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FLOWS IN NEAR ALGEBRAS WITH APPLICATIONS TO HARNESSES

BY

WŁODZIMIERZ BRYC (Cincinnati, OH), JACEK WESOŁOWSKI (Warszawa) and AGNIESZKA ZIĘBA (Warszawa)

Abstract. We introduce one-way flows in near algebras and two-way flows in double near algebras with two interrelated multiplications. We establish parametric representations of the one-way and two-way flows in terms of a single element of the algebra that we call a flow generator. We indicate probabilistic applications of the one-way flows to a study of polynomial stochastic processes. We apply our results on the two-way flows to harnesses and quadratic harnesses in probability theory, generalizing some previous results.

1. Introduction. In this paper we study near algebras and related algebraic structures motivated by the probabilistic concepts of polynomial processes, harnesses and quadratic harnesses with martingales being their common prefiguration. Martingales are fundamental in stochastic analysis, so we first explain how algebraic techniques arise in the study of martingale-like stochastic processes.

Recall that an integrable stochastic process $(X_t)_{t\geq 0}$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a martingale if $\mathbb{E}(X_t\,|\,\mathcal{F}_s) = X_s$ for all $0\leq s\leq t$. Consider a potentially wider family of stochastic processes $(X_t)_{t\geq 0}$ satisfying the condition $\mathbb{E}(X_t\,|\,\mathcal{F}_s) = \alpha_{st}X_s+\beta_{st}$, for some nonrandom coefficients α_{st},β_{st}, $0\leq s\leq t$. Are martingales the only examples of this family? If other choices of coefficients besides $(\alpha_{st},\beta_{st})\equiv(1,0)$ (the martingale case) are possible, can we describe somehow their generic form? These questions, due to the tower property of conditional expectation, can be reformulated through the system of equations

$$(\alpha_{su},\beta_{su}) = (\alpha_{tu}\alpha_{st},\alpha_{tu}\beta_{st}+\beta_{tu}) =: (\alpha_{st},\beta_{st})\boxdot(\alpha_{tu},\beta_{tu}), \tag{1.1}$$

holding for $0\leq s<t<u$, where $\boxdot$ denotes the binary operation in $\mathbb{R}^2$

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defined through the middle term in (1.1). Actually, $(\mathbb{R}^2, +, \boxdot)$ is a simple example of an algebraic structure known as a near algebra. This is a toy example of a flow in abstract near algebras, which we analyze, as a warm-up, in Section 2.

It appears that near algebras provide a specially useful base for studying properties of a general class of stochastic processes with polynomial conditional moments when conditioning is with respect to the past-future filtration $(\mathcal{F}_{s,u})_{0\le s<u}$, as it is for harnesses and quadratic harnesses that we concentrate on in this paper. The algebraic structure which is convenient for identification of polynomial conditional moments is a double near algebra (i.e., two near algebras on the same linear space with suitably related multiplications).

In the simplest case, i.e., when $\mathbb{E}(X_t \,|\, \mathcal{F}_{s,u}) = \alpha_{tsu}X_s + \beta_{tsu}X_u$, $0 \le s < t < u$, the analog of (1.1) is

$$(\alpha_{tru}, \beta_{tru}) = (\alpha_{tsu}\alpha_{sru}, \alpha_{tsu}\beta_{sru} + \beta_{tsu}) =: (\alpha_{sru}, \beta_{sru}) \boxdot_1 (\alpha_{tsu}, \beta_{tsu}),$$
$$(\alpha_{sru}, \beta_{sru}) = (\alpha_{srt} + \alpha_{tru}\beta_{srt}, \beta_{srt}\beta_{tru}) =: (\alpha_{tru}, \beta_{tru}) \boxdot_2 (\alpha_{srt}, \beta_{srt}),$$

holding for $0 \le r < s < t < u$, where $\boxdot_1$ and $\boxdot_2$ are binary operations on $\mathbb{R}^2$. Actually, $(\mathbb{R}^2, +, \boxdot_i)$, $i = 1, 2$, are two near algebras on the same real linear space $(\mathbb{R}^2, +)$ with interrelated multiplications. This is a toy example of a (two-way) flow in abstract double near algebras, which we introduce and analyze in Section 3.

As already mentioned, near algebra is the starting point of our considerations, so let us recall the definition of this algebraic structure:

DEFINITION 1.1. Let V be a set with two binary operations $+$ and $\boxdot$. We say that $(V, +, \boxdot)$ is a *near algebra* if

(i) $(V, +)$ is a linear space over $\mathbb{R}$,
(ii) multiplication $\boxdot$ is associative, i.e.,

$$(\mathbf{x} \boxdot \mathbf{y}) \boxdot \mathbf{z} = \mathbf{x} \boxdot (\mathbf{y} \boxdot \mathbf{z}) \quad \text{for all } \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, \tag{1.2}$$

(iii) multiplication $\boxdot$ is left-distributive with respect to addition, i.e.,

$$\mathbf{x} \boxdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \boxdot \mathbf{y} + \mathbf{x} \boxdot \mathbf{z} \quad \text{for all } \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, \tag{1.3}$$

(iv) multiplication $\boxdot$ is left-homogeneous, i.e.,

$$\mathbf{x} \boxdot (\lambda\mathbf{y}) = \lambda(\mathbf{x} \boxdot \mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in V \quad \text{for all } \lambda \in \mathbb{R}. \tag{1.4}$$

We consider only near algebras with multiplicative identity, i.e., with a $\boxdot$-neutral element $\mathbf{e}_{\boxdot} \in V$ which satisfies $\mathbf{x} \boxdot \mathbf{e}_{\boxdot} = \mathbf{e}_{\boxdot} \boxdot \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$. We denote by $\mathbf{x}^{-\boxdot} \in V$ the $\boxdot$-inverse of $\mathbf{x} \in V$, if it exists, as the unique element in V satisfying $\mathbf{x} \boxdot \mathbf{x}^{-\boxdot} = \mathbf{x}^{-\boxdot} \boxdot \mathbf{x} = \mathbf{e}_{\boxdot}$.

Near algebras, introduced independently by Yamamuro [19] to study mappings on Banach spaces, and later by Brown [1], fall into a hierarchy of algebraic systems of mappings on different algebraic structures including near-rings and near-fields; see Pilz [15, Sect. 1]. For a recent exposition and additional references see [16].

Our goal in this paper is to use near algebras to study mappings that arise in the theory of stochastic processes which we call harnesses following Hammersley [10]. In the simplest case harnesses are integrable processes adapted to the past-future filtration $(\mathcal{F}_{su})_{0\leq s<u}$ satisfying

$$\mathbb{E}\big(\tfrac{X_u-X_t}{u-t} \mid \mathcal{F}_{s,u}\big) = \tfrac{X_u-X_s}{u-s}, \quad 0 \leq s < t < u. \tag{1.5}$$

Williams [18] analyzed harnesses with countable parameter set under the finite second moments assumption and related them to random walks; see also [14]. Kingman [11, 12] studied somewhat paradoxical properties of harnesses in the absence of second moments. In an unpublished note Williams characterized a Wiener process as a harness with continuous trajectories; several authors [8, 9, 20, 21] extended Williams' result to the multi-parameter setting.

Quadratic harnesses are square-integrable harnesses satisfying additionally

$$\mathbb{E}(X_t^2 \,|\, \mathcal{F}_{s,u}) = Q_{tsu}(X_s, X_u), \tag{1.6}$$

where $Q_{tsu}(x, y)$ is a polynomial of degree 2 in x and y (see, e.g., [2, 6, 17].

We emphasize that the concrete form of the coefficients in polynomial conditional expectations determines properties of the process. For example, quadratic harnesses are often uniquely determined by conditions (1.5) and (1.6). Moreover, harnesses and quadratic harnesses include important families of stochastic processes such as the Wiener, Gamma, and the Poisson processes. Other examples of quadratic harnesses are related to noncommutative probability [3] and to Askey–Wilson polynomials [4].

The paper is organized as follows. In Section 2 we analyze one-way flows in near algebras. In Section 3 we introduce double near algebras and two-way flows. The main results of the paper, Theorem 3.3 and its converse Theorem 3.9, describe the two-way flow in terms of a single element of the double near algebra (called the flow generator). In Section 4 we apply abstract algebraic results of Sections 2 and 3 to stochastic processes with polynomial conditional moments.

2. Flows in near algebras: a warm-up. In this section we shall write $\mathbf{e} = \mathbf{e}_{\boxdot}$ for the identity and $\mathbf{x}^{-1} = \mathbf{x}^{-\boxdot}$ for the inverse of $\mathbf{x}$.

Definition 2.1. We will say that $\mathbf{f} \in V$ is a $\boxdot$-*null element* if for all $\mathbf{x} \in V$ we have

$$\mathbf{x} \boxdot \mathbf{f} = \mathbf{f}. \tag{2.1}$$

Referring to $\boxdot$-null elements we will also omit the symbol of multiplication in the notation in this section.

Note that zero of $(V,+)$ is a null element: by (1.4), for all $\mathbf{x}\in V$,

$$\mathbf{x}\boxdot\mathbf{0}=\mathbf{x}\boxdot(0\cdot\mathbf{0})=0(\mathbf{x}\boxdot\mathbf{0})=\mathbf{0},$$

though for $\mathbf{x}\neq\mathbf{0}\in V$ it may happen (see e.g. Section 2.1) that $\mathbf{0}\boxdot\mathbf{x}\neq\mathbf{0}$. It is easy to see that a null element cannot be invertible. In Section 2.1 we consider a near algebra where all noninvertible elements are $\boxdot$-null.

The following formulas will be used here and in Section 3.

PROPOSITION 2.2. *Suppose* $\mathbf{x}$ *is invertible,* $\mathbf{f}$ *is null and* $\alpha,\beta,\gamma\in\mathbb{R}$.

(i) *Then* $\mathbf{x}+\mathbf{f}$ *is invertible and*

$$(\mathbf{x}+\mathbf{f})^{-1}=(\mathbf{e}-\mathbf{f})\boxdot\mathbf{x}^{-1}. \tag{2.2}$$

(ii) *If* $\alpha\mathbf{x}+\beta\mathbf{e}$ *or* $\beta\mathbf{x}^{-1}+\alpha\mathbf{e}$ *is invertible, then* $\alpha\mathbf{x}+\beta\mathbf{e}+\gamma\mathbf{f}$ *is invertible and*

$$\beta(\alpha\mathbf{x}+\beta\mathbf{e}+\gamma\mathbf{f})^{-1}+\alpha(\beta\mathbf{x}^{-1}+\alpha\mathbf{e}+\gamma\mathbf{f})^{-1}+\gamma\mathbf{f}=\mathbf{e}. \tag{2.3}$$

Proof. Since formula (2.2) is straightforward, we will prove only (2.3). Since $\gamma\mathbf{f}$ is a null element (see (1.4)), it follows by (i) that at least one of $\alpha\mathbf{x}+\beta\mathbf{e}+\gamma\mathbf{f}$ and $\beta\mathbf{x}^{-1}+\alpha\mathbf{e}+\gamma\mathbf{f}$ is invertible. Since $\beta\mathbf{x}^{-1}+\alpha\mathbf{e}+\gamma\mathbf{f}=\mathbf{x}^{-1}\boxdot(\beta\mathbf{e}+\alpha\mathbf{x}+\gamma\mathbf{f})$ it follows that both are invertible. Using (1.4) we get

$$\alpha(\beta\mathbf{x}^{-1}+\alpha\mathbf{e}+\gamma\mathbf{f})^{-1}=(\alpha\mathbf{x}+\beta\mathbf{e}+\gamma\mathbf{f})^{-1}\boxdot(\alpha\mathbf{x}).$$

By (2.1) we also have

$$\gamma\mathbf{f}=(\alpha\mathbf{x}+\beta\mathbf{e}+\gamma\mathbf{f})^{-1}\boxdot(\gamma\mathbf{f}).$$

Plugging these two expressions into the left hand side of (2.3) and using left distributivity of $\boxdot$ we rewrite the left hand side of (2.3) as

$$(\alpha\mathbf{x}+\beta\mathbf{e}+\gamma\mathbf{f})^{-1}\boxdot(\alpha\mathbf{x}+\beta\mathbf{e}+\gamma\mathbf{f}),$$

and the proof is complete. ∎

A family $(\mathbf{x}_{st})_{0\le s<t}$ of invertible elements of a near algebra $(V,+,\boxdot)$ that satisfies the system of equations

$$\mathbf{x}_{st}\boxdot\mathbf{x}_{tu}=\mathbf{x}_{su},\qquad 0\le s<t<u, \tag{2.4}$$

will be called a *one-way* flow on $[0,\infty)$ in V if for all $0\le s<u$,

$$\tfrac{\mathbf{x}_{su}-\mathbf{e}}{u-s}\quad\text{does not depend on } u. \tag{2.5}$$

Our goal is to show that such a family is uniquely determined by a single element $\mathbf{h}\in V$ which one could call a flow generator.

THEOREM 2.3. *Suppose that* $(\mathbf{x}_{st})_{0\le s<t}$ *is a one-way flow in* V. *Then there exists a unique* $\mathbf{h}\in V$ *such that* $\mathbf{e}+s\mathbf{h}$ *is* $\boxdot$*-invertible for all* $s\ge 0$ *and*

$$\mathbf{x}_{st}=(\mathbf{e}+s\mathbf{h})^{-1}\boxdot(\mathbf{e}+t\mathbf{h})\quad \textit{for all } 0\le s<t. \tag{2.6}$$

Conversely, if $\mathbf{h} \in V$ *is such that* $\mathbf{e} + s\mathbf{h}$ *is* $\boxdot$*-invertible for all* $s \geq 0$*, then* $(\mathbf{x}_{st})_{0 \leq s < t}$ *given by* (2.6) *is a one-way flow.*

Proof. We show a slightly stronger result that (2.6) follows from (2.4) and (2.5) used for $s = 0$ only. For $t > 0$, denote $\mathbf{h}_t = \mathbf{x}_{0t}$ and let $\mathbf{h}_0 = \mathbf{e}$. By assumption, $\mathbf{h}_t^{-1}$ exists, so from (2.4) we see that for $0 \leq t < u$ we have

$$\mathbf{x}_{tu} = \mathbf{h}_t^{-1} \boxdot \mathbf{h}_u. \tag{2.7}$$

Inserting $\mathbf{h}_u$ into (2.5) with $s = 0$, we see that there exists $\mathbf{h}$ such that $\frac{1}{u}(\mathbf{h}_u - \mathbf{e}) = \mathbf{h}$, i.e., $\mathbf{h}_u = \mathbf{e} + u\mathbf{h}$. In particular, $\mathbf{e} + u\mathbf{h}$ is invertible for all $u \geq 0$ and (2.6) follows from (2.7).

