \in hA_g(W_0)$.

Now we assume that every integer l with $h \leq l < n$ $(n \geq h + g(g-1))$ is in $hA_g(W_0)$. We will prove that $n \in hA_g(W_0)$.

Let k be the integer such that

$$(g-1)g^{mk} \le n-h < (g-1)g^{m(k+1)}. \tag{3.1}$$

Then $k \ge 0$. Let i be the integer such that

$$(g-1)g^{mk+i} \le n-h < (g-1)g^{mk+i+1}. \tag{3.2}$$

Then $0 \le i \le m-1$ and

$$h \le n-(g-1)g^{mk+i} < (g-1)^2 g^{mk+i} + h. \tag{3.3}$$

By (3.2) and $n-h \ge g(g-1)$, we have $mk+i \ge 1$. We consider the following two cases:

Case 1: $0 \le i \le m-t-1$. By the induction hypothesis, $n-(g-1)g^{mk+i} \in hA_g(W_0)$. Let

$$n-(g-1)g^{mk+i} = a_1 + \cdots + a_h, \quad a_j \in A_g(W_0), j = 1, \ldots, h. \tag{3.4}$$

If $a_j \ge (g-1)g^{mk+i}$ for all $1 \le j \le h$, then from $mk+i \ge 1$ and $h > g^t(g-1)$ we have

$$n-(g-1)g^{mk+i} = a_1 + \cdots + a_h \ge hg^{mk+i}(g-1) > (g-1)^2 g^{mk+i} + h,$$

which contradicts (3.3). So at least one of a_j's is less than $(g-1)g^{mk+i}$.

Subcase 1.1: $0 \le i \le m-t-2$. Choose an integer $a_j < (g-1)g^{mk+i}$. Write

$$a_j = \sum_{f \in F_0} a_f g^f, \quad 1 \le a_f \le g-1.$$

Then $f_{\max} := \max\{f : f \in F_0\} \le mk+i$.

If $f_{\max} < mk+i$, then $a_j + (g-1)g^{mk+i} \in A_g(W_0)$.

If $f_{\max} = mk+i$, then

$$\begin{aligned} a_j + (g-1)g^{mk+i} &= \sum_{f \in F_0 \setminus \{mk+i\}} a_f g^f + a_{mk+i} g^{mk+i} + (g-1)g^{mk+i} \\ &= \sum_{f \in F_0 \setminus \{mk+i\}} a_f g^f + (a_{mk+i}-1) \cdot g^{mk+i} + g^{mk+i+1}, \end{aligned}$$

we have $a_j + (g-1)g^{mk+i} \in A_g(W_0)$. Noting that

$$n = (n-(g-1)g^{mk+i}) + (g-1)g^{mk+i},$$

by (3.4) we have $n \in hA_g(W_0)$.

Subcase 1.2: $i = m-t-1$. There exist at least $g-1$ a_j's which are less than $(g-1)g^{mk+m-t-1}$. Otherwise, we have

$$\begin{aligned} a_1 + \cdots + a_h &\geq (h-(g-2))(g-1)g^{mk+m-t-1} + (g-2) \\ &= (h-(g-1))(g-1)g^{mk+m-t-1} \\ &\quad + (g-1)g^{mk+m-t-1} + (g-2) \\ &> (g-1)^2 g^{mk+m-t-1} + h, \end{aligned}$$

which contradicts (3.3). Without loss of generality, we may assume that

$$a_j < (g-1)g^{mk+m-t-1}, \quad j = 1, \ldots, g-1.$$

Fix a $j \in \{1, \ldots, g-1\}$, and write

$$a_j = \sum_{f \in F_0} a_f g^f, \quad 1 \leq a_f \leq g-1.$$

Then $f_{\max} := \max\{f : f \in F_0\} \leq mk+m-t-1$.

If $f_{\max} < mk+m-t-1$, then $a_j + g^{mk+m-t-1} \in A_g(W_0)$.

If $f_{\max} = mk+m-t-1$, then $1 \leq a_{mk+m-t-1} \leq g-2$ and

$$\begin{aligned} &a_j + g^{mk+m-t-1} \\ &\quad = \sum_{f \in F_0 \setminus \{mk+m-t-1\}} a_f g^f + a_{mk+m-t-1} g^{mk+m-t-1} + g^{mk+m-t-1} \\ &\quad = \sum_{f \in F_0 \setminus \{mk+m-t-1\}} a_f g^f + (a_{mk+m-t-1}+1) \cdot g^{mk+m-t-1}, \end{aligned}$$

we have $a_j + g^{mk+m-t-1} \in A_g(W_0)$. Thus

$$\begin{aligned} n &= a_1 + \cdots + a_h + (g-1)g^{mk+m-t-1} \\ &= (a_1 + g^{mk+m-t-1}) + \cdots + (a_j + g^{mk+m-t-1}) + a_{j+1} + \cdots + a_h. \end{aligned}$$

Hence $n \in hA_g(W_0)$.

