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A TRISECTION IN THE AUSLANDER–REITEN QUIVER

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Abstract. We discuss the existence of a trisection in the components of the Auslander–Reiten quiver of an algebra.

1. Introduction. For a given artin algebra Λ, we denote by $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules and by $\operatorname{ind}\Lambda$ its subcategory consisting of isoclasses of indecomposable modules. In order to understand $\operatorname{ind}\Lambda$ one can, for instance, represent it by the so-called Auslander–Reiten quiver Γ_Λ. The points of such quiver represent the objects of $\operatorname{ind}\Lambda$ while the arrows are related to irreducible morphisms. For a formal definition we indicate, for instance, the books [AC, ASS] but we stress here the importance of knowing the structure of such quivers for the study of $\operatorname{ind}\Lambda$.

Another well successful strategy to understand $\operatorname{ind}\Lambda$ started with the introduction of tilting theory by Brenner–Butler [BB] and Happel–Ringel [HR]. The important class of tilted algebras was defined in the latter work. Without entering into the history of this line of research, a key step was also the introduction of quasitilted algebras by Happel–Reiten–Smalø [HRS]. There, the authors defined the following subcategories of $\operatorname{ind}\Lambda$:

$$\mathcal{L}_\Lambda = \{X \in \operatorname{ind}\Lambda\colon \operatorname{pd}_\Lambda Y \le 1 \text{ whenever there is a path } Y \rightsquigarrow X\},$$
$$\mathcal{R}_\Lambda = \{X \in \operatorname{ind}\Lambda\colon \operatorname{id}_\Lambda Y \le 1 \text{ whenever there is a path } X \rightsquigarrow Y\},$$

where $M \rightsquigarrow N$ means that there exists a path of nonzero morphisms from M to N. Recall also that $\operatorname{pd}_\Lambda N$ and $\operatorname{id}_\Lambda N$ indicate, respectively, the projective and the injective dimensions of a module N.

Happel, Reiten and Smalø showed that if Λ is quasitilted, then the triple $(\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda, \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda, \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda)$ is a trisection of $\operatorname{ind}\Lambda$ (see below for the definition).

In [CL], in a joint work with M. Lanzilotta, we characterised the algebras such that $(\mathcal{L}_\Lambda\setminus\mathcal{R}_\Lambda, \mathcal{L}_\Lambda\cap\mathcal{R}_\Lambda, \mathcal{R}_\Lambda\setminus\mathcal{L}_\Lambda)$ is a trisection of $\operatorname{ind}\Lambda$ and called them

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shod algebras. However, this trisection of $\operatorname{ind} \Lambda$ does not necessarily induce a trisection of the connected components of the Auslander–Reiten quiver Γ_Λ of Λ in the sense that each component lies entirely in one of these parts. For instance, if Λ is a representation-finite shod algebra which is not hereditary, then there exist indecomposable modules X, Y such that $\operatorname{pd}_\Lambda X \geq 2$ and $\operatorname{id}_\Lambda Y \geq 2$ and so Γ_Λ (which is connected if we assume Λ indecomposable) will contain modules not in $\mathcal{L}_\Lambda$ and modules not in $\mathcal{R}_\Lambda$.

The purpose of this note is to characterise those algebras such that the triple $(\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda, \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda, \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda)$ induces a trisection in their Auslander–Reiten quivers.

2. Preliminaries. We dedicate this section to clarify the notions needed to prove our main result. We refer the reader to [AC, ASS] for basics on the representation theory of algebras.

2.1. Components of Auslander–Reiten quivers. Let Γ be a (connected) component of the Auslander–Reiten quiver Γ_Λ of an algebra Λ. We say that Γ is *regular* if it contains neither projective nor injective modules. Also, Γ is *semiregular* if it does not contain projectives and injectives simultaneously.

2.2. Tilted algebras. A *tilted algebra* is an endomorphism algebra of the type $\operatorname{End}_H(T)$ where H is a hereditary algebra and T is a *tilting H-module*, that is, it satisfies:

(i) $\operatorname{pd}_H T \leq 1$;
(ii) $\operatorname{Ext}^1_H(T, T) = 0$; and
(iii) there exists a short exact sequence $0 \to H_H \to T_1 \to T_2 \to 0$ with $T_1, T_2 \in \operatorname{add} T$.

A tilted algebra is characterised by the existence of a *complete slice* in its Auslander–Reiten quiver. We recall that any complete slice $\mathcal{S}$ separates $\operatorname{ind} \Lambda$ into two parts: the modules which are generated by modules in $\mathcal{S}$ and those which are cogenerated by modules in $\mathcal{S}$, the intersection of the two parts being $\mathcal{S}$ itself. A component of Γ_Λ containing a complete slice is called *connecting* and there are at most two of them. There are two such components only when Λ is a *concealed algebra*, and in this case one is postprojective and the other is preinjective.