To prove uniqueness, suppose (2.6) holds with $\mathbf{h}$ and $\mathbf{h}'$, i.e., $(\mathbf{e}+s\mathbf{h})^{-1} \boxdot (\mathbf{e} + t\mathbf{h}) = (\mathbf{e} + s\mathbf{h}')^{-1} \boxdot (\mathbf{e} + t\mathbf{h}')$. Taking $s = 0$ gives $\mathbf{h} = \mathbf{h}'$.

To prove the converse, it is clear that expression (2.6) solves (2.4) for all $0 \leq s < t < u$ and that, as a product of invertible elements, $\mathbf{x}_{st}$ is invertible. To verify (2.5), we write its left hand side as

$$\begin{aligned} \frac{1}{u-s}\big((\mathbf{e}+s\mathbf{h})^{-1} \boxdot (\mathbf{e}+u\mathbf{h}) - \mathbf{e}\big) &= \frac{1}{u-s}(\mathbf{e}+s\mathbf{h})^{-1} \boxdot ((\mathbf{e}+u\mathbf{h}) - (\mathbf{e}+s\mathbf{h})) \\ &= (\mathbf{e} + s\mathbf{h})^{-1} \boxdot \mathbf{h}, \end{aligned}$$

which does not depend on u. ∎

2.1. Flows in the near algebra of linear maps. Let V be a linear space over $\mathbb{R}$. For any $\mathbf{a} = (\alpha, \underline{a}), \mathbf{b} = (\beta, \underline{b}) \in \mathbb{R} \times V$ define

$$\mathbf{a} \boxdot \mathbf{b} = (\beta\alpha, \beta\underline{a} + \underline{b}). \tag{2.8}$$

One can check that $\mathcal{A}_V := (\mathbb{R} \times V, +, \boxdot)$ is a near algebra with $\mathbf{e} = (1, \underline{0})$.

We observe that any $\mathbf{a} = (\alpha, \underline{a}) \in \mathbb{R} \times V$ with $\alpha \neq 0$ is invertible with inverse

$$\mathbf{a}^{-1} = (\alpha^{-1}, -\alpha^{-1}\underline{a}).$$

Noninvertible $\mathbf{a} = (\alpha, \underline{a}) \in \mathbb{R} \times V$ have $\alpha = 0$, so are null, as (2.8) gives (2.1).

Note that $\mathcal{A}_V$ can be viewed as an algebra of linear maps $f_\mathbf{a} : \mathbb{R} \times V^* \to \mathbb{R}$, where V^* is the algebraic dual of V, $\mathbf{a} \in \mathbb{R} \times V$. For $\mathbf{a} = (\alpha, \underline{a}) \in \mathbb{R} \times V$ the map $f_\mathbf{a}$ is defined by

$$f_\mathbf{a}(\lambda, \Lambda) = \lambda\alpha + \Lambda\underline{a}, \qquad (\lambda, \Lambda) \in \mathbb{R} \times V^*.$$

Then, for $\mathbf{a}, \mathbf{b} \in \mathbb{R} \times V$ we define a composition $f_\mathbf{a} \circ f_\mathbf{b}$ by

$$(f_\mathbf{a} \circ f_\mathbf{b})(\lambda, \Lambda) := f_\mathbf{b}(f_\mathbf{a}(\lambda, \Lambda), \Lambda), \qquad (\lambda, \Lambda) \in \mathbb{R} \times V^*.$$

This composition corresponds to operation (2.8) on pairs $\mathbf{a}, \mathbf{b}$, i.e.,

$$f_\mathbf{a} \circ f_\mathbf{b} = f_{\mathbf{a} \boxdot \mathbf{b}}.$$

Proposition 2.4. *Let* $\mathbf{x}_{st} = (\xi_{st}, \underline{x}_{st}) \in \mathbb{R} \times V$*,* $0 \leq s < t$*, be such that* $\xi_{st} \neq 0$ *for all* $0 \leq s < t$*. Assume that* (2.4) *and* (2.5) *are satisfied. Then* *there exist* $\alpha \geq 0$ *and* $\underline{a} \in V$ *such that for any* $0 \leq s < t$,

$$\mathbf{x}_{st} = \left(\tfrac{1+\alpha t}{1+\alpha s}, \tfrac{t-s}{1+\alpha s}\underline{a}\right). \tag{2.9}$$

Proof. We apply Theorem 2.3 with $\mathbf{h} = (\alpha, \underline{a})$. It is clear that $\mathbf{e} + s\mathbf{h}$ is invertible for all $s \geq 0$ if and only if $\alpha \geq 0$. Formula (2.9) follows by calculation. ∎

REMARK 2.5. If we assume that the flow equation (2.4) and condition (2.5) are satisfied only for $0 < s < t < u$ then, as shown in the proof of Proposition 4.1, additional solutions arise.

Another example is related to endomorphisms. Let $L := L(V)$ be the space of endomorphisms of a linear space V. For any $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2) \in L^2$ define

$$\mathbf{A} \boxdot \mathbf{B} = (A_1B_1, A_2B_1 + B_2).$$

One can easily check that $(L^2, +, \boxdot)$ is a near algebra with $\mathbf{e} = (\mathrm{Id}, 0)$. Any $\mathbf{A} = (A_1, A_2) \in L^2$ with A_1 invertible has inverse of the form

$$\mathbf{A}^{-\boxdot} = (A_1^{-1}, -A_2A_1^{-1}).$$

Elements of L^2 with $A_1 = 0$ are null and they form a proper subclass of noninvertible elements of this near algebra. An application of Theorem 2.3 gives the solution of the flow equation for a family $\mathbf{x}_{st}$, $0 \leq s < t$, of invertible elements satisfying condition (2.5):

There exist $G, H \in L$ such that $\mathrm{Id} + sH$ is invertible for all $s \geq 0$ and

$$\mathbf{x}_{st} = ((\mathrm{Id} + tH)(\mathrm{Id} + sH)^{-1}, (t-s)G(\mathrm{Id} + sH)^{-1}), \quad 0 \leq s < t.$$

We will consider a related example more thoroughly when discussing double near algebras in Section 3.1.

3. Flows in double near algebras. We now introduce our main algebraic object.

DEFINITION 3.1. We say that $(V, +, \ltimes, \rtimes)$ is a *double near algebra* (DNA) if

(i) $(V, +, \ltimes)$ and $(V, +, \rtimes)$ are near algebras with neutral elements $\mathbf{e}_\ltimes$ and $\mathbf{e}_\rtimes$, respectively,

(ii) $\mathbf{e}_\ltimes$ is $\rtimes$-null and $\mathbf{e}_\rtimes$ is $\ltimes$-null, i.e., for every $\mathbf{x} \in V$,

$$\mathbf{x} \rtimes \mathbf{e}_\ltimes = \mathbf{e}_\ltimes \quad \text{and} \quad \mathbf{x} \ltimes \mathbf{e}_\rtimes = \mathbf{e}_\rtimes. \tag{3.1}$$

We denote by $\mathbf{x}^{-\ltimes}$ and $\mathbf{x}^{-\rtimes}$ the inverse elements of $\mathbf{x}$ with respect to multiplications $\ltimes$ and $\rtimes$, respectively, if they exist.

DEFINITION 3.2. A family $(\mathbf{x}_{sru})_{0 \leq r \leq s < u}$ of $\ltimes$- and $\rtimes$-invertible elements of V is called a *two-way flow* if it satisfies the flow equations

$$\mathbf{x}_{sru} \rtimes \mathbf{x}_{tsu} = \mathbf{x}_{tru}, \quad \mathbf{x}_{tru} \ltimes \mathbf{x}_{srt} = \mathbf{x}_{sru}, \quad 0 \leq r < s < t < u, \tag{3.2}$$

and the structure condition: for all $0 \le r < u$,

$$\frac{\mathbf{x}_{sru}-\mathbf{e}_{\ltimes}}{u-s} + \frac{\mathbf{x}_{sru}-\mathbf{e}_{\rtimes}}{s-r} \quad \text{does not depend on } s \in (r,u). \tag{3.3}$$

Note that the family

$$\mathbf{x}_{sru} = \tfrac{u-s}{u-r}\mathbf{e}_{\rtimes} + \tfrac{s-r}{u-r}\mathbf{e}_{\ltimes}, \qquad 0 \le r < s < u, \tag{3.4}$$

is a two-way flow in any DNA. To see $\ltimes$- and $\rtimes$-invertibility we use (2.2). The flow equations follow by straightforward calculation. The expression in (3.3) is zero.

We will use the following notation. For any $\mathbf{h} \in V$ and $r \ge 0$ denote

$$\mathbf{h}_r = r(1-r)\mathbf{h} + (1-r)\mathbf{e}_{\rtimes} + r\mathbf{e}_{\ltimes}. \tag{3.5}$$

The following result is a parametric description of all two-way flows.

THEOREM 3.3. *If a family $(\mathbf{x}_{sru})_{0\le r<s<u}$ is a two-way flow, then there exists a unique $\mathbf{h} \in V$ such that for all $0 \le r < s < u$,*

$$\mathbf{x}_{sru} = (\mathbf{h}_u^{-\ltimes} \ltimes \mathbf{h}_r)^{-\rtimes} \rtimes (\mathbf{h}_u^{-\ltimes} \ltimes \mathbf{h}_s). \tag{3.6}$$

For later reference it will be convenient to rewrite (3.6) as

$$\mathbf{x}_{sru} = \mathbf{w}_{ru}^{-\rtimes} \rtimes \mathbf{w}_{su}, \tag{3.7}$$

where for $0 \le r < u$,

$$\mathbf{w}_{ru} = \mathbf{h}_u^{-\ltimes} \ltimes \mathbf{h}_r. \tag{3.8}$$

We remark that $\mathbf{w}_{st} := \mathbf{x}_{s0t}$ solves the induced flow equation $\mathbf{w}_{tu} \ltimes \mathbf{w}_{st} = \mathbf{w}_{su}$ but although its solution (3.8) looks deceptively similar to (2.6), the simple argument we used to derive (2.6) is not applicable here, as in our solution $\mathbf{w}_{0u} = \mathbf{e}_{\rtimes}$ is not $\ltimes$-invertible.

A computation shows that (3.5) and (3.6) with $\mathbf{h} = 0$ give the two-way flow (3.4).

Proof of Theorem 3.3. Denote

$$\mathbf{w}_{su} := \begin{cases} \mathbf{x}_{s0u} & \text{for } 0 < s < u, \\ \mathbf{e}_{\rtimes} & \text{for } 0 = s < u. \end{cases}$$

From the first flow equation in (3.2) we have

$$\mathbf{x}_{sru} = \mathbf{w}_{ru}^{-\rtimes} \rtimes \mathbf{w}_{su} \tag{3.9}$$

for all $0 \le r < s < u$. Furthermore,

$$\mathbf{w}_{su} = \mathbf{v}_u^{-\ltimes} \ltimes \mathbf{v}_s \tag{3.10}$$

for all $0 \le s < u$, where

$$\mathbf{v}_s := \begin{cases} \mathbf{w}_{1s}^{-\ltimes} & \text{for } s > 1, \\ \mathbf{e}_{\ltimes} & \text{for } s = 1, \\ \mathbf{w}_{s1} & \text{for } 0 < s < 1, \\ \mathbf{e}_{\rtimes} & \text{for } s = 0, \end{cases}$$

where the case $0 < s < u$ follows from the second equation in (3.2) and the case $s = 0 < u$ is a simple consequence of (3.1). So from the above considerations it is sufficient to give a parametrization of $\mathbf{x}_{s01}$ and $\mathbf{x}_{10s}$ for $s < 1$ and $s > 1$ respectively. Firstly, note that after a simple rewriting of (3.3), we find that there exists $\mathbf{g}_u \in V$ such that

$$\mathbf{g}_u := \tfrac{u}{(u-t)t}\mathbf{x}_{t0u} - \tfrac{1}{u-t}\mathbf{e}_{\ltimes} - \tfrac{1}{t}\mathbf{e}_{\rtimes} \in V. \tag{3.11}$$

Thus for all $0 < t < u$ we obtain

$$\mathbf{x}_{t0u} = \tfrac{(u-t)t}{u}\mathbf{g}_u + \tfrac{t}{u}\mathbf{e}_{\ltimes} + \tfrac{u-t}{u}\mathbf{e}_{\rtimes}. \tag{3.12}$$

Inserting expression (3.12) for $\mathbf{x}_{s0t}$ and $\mathbf{x}_{s0u}$ into the second equation in (3.2) and then using (1.3), (1.4) and (3.1) gives

$$\tfrac{(t-s)s}{t}\mathbf{x}_{t0u} \ltimes \mathbf{g}_t + \tfrac{t-s}{t}\mathbf{e}_{\rtimes} + \tfrac{s}{t}\mathbf{x}_{t0u} = \tfrac{s(u-s)}{u}\mathbf{g}_u + \tfrac{u-s}{u}\mathbf{e}_{\rtimes} + \tfrac{s}{u}\mathbf{e}_{\ltimes}.$$

Substituting $\mathbf{x}_{t0u}$ in the third component on the left hand side above according to (3.12), after a simple calculation we get

$$\tfrac{1}{t}\mathbf{x}_{t0u} \ltimes \mathbf{g}_t = \tfrac{1}{u}\mathbf{g}_u, \qquad 0 < t < u.$$

Using (1.3), (1.4), (3.1), (3.11) and the $\ltimes$-invertibility of $\mathbf{x}_{t0u}$ we obtain

$$\mathbf{x}_{t0u} \ltimes \left(\tfrac{1}{t}\mathbf{g}_t - \tfrac{1}{(u-t)t}\mathbf{e}_{\ltimes} + \tfrac{1}{u(u-t)}\mathbf{x}_{t0u}^{-\ltimes} + \tfrac{1}{tu}\mathbf{e}_{\rtimes}\right) = \mathbf{0} \in V.$$

Applying $\mathbf{x}_{t0u}^{-\ltimes}\ltimes$ to the above (recall that $\mathbf{0}$ is $\ltimes$-null) we get

$$\tfrac{1}{t}\mathbf{g}_t - \tfrac{1}{(u-t)t}\mathbf{e}_{\ltimes} + \tfrac{1}{u(u-t)}\mathbf{x}_{t0u}^{-\ltimes} + \tfrac{1}{tu}\mathbf{e}_{\rtimes} = \mathbf{0}.$$

Therefore, for all $0 < t < u$ we have

$$\mathbf{x}_{t0u}^{-\ltimes} = -\tfrac{(u-t)u}{t}\mathbf{g}_t + \tfrac{u}{t}\mathbf{e}_{\ltimes} - \tfrac{u-t}{t}\mathbf{e}_{\rtimes}. \tag{3.13}$$

Let $\mathbf{h} := \mathbf{g}_1$. Then (3.12) gives

$$\mathbf{x}_{t01} = t(1-t)\mathbf{h} + t\mathbf{e}_{\ltimes} + (1-t)\mathbf{e}_{\rtimes} = \mathbf{h}_t, \quad t < 1,$$

where the last equality is due to the definition of $\mathbf{h}_t$. Moreover, from (3.13) we get

$$\mathbf{x}_{10u}^{-\ltimes} = u(1-u)\mathbf{h} + u\mathbf{e}_{\ltimes} + (1-u)\mathbf{e}_{\rtimes} = \mathbf{h}_u, \quad u > 1.$$

To sum up, we have shown that $\mathbf{v}_s = \mathbf{h}_s$ for all $s \ge 0$. This identity together with (3.9) and (3.10) gives the conclusion (3.6).