Case 2: $m-t \leq i \leq m-1$. Let u be the integer such that

$$h(g-1)g^{mk+u} \leq n-h < h(g-1)g^{mk+u+1}. \tag{3.5}$$

Since $h > g^t$ and $g^m > g^{t+2}h$, we have

$$g^{mk+i+1} \leq g^{mk+m} < hg^{mk+m-t}, \tag{3.6}$$

$$g^{mk+i} \geq g^{mk+m-t} > hg^{mk+2}. \tag{3.7}$$

By (3.2), (3.5) and (3.6), we have

$$h(g-1)g^{mk+u} \leq (g-1)g^{mk+i+1} < h(g-1)g^{mk+m-t},$$

thus $u \leq m-t-1$.

By (3.2), (3.5) and (3.7), we have

$$h(g-1)g^{mk+u+1} \geq (g-1)g^{mk+i} > h(g-1)g^{mk+2},$$

thus $u \geq 2$.

So $mk+u \in W_0$ and $(g-1)g^{mk+u} \in A_g(W_0)$.

SUBCASE 2.1: $u = m - t - 1$. Then by (3.1) and $h > g^t(g-1)$, we have

$$\begin{aligned} n - \sum_{j=2}^{m-t-1} h(g-1)g^{mk+j} &< (g-1)g^{mk+m} + h - \sum_{j=2}^{m-t-1} h(g-1)g^{mk+j} \\ &= (g-1)g^{mk+m} + h - hg^{mk}(g^{m-t} - g^2) \\ &= g^{mk}(g^m(g-1-hg^{-t}) + g^2h) + h \\ &< g^{mk}(g^{t+2}h(g-1-hg^{-t}) + g^2h) + h \\ &= g^{mk}(g^2h(g^t(g-1) - h) + g^2h) + h \\ &\le g^{mk}(-g^2h + g^2h) + h = h. \end{aligned}$$

Thus

$$n - h < \sum_{j=2}^{m-t-1} h(g-1)g^{mk+j}.$$

Let

$$\sum_{j=v}^{m-t-1} h(g-1)g^{mk+j} \le n - h < \sum_{j=v-1}^{m-t-1} h(g-1)g^{mk+j}.$$

Then $3 \le v \le m - t - 1$ and

$$0 \le n - h - \sum_{j=v}^{m-t-1} h(g-1)g^{mk+j} < h(g-1)g^{mk+v-1}.$$

Let

$$s(g-1)g^{mk+v-1} \le n - h - \sum_{j=v}^{m-t-1} h(g-1)g^{mk+j} < (s+1)(g-1)g^{mk+v-1}.$$

Then $0 \le s \le h - 1$ and

$$h \le n - s(g-1)g^{mk+v-1} - \sum_{j=v}^{m-t-1} h(g-1)g^{mk+j} < (g-1)g^{mk+v-1} + h. \tag{3.8}$$

By the induction hypothesis, we may assume that

$$n - s(g-1)g^{mk+v-1} - \sum_{j=v}^{m-t-1} h(g-1)g^{mk+j} = b_1 + \cdots + b_h, \tag{3.9}$$

where $b_j \in A_g(W_0)$, $j = 1, \ldots, h$.

By (3.8) and (3.9), we have

$$b_j < g^{mk+v}, \quad j = 1, \ldots, h. \tag{3.10}$$

Otherwise, we have

$$b_1 + \cdots + b_h \ge g^{mk+v} + h - 1 > (g-1)g^{mk+v-1} + h,$$

which contradicts (3.8). Moreover, we know that there exist at most $g-1$ b_j's which are greater than or equal to g^{mk+v-1}. Otherwise, we have

$$\begin{aligned} b_1+\cdots+b_h &\ge gg^{mk+v-1}+(h-g)\\ &=(g-1)g^{mk+v-1}+g^{mk+v-1}-g+h\\ &>(g-1)g^{mk+v-1}+h, \end{aligned}$$

which contradicts (3.8).