Any tilted algebra Λ satisfies: (i) the global dimension $\operatorname{gl.dim} \Lambda$ of Λ is at most 2; and (ii) for each $X \in \operatorname{ind} \Lambda$, either $\operatorname{pd}_\Lambda X \leq 1$ or $\operatorname{id}_\Lambda X \leq 1$.

2.3. Quasitilted algebras. Quasitilted algebras were introduced as a generalisation of tilted algebras: they are endomorphism algebras of tilting objects in abelian categories. We shall use, however, the following characterisation proven in [HRS]. An algebra Λ is *quasitilted* if and only if

(i) gl.dim $\Lambda \le 2$, and
(ii) for each $X \in \operatorname{ind} \Lambda$, either $\operatorname{pd}_\Lambda X \le 1$ or $\operatorname{id}_\Lambda X \le 1$.

Also, using the notation introduced above, if Λ is quasitilted, then $\operatorname{ind} \Lambda = \mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda$. Representation-finite quasitilted algebras are tilted (see [HRS]).

2.4. Shod algebras. The notion of a shod algebra was introduced in [CL] as an extension of quasitilted algebras. We say that an algebra Λ is *shod* (for **s**mall **ho**mological **d**imensions) if for each $X \in \operatorname{ind} \Lambda$, either $\operatorname{pd}_\Lambda X \le 1$ or $\operatorname{id}_\Lambda X \le 1$ (that is, the second conditon in the characterisation of quasitilted algebras recalled above). There are shod algebras with global dimension three, so the class of shod algebras properly includes the class of quasitilted ones. It has been proven in [CL] that an algebra Λ is shod if and only if $\operatorname{ind} \Lambda = \mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda$.

2.5. Trisections. Because of the last result, if Λ is a shod algebra, then the triple $(\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda, \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda, \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda)$ is a trisection of $\operatorname{ind} \Lambda$ in the sense that $\operatorname{ind} \Lambda$ is the (disjoint) union of the subcategories $\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda$, $\mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda$ and $\mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda$ and there are no morphisms *from right to left* in the chosen order, that is, $\operatorname{Hom}_\Lambda(\mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda, \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda) = \operatorname{Hom}_\Lambda(\mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda, \mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda) = \operatorname{Hom}_\Lambda(\mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda, \mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda) = 0$.

As mentioned in the introduction, we are here interested in another kind of trisection. It is not a trisection of $\operatorname{ind} \Lambda$ considering the splitting of their objects into three parts. Our main interest here is to study when the triple $(\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda, \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda, \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda)$ divides the components of Γ_Λ into three separated parts. Let us formalise this.

Definition 2.1. Let Λ be an artin algebra. A triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of subcategories of $\operatorname{ind} \Lambda$ is a *trisection of the components of the AR-quiver* Γ_Λ if:

(i) $\operatorname{ind} \Lambda$ is a disjoint union of $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$;
(ii) each (connected) component of Γ_Λ is completely contained in one of the parts $\mathcal{A}, \mathcal{B}$ or $\mathcal{C}$;
(iii) $\operatorname{Hom}_\Lambda(\mathcal{C}, \mathcal{B}) = \operatorname{Hom}_\Lambda(\mathcal{C}, \mathcal{A}) = \operatorname{Hom}_\Lambda(\mathcal{B}, \mathcal{A}) = 0$.

3. The result. We now state and prove our main result.

Theorem 3.1. *The following are equivalent for an artin algebra* Λ*:*

(a) $(\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda, \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda, \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda)$ *is a trisection of the components of the AR-quiver* Γ_Λ.
(b) Λ *is one of the following types:*
  - *a hereditary algebra,*
  - *a representation-infinite tilted algebra with a regular connected component,*
  - *a quasitilted algebra which is not tilted.*

Proof. Assume (a). Then clearly

$$\operatorname{ind} A = (\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda) \cup (\mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda) \cup (\mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda) = \mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda$$

and so, as mentioned above, Λ is a shod algebra (see [CL, Theorem 2.1]). If gl.dim $\Lambda = 3$, then, because of [CL, Proposition 2.4], there exists a path of irreducible morphisms

$$I = X_0 \to X_1 \to \cdots \to \tau Z \to Y \to Z \to \cdots \to W_1 \to W_0 = P$$