To see that $\mathbf{h}$ is unique, assume that there exists $\widetilde{\mathbf{h}} \neq \mathbf{h}$ such that plugging $\widetilde{\mathbf{h}}$ instead of $\mathbf{h}$ in (3.5) we obtain $\widetilde{\mathbf{h}}_r$ and then from (3.6) another expression for $\mathbf{x}_{sru}$. Then, comparing these two expressions for $\mathbf{x}_{sru}$, we get

$$(\widetilde{\mathbf{h}}_u^{-\ltimes} \ltimes \widetilde{\mathbf{h}}_r)^{-\rtimes} \rtimes (\widetilde{\mathbf{h}}_u^{-\ltimes} \ltimes \widetilde{\mathbf{h}}_s) = (\mathbf{h}_u^{-\ltimes} \ltimes \mathbf{h}_r)^{-\rtimes} \rtimes (\mathbf{h}_u^{-\ltimes} \ltimes \mathbf{h}_s), \quad 0 \le r < s < u.$$

Since $\widetilde{\mathbf{h}}_0 = \mathbf{h}_0 = \mathbf{e}_\rtimes$, the above equality with $r = 0$ yields

$$\widetilde{\mathbf{h}}_u^{-\ltimes} \ltimes \widetilde{\mathbf{h}}_s = \mathbf{h}_u^{-\ltimes} \ltimes \mathbf{h}_s, \quad 0 < s < u,$$

or equivalently

$$\mathbf{h}_u \ltimes \widetilde{\mathbf{h}}_u^{-\ltimes} = \mathbf{h}_s \ltimes \widetilde{\mathbf{h}}_s^{-\ltimes}.$$

Thus $\mathbf{h}_u \ltimes \widetilde{\mathbf{h}}_u^{-\ltimes}$ does not depend on u. Since $\mathbf{h}_1 = \widetilde{\mathbf{h}}_1 = \mathbf{e}_\ltimes$ it follows that $\mathbf{h}_u \ltimes \widetilde{\mathbf{h}}_u^{-\ltimes} = \mathbf{e}_\ltimes$ for all $u > 0$ and consequently $\mathbf{h} = \widetilde{\mathbf{h}}$. ∎

REMARK 3.4. It follows from the proof above that to get representations (3.5), (3.6), it suffices to assume that

(i) $\mathbf{x}_{s0u}$ is $\ltimes$- and $\rtimes$-invertible for $0 < s < u$,
(ii) $(\mathbf{x}_{s0u})_{0<s<u}$ satisfies (3.3) and the second flow equation (3.2) for $r = 0$,
(iii) $(\mathbf{x}_{sru})_{0\le r<s<u}$ satisfies (3.2).

Note also that in view of (i) it follows from (iii) that $\mathbf{x}_{sru}$ is $\rtimes$-invertible for any $0 \le r < s < u$.

Now we will focus on a converse to Theorem 3.3, i.e., we will study the following question: for which $\mathbf{h} \in V$ is the family given by (3.6) a two-way flow? To do that we need some identities for the elements $\mathbf{h}_r$, $r \ge 0$.

LEMMA 3.5. *Let* $0 \le r < s < u$. *Then*

(3.14) $\quad \mathbf{h}_s = \frac{u-s}{u-r}\mathbf{h}_r + \frac{s-r}{u-r}\mathbf{h}_u + (u-s)(s-r)\mathbf{h},$

(3.15) $\quad ru\mathbf{h}_s = \frac{u-s}{u-r}su\mathbf{h}_r + \frac{s-r}{u-r}rs\mathbf{h}_u - (u-s)(s-r)\mathbf{e}_\rtimes,$

(3.16) $\quad \frac{\mathbf{h}_u}{(u-t)(u-s)(u-r)} + \frac{\mathbf{h}_s}{(u-s)(t-s)(s-r)} = \frac{\mathbf{h}_r}{(u-r)(t-r)(s-r)} + \frac{\mathbf{h}_t}{(u-t)(t-s)(t-r)}.$

Proof. All these identities are easily obtained by direct calculations based on the coefficients in (3.5). ∎

We now give a partial converse to Theorem 3.3, which implies a converse to Remark 3.4.

PROPOSITION 3.6. *If* $\mathbf{h} \in V$ *is such that the right hand side of* (3.6) *is well defined then the family* $(\mathbf{x}_{sru})_{0\le r<s<u}$ *defined by* (3.5) *and* (3.6) *satisfies the structure condition* (3.3), *is* $\rtimes$*-invertible and the first flow equation* (3.2) *holds. In addition,* $\mathbf{x}_{s0u}$ *is* $\ltimes$*-invertible, and the second flow equation* (3.2) *holds for* $r = 0 < s < t < u$.

Proof. As a product (3.6) of $\rtimes$-invertible elements, $\mathbf{x}_{sru}$ is $\rtimes$-invertible for $r \ge 0$. Furthermore, by associativity of $\rtimes$ the first equality of (3.2) holds. Moreover, (3.7) gives $\mathbf{x}_{s0u} = \mathbf{w}_{su}$, which is $\ltimes$-invertible from (3.8), and associativity of $\ltimes$ gives the second equality of (3.2) for $r=0$. To verify the structure condition (3.3), note that

$$\begin{aligned}\tfrac{\mathbf{x}_{sru}-\mathbf{e}_{\ltimes}}{u-s}+\tfrac{\mathbf{x}_{sru}-\mathbf{e}_{\rtimes}}{s-r}&=\mathbf{w}_{ru}^{-\rtimes}\rtimes\big(\tfrac{u-r}{(u-s)(s-r)}\mathbf{w}_{su}-\tfrac{1}{u-s}\mathbf{e}_{\ltimes}-\tfrac{1}{s-r}\mathbf{w}_{ru}\big)\\&=\mathbf{w}_{ru}^{-\rtimes}\rtimes\big\{\mathbf{h}_u^{-\ltimes}\ltimes\big[\tfrac{u-r}{(u-s)(s-r)}\mathbf{h}_s-\tfrac{1}{u-s}\mathbf{h}_u-\tfrac{1}{s-r}\mathbf{h}_r\big]\big\}\\&=\mathbf{w}_{ru}^{-\rtimes}\rtimes\{\mathbf{h}_u^{-\ltimes}\ltimes((u-r)\mathbf{h})\},\end{aligned}$$

where the last equality follows from (3.14). Thus (3.3) is satisfied for all $0\le r<s<u$. ∎

As Example 3.12 shows, the full converse of Theorem 3.3 requires additional assumptions. We introduce the following concept.

Definition 3.7. We say that $\mathbf{h}\in V$ is a *flow generator* if $\mathbf{h}_r$ in (3.5) is $\ltimes$-invertible for every $r>0$ and defines by (3.8) the $\rtimes$-invertible family $(\mathbf{w}_{ru})_{0<r<u}$ satisfying, for all $0<r<s<u$,

$$\mathbf{w}_{ru}^{-\rtimes}\rtimes\mathbf{w}_{su}=(\mathbf{w}_{rs}^{-\rtimes}\rtimes\mathbf{w}_{su}^{-\ltimes})^{-\ltimes}. \tag{3.17}$$

(In particular, we assume that the inverse on the right hand side of (3.17) exists.)

Since this definition is rather cumbersome, we give a somewhat simpler sufficient condition.

Proposition 3.8. *Suppose that* $\mathbf{h}\in V$ *is such that expression* (3.8) *is well defined and the corresponding family* $(\mathbf{w}_{ru})_{0<r<u}$ *is such that for all* $0<r<u$ *the expression*

$$\tfrac{1}{u(u-r)}(\mathbf{w}_{ru}^{-\rtimes})^{-\ltimes}+\tfrac{1}{ru}\mathbf{e}_{\ltimes}-\tfrac{1}{r(u-r)}\mathbf{e}_{\rtimes} \tag{3.18}$$

is well defined and does not depend on u. *Then* $\mathbf{h}$ *is a flow generator.*

Proof. Since (3.18) does not depend on u, for all $0<r<s<u$ we have

$$\tfrac{1}{u(u-r)}(\mathbf{w}_{ru}^{-\rtimes})^{-\ltimes}+\tfrac{1}{ru}\mathbf{e}_{\ltimes}-\tfrac{1}{r(u-r)}\mathbf{e}_{\rtimes}=\tfrac{1}{s(s-r)}(\mathbf{w}_{rs}^{-\rtimes})^{-\ltimes}+\tfrac{1}{rs}\mathbf{e}_{\ltimes}-\tfrac{1}{r(s-r)}\mathbf{e}_{\rtimes}.$$

Consequently, solving this for $(\mathbf{w}_{rs}^{-\rtimes})^{-\ltimes}$ by straightforward algebra we have

$$\mathbf{w}_{rs}^{-\rtimes}=\big(\tfrac{s(s-r)}{u(u-r)}(\mathbf{w}_{ru}^{-\rtimes})^{-\ltimes}-\tfrac{(u-s)(s-r)}{ru}\mathbf{e}_{\ltimes}+\tfrac{s(u-s)}{r(u-r)}\mathbf{e}_{\rtimes}\big)^{-\ltimes}.$$

Using (2.3) with $\mathbf{x}=\mathbf{w}_{ru}^{-\rtimes}$, $\mathbf{e}=\mathbf{e}_{\ltimes}$, $\mathbf{f}=\mathbf{e}_{\rtimes}$, $\alpha=-\frac{(u-s)(s-r)}{ru}$, $\beta=\frac{s(s-r)}{u(u-r)}$ and $\gamma=\frac{s(u-s)}{r(u-r)}$ we get

$$\begin{aligned}&\tfrac{(u-r)(u-s)}{rs}\mathbf{w}_{rs}^{-\rtimes}\\&\quad=\big(\tfrac{s(s-r)}{u(u-r)}\mathbf{e}_{\ltimes}+\tfrac{s(u-s)}{r(u-r)}\mathbf{e}_{\rtimes}-\tfrac{(u-s)(s-r)}{ru}\mathbf{w}_{ru}^{-\rtimes}\big)^{-\ltimes}-\tfrac{u(u-r)}{s(s-r)}\mathbf{e}_{\ltimes}+\tfrac{u(u-s)}{r(s-r)}\mathbf{e}_{\rtimes}.\end{aligned}$$

So

$$\mathbf{w}_{rs}^{-\rtimes} \rtimes \big(\tfrac{(u-r)(u-s)}{rs}\mathbf{e}_{\rtimes} + \tfrac{u(u-r)}{s(s-r)}\mathbf{e}_{\ltimes} - \tfrac{u(u-s)}{r(s-r)}\mathbf{w}_{rs}\big) = \big(\mathbf{w}_{ru}^{-\rtimes} \rtimes \big(\tfrac{s(s-r)}{u(u-r)}\mathbf{e}_{\ltimes} - \tfrac{(u-s)(s-r)}{ru}\mathbf{e}_{\rtimes} + \tfrac{s(u-s)}{r(u-r)}\mathbf{w}_{ru}\big)\big)^{-\ltimes}. \tag{3.19}$$

Next we note that since $\mathbf{w}_{rs} = \mathbf{h}_s^{-\ltimes} \ltimes \mathbf{h}_r$ (see (3.8)), and $\mathbf{e}_{\rtimes}$ is $\ltimes$-null, the expression in the parentheses on the left hand side of (3.19) can be rewritten as

$$\tfrac{(u-r)(u-s)}{rs}\mathbf{e}_{\rtimes} + \tfrac{u(u-r)}{s(s-r)}\mathbf{e}_{\ltimes} - \tfrac{u(u-s)}{r(s-r)}\mathbf{w}_{rs} = \mathbf{h}_s^{-\ltimes} \ltimes \Big(\tfrac{u(u-r)}{s(s-r)}\mathbf{h}_s - \tfrac{u(u-s)}{r(s-r)}\mathbf{h}_r + \tfrac{(u-r)(u-s)}{rs}\mathbf{e}_{\rtimes}\Big) \overset{(3.15)}{=} \mathbf{h}_s^{-\ltimes} \ltimes \mathbf{h}_u = \mathbf{w}_{su}^{-\ltimes}.$$

Analogously, we rewrite the expression in the inner parentheses on the right hand side of (3.19). We get

$$\tfrac{s(s-r)}{u(u-r)}\mathbf{e}_{\ltimes} - \tfrac{(u-s)(s-r)}{ru}\mathbf{e}_{\rtimes} + \tfrac{s(u-s)}{r(u-r)}\mathbf{w}_{ru} = \mathbf{w}_{su}.$$

Thus (3.19) simplifies to

$$\mathbf{w}_{rs}^{-\rtimes} \rtimes \mathbf{w}_{su}^{-\ltimes} = (\mathbf{w}_{ru}^{-\rtimes} \rtimes \mathbf{w}_{su})^{-\ltimes}.$$

Taking the $\ltimes$-inverse yields (3.17). ■

THEOREM 3.9. *Let* $\mathbf{h} \in V$ *and let* $(\mathbf{x}_{sru})_{0\le r<s<u}$ *be generated by* $\mathbf{h}$ *according to* (3.5) *and* (3.6). *The family* $(\mathbf{x}_{sru})_{0\le r<s<u}$ *is a two-way flow if and only if* $\mathbf{h}$ *is a flow generator.*

