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A NOTE ON THE JACOBIAN CONJECTURE

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Abstract. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with non-vanishing Jacobian. If the set S_F of non-properness of F is smooth, then F is a surjective mapping. Moreover, if S_F is connected, then $\chi(S_F) > 0$. Additionally, if $n = 2$, then S_F cannot be a curve without self-intersections.

1. Introduction. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a generically finite polynomial mapping. The *set of non-properness* of F is $S_F = \{y \in \mathbb{C}^n$: there is a sequence $x_n \to \infty$ such that $F(x_n) \to y\}$, which was introduced by the author in [6], [8]. This set if non-empty is a $\mathbb{C}$-uniruled hypersurface (see [6], [8]). If F is an etale mapping (as in the Jacobian Conjecture), then $S_F = \{y \in \mathbb{C}^n : \#f^{-1}(y) < \mu(F)\}$, where $\mu(F)$ denotes the number of points in a general fiber of F. In particular, in this case the mapping F is a topological covering over $\mathbb{C}^n \setminus S_F$. In order to prove the Jacobian Conjecture, it is enough to show that $S_F = \emptyset$. As the first step in this direction Nollet, Taylor and Xavier [9] have proved that S_F cannot be a connected non-bifurcated algebraic hypersurface. More generally, Nollet and Xavier [10] have proved that the group $\pi_1(\mathbb{C}^n \setminus S_F)$ cannot be abelian. Note that by [11] (see also [15]) the complement of a connected non-bifurcated hypersurface has a cyclic fundamental group, hence in fact [10] implies [9]. Since there exist smooth connected hypersurfaces V in $\mathbb{C}^n$ with $\pi_1(\mathbb{C}^n \setminus V)$ non-abelian, the following conjecture is still open (see [10]):

NOLLET–XAVIER CONJECTURE. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with non-vanishing Jacobian. Then the set S_F of non-properness of F cannot be a smooth connected hypersurface.*

We give a partial answer to the Conjecture above:

THEOREM 1.1. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with non-vanishing Jacobian. Assume that the set S_F is smooth. Then F is a surjective mapping. Moreover, if S_F is connected, then $\chi(S_F) > 0$.*

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Corollary 1.2. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with non-vanishing Jacobian. If $a \in \mathbb{C}^n \setminus F(\mathbb{C}^n)$, then a is a singular point of S_F.*

The conjecture is true in dimension $n = 2$. Indeed, in this case S_F has to be a parametric curve (see [8]), so it is isomorphic to $\mathbb{C}$. But it is well known that the complement of such a curve has cyclic fundamental group (see [1], [13]) and so the conjecture is true in dimension 2. At the end we generalize this result and we prove:

Theorem 1.3. *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping with non-vanishing Jacobian. Then S_F cannot be a curve without self-intersections.*

Note that this is not true in the real case. In fact, Pinchuk has constructed an example of a polynomial mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ with non-vanishing Jacobian for which S_F is a semi-algebraic curve homeomorphic to $\mathbb{R}$ (see [12], [3], [5]).

2. Proof of Theorem 1.1. We start with:

Theorem 2.1. *Let $(V, a) \subset (\mathbb{C}^N, a)$ be a germ of n-dimensional normal variety. Assume that $F : (V_a, a) \to (\mathbb{C}^n, 0)$ is a proper holomorphic mapping with critical set (C, a) and discriminant $(D, 0) = (F(C), 0)$. If $(D, 0)$ is smooth, then V_a is a smooth germ and there is an integer r such that in some coordinates we have*

$$F : (V_a, a) \ni (x_1, x_2, \ldots, x_n) \mapsto (x_1^r, x_2, \ldots, x_n) \ni (\mathbb{C}^n, 0).$$

Proof. Take a ball $B(0, r) =: B$ so small that in some coordinates we have $D = \{x \in B : x_1 = 0\}$. We can assume that the mapping $F : V \to B$ is well defined and proper. In particular, the mapping $F : V \setminus F^{-1}(D) \to B \setminus D$ is a topological covering. Since $\pi_1(B \setminus D) = \mathbb{Z}$ we have $F_*(\pi_1(V \setminus F^{-1}(D)) = (r) \subset \mathbb{Z}$, where r is a natural number. Let $g : \mathbb{C}^n \ni (x_1, x_2, \ldots, x_n) \mapsto (x_1^r, x_2, \ldots, x_n) \in \mathbb{C}^n$ and take $B' = g^{-1}(B)$. Consider $g : B' \setminus \{x : x_1 = 0\} \ni x \mapsto g(x) \in B \setminus D$.