Without loss of generality, we may assume that $b_j \ge g^{mk+v-1}$, $j=1,\dots,l$, where $l \le g-1$. Let

$$b_j=b'_j+r_jg^{mk+v-1}, \quad j=1,\dots,l,$$

be the g-adic expansion of b_j according to the fact that $b_j<g^{mk+v}$, $j=1,\dots,h$. We have $r_1+\cdots+r_l\le g-1$. Otherwise,

$$b_1+\cdots+b_h \ge g\cdot g^{mk+v-1}+h-l>(g-1)g^{mk+v-1}+h,$$

which contradicts (3.8).

Noting that $0\le s\le h-1$, we have

$$\begin{aligned} n &= b_1+\cdots+b_h+s(g-1)g^{mk+v-1}+\sum_{j=v}^{m-t-1}h(g-1)g^{mk+j}\\ &=\Big(b'_1+(r_1+\cdots+r_l)g^{mk+v-1}+\sum_{j=v}^{m-t-1}(g-1)g^{mk+j}\Big)\\ &\quad+\Big(b'_2+(g-1)g^{mk+v-1}+\sum_{j=v}^{m-t-1}(g-1)g^{mk+j}\Big)\\ &\quad+\cdots+\Big(b'_l+(g-1)g^{mk+v-1}+\sum_{j=v}^{m-t-1}(g-1)g^{mk+j}\Big)\\ &\quad+\Big(b_{l+1}+(g-1)g^{mk+v-1}+\sum_{j=v}^{m-t-1}(g-1)g^{mk+j}\Big)\\ &\quad+\cdots+\Big(b_{s+1}+(g-1)g^{mk+v-1}+\sum_{j=v}^{m-t-1}(g-1)g^{mk+j}\Big)\\ &\quad+\Big(b_{s+2}+\sum_{j=v}^{m-t-1}(g-1)g^{mk+j}\Big)+\cdots+\Big(b_h+\sum_{j=v}^{m-t-1}(g-1)g^{mk+j}\Big). \end{aligned}$$

Hence $n\in hA_g(W_0)$.

SUBCASE 2.2: $1\le u<m-t-1$. Let

$$s(g-1)g^{mk+u+1}\le n-h<(s+1)(g-1)g^{mk+u+1}.$$

By (3.5) and $h > g^t(g-1)$, we have

$$g^{t-1}(g-1)^2 g^{mk+u+1} < n - h < h(g-1)g^{mk+u+1}.$$

Thus $g^{t-1}(g-1) \le s \le h-1$. Let

$$qg^{mk+u} \le n - h - s(g-1)g^{mk+u+1} < (q+1)g^{mk+u}.$$

Noting that

$$0 \le n - h - s(g-1)g^{mk+u+1} < g(g-1)g^{mk+u},$$

we have $0 \le q \le g(g-1) - 1 \le h - 2$ and

$$h \le n - qg^{mk+u} - s(g-1)g^{mk+u+1} < g^{mk+u} + h. \tag{3.11}$$

By the induction hypothesis, we may assume that

$$n - qg^{mk+u} - s(g-1)g^{mk+u+1} = c_1 + \cdots + c_h,$$

where $c_j \in A_g(W_0)$ $(1 \le j \le h)$. By (3.11) we know that there exists at most one c_j greater than or equal to g^{mk+u}. Hence, we may assume that

$$c_j < g^{mk+u}, \quad j = 1, \ldots, h,$$

or

$$c_1 \ge g^{mk+u} \quad \text{and} \quad c_j < g^{mk+u} \quad (j = 2, \ldots, h).$$

Noting that $q \le s$, $g^{t-1}(g-1) \le s \le h-1$ and $0 \le q \le g(g-1) - 1 \le h-2$, we have

$$\begin{aligned} n &= c_1 + \cdots + c_h + qg^{mk+u} + s(g-1)g^{mk+u+1} \\ &= c_1 + (c_2 + g^{mk+u} + (g-1)g^{mk+u+1}) + \cdots + (c_{q+1} + g^{mk+u} \\ &\quad + (g-1)g^{mk+u+1}) \\ &\quad + (c_{q+2} + (g-1)g^{mk+u+1}) + \cdots + (c_{s+1} + (g-1)g^{mk+u+1}) \\ &\quad + c_{s+2} + \cdots + c_h. \end{aligned}$$

Hence $n \in hA_g(W_0)$.

This completes the proof of Theorem 1.4.

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