Proof. Sufficiency. If $\mathbf{h} \in V$ is a flow generator, then in view of Proposition 3.8, it remains to verify $\ltimes$-invertibility and the second flow equation for $r > 0$. We see that the left hand side of (3.17) has $\ltimes$-inverse, so (3.7) implies that $\mathbf{x}_{sru}$ is $\ltimes$-invertible for $0 < r < s < u$ with inverse

$$\mathbf{x}_{sru}^{-\ltimes} = \mathbf{w}_{rs}^{-\rtimes} \rtimes \mathbf{w}_{su}^{-\ltimes}. \tag{3.20}$$

We rewrite the structure condition (3.3) (holding by Proposition 3.6) as

$$\tfrac{u-t}{u-s}\mathbf{x}_{sru} + \tfrac{(t-s)(s-r)}{(u-s)(u-r)}\mathbf{e}_{\ltimes} = \tfrac{(t-s)(u-t)}{(t-r)(u-r)}\mathbf{e}_{\rtimes} + \tfrac{s-r}{t-r}\mathbf{x}_{tru} \tag{3.21}$$

for $0 < r < s < t < u$. Combining this with (3.20) gives

$$\begin{aligned}\mathbf{x}_{sru}^{-\ltimes} \ltimes \mathbf{x}_{tru} &= \tfrac{(t-s)(t-r)}{(u-s)(u-r)}\mathbf{x}_{sru}^{-\ltimes} + \tfrac{(t-r)(u-t)}{(s-r)(u-s)}\mathbf{e}_{\ltimes} - \tfrac{(t-s)(u-t)}{(s-r)(u-r)}\mathbf{e}_{\rtimes} \\ &= \mathbf{w}_{rs}^{-\rtimes} \rtimes \big(\tfrac{(t-s)(t-r)}{(u-s)(u-r)}\mathbf{w}_{su}^{-\ltimes} + \tfrac{(t-r)(u-t)}{(s-r)(u-s)}\mathbf{e}_{\ltimes} - \tfrac{(t-s)(u-t)}{(s-r)(u-r)}\mathbf{w}_{rs}\big) \\ &= \mathbf{w}_{rs}^{-\rtimes} \rtimes \big[\mathbf{h}_s^{-\ltimes} \ltimes \big(\tfrac{(t-s)(t-r)}{(u-s)(u-r)}\mathbf{h}_u + \tfrac{(t-r)(u-t)}{(s-r)(u-s)}\mathbf{h}_s - \tfrac{(t-s)(u-t)}{(s-r)(u-r)}\mathbf{h}_r\big)\big].\end{aligned}$$

Thus (3.16) yields

$$\mathbf{x}_{sru}^{-\ltimes} \ltimes \mathbf{x}_{tru} = \mathbf{w}_{rs}^{-\rtimes} \rtimes \big(\mathbf{h}_s^{-\ltimes} \ltimes \mathbf{h}_t\big) = \mathbf{x}_{srt}^{-\ltimes},$$

whence the second flow equation (3.2) for $r > 0$ follows.

Necessity. If $(\mathbf{x}_{sru})_{0\le r<s<u}$ is a two-way flow, then as in the proof of Proposition 3.6, we get

$$\mathbf{x}_{sru} = (u-s)(s-r)\mathbf{w}_{ru}^{-\rtimes} \rtimes (\mathbf{h}_u^{-\ltimes} \ltimes \mathbf{h}) + \tfrac{s-r}{u-r}\mathbf{e}_{\ltimes} + \tfrac{u-s}{u-r}\mathbf{e}_{\rtimes}. \tag{3.22}$$

Inserting this formula for $\mathbf{x}_{srt}$ into the second flow equation yields

$$\mathbf{x}_{sru} = (t-s)(s-r)\mathbf{x}_{tru} \ltimes \{\mathbf{w}_{rt}^{-\rtimes} \rtimes (\mathbf{h}_t^{-\ltimes} \ltimes \mathbf{h})\} + \tfrac{s-r}{t-r}\mathbf{x}_{tru} + \tfrac{t-s}{t-r}\mathbf{e}_{\rtimes}.$$

Substituting $\mathbf{x}_{sru}$ and the second $\mathbf{x}_{tru}$ above according to (3.22), after simplifications we obtain

$$\begin{aligned}\mathbf{x}_{tru} \ltimes \{\mathbf{w}_{rt}^{-\rtimes} \rtimes (\mathbf{h}_t^{-\ltimes} \ltimes \mathbf{h})\} &= \mathbf{w}_{ru}^{-\rtimes} \rtimes (\mathbf{h}_u^{-\ltimes} \ltimes \mathbf{h}) \\ &= \tfrac{1}{(u-t)(t-r)}\mathbf{x}_{tru} - \tfrac{1}{(u-t)(u-r)}\mathbf{e}_{\ltimes} - \tfrac{1}{(t-r)(u-r)}\mathbf{e}_{\rtimes},\end{aligned}$$

where the last equality holds due to (3.22). Applying $\mathbf{x}_{tru}^{-\ltimes}\ltimes$ to the equality of the first and the third expressions above and solving for $\mathbf{x}_{tru}^{-\ltimes}$ we get

$$\begin{aligned}\mathbf{x}_{tru}^{-\ltimes} &= \tfrac{u-r}{t-r}\mathbf{e}_{\ltimes} - \tfrac{u-t}{t-r}\mathbf{e}_{\rtimes} - (u-t)(u-r)\mathbf{w}_{rt}^{-\rtimes} \rtimes (\mathbf{h}_t^{-\ltimes} \ltimes \mathbf{h}) \\ &= \mathbf{w}_{rt}^{-\rtimes} \rtimes \left(\tfrac{u-r}{t-r}\mathbf{e}_{\ltimes} - \tfrac{u-t}{t-r}\mathbf{w}_{rt} - (u-t)(u-r)\mathbf{h}_t^{-\ltimes} \ltimes \mathbf{h}\right) \\ &= \mathbf{w}_{rt}^{-\rtimes} \rtimes \left[\mathbf{h}_t^{-\ltimes} \ltimes \left(\tfrac{u-r}{t-r}\mathbf{h}_t - \tfrac{u-t}{t-r}\mathbf{h}_r - (u-t)(u-r)\mathbf{h}\right)\right].\end{aligned}$$

Thus, by (3.14),

$$\mathbf{x}_{tru}^{-\ltimes} = \mathbf{w}_{rt}^{-\rtimes} \rtimes (\mathbf{h}_t^{-\ltimes} \ltimes \mathbf{h}_u) = \mathbf{w}_{rt}^{-\rtimes} \rtimes \mathbf{w}_{tu}^{-\ltimes}.$$

Recalling that $\mathbf{x}_{tru} = \mathbf{w}_{ru}^{-\rtimes} \rtimes \mathbf{w}_{tu}$, this gives (3.17), so $\mathbf{h}$ is a flow generator. ∎

3.1. Two-way flows in the DNA of linear maps. Let $L = L(V)$ be the space of endomorphisms on a linear space V. For $A_1, A_2, B_1, B_2 \in L$ we define

$$\begin{aligned}(A_1, A_2) \ltimes (B_1, B_2) &= (B_1 + A_1B_2, A_2B_2), \\ (A_1, A_2) \rtimes (B_1, B_2) &= (A_1B_1, A_2B_1 + B_2).\end{aligned}$$

Then $\mathcal{L}_V := (L^2, +, \ltimes, \rtimes)$ is a DNA with $\mathbf{e}_{\ltimes} = (0, \mathrm{Id})$ and $\mathbf{e}_{\rtimes} = (\mathrm{Id}, 0)$. Let $\mathbf{A} = (A_1, A_2)$ be an element of this DNA. If A_2 is invertible then $\mathbf{A}^{-\ltimes} = (-A_1A_2^{-1}, A_2^{-1})$ exists. Similarly, if A_1 is invertible then $\mathbf{A}^{-\rtimes} = (A_1^{-1}, -A_2A_1^{-1})$.

This DNA can be interpreted as the family of linear maps $f_{\mathbf{A}} : L^2 \to L$ with suitable compositions. For $\mathbf{A} = (A_1, A_2) \in L^2$ we define

$$f_{\mathbf{A}}(X, Y) = XA_1 + YA_2, \quad X, Y \in L.$$

For the family $\{f_{\mathbf{A}} : \mathbf{A} \in L^2\}$ of maps, there are two natural compositions:

$$\begin{aligned}(f_{\mathbf{A}} \circ_1 f_{\mathbf{B}})(X, Y) &:= f_{\mathbf{B}}(f_{\mathbf{A}}(X, Y), Y), \\ (f_{\mathbf{A}} \circ_2 f_{\mathbf{B}})(X, Y) &:= f_{\mathbf{B}}(X, f_{\mathbf{A}}(X, Y)).\end{aligned}$$

Note that they are directly related to the products $\rtimes$ and $\ltimes$ in $\mathcal{L}_V$:

$$f_{\mathbf{A}} \circ_1 f_{\mathbf{B}} = f_{\mathbf{A}\rtimes\mathbf{B}} \quad \text{and} \quad f_{\mathbf{A}} \circ_2 f_{\mathbf{B}} = f_{\mathbf{A}\ltimes\mathbf{B}}.$$

We now apply Theorem 3.3 in this setting.

Proposition 3.10. *The following conditions are equivalent:*

(i) *A family* $(\mathbf{x}_{sru})_{0\le r<s<u}$ *in* $\mathcal{L}_V$ *is a two-way flow.*
(ii) *There exist* $H, G \in L$ *such that* $(1-r)(1-s)H+rsG+\mathrm{Id}$ *is an invertible element of* L *for all* $0 \le r < s$ *and*

$$\mathbf{x}_{sru} = \left(\tfrac{u-s}{u-r}B_{ru}^{-1}B_{su}, \tfrac{s-r}{u-r}B_{ur}^{-1}B_{sr}\right), \qquad 0 \le r < s < u, \tag{3.23}$$

with $B_{\alpha\beta} = (1-\alpha)(1-\beta)H+\alpha\beta H_\beta G H_\beta^{-1}+\mathrm{Id}$, *where* $H_\beta = (1-\beta)H+\mathrm{Id}$.

Proof. (i)⇒(ii): Following (3.5) we get

$$\mathbf{h}_r = r(1-r)(G,H) + (1-r)(\mathrm{Id},0) + r(0,\mathrm{Id}) = ((1-r)(rG+\mathrm{Id}), rH_r).$$

Thus

$$\mathbf{h}_r^{-\ltimes} = \left(-\tfrac{1-r}{r}(rG+\mathrm{Id})H_r^{-1}, \tfrac{1}{r}H_r^{-1}\right).$$

Consequently, (3.8) yields

$$\begin{aligned}\mathbf{w}_{ru} = \mathbf{h}_u^{-\ltimes} \ltimes \mathbf{h}_r &= \left(-\tfrac{1-u}{u}(uG+\mathrm{Id})H_u^{-1}, \tfrac{1}{u}H_u^{-1}\right) \ltimes \left((1-r)(rG+\mathrm{Id}), rH_r\right)\\ &= \left(\tfrac{u-r}{u}b_{ru}H_u^{-1}, \tfrac{r}{u}H_rH_u^{-1}\right),\end{aligned}$$

for $b_{ru} = b_{ur} = (1-r)(1-u)H + ruG + \mathrm{Id}$. Thus,

$$\mathbf{w}_{ru}^{-\rtimes} = \left(\tfrac{u}{u-r}H_u b_{ru}^{-1}, -\tfrac{r}{u-r}H_r b_{ru}^{-1}\right)$$

and

$$\begin{aligned}\mathbf{w}_{ru}^{-\rtimes} \rtimes \mathbf{w}_{su} &= \left(\tfrac{u}{u-r}H_u b_{ru}^{-1}, -\tfrac{r}{u-r}H_r b_{ru}^{-1}\right) \rtimes \left(\tfrac{u-s}{u}b_{su}H_u^{-1}, \tfrac{s}{u}H_sH_u^{-1}\right)\\ &= \left(\tfrac{u-s}{u-r}H_u b_{ru}^{-1}b_{su}H_u^{-1}, \tfrac{s}{u}H_sH_u^{-1} - \tfrac{r(u-s)}{u(u-r)}H_r b_{ru}^{-1}b_{su}H_u^{-1}\right).\end{aligned} \tag{3.24}$$

Since

$$s(u-r)b_{ru}H_s = u(s-r)b_{rs}H_u + r(u-s)b_{su}H_r,$$

the second coordinate of (3.24) can be written as $\tfrac{s-r}{u-r}H_r b_{ru}^{-1}b_{rs}H_r^{-1}$. Thus (3.23) follows on noting that $H_\beta b_{\alpha\beta}H_\beta^{-1} = B_{\alpha\beta}$ is invertible in $L(V)$.

(ii)⇒(i): Note that (G,H), for which b_{rs} is invertible for all $0 \le r < s$, defines a family $(\mathbf{w}_{ru})_{0\le r<u}$ of $\rtimes$-invertible elements. Furthermore $\mathbf{w}_{ru}^{-\rtimes}$ has $\ltimes$-inverse for all $0 < r < u$ and

$$\begin{aligned}&\tfrac{1}{u(u-r)}(\mathbf{w}_{ru}^{-\rtimes})^{-\ltimes} + \tfrac{1}{ru}\mathbf{e}_\ltimes - \tfrac{1}{r(u-r)}\mathbf{e}_\rtimes\\ &\qquad= \tfrac{1}{u(u-r)}\left(\tfrac{u}{r}H_uH_r^{-1}, -\tfrac{u-r}{r}b_{ru}H_r^{-1}\right) + \tfrac{1}{ru}\mathbf{e}_\ltimes - \tfrac{1}{r(u-r)}\mathbf{e}_\rtimes\\ &\qquad= \left(\tfrac{1}{r(u-r)}(H_u - H_r)H_r^{-1}, \tfrac{1}{ru}(H_r - b_{ru})H_r^{-1}\right).\end{aligned}$$

Since $b_{ru} = (1-u)H_r + u(\mathrm{Id}+rG)$, the element

$$\tfrac{1}{u(u-r)}(\mathbf{w}_{ru}^{-\rtimes})^{-\ltimes} + \tfrac{1}{ru}\mathbf{e}_\ltimes - \tfrac{1}{r(u-r)}\mathbf{e}_\rtimes = \left(-\tfrac{1}{r}HH_r^{-1}, \tfrac{1}{r}\mathrm{Id} - \tfrac{1}{r}H_r^{-1} - GH_r^{-1}\right)$$

does not depend on u. So from Proposition 3.8 we infer that (G, H) is a flow generator and Theorem 3.9 shows that $(\mathbf{x}_{sru})_{0\le r<s<u}$ is a two-way flow. ∎

The following generalization of (3.4) illustrates the fact that the definition of the two-way flow on $[0,\infty)$ is not symmetric with respect to $\ltimes$ and $\rtimes$.