We show that $F : V \to B$ is holomorphically equivalent to $g : B' \to B$. Since $F_*(\pi_1(V \setminus F^{-1}(D))) = g_*(\pi_1(B' \setminus \{x : x_1 = 0\})) = (r)$ we can lift the covering g to a holomorphic mapping $\phi : B' \setminus \{x : x_1 = 0\} \to V \setminus F^{-1}(D)$ such that the following diagram commutes:

$$\begin{array}{ccc} & & V \setminus F^{-1}(D) \\ & \overset{\phi}{\nearrow} & \downarrow F \\ B' \setminus \{x : x_1 = 0\} & \xrightarrow{\ g\ } & B \setminus D \end{array}$$

Since the mappings F and g are proper, the mapping ϕ is locally bounded and by the Riemann extension theorem it can be extended to a holomorphic mapping Φ on the whole of B'. Since V is normal we can also extend ϕ^{-1} to a holomorphic mapping $\Psi : V \to B'$. Since $\psi \circ \phi =$ identity we also have $\Psi \circ \Phi =$ identity. In particular, V is smooth at a. ■

COROLLARY 2.2. *Under the assumptions above, the germ* (C, a) (*with the reduced structure*) *is smooth and* $F^{-1}((D, 0)) = (C, a)$.

Now we can prove Theorem 1.1. By the Zariski Main Theorem in Grothendieck's version, there exists a normal variety V and a finite mapping $\overline{F} : V \to \mathbb{C}^n$ such that

(a) $\mathbb{C}^n \subset V$,
(b) $\overline{F}|_{\mathbb{C}^n} = F$.

Take $S = S_F$ and let $T = F^{-1}(S)$. Since the mapping F is quasi-finite, the set $F(T)$ is dense in S. It is enough to prove that T is closed in V, since $F|_T : T \to S$ is then finite and hence surjective. Assume that T is not closed and let $a \in \overline{T} \cap (V \setminus \mathbb{C}^n)$. Let $b = \overline{F}(a)$. Consider the mapping $\overline{F} : (V, a) \to (\mathbb{C}^n, b)$. It is easy to see that the discriminant of this mapping is $\mathbb{S}_b$ and it is smooth by assumption. Hence by Theorem 2.1 the germ $\mathbb{L}_a := (V \setminus \mathbb{C}^n)_a$ is a germ of smooth hypersurface and $\overline{F}^{-1}(\mathbb{S}_b) = \mathbb{L}_a$. But $\mathbb{T}_a \subset \overline{F}^{-1}(\mathbb{S}_b)$. This is a contradiction.

Now assume that S is connected and let $a = \chi(S)$. Let q be the topological degree of F. Since F is not proper we have $q > 1$. Outside S the mapping F is a topological covering of degree q, hence we have

$$\chi(\mathbb{C}^n) = 1 = (1 - a)q + sa = q - a(q - s),$$

where s is the topological degree of a covering $T \to S$. Note that $q - s > 0$. Hence if $a < 0$ we have $1 > q$, a contradiction. If $a = 0$ we have $q = 1$, so F is a diffeomorphism and $S_F = \emptyset$, a contradiction again. Hence $a > 0$ as asserted.

The proof of Corollary 1.2 is the same as the first part of the proof of Theorem 1.1.

3. Proof of Theorem 1.3. We will need the topological characterization of irreducible algebraic curves with Euler characteristic 1. Let us begin with the following simple observation (see e.g. [7]). The Euler characteristic of a set X will be denoted by $\chi(X)$.

THEOREM 3.1. *Let X be an irreducible affine curve of genus g (this means that a smooth model X_1 of a compactification of X has genus g). Suppose that X has n points at infinity (i.e., n is the number of points in $X_1 \setminus X_0$, where X_0 is the normalization of X). Let* $\mathrm{Sing}(X) = \{a_1, \ldots, a_r\}$ *be the* *singular locus. Further suppose that for each* i*, the germ* X_{a_i} *has* k_i *irreducible components. Then*

$$\chi(X) = 2(1-g) - n - \sum_{i=1}^{r}(k_i - 1).$$

Corollary 3.2. *Let* X *be an irreducible affine curve. Then* $\chi(X) \le 1$ *and equality holds if and only if* X *is homeomorphic to* $\mathbb{C}$.

Plane curves homeomorphic to the complex line have a very nice description due to Zaidenberg and Lin [16]:

Theorem 3.3. *Let* $X \subset \mathbb{C}^2$ *be an affine algebraic curve homeomorphic to a complex line. Then in suitable coordinates,* X *may be written as*

$$X = \{(x,y) \in \mathbb{C}^2 : x^k = y^l,\ (k,l) = 1\}.$$

For plane curves isomorphic to $\mathbb{C}$ we have the famous Abhyankar–Moh–Suzuki theorem (see [1], [13]):

Theorem 3.4. *If* $\Gamma \subset \mathbb{C}^2$ *is a curve isomorphic to* $\mathbb{C}$ *then in some coordinates we have* $\Gamma = \{(x,y) \in \mathbb{C}^2 : x = 0\}$.

The following proposition, proved in [14], is important for our study:

Theorem 3.5. *Let* $f : \mathbb{C}^2 \to \mathbb{C}$ *be a primitive polynomial and let* χ_p *denote the Euler characteristic of a general fiber. Let* S *denote the set of all atypical points. For every* $s \in S$ *we have* $\chi(\Gamma_s) > \chi_p$ *and*

$$\sum_{s\in S}\{\chi(\Gamma_s) - \chi_p\} = 1 - \chi_p.$$

We get at once the following interesting

Corollary 3.6. *If a singular fiber* Γ_s *of a family* f *has Euler characteristic* 1*, then all other fibers of* f *have Euler characteristic different from* 1.