between indecomposable modules, where I is an injective module, P is a projective and the subpaths $I \rightsquigarrow \tau Z$ and $Z \rightsquigarrow P$ are sectional. In particular, $\operatorname{Hom}_\Lambda(I, \tau Z) \neq 0$ (because sectional paths have nonzero composition, see for instance, [AC, Theorem IV.3.5]) and so $\operatorname{pd}_\Lambda Z \geq 2$ (see, for instance, [AC, Theorem III.1.11]). Hence the component Γ of Γ_Λ which contains such a path has a module which is not in $\mathcal{L}_\Lambda$. On the other hand, the path $Z \rightsquigarrow P$ being sectional implies that $\operatorname{Hom}_\Lambda(Z, P) \neq 0$ and we see that $\operatorname{id}_\Lambda \tau Z \geq 2$ (see, for instance, [AC, Theorem III.1.11]). Hence Γ has a module not in $\mathcal{R}_\Lambda$. This contradicts condition (a). Hence gl.dim $\Lambda \leq 2$ and so, by [CL], Λ is a quasitilted algebra.

Suppose now that Λ is a tilted algebra but not hereditary and let Γ' be a connected component of Γ_Λ. We first observe that a complete slice lies in $\mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda$. So, because of (a), we infer that the whole component Γ' lies in $\mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda$. We shall show that Γ' is regular. Suppose Γ' has projectives. Now, if all projective modules of Γ' are hereditary (that is, their radicals are also projectives), then Γ' is postprojective and Λ is itself hereditary, a contradiction to our assumptions. So, there exists an indecomposable projective P in Γ' such that $\operatorname{rad} P$ is not projective. In particular, $\operatorname{rad} P$ has an indecomposable summand X such that $\tau X \neq 0$. Clearly, using the same arguments as above, $\operatorname{id}_\Lambda \tau X = 2$ and so $\tau X \notin \mathcal{R}_\Lambda$, a contradiction. Therefore Γ' has no projectives. A dual argument shows that Γ' has no injectives and so it is regular. Therefore, Λ is a representation-infinite tilted algebra with a regular connected component (and so, the unique connected component). This proves (b).

Let us now assume (b). If Λ is hereditary, then $\mathcal{L}_\Lambda = \mathcal{R}_\Lambda = \operatorname{ind} A$ and so $\Gamma_\Lambda \subset \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda$ and (a) is satisfied in this case.

Now, assume that Λ is quasitilted but not tilted. We first observe that since representation-finite quasitilted algebras are tilted, Λ is representation-infinite. Let Γ be a component of Γ_Λ. By [CS, Corollary E], Γ is semiregular. Also, using [CS, Theorem D], we infer that $\Gamma \subset \mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda$ if Γ has projectives or $\Gamma \subset \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda$ if it contains an injective module. It remains to consider the case when Γ is regular. Suppose first that $\Gamma \cap \mathcal{R}_\Lambda \neq \emptyset$. Then, by [CS, Theorem B(a)], $\Gamma \subset \mathcal{R}_\Lambda$. Now, if $\Gamma \cap \mathcal{L}_\Lambda = \emptyset$, then $\Gamma \subset \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda$. Otherwise, we infer by [CS, Theorem B(b)] that $\Gamma \subset \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda$. Dually, if $\Gamma \cap \mathcal{L}_\Lambda \neq \emptyset$, then either $\Gamma \subset \mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda$ or $\Gamma \subset \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda$. Hence, condition (a) is also satisfied in this case.

It remains to consider the case where Λ is a representation-infinite tilted algebra with a regular connected component. Let Γ' be such a regular connected component of Γ_Λ. Since, in this case, any module of Γ' lies in a complete slice (which lies in $\mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda$), we infer that $\Gamma' \subset \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda$. Suppose Γ is a component of Γ_Λ distinct from Γ'. Then $\mathrm{Hom}_\Lambda(\Gamma, \Gamma') \neq 0$ or $\mathrm{Hom}_\Lambda(\Gamma', \Gamma) \neq 0$ but not both. Suppose the former. If $X \in \Gamma$, then $\mathrm{Hom}_\Lambda(X, \Gamma') \neq 0$ because Γ' is connected. Then $\Gamma \subset \mathcal{L}_\Lambda$ because $\mathcal{L}_\Lambda$ is closed under predecessors and $\Gamma' \subset \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda$. Also, $\mathrm{Hom}_\Lambda(\mathrm{D}\Lambda, \Gamma') = 0$, and so Γ has no injective modules. If Γ is semiregular, then it has projectives. Hence, by [CS, Theorem C], $\Gamma \subset \mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda$. There remains the case where Γ is regular. However, in this case, the result follows from [CS, Theorem B]. A dual argument for components Γ such that $\mathrm{Hom}_\Lambda(\Gamma', \Gamma) \neq 0$ implies that Γ lies either in $\mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda$ or in $\mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda$, and the result is proven. ∎

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