PROPOSITION 3.11. *In a DNA, consider* $\mathbf{h} = \beta\mathbf{e}_{\rtimes} + \gamma\mathbf{e}_{\ltimes}$. *Then* $\mathbf{h}$ *is a flow generator if and only if* $\gamma \in [-1, 0]$ *and* $\beta + \gamma \ge 0$.

Equivalently, $\{\mathbf{x}_{sru} \in \operatorname{span}\{\mathbf{e}_{\rtimes}, \mathbf{e}_{\ltimes}\} : 0 \le r < s < t\}$ *is a two-way flow if and only if there exist* $\rho \in [0, 1]$ *and* $\alpha \ge 0$ *such that*

$$\mathbf{x}_{sru} = \tfrac{(u-s)b(s,u)}{(u-r)b(r,u)}\mathbf{e}_{\rtimes} + \tfrac{(s-r)b(r,s)}{(u-r)b(r,u)}\mathbf{e}_{\ltimes} \tag{3.25}$$

with $b(s, u) = \alpha su + \rho(s + u) + 1 - \rho$.

Proof. We note that $\operatorname{span}\{\mathbf{e}_{\rtimes}, \mathbf{e}_{\ltimes}\}$ with induced multiplications forms a sub-DNA that is isomorphic to $\mathcal{L}_{\mathbb{R}}$. By Proposition 3.10, $\mathbf{h}$ is a flow generator if and only if $(1 - r)(1 - s)\gamma + rs\beta + 1 \ne 0$ for all $0 \le r < s$. Taking $r = 0$ we see that we must have $(1 - s)\gamma + 1 > 0$ for all $s > 0$, i.e. $\gamma \in [-1, 0]$. Considering now large r, we see that $\beta + \gamma \ge 0$.

Conversely, if $\gamma \in [-1, 0]$ and $\beta + \gamma \ge 0$ then $(1 - r)(1 - s)\gamma + rs\beta + 1 = b(r, s) > 0$ for all $s > r \ge 0$ as $\rho = -\gamma \in [0, 1]$ and $\alpha = \beta + \gamma \ge 0$. Formula (3.23) gives (3.25). ∎

3.2. Two-way flows in the DNA for quadratic harnesses. The following DNA is implicit in the study of conditional variances in [2]. Consider the linear space $(V, +) := (\mathbb{R}^6 \times \mathbb{R}^2, +)$, whose elements will be written in the form

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix} = \begin{pmatrix}x_1, \dots, x_6\\ u_1, u_2\end{pmatrix}. \tag{3.26}$$

For $\mathbf{x} = (x_1, \dots, x_6)$, $\mathbf{y} = (y_1, \dots, y_6) \in \mathbb{R}^6$ and $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, we define multiplication operations $\ltimes$ and $\rtimes$ by

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix} \rtimes \begin{pmatrix}\mathbf{y}\\ \mathbf{v}\end{pmatrix} := \tag{3.27}$$
$$\begin{pmatrix}x_1y_1, x_2y_1+u_1y_2, x_3y_1+u_2y_2+y_3, x_4y_1+u_1y_4, x_5y_1+u_2y_4+y_5, x_6y_1+y_6\\ u_1v_1, u_2v_1+v_2\end{pmatrix},$$

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix} \ltimes \begin{pmatrix}\mathbf{y}\\ \mathbf{v}\end{pmatrix} := \tag{3.28}$$
$$\begin{pmatrix}y_1+u_1y_2+x_1y_3, u_2y_2+x_2y_3, x_3y_3, x_4y_3+y_4+u_1y_5, x_5y_3+u_2y_5, x_6y_3+y_6\\ v_1+u_1v_2, u_2v_2\end{pmatrix}.$$

It is easy to check that $Q := (V, +, \ltimes, \rtimes)$ is a DNA with neutral elements

$$\mathbf{e}_{\ltimes} = \begin{pmatrix} 0,0,1,0,0,0 \\ 0,1 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_{\rtimes} = \begin{pmatrix} 1,0,0,0,0,0 \\ 1,0 \end{pmatrix}. \tag{3.29}$$

The $\ltimes$-inverse of $\binom{\mathbf{x}}{\mathbf{u}}$ exists if and only if $x_3 \neq 0$ and $u_2 \neq 0$, and the $\rtimes$-inverse exists if and only if $x_1 \neq 0$ and $u_1 \neq 0$. These inverses are

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}^{-\ltimes} = \begin{pmatrix} \frac{x_2u_1 - x_1u_2}{x_3u_2}, -\frac{x_2}{x_3u_2}, \frac{1}{x_3}, \frac{x_5u_1 - x_4u_2}{x_3u_2}, -\frac{x_5}{x_3u_2}, -\frac{x_6}{x_3} \\ -\frac{u_1}{u_2}, \frac{1}{u_2} \end{pmatrix}, \tag{3.30}$$

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}^{-\rtimes} = \begin{pmatrix} \frac{1}{x_1}, -\frac{x_2}{x_1u_1}, \frac{x_2u_2 - x_3u_1}{x_1u_1}, -\frac{x_4}{x_1u_1}, \frac{x_4u_2 - x_5u_1}{x_1u_1}, -\frac{x_6}{x_1} \\ \frac{1}{u_1}, -\frac{u_2}{u_1} \end{pmatrix}. \tag{3.31}$$

We remark that it is difficult to determine general conditions for

$$\mathbf{h} = \begin{pmatrix} h_1, h_2, h_3, h_4, h_5, h_6 \\ g_1, g_2 \end{pmatrix} \in Q \tag{3.32}$$

to be a flow generator. But the lower coordinates (u_1, u_2) in (3.26) under the multiplications $\ltimes$ and $\rtimes$ from Section 3.1 behave like elements of $\mathcal{L}_{\mathbb{R}}$, so by Proposition 3.11 a necessary condition for $\mathbf{h}$ to be a flow generator is that $g_2 \in [-1, 0]$ and $g_1 + g_2 \geq 0$.

The following example shows that these conditions are not sufficient and the converse of Theorem 3.3 does not hold without additional assumptions.

Example 3.12. Consider $\mathbf{h} := \binom{1,1,0,0,0,0}{1,0} \in V$. One can check that (3.7) gives a well defined family $\{\mathbf{x}_{sru} : \ 0 \leq r < s < u\}$ with both inverses, but the second flow equation in (3.2) fails, e.g., for $r = 1$, $s = 2$, $t = 3$, $u = 4$.

We now apply Theorems 3.3 and 3.9 to a family of a special form.

Theorem 3.13. *A family*

$$\left\{ \mathbf{x}_{sru} = \begin{pmatrix} x_{sru}^{(1)}, \ldots, x_{sru}^{(6)}, \\ \frac{u-s}{u-r}, \frac{s-r}{u-r} \end{pmatrix} : 0 \leq r < s < u \right\}$$

is a two-way flow if and only if there exist $\alpha \geq 0$, $\rho \in [0,1]$, $\beta \geq -2\sqrt{\alpha(1-\rho)}$, and $h_4, h_5, h_6 \in \mathbb{R}$ such that

$$\begin{aligned} x_{sru}^{(1)} &= \frac{(u-s)c(s,u)}{(u-r)c(r,u)}, & x_{sru}^{(4)} &= \frac{(u-s)(s-r)(h_4u - h_5(1-u))}{(u-r)c(r,u)}, \\ x_{sru}^{(2)} &= \frac{(s-r)(u-s)(2\rho - \beta)}{(u-r)c(r,u)}, & x_{sru}^{(5)} &= \frac{(u-s)(s-r)(h_5(1-r) - h_4r)}{(u-r)c(r,u)}, \\ x_{sru}^{(3)} &= \frac{(s-r)c(r,s)}{(u-r)c(r,u)}, & x_{sru}^{(6)} &= \frac{(u-s)(s-r)h_6}{c(r,u)}, \end{aligned} \tag{3.33}$$

where $c(s,u) = \alpha su + (\beta - \rho)s + \rho u + 1 - \rho$, $0 \leq s < u$.

Equivalently,

$$\mathbf{h} = \begin{pmatrix} \alpha + \beta - \rho, 2\rho - \beta, -\rho, h_4, h_5, h_6 \\ 0, 0 \end{pmatrix} \tag{3.34}$$

is a flow generator if and only if

$$\alpha \geq 0, \qquad \rho \in [0,1], \qquad \beta \geq -2\sqrt{\alpha(1-\rho)}.$$

Proof. "⇒" With $\mathbf{h}$ as in (3.32) we get

$$\mathbf{h}_u = (1-u)\begin{pmatrix} 1+uh_1, uh_2, u\big(h_3+\frac{1}{1-u}\big), uh_4, uh_5, uh_6 \\ 1+ug_1, u\big(g_2+\frac{1}{1-u}\big) \end{pmatrix}.$$

We now use Theorem 3.3. Since the lower coordinates of the flow from actually a flow in the DNA $\mathcal{L}_{\mathbb{R}}$ of Section 3.1, we conclude from (3.25) with appropriate parameters that the lower coordinates of $\mathbf{x}_{10u}$ are

$$\tfrac{(u-1)(g_1u+1)}{u(1-g_2(u-1))} \quad \text{and} \quad \tfrac{1}{u(1-g_2(u-1))}$$

for all $u > 1$. On the other hand, by our assumptions they are $(u-1)/u$ and $1/u$, respectively. Therefore, $g_1 = g_2 = 0$.

Denoting now $\alpha := h_1 + h_2 + h_3$, $\beta := -h_2 - 2h_3$, $\rho := -h_3$ and again using (3.7) and (3.8) to calculate $\mathbf{x}_{sru}$ we see that the lower coordinates are as they should be and the upper coordinates are as given in (3.33).

By $\rtimes$-invertibility of $\mathbf{h}_u$ we have $1-(1-u)\rho \neq 0$ for any $u > 0$ and thus $\rho \in [0,1]$.

Since $\mathbf{x}_{sru}$ is $\ltimes$-invertible it follows that $c(s,u) \neq 0$. Consequently, by definition $c(s,u) > 0$ for small $0 \leq s < u$ and by continuity the inequality holds for any $0 \leq s < u$. Taking large values of $0 \leq s < u$ we conclude that $\alpha \geq 0$. Note that $c(s,u)$ can be rewritten as $c(s,u) = \alpha s^2 + \beta s + 1 - \rho + (u-s)(\alpha s + \rho)$. Since the last term is nonnegative and vanishes at $u = s$, considering separately the case $\alpha = \rho = 0$ and $\alpha + \rho > 0$ we conclude that $c(s,u) > 0$ for all $0 \leq s < u$ if $\alpha s^2 + \beta s + 1 - \rho \geq 0$ for all $s \geq 0$, i.e., when $\beta \geq -2\sqrt{\alpha(1-\rho)}$.

"⇐" Under the assumed constraints on the parameters, $c(s,u) > 0$ for all $0 \leq s < u$, so formulas in (3.33) show that $\mathbf{x}_{sru}$ is $\ltimes$-invertible and $\rtimes$-invertible for all $0 \leq r < s < u$. A lengthy calculation shows that the flow equations (3.2) are satisfied for all $0 \leq r < s < u$. To verify condition (3.3) we confirm by calculation that the element

$$\tfrac{\mathbf{x}_{sru}-\mathbf{e}_{\ltimes}}{u-s} + \tfrac{\mathbf{x}_{sru}-\mathbf{e}_{\rtimes}}{s-r} = \tfrac{1}{c(r,u)} \times \begin{pmatrix} \beta-\rho+\alpha u, 2\rho-\beta, -\alpha r-\rho, h_4u+h_5(u-1), h_5(1-r)-h_4r, h_6(u-r) \\ 0, 0 \end{pmatrix}$$

does not depend on $s \in (r,u)$. ∎

4. Applications to stochastic processes. In this section we give some probabilistic applications of the previous results. We encounter a technical difficulty that the arising flows are on $(0,\infty)$ instead of $[0,\infty)$, which we overcome by replacing the time variables r, s, t with $r+p, s+p, t+p$ for

$p > 0$. This method introduces additional solutions of flow equations that do not extend to $[0, \infty)$.

4.1. Harnesses. Let $(X_t)_{t\geq 0}$ be a separable integrable stochastic process with complete σ-fields $\mathcal{F}_{r,u} := \sigma\{X_t : t \in [0,r] \cup [u,\infty)\}$ and linear regressions

$$\mathbb{E}(X_s \,|\, \mathcal{F}_{r,u}) = a_{sru} X_r + b_{sru} X_u, \tag{4.1}$$

where the coefficients a_{sru} and b_{sru} are deterministic and depend only on $0 \leq r < s < u$. Following Mansuy and Yor [13] we say that $(X_t)_{t\geq 0}$ is a *harness* if (4.1) holds with

$$a_{sru} = \tfrac{u-s}{u-r} \quad \text{and} \quad b_{sru} = \tfrac{s-r}{u-r} \tag{4.2}$$

(see (1.5)). Here is a probabilistic application of Proposition 2.4 to harnesses with quadratic regressions (compare [7]).

Proposition 4.1. *Suppose that a square integrable process $(X_t)_{t\geq 0}$ is a harness with moments $\mathbb{E}\, X_t = 0$, $\mathbb{E}\, X_s X_t = s \wedge t$, and that $X_s^2, X_s, 1$ are linearly independent for any $s > 0$. Let $\mathcal{F}_t = \sigma\{X_s : s \in [0,t]\}$, and assume that for any $0 < s < t$ there exist nonrandom $a_{ts} \neq 0$, b_{ts} and c_{ts} such that*

$$\mathbb{E}(X_t^2 \,|\, \mathcal{F}_s) = a_{ts} X_s^2 + b_{ts} X_s + c_{ts}. \tag{4.3}$$

(i) *If $s \mapsto a_{ts}$ is bounded on $(0,t]$ for some $t > 0$, then there exist $b \in \mathbb{R}$ and $a \geq 0$ such that*

$$a_{ts} = \tfrac{1+ta}{1+sa}, \qquad b_{ts} = \tfrac{(t-s)b}{1+sa}, \qquad c_{ts} = \tfrac{t-s}{1+sa}. \tag{4.4}$$

(ii) *If $s \mapsto a_{ts}$ is unbounded on $(0,t]$ for some $t > 0$, then there exists $b \in \mathbb{R}$ such that*

$$a_{ts} = \tfrac{t}{s}, \qquad b_{ts} = \tfrac{(t-s)b}{s}, \qquad c_{ts} = 0. \tag{4.5}$$

Proof. We first observe that passing to the limit in (4.1) as $u \to \infty$ (see [3, proof of (4)]), we get

$$\mathbb{E}(X_t \,|\, \mathcal{F}_s) = X_s. \tag{4.6}$$

We will relate the problem to a flow in the near-algebra $\mathcal{A}_V$ from Section 2.1 with $V = \mathbb{R}^2$ and multiplication given in (2.8). Within this framework we will apply Proposition 2.4. There is however a technical difficulty caused by lack of linear independence for $s = 0$, as $X_0 = 0$, so we consider $\mathbf{x}_{st} = (a_{ts}, b_{ts}, c_{ts})$ for $0 < s < t < u$, i.e., we exclude $s = 0$. Since $a_{ts} \neq 0$ by assumption, $\mathbf{x}_{st}$ is $\boxdot$-invertible.