Finally, we need the following nice result of Gwoździewicz (see [4]):

Theorem 3.7. *Let* $F : \mathbb{C}^2 \to \mathbb{C}^2$ *be a polynomial mapping with non-vanishing Jacobian. Assume that* F *is injective on some line* $L \subset \mathbb{C}$*. Then* F *is an isomorphism.*

Now we turn to the proof of Theorem 1.3. Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping with non-vanishing Jacobian and assume that S_F is a curve without self-intersections. First we show that S_F is irreducible. Assume that S_F has at least two components A and B. By [8] the curves A, B have polynomial parametrizations. In particular, their normalizations are isomorphic to $\mathbb{C}$. Since S_F has no self-intersections we see that A, B are homeomorphic to $\mathbb{C}$ and consequently they have Euler characteristic 1.

Let h_A, h_B be irreducible equations of A and B. Asume $\deg h_A \ge \deg h_B$. Consider h_B as a regular function on A. Since A and B have no common points, the function h_B is non-zero on A. But A is a parametric curve (see [8]), hence the function h_B restricted to A has to be a constant c. In particular, h_A divides $h_B - c$ and since $\deg h_A \geq \deg h_B$ we have $h_A = k(h_B - c)$, where $k \in \mathbb{C}^*$. Consequently, B is a fiber in the family given by $h_a = t$. If A or B is a singular curve, then by Corollary 3.6 we get a contradiction. If A is smooth, so is B and all other components of S_F by Theorem 3.4. In particular, in some system of coordinates we have $S_F = \{(x, y) \in \mathbb{C}^2 : \prod_{i=1}^r (x - a_i) = 0\}$. By [2] this is a contradiction.

Thus we can assume that S_F is an irreducible curve and it is homeomorphic to $\mathbb{C}$. In particular, $\chi(S_F) = 1$. Take $U = \mathbb{C}^2 \setminus S_F$. We have $\chi(U) = \chi(\mathbb{C}^2 \setminus S_F) = \chi(\mathbb{C}^2) - \chi(S_F) = 1 - 1 = 0$. Take $U' = F^{-1}(U)$. Since $S_F \cap U = \emptyset$, the mapping $F : U' \to U$ is proper, in particular it is a topological covering of degree $d = \mu(F)$ (here $\mu(F)$ is the geometric degree of F). In particular, $\chi(U') = d\chi(U) = 0$. Note that $F^{-1}(S_F) = \mathbb{C}^2 \setminus U'$. Hence $\chi(F^{-1}(S_F)) = \chi(\mathbb{C}^2) - \chi(U') = 1 - 0 = 1$. Let $F^{-1}(S_F) = \bigcup_{i=1}^r A_i$ be the decomposition of $F^{-1}(S_F)$ into irreducible components. Of course $A_i \cap A_j = \emptyset$ for $i \neq j$ and so $\chi(\bigcup_{i=1}^r A_i) = \sum_{i=1}^r \chi(A_i)$. By Corollary 3.2, there exists A_i such that $\chi(A_i) = 1$. In particular, A_i is homeomorphic to $\mathbb{C}$, and consequently it has only one place at infinity. Thus the mapping $g = F|_{A_i} : A_i \to S_F$ is proper, hence a topological covering. Since S_F is homeomorphic to $\mathbb{C}$ this mapping is in fact a homeomorphism. Using Theorem 3.3 we can assume that the curve S_F is given by the equation $x^p = y^q$, where $(p, q) = 1$. Denote $O = (0, 0)$. Let $O' = g^{-1}(O)$ and let L be the line $\{x = 0\}$. Note that $L \cap S_F = \{O\}$. Denote by L' the component of the curve $F^{-1}(L)$ which contains O' and let $h = F|_{L'}$.

We will compute $\chi(L')$. If $V = L \setminus \{O\}$, then the mapping $h : h^{-1}(V) =: V' \to V$ is proper, hence a topological covering. Since $\chi(V) = \chi(L \setminus \{O\}) = \chi(L) - \chi(\{O\}) = 1 - 1 = 0$, we have as before $\chi(h^{-1}(V)) = 0$. In particular, $\chi(L') \geq \chi(V') + \chi(O') = 1$. Consequently, by Corollary 3.2 we have $\chi(L') = 1$. Since the curve L' is smooth, it is isomorphic to $\mathbb{C}$. Since L' has only one place at infinity, the mapping $h : L' \to L$ is proper and of course it has no ramifications. Hence it is an isomorphism. By Theorem 3.4 we can assume that the curve L' is a line in $\mathbb{C}^2$. Since the mapping $F|_{L'}$ is injective, Theorem 3.7 shows that F is an isomorphism, so $S_F = \emptyset$, a contradiction.

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