Note that (4.6) gives

$$a_{us} X_s^2 + b_{us} X_s + c_{us} = \mathbb{E}(X_u^2 \,|\, \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(X_u^2 \,|\, \mathcal{F}_t) \,|\, \mathcal{F}_s)$$
$$= a_{ut}\, \mathbb{E}(X_t^2 \,|\, \mathcal{F}_s) + b_{ut}\, \mathbb{E}(X_t \,|\, \mathcal{F}_s) + c_{ut} = a_{ut}(a_{ts} X_s^2 + b_{ts} X_s + c_{ts}) + b_{ut} X_s + c_{ut}.$$

Comparing the coefficients of the linearly independent functions X_s^2, X_s and 1, we see that the flow equation (2.4) holds for all $0 < s < t < u$.

Using (4.6) again, we have

$$\mathbb{E}(X_t X_u \,|\, \mathcal{F}_s) = \mathbb{E}(X_t^2 \,|\, \mathcal{F}_s) = a_{ts} X_s^2 + b_{ts} X_s + c_{ts}. \tag{4.7}$$

On the other hand, the harness property (4.1) with (4.2) implies

$$\begin{aligned} \mathbb{E}(X_t X_u \,|\, \mathcal{F}_s) &= \mathbb{E}(\mathbb{E}(X_t \,|\, \mathcal{F}_{s,u}) X_u \,|\, \mathcal{F}_s) \\ &= \tfrac{u-t}{u-s} X_s \, \mathbb{E}(X_u \,|\, \mathcal{F}_s) + \tfrac{t-s}{u-s} \, \mathbb{E}(X_u^2 \,|\, \mathcal{F}_s) \\ &= \tfrac{u-t}{u-s} X_s^2 + \tfrac{t-s}{u-s} (a_{us} X_s^2 + b_{us} X_s + c_{us}). \end{aligned} \tag{4.8}$$

Comparing (4.7) and (4.8) we obtain (2.5) for all $0 < s < u$. We now fix $p > 0$ and apply Proposition 2.4 to the flow $\{\widetilde{\mathbf{x}}_{st} : 0 \leq s < t\}$ defined by $\widetilde{\mathbf{x}}_{st} = \mathbf{x}_{s+p,t+p}$. Formula (2.9) shows that there exist constants $a_p \geq 0$, $b_p, c_p \in \mathbb{R}$ such that for all $t > s > p$ we have

$$a_{ts} = \frac{1 + a_p(t-p)}{1 + a_p(s-p)}, \qquad b_{ts} = \frac{(t-s) b_p}{1 + a_p(s-p)}, \qquad c_{ts} = \frac{(t-s) c_p}{1 + a_p(s-p)}. \tag{4.9}$$

Recall that the original family $\mathbf{x}_{st} = (a_{ts}, b_{ts}, c_{ts})$, $0 \leq s < t$, does not depend on p. We now observe that

$$\tfrac{t - s a_{ts}}{t-s} = \tfrac{p a_p - 1}{a_p(p-s) - 1}, \tag{4.10}$$

where the left hand side does not depend on $p \in (0, s)$. Since we can take $s > 0$ arbitrarily large, this means that we have two cases:

- Case A: $p a_p \neq 1$ for all $p > 0$.
- Case B: $p a_p = 1$ for all $p > 0$.

In Case A, we use (4.10) to write $(t-s)/(t - s a_{ts})$ as $1 + s a_p/(1 - p a_p)$. Since this expression does not depend on p when $s > p$ is arbitrary, there is a constant a such that $a_p/(1 - p a_p) = a$ for all $p > 0$. We get $a_p = a/(1 + ap)$ for all small enough $p > 0$. As $a_p \geq 0$, this implies that $a \geq 0$ and hence $a_p = a/(1 + ap)$ for all $p > 0$. Dividing the last two equalities in (4.9) by $t - s$ and applying (4.10) we see that $(1 + ap) b_p = b$ and $(1 + ap) c_p = c$ do not depend on p. Inserting these expressions into (4.9) for all $0 < s < t$ we get

$$a_{ts} = \tfrac{1 + at}{1 + as}, \qquad b_{ts} = \tfrac{(t-s) b}{1 + as}, \qquad c_{ts} = \tfrac{(t-s) c}{1 + as}. \tag{4.11}$$

From (4.11) it is clear that Case A does not arise if the coefficients a_{st} are unbounded for some t.

Since $\mathbb{E}\, X_t^2 = t$, taking the expected value of (4.3), at some $t > s > 0$ we get

$$t = \tfrac{1 + at}{1 + as} s + \tfrac{(t-s) c}{1 + as},$$

which gives $c = 1$. As $X_0 = 0$, the coefficients of the quadratic form on the right hand side of (4.7) are given by (4.4) for all $0 \le s < t$.

In Case B we have $a_p = 1/p$ for all $p > 0$. From (4.9) we get $a_{ts} = t/s$ for all $t > s$. It is therefore clear that Case B does not arise if the coefficient a_{st} is bounded near $s = 0$ for some t.

Taking the expected value of both sides of (4.3), we get $c_{ts} = 0$ for all $t > s > 0$. Formula (4.9) gives $b_{ts} = (t-s)pb_p/s$, so $pb_p = b$ does not depend on $p > 0$ and (4.5) follows. ∎

4.2. More general linear regressions. In this section we are interested in a separable integrable stochastic process such that $\mathbb{E}\, X_t = 0$ with the property that for all $0 < r < s < u$ we have

$$\mathbb{E}\big(\tfrac{X_s - X_r}{s-r} - \tfrac{X_u - X_s}{u-s} \;\big|\; \mathcal{F}_{r,u}\big) = A_{ru}X_r + B_{ru}X_u, \tag{4.12}$$

where A_{ru} and B_{ru} are deterministic functions of r and u but not of s. Note that (4.12) defines a harness (see (4.1) and (4.2)) when $A_{ru} = B_{ru} \equiv 0$.

Theorem 4.2. *Let $(X_t)_{t \ge 0}$ be an integrable centered stochastic process such that* (4.12) *holds. Suppose that X_r and X_s are linearly independent as functions on the probability space Ω for $0 < r < s$.*

(i) *If $r \mapsto A_{ru}$ is bounded on $(0, u]$ for some $u > 0$, then there exist $\alpha \ge 0$ and $\rho \in [0, 1]$ such that* (4.1) *holds for all $0 \le r < s < u$ with*

$$a_{sru} = \tfrac{(u-s)b(s,u)}{(u-r)b(r,u)}, \quad b_{sru} = \tfrac{(s-r)b(r,s)}{(u-r)b(r,u)}, \tag{4.13}$$

and with $b(s, u)$ defined in Proposition 3.11.

(ii) *If $r \mapsto A_{ru}$ is unbounded on $(0, u]$ for some $u > 0$, then* (4.1) *holds for all $0 < r < s < u$ with*

$$a_{sru} = \tfrac{s(u-s)}{r(u-r)}, \quad b_{sru} = \tfrac{s(s-r)}{u(u-r)}. \tag{4.14}$$

Remark 4.3. Recalculating A_{ru} and B_{ru} from either (4.13) or (4.14), in view of (4.15), we get respectively

(i) $A_{ru} = \frac{\alpha u + \rho}{b(r,u)}$, $B_{ru} = -\frac{\alpha r + \rho}{b(r,u)}$;

(ii) $A_{ru} = \frac{1}{r}$, $B_{ru} = -\frac{1}{u}$.

Proof of Theorem 4.2. Clearly (4.12) implies that (4.1) holds with

$$a_{sru} = \tfrac{(u-s)(s-r)}{u-r}\big(\tfrac{1}{s-r} + A_{ru}\big), \quad b_{sru} = \tfrac{(u-s)(s-r)}{u-r}\big(\tfrac{1}{u-s} + B_{ru}\big) \tag{4.15}$$

for all $0 < r < s < u$. From the tower property of conditional expectation we have, for $0 < r < s < t < u$,

$$\begin{aligned} \mathbb{E}(X_t \,|\, \mathcal{F}_{r,u}) &= \mathbb{E}(\mathbb{E}(X_t \,|\, \mathcal{F}_{s,u}) \,|\, \mathcal{F}_{r,u}) \\ &= \mathbb{E}(a_{tsu}X_s + b_{tsu}X_u \,|\, \mathcal{F}_{r,u}) = a_{tsu}a_{sru}X_r + (a_{tsu}b_{sru} + b_{tsu})X_u \end{aligned} \tag{4.16}$$

and

$$\begin{aligned}(4.17)\qquad \mathbb{E}(X_s \mid \mathcal{F}_{r,u}) &= \mathbb{E}(\mathbb{E}(X_s \mid \mathcal{F}_{r,t}) \mid \mathcal{F}_{r,u}) \\ &= \mathbb{E}(a_{srt}X_r + b_{srt}X_t \mid \mathcal{F}_{r,u}) = (a_{srt} + a_{tru}b_{srt})X_r + b_{srt}b_{tru}X_u.\end{aligned}$$

Consider the DNA $\mathcal{L}_\mathbb{R}$ from Section 3.1, which is isomorphic to the span of $\mathbf{e}_\rtimes = (1,0), \mathbf{e}_\ltimes = (0,1)$, and recall Proposition 3.11. Define a family $(\mathbf{x}_{sru})_{0<r<s<u}$ by $\mathbf{x}_{sru} = (a_{sru}, b_{sru})$. By linear independence of X_r and X_u, from (4.16) and (4.17) we deduce that this family satisfies the flow equations (3.2) for all $0 < r < s < t < u$.

We note that

$$\mathbb{E}\left(\tfrac{X_s-X_r}{s-r} - \tfrac{X_u-X_s}{u-s} \;\middle|\; \mathcal{F}_{r,u}\right) = \left(\tfrac{a_{sru}-1}{s-r} + \tfrac{a_{sru}}{u-s}\right)X_r + \left(\tfrac{b_{sru}}{s-r} + \tfrac{b_{sru}-1}{u-s}\right)X_u.$$

Condition (4.12) implies that the left hand side does not depend on $s \in (r,u)$. Therefore

$$\tfrac{(a_{sru},b_{sru})-(1,0)}{s-r} + \tfrac{(a_{sru},b_{sru})-(0,1)}{u-s}$$

does not depend on $s \in (r,u)$. Consequently, $(\mathbf{x}_{sru})_{0<r<s<u}$ satisfies the structure condition (3.3).

To ensure $\rtimes$-invertibility and $\ltimes$-invertibility, we need to verify that $a_{sru} \neq 0$ and $b_{sru} \neq 0$. Suppose that $a_{sru} = 0$ for some $0 < r < s < u$. Then from (4.16) by linear independence we have $a_{tru} = 0$ for all $t \in (s,u)$. Hence from (4.12) we obtain

$$A_{ru}X_r + B_{ru}X_u = \mathbb{E}\left(\tfrac{X_t-X_r}{t-r} - \tfrac{X_u-X_t}{u-t} \;\middle|\; \mathcal{F}_{r,u}\right) = \tfrac{u-r}{(t-r)(u-t)}b_{tru}X_u - \tfrac{X_r}{t-r} - \tfrac{X_u}{u-t},$$

and again by linear independence, $A_{ru} = -1/(t-r)$ for $s < t < u$, which is impossible as A_{ru} does not depend on t. Therefore $a_{sru} \neq 0$ for every $0 < r < s < u$. Analogously from (4.17) we can show that $b_{sru} \neq 0$ for all $0 < r < s < u$.

This means that $(\mathbf{x}_{sru})_{0<r<s<u}$ is a two-way flow on $(0,\infty)$. We now proceed as in the proof of Proposition 4.1. For arbitrary $p > 0$, define $r' = r + p, s' = s + p, u' = u + p$, and consider $(\widetilde{\mathbf{x}}_{sru})_{0\le r<s<u}$ given by $\widetilde{\mathbf{x}}_{sru} = (a_{s'r'u'}, b_{s'r'u'})$. Proposition 3.11 shows that there exist $\alpha_p \geq 0$ and $\rho_p \in [0,1]$ such that with

$$b_p(r,u) = \alpha_p ru + (\rho_p - p\alpha_p)(r+u) + 1 - \rho_p(1+2p) + p^2\alpha_p$$

we have

$$(4.18)\qquad a_{sru} = \tfrac{(u-s)b_p(s,u)}{(u-r)b_p(r,u)}, \quad b_{sru} = \tfrac{(s-r)b_p(r,s)}{(u-r)b_p(r,u)}, \quad p \le r < s < u.$$

Thus

$$A_{ru} = \tfrac{\alpha_p u + \rho_p - p\alpha_p}{b_p(r,u)} \quad \text{and} \quad B_{ru} = -\tfrac{\alpha_p r + \rho_p - p\alpha_p}{b_p(r,u)}.$$

Recall that the original family $\mathbf{x}_{sru} = (a_{sru}, b_{sru})$, $0 \le s < t$, does not depend on $p \in (0,r)$ and therefore in view of (4.15) we conclude that A_{ru} and B_{ru}

also do not depend on $p \in (0, r)$. Furthermore we have

$$\frac{A_{ru} + B_{ru}}{u - r} = \frac{\alpha_p}{b_p(r, u)}, \tag{4.19}$$

$$\frac{uB_{ru} + rA_{ru}}{u - r} = \frac{p\alpha_p - \rho_p}{b_p(r, u)}, \tag{4.20}$$

and the left hand sides above do not depend on $p \in (0, r)$. Since $r > 0$ can be arbitrarily large, in view of (4.19) it follows that either $\alpha_p = 0$ for all $p > 0$ or $\alpha_p \neq 0$ for all $p > 0$. We consider these two cases separately.

CASE A: $\alpha_p = 0$ *for all* $p > 0$. In this case, the fact that (4.20) does not depend on $p \in (0, r)$ and r can be taken arbitrarily large implies that either $\rho_p = 0$ for all $p > 0$ and (4.13) holds with $\alpha = 0$, $\rho = 0$, or after factoring out ρ_p from the denominator on the right hand side of (4.20), the term $1/\rho_p - 2p = C \geq 1$ does not depend on $p > 0$. In the latter case, (4.13) holds with $\alpha = 0, \rho = 1/C \in (0, 1]$.

CASE B: $\alpha_p > 0$ *for all* $p > 0$. In this case, dividing (4.20) by (4.19) we see that $\rho_p/\alpha_p - p = C_1$ does not depend on $p > 0$. Putting this into (4.19) with α_p factored out from the denominator, we see that the expression

$$\tfrac{1}{\alpha_p} - (2C_1 + 1)p - p^2 = C_2$$

in the denominator cannot depend on $p > 0$. Hence

$$\alpha_p = \tfrac{1}{2C_1 p + p + p^2 + C_2}, \quad \rho_p = \tfrac{C_1 + p}{2C_1 p + p + p^2 + C_2}. \tag{4.21}$$

Note that since $\alpha_p > 0$ and $\rho_p \in [0, 1]$ (see Proposition 3.11), taking the limit as $p \searrow 0$ in both expressions (4.21) we see that $0 \leq C_1 \leq C_2$.

Substituting (4.21) into (4.18) we get the answer which does not depend on $p > 0$, with

$$\tfrac{b_p(s,u)}{b_p(r,u)} = \tfrac{C_1(s+u-1)+C_2+su}{C_1(r+u-1)+C_2+ru}, \quad \tfrac{b_p(r,s)}{b_p(r,u)} = \tfrac{C_1(r+s-1)+C_2+rs}{C_1(r+u-1)+C_2+ru}.$$

If $C_1 = C_2 = 0$, we get (4.14) with unbounded A_{ru}. If $C_2 > 0$, we get (4.13) with

$$\alpha = 1/C_2 > 0, \quad \rho = C_1/C_2 \in [0, 1].$$

Combining the two cases, we see that either A_{ru} is unbounded as $r \to 0$ and then (4.14) holds, or A_{ru} is bounded as $r \to 0$ and then (4.13) holds with $\alpha \geq 0$ and $\rho \in [0, 1]$. ∎

4.3. Quadratic harnesses. Following [2], we say that a separable square integrable stochastic process $(X_t)_{t \geq 0}$ is a *standard quadratic harness* if it is a harness, $\mathbb{E}\, X_t = 0$ and $\mathbb{E}\, X_s X_t = s \wedge t$ for $s, t \geq 0$, and

$$\mathbb{E}(X_s^2 \mid \mathcal{F}_{r,u}) = A_{sru} X_r^2 + B_{sru} X_r X_u + C_{sru} X_u^2 + D_{sru} X_r + E_{sru} X_u + F_{sru}, \tag{4.22}$$

where the deterministic coefficients $A_{sru}, \dots, F_{sru}$ depend only on $0 \le r < s < u$ (see (1.6)).

The following result extends [2, Theorem 2.2], which concerned only the case $\chi = 1$.

THEOREM 4.4. *Let $(X_t)_{t\ge 0}$ be a standard quadratic harness. Suppose that 1, X_r, X_u, X_rX_u, X_r^2, X_u^2 are linearly independent for all $0 < r < u$ as functions on Ω. Then there exist constants $\chi \in \{0,1\}$, $\sigma \ge 0$, $\tau \ge 0$, $\gamma \le \chi + 2\sqrt{\sigma\tau}$ and $\eta, \theta \in \mathbb{R}$ such that $\gamma, \sigma, \tau, \chi$ are not all zero and*

$$\mathbb{V}\mathrm{ar}(X_s \,|\, \mathcal{F}_{r,u}) = \tfrac{(s-r)(u-s)}{c_{\tau,\sigma,\gamma,\chi}(r,u)} Q_{\theta,\eta,\tau,\sigma,\gamma-\chi,\chi}\big(\tfrac{X_u - X_r}{u-r}, \tfrac{uX_r - rX_u}{u-r}\big), \qquad 0 < r < u, \tag{4.23a}$$

with

$$c_{\tau,\sigma,\gamma,\chi}(r,u) = \tau + \sigma r u - \gamma r + \chi u, \tag{4.23b}$$

$$Q_{\theta,\eta,\tau,\sigma,\rho,\chi}(x,y) = \tau x^2 + \sigma y^2 + \rho x y + \theta x + \eta y + \chi. \tag{4.23c}$$

REMARK 4.5. The values of the two-valued parameter χ are determined by the last term in (4.22) as follows:

(i) If $F_{sru} \ne 0$ for some $0 < r < s < u$ then $\chi = 1$ and hence $F_{sru} \ne 0$ for all $0 < r < s < u$.
(ii) If $F_{sru} = 0$ for some $0 < r < s < u$ then $\chi = 0$ and hence $F_{sru} = 0$ for all $0 < r < s < u$.

By homogeneity of (4.23a) if $\chi = 0$, the remaining parameters are defined only up to an arbitrary multiplicative factor. Therefore, if $\tau > 0$, we can take $\tau = 1$ as in Lemma 4.7; or if $\sigma > 0$, we can take $\sigma = 1$; or if $\gamma < 0$, we can take $\gamma = -1$. (These three cases arise in the proof below.)

REMARK 4.6. If $\chi = \tau = 0$ and $\sigma > 0$, or if $\chi = \tau = \sigma = 0$ and $\gamma < 0$, then the conditional variance and most of the coefficients in (4.22) are unbounded near $r = 0$.

Proof of Theorem 4.4. We apply Theorem 3.13, but as in the proofs of Proposition 4.1 and Theorem 4.2 we encounter technical difficulties as the arising two-way flows are on $(0,\infty)$ instead of $[0,\infty)$ and, as indicated in Remark 4.6, may not extend to $[0,\infty)$. We therefore begin with the following direct application of Theorem 3.13.

LEMMA 4.7. *For every $p > 0$ there exist constants $\eta_p, \theta_p, \gamma_p \in \mathbb{R}$, $\chi_p \in \{0,1\}$, $\sigma_p \ge 0$, $\tau_p \ge 0$, where $\tau_p = 1$ when $\chi_p = 0$ and $\gamma_p \le \chi_p + 2\sqrt{\sigma_p\tau_p}$, such that for all $p < r < s < u$,*

$$\mathbb{V}\mathrm{ar}(X_s \,|\, \mathcal{F}_{r,u}) = \tfrac{(s-r)(u-s)}{c_{\tau_p^*,\sigma_p,\gamma_p^*,\chi_p^*}(r,u)} Q_{\theta_p^*,\eta_p,\tau_p^*,\sigma_p,\gamma_p^*-\chi_p^*,\chi_p^*}\big(\tfrac{X_u - X_r}{u-r}, \tfrac{uX_r - rX_u}{u-r}\big), \tag{4.24}$$

where

$$\tau_p^* = \tau_p + p(\gamma_p - \chi_p) + p^2\sigma_p, \quad \chi_p^* = \chi_p - p\sigma_p,$$
$$\gamma_p^* = \gamma_p + p\sigma_p, \quad \theta_p^* = \theta_p + p\eta_p, \tag{4.25}$$

while c *and* Q *are defined in* (4.23b) *and* (4.23c), *respectively.*

Proof. Recall the notation (4.2), (4.22) and the DNA from Section 3.2. For $0 < s < t < u$ let

$$\mathbf{x}_{tsu} := \begin{pmatrix} A_{tsu}, B_{tsu}, C_{tsu}, D_{tsu}, E_{tsu}, F_{tsu} \\ a_{tsu}, b_{tsu} \end{pmatrix} \in \mathbb{R}^6 \times \mathbb{R}^2.$$

We proceed to verify that $\{\mathbf{x}_{sru}\}$ is a two-way flow on $(0, \infty)$ with respect to the multiplications (3.27) and (3.28).

Because

$$\mathbb{E}(X_s X_t \mid \mathcal{F}_{r,u}) = \mathbb{E}(X_s \, \mathbb{E}(X_t \mid \mathcal{F}_{s,u}) \mid \mathcal{F}_{r,u}) = \mathbb{E}(\mathbb{E}(X_s \mid \mathcal{F}_{r,t}) X_t \mid \mathcal{F}_{r,u}),$$

by linear independence of 1, X_s, X_t, X_sX_t, X_s^2, X_t^2 for $0 < r < s < t < u$ we have (see e.g. [2, proof of Claim 3.1])

$$\begin{aligned} a_{tsu}A_{sru} &= b_{srt}A_{tru} + a_{srt}a_{tru}, & a_{tsu}B_{sru} &= b_{srt}B_{tru}, \\ b_{tsu}b_{sru} + a_{tsu}C_{sru} &= b_{srt}C_{tru}, & a_{tsu}D_{sru} &= b_{srt}D_{tru}, \\ a_{tsu}E_{sru} &= b_{srt}E_{tru}, & a_{tsu}F_{sru} &= b_{srt}F_{tru}. \end{aligned} \tag{4.26}$$

Since $a_{srt}a_{tru} = \frac{(t-s)(u-t)}{(t-r)(u-r)}$, $b_{tsu}b_{sru} = \frac{(t-s)(s-r)}{(u-s)(u-r)}$ and

$$\tfrac{u-t}{u-r} = \tfrac{(t-s)(u-t)}{(t-r)(u-r)} + \tfrac{(s-r)(u-t)}{(t-r)(u-r)}, \quad \tfrac{(u-t)(s-r)}{(u-s)(u-r)} + \tfrac{(t-s)(s-r)}{(u-s)(u-r)} = \tfrac{s-r}{u-r},$$

we see that

$$\tfrac{u-t}{u-s}\mathbf{x}_{sru} + \tfrac{(t-s)(s-r)}{(u-s)(u-r)}\mathbf{e}_{\ltimes} = \tfrac{(t-s)(u-t)}{(t-r)(u-r)}\mathbf{e}_{\rtimes} + \tfrac{s-r}{t-r}\mathbf{x}_{tru}$$

for all $0 \le s < t < u$, with $\mathbf{e}_{\ltimes}$ and $\mathbf{e}_{\rtimes}$ defined by (3.29). Hence

$$\tfrac{\mathbf{x}_{sru} - \mathbf{e}_{\ltimes}}{u-s} + \tfrac{\mathbf{x}_{sru} - \mathbf{e}_{\rtimes}}{s-r} = \tfrac{\mathbf{x}_{tru} - \mathbf{e}_{\ltimes}}{u-t} + \tfrac{\mathbf{x}_{tru} - \mathbf{e}_{\rtimes}}{t-r}$$

and the structural condition (3.3) is satisfied.

Moreover, because

$$\mathbb{E}(X_t^2 \mid \mathcal{F}_{r,u}) = \mathbb{E}(\mathbb{E}(X_t^2 \mid \mathcal{F}_{s,u}) \mid \mathcal{F}_{r,u}),$$

we have (see e.g. [2, proof of Claim 3.2])

$$\begin{aligned} A_{tru} &= A_{tsu}A_{sru}, & B_{tru} &= A_{t,s,u}B_{sru} + B_{tsu}a_{sru}, \\ C_{tru} &= A_{tsu}C_{sru} + B_{tsu}b_{sru} + C_{tsu}, & D_{tru} &= A_{tsu}D_{sru} + D_{tsu}a_{sru}, \\ E_{tru} &= A_{tsu}E_{sru} + D_{tsu}b_{sru} + E_{tsu}, & F_{tru} &= A_{tsu}F_{sru} + F_{tsu}. \end{aligned}$$

The above conclusion holds for $r > 0$ by linear independence of 1, X_s, X_t, X_sX_t, X_s^2, X_t^2, leaving the constants A_{s0u}, B_{s0u} and D_{s0u} arbitrary or undefined as $X_0 = 0$. Hence the first flow equation in (3.2) is satisfied on $(0, \infty)$.

If $A_{s_0 r_0 u_0} = 0$ for some $0 < r_0 < s_0 < u_0$, then from the first equality above we get $A_{t_0 r_0 u_0} = 0$ for all t_0 such that $s_0 < t_0 < u_0$. Then from the first equality in (4.26) we obtain $a_{s_0 r_0 t_0} a_{t_0 r_0 u_0} = 0$, which contradicts (4.2). Hence $A_{sru} \neq 0$ for all $0 < r < s < u$, and $\mathbf{x}_{sru}$ is $\rtimes$-invertible.

Because

$$\mathbb{E}(X_s^2 \mid \mathcal{F}_{r,u}) = \mathbb{E}(\mathbb{E}(X_s^2 \mid \mathcal{F}_{r,t}) \mid \mathcal{F}_{r,u}),$$

analogously we can show that the second flow equation (3.2) is satisfied on $(0, \infty)$, where $\ltimes$ is defined in (3.28). Furthermore, by a similar argument, $C_{sru} \neq 0$ for every $0 < r < s < u$, so $\mathbf{x}_{sru}$ is $\ltimes$-invertible.

This means that $(\mathbf{x}_{sru})_{0<r<s<u}$ is a two-way flow on $(0, \infty)$. We now proceed as in the proofs of Proposition 4.1 and Theorem 4.2. For arbitrary $p > 0$, define $r' = r + p, s' = s + p, u' = u + p$, and consider $(\widetilde{\mathbf{x}}_{sru})_{0 \le r<s<u}$ given by $\widetilde{\mathbf{x}}_{sru} = \mathbf{x}_{s'r'u'}$. Then $(\widetilde{\mathbf{x}}_{tsu})_{0 \le s<t<u}$ is a two-way flow on $[0, \infty)$ with (unique) flow generator (3.34) that was determined in the proof of Theorem 3.13. Of course, $\mathbf{h}$ now depends on $p > 0$. We now reparametrize $\mathbf{h}$ using $\eta_p, \theta_p, \sigma_p, \tau_p, \gamma_p, \phi_p \in \mathbb{R}$ such that

$$\alpha + \beta - \rho = \tfrac{\sigma_p - \gamma_p}{\chi_p + \tau_p}, \quad 2\rho - \beta = \tfrac{\gamma_p + \chi_p}{\chi_p + \tau_p}, \quad \rho = \tfrac{\chi_p}{\chi_p + \tau_p},$$
$$h_4 = \tfrac{\eta_p - \theta_p}{\chi_p + \tau_p}, \quad h_5 = \tfrac{\theta_p}{\chi_p + \tau_p}, \quad h_6 = \tfrac{\phi_p}{\chi_p + \tau_p}.$$

Here $\chi_p = 1_{\rho=0}$ takes only two values $0, 1$; without loss of generality we may assume $\chi_p + \tau_p > 0$, taking $\tau_p = 1$ when $\chi_p = 0$. In this parametrization $\mathbf{h}$ takes the form

$$\mathbf{h} = \frac{1}{\tau_p + \chi_p} \begin{pmatrix} \sigma_p - \gamma_p, \gamma_p + \chi_p, -\chi_p, \eta_p - \theta_p, \theta_p, \phi_p \\ 0, 0 \end{pmatrix}.$$

The condition $\rho \in [0, 1]$ is then equivalent to $\tau_p \ge 0$, the condition $\alpha \ge 0$ is equivalent to $\sigma_p \ge 0$ and the condition $\beta \ge -2\sqrt{\alpha(1-\rho)}$ is equivalent to $\gamma_p \le \chi_p + 2\sqrt{\sigma_p \tau_p}$.

Formulas in (3.33) give

$$A_{sru} = \tfrac{(u-s) c_{\tau_p^*, \sigma_p, \gamma_p^*, \chi_p^*}(s,u)}{(u-r) c_{\tau_p^*, \sigma_p, \gamma_p^*, \chi_p^*}(r,u)}, \tag{4.27}$$

$$B_{sru} = \tfrac{(s-r)(u-s)(\gamma_p^* + \chi_p^*)}{(u-r) c_{\tau_p^*, \sigma_p, \gamma_p^*, \chi_p^*}(r,u)}, \tag{4.28}$$

$$C_{sru} = \tfrac{(s-r) c_{\tau_p^*, \sigma_p, \gamma_p^*, \chi_p^*}(r,s)}{(u-r) c_{\tau_p^*, \sigma_p, \gamma_p^*, \chi_p^*}(r,u)}, \tag{4.29}$$

$$D_{sru} = \tfrac{(s-r)(u-s)(\eta_p u - \theta_p^*)}{(u-r) c_{\tau_p^*, \sigma_p, \gamma_p^*, \chi_p^*}(r,u)}, \tag{4.30}$$

$$E_{sru} = \tfrac{(s-r)(u-s)(\theta_p^* - \eta_p r)}{(u-r) c_{\tau_p^*, \sigma_p, \gamma_p^*, \chi_p^*}(r,u)}, \tag{4.31}$$

$$F_{sru} = \tfrac{(s-r)(u-s)\phi_p}{c_{\tau_p^*, \sigma_p, \gamma_p^*, \chi_p^*}(r,u)}, \tag{4.32}$$

with τ_p^*, γ_p^*, χ_p^* and θ_p^* as given in (4.25). Moreover, from the assumptions that $\mathbb{E}\, X_t = 0$ and $\mathbb{E}\, X_s X_t = s \wedge t$ we get

$$s = \mathbb{E}\, X_s^2 = \mathbb{E}\,\mathbb{E}(X_s^2 \,|\, \mathcal{F}_{r,u}) = A_{sru} r + B_{sru} r + C_{sru} u + F_{sru}.$$

Using (4.27)–(4.29) and (4.32), after some algebra, we get

$$\phi_p = \chi_p - p\sigma_p = \chi_p^*. \tag{4.33}$$

Since $\mathbb{V}\mathrm{ar}(X_s \,|\, \mathcal{F}_{r,u}) = \mathbb{E}(X_s^2 \,|\, \mathcal{F}_{r,u}) + [\mathbb{E}(X_s \,|\, \mathcal{F}_{r,u})]^2$, elementary but tedious calculations give (4.24). ∎

By linear independence of 1, X_r, X_u, X_r^2, $X_r X_u$, X_u^2 for $0 < r < u$ it follows from (4.24) that the vector

$$v_p^* := \frac{(\sigma_p, \tau_p^*, \gamma_p^*, \eta_p, \theta_p^*, \chi_p^*)}{c_{\tau_p^*, \sigma_p, \gamma_p^*, \chi_p^*}(r,u)} \tag{4.34}$$

does not depend on $p > 0$, in the sense that $v_{p_1}^* = v_{p_2}^*$ for all positive p_1, p_2 when $r < u$ are arbitrary in $[p_1 \vee p_2, \infty)$.

Using this fact we conclude that if either $\chi_p^* = 0$, or $\sigma_p = 0$, or $\gamma_p^* = 0$ for some $p > 0$, then, respectively, $\chi_p^* = 0$, or $\sigma_p = 0$, or $\gamma_p^* = 0$ for all $p > 0$. We therefore have the following four cases which we need to consider separately:

- Case A: $\chi_p^* \neq 0$ for all $p > 0$.
- Case B: $\chi_p^* = 0$ and $\sigma_p > 0$ for all $p > 0$.
- Case C: $\chi_p^* = 0$, $\sigma_p = 0$ and $\gamma_p^* = 0$ for all $p > 0$.
- Case D: $\chi_p^* = 0$, $\sigma_p = 0$ and $\gamma_p^* \neq 0$ for all $p > 0$.

Case A: $\chi_p^* \neq 0$ *for all* $p > 0$.

Claim. *There exist constants* $\sigma \geq 0$, $\tau \geq 0$ *and* $\gamma \leq 1 + 2\sqrt{\sigma\tau}$, $\eta, \theta \in \mathbb{R}$ *such that for all* $p > 0$,

$$(\sigma_p, \tau_p^*, \gamma_p^*, \eta_p, \theta_p^*, \chi_p^*) = (1 + p\sigma)(\sigma, \tau, \eta, \theta, \gamma, 1). \tag{4.35}$$

Proof. In this case, after dividing the first five components of v_p^* (see (4.34)), by its last component we see that there exist constants $\sigma, \tau, \gamma, \eta, \theta \in \mathbb{R}$ (independent of p) such that

$$(\sigma_p, \tau_p^*, \gamma_p^*, \eta_p, \theta_p^*) = \chi_p^*(\sigma, \tau, \gamma, \eta, \theta).$$

Plugging the relation $\sigma_p = \sigma\chi_p^*$ into (4.33) we get $\chi_p^*(1 + p\sigma) = \chi_p$. Consequently, since $\chi_p \in \{0, 1\}$ and $\chi_p^* \neq 0$ it follows that $\chi_p := \chi = 1$. Thus $\chi_p^* > 0$ at least for small $p > 0$. Combining this with $\sigma_p = \sigma\chi_p^*$ and with the fact that $\sigma_p \geq 0$ we get $\sigma \geq 0$, so $1 + p\sigma > 0$ for any $p > 0$. Hence we conclude that (4.35) hold for all $p > 0$.

Note that $\lim_{p\to 0^+} \tau_p^* = \lim_{p\to 0^+} \tau_p = \tau$. Since $\tau_p \geq 0$ for all $p \geq 0$, we get $\tau \geq 0$. Similarly, $\lim_{p\to 0^+} \gamma_p^* = \lim_{p\to 0^+} \gamma_p = \gamma$ and $\lim_{p\to 0^+} \sigma_p = \sigma$. Since $\gamma_p \leq 1 + 2\sqrt{\sigma_p \tau_p}$ for all $p > 0$, we get $\gamma \leq 1 + 2\sqrt{\sigma\tau}$. ∎

Substituting (4.35) into (4.24) we immediately get (4.23a)–(4.23c) with $\chi = 1$.

CASE B: *$\chi_p^* = 0$ and $\sigma_p > 0$ for all $p > 0$.*

CLAIM. *There exist constants $\eta, \theta \in \mathbb{R}$, $\tau \geq 0$, $\gamma \leq 2\sqrt{\tau}$ such that for all $p > 0$,*

$$(\sigma_p, \tau_p^*, \gamma_p^*, \eta_p, \theta_p^*, \chi_p^*) = \tfrac{1}{p}(\sigma, \tau, \gamma, \eta, \theta, 0). \tag{4.36}$$

Proof. From (4.33) we get $\chi_p = p\sigma_p > 0$ for all $p > 0$. As $\chi_p \in \{0, 1\}$, this means that $\chi_p = 1$ and $\sigma_p = 1/p$. Dividing components 2–5 of v_p^* by the first component (see (4.34)), we see that there exist constants $\tau, \gamma, \eta, \theta \in \mathbb{R}$ such that

$$p(\tau_p^*, \gamma_p^*, \eta_p, \theta_p^*) = (\tau, \gamma, \eta, \theta).$$

Thus we get (4.36). Since $\tau_p \geq 0$ we infer that $\tau \geq 0$. Furthermore, $\gamma_p \leq 2\sqrt{\tau_p \sigma_p}$ yields $\gamma - p \leq 2\sqrt{\tau + p^2 - p\gamma}$. Taking $p \to 0$ we get $\gamma \leq 2\sqrt{\tau}$. ∎

Substituting (4.36) into (4.24) we immediately get (4.23a)–(4.23c) with $\chi = \sigma = 1$.

CASE C: *$\chi_p^* = 0$, $\sigma_p = 0$ and $\gamma_p^* = 0$ for all $p > 0$.*

CLAIM. *There are constants $\eta, \theta \in \mathbb{R}$ such that for all $p > 0$,*

$$(\sigma_p, \tau_p^*, \gamma_p^*, \eta_p, \theta_p^*, \chi_p^*) = (0, 1, 0, \eta, \theta, 0). \tag{4.37}$$

Proof. Note $\gamma_p^* = \gamma_p = 0$. Moreover, $\tau_p = 1$ yields $\tau_p^* = 1$. Thus (4.37) follows from the fact that in (4.34) the vector $v_p^* = (0, 1, 0, \eta_p, \theta_p^*, 0)$ does not depend on $p > 0$. ∎

Substituting (4.37) into (4.24) we get (4.23a)–(4.23c) with $\chi = 0$, $\sigma = 0$, $\gamma = 0$, $\tau = 1$.

CASE D: *$\chi_p^* = 0$, $\sigma_p = 0$ and $\gamma_p^* \neq 0$ for all $p > 0$.*

CLAIM. *There exist $\tau \geq 0$ and $\eta, \theta \in \mathbb{R}$ such that for all $p > 0$,*

$$(\sigma_p, \tau_p^*, \gamma_p^*, \eta_p, \theta_p^*, \chi_p^*) = \tfrac{1}{p+\tau}(0, \tau, -1, \eta, \theta, 0). \tag{4.38}$$

Proof. By Lemma 4.7 we have $\tau_p = 1$, $\gamma_p^* = \gamma_p$ and $\chi_p = 0$. Dividing the remaining components of v_p^* by its third component (see (4.34)), we get

$$(\tau_p^*, \eta_p, \theta_p^*) = -\gamma_p^*(\tau, \eta, \theta)$$

for some constants τ, η, θ. Since $\tau_p^* = 1 + p\gamma_p$, comparing the first components in the above equation we get $\gamma_p = -\frac{1}{\tau + p}$, at least for small $p > 0$. This identity together with the fact that $\gamma_p \leq 0$ for all $p > 0$ yields $\tau \geq 0$. Hence the identity holds for all $p > 0$, and (4.38) follows. ∎

Substituting (4.38) into (4.24) we get (4.23a)–(4.23c) with $\chi = 0$, $\sigma = 0$, $\gamma = -1$.

REMARK 4.8. A conjecture in [5] says that when $\chi = 1$, the remaining parameters determine the distribution of the quadratic harness uniquely. The following example shows that uniqueness may fail when $\chi = 0$.

Let $(X_t)_{t\geq 0}$ be a Markov process defined in [3, Section 4.1] as the q-Brownian process with $q = -1$, i.e., (4.23a) holds with parameters $\chi = 1$, $\theta = \eta = \sigma = \tau = 0$ and $\gamma = -1$. It is known that $(X_t)_{t\geq 0}$ is a standard harness and $|X_t| = \sqrt{t}$ for all $t \geq 0$. Hence the linear independence assumption fails, and we can write the quadratic form (4.23c) in many ways.

Let $(Z_t)_{t\geq 0}$ be a stochastic process given by $Z_t := Y \cdot X_t$, where Y is a random variable such that Y and $(X_t)_{t\geq 0}$ are independent and $\mathbb{E}\, Y^2 = 1$. Then $\mathbb{E}\, Z_t = 0$, $\mathbb{E}\, Z_s Z_t = s \wedge t$. Let $(\mathcal{F}_{s,u})_{0\leq s<u}$ and $(\mathcal{G}_{s,u})_{0\leq s<u}$ be filtrations for $(X_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ respectively. Then

$$\begin{aligned}\mathbb{E}(Z_t \,|\, \mathcal{G}_{s,u}) &= \mathbb{E}(Y\, \mathbb{E}(X_t \,|\, Y, \mathcal{F}_{s,u}) \,|\, \mathcal{G}_{s,u}) = \mathbb{E}(Y\, \mathbb{E}(X_t \,|\, \mathcal{F}_{s,u}) \,|\, \mathcal{G}_{s,u}) \\ &= \mathbb{E}\left(Y\left(\tfrac{u-t}{u-s} X_s + \tfrac{t-s}{u-s} X_u\right) \,\middle|\, \mathcal{G}_{s,u}\right) = \tfrac{u-t}{u-s} Z_s + \tfrac{t-s}{u-s} Z_u,\end{aligned}$$

hence $(Z_t)_{t\geq 0}$ is a harness. Because $X_t^2 = t = \frac{u-t}{u-s}s + \frac{t-s}{u-s}u = \frac{u-t}{u-s}X_s^2 + \frac{t-s}{u-s}X_u^2$ for all $0 \leq s < t < u$ we have

$$\mathbb{E}(Z_t^2 \,|\, \mathcal{G}_{s,u}) = \mathbb{E}(tY^2 \,|\, \mathcal{G}_{s,u}) = \tfrac{u-t}{u-s} Z_s^2 + \tfrac{t-s}{u-s} Z_u^2,$$

so $(Z_t^2)_{t\geq 0}$ is also a harness. From the above form, it follows that $(Z_t)_{t\geq 0}$ is a quadratic harness with parameters $\chi = \theta = \eta = \gamma = \sigma = 0$ and $\tau = 1$. The distribution of Z_t is not unique as it depends on the choice of Y.

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Włodzimierz Bryc
Department of Mathematical Sciences
University of Cincinnati
Cincinnati, OH 45221-0025, USA
E-mail: Wlodzimierz.Bryc@UC.edu

Jacek Wesołowski, Agnieszka Zięba
Faculty of Mathematics
and Information Science
Warsaw University of Technology
Warszawa, Poland
E-mail: wesolo@mini.pw.edu.pl
agnieszka.zieba@mini.pw.edu.pl