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SPECTRAL MULTIPLIER THEOREM AND SUB-GAUSSIAN HEAT KERNEL ESTIMATES

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Abstract. We study general spectral multiplier theorems for nonnegative self-adjoint operators on spaces of homogeneous type. We show that a Hörmander type spectral multiplier theorem follows from sub-Gaussian upper bounds for the corresponding heat kernel. Our result can be applied to fractal manifolds and quantum graphs.

1. Introduction. Let (X, d, μ) be a metric measure space of homogeneous type. Let L be a nonnegative self-adjoint operator acting on $L^2(X, d\mu)$ and let $dE_L(\lambda)$ be the spectral resolution of L. Then for any bounded function $F\colon [0,\infty) \to \mathbb{R}$ one can define the multiplier

$$F(L) = \int_0^\infty F(\lambda)\, dE_L(\lambda).$$

By the spectral theorem the operator $F(L)$ is bounded on $L^2(X)$. The theory of spectral multipliers concerns conditions on F which ensure that $F(L)$ can be extended to a bounded operator on $L^p(X, d\mu)$ for some range of exponents $p \neq 2$. Spectral multiplier results are modeled on Fourier multiplier results described in the fundamental works of Mikhlin [M65] and Hörmander [H60].

In the last fifty years, spectral multipliers have been studied by many authors in different contexts, including differential or pseudo-differential operators on manifolds, sub-Laplacians on Lie groups, Schrödinger operators as well as operators in abstract settings. We refer the reader to [A94, B03, C91, COSY20, COSY16, CS01, DOS02, JN95, KU15, M17, M65, SYY14] and references therein. We mention in particular the paper [DOS02] where the authors show that a Hörmander type spectral multiplier theorem follows from the following classical Gaussian upper bounds for the heat kernel: there exist constants $m \geq 2$ and $C, c > 0$ such that the heat kernel $p_t(x, y)$ of the

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semigroup e^{-tL} satisfies

$$(\mathrm{GE}_m) \qquad |p_t(x,y)| \le C\mu(B(x,\sqrt[m]{t}))^{-1}\exp\Big(-c\Big(\frac{d(x,y)^m}{t}\Big)^{\frac{1}{m-1}}\Big).$$

Assume X satisfies the doubling property with homogeneous dimension n, that is, there exist constants $C>0$ and $n>0$ such that

$$\mu(B(x,\lambda r)) \le C\lambda^n \mu(B(x,r)), \quad \forall \lambda \ge 1,\, x\in X,\, r>0.$$

See (DP) and (2.1) below. In [DOS02], Duong, Ouhabaz and Sikora obtained the following Hörmander type spectral multiplier theorem.

THEOREM 1.1 ([DOS02]). *Let (X,d,μ) be a metric measure space satisfying the doubling condition with homogeneous dimension n. Suppose that $s>n/2$ and assume that L satisfies the Gaussian heat kernel estimate* (GE_m)*. Then for any Borel bounded function F such that*

$$\sup_{a>0}\|\varphi\delta_a F\|_{W_s^\infty}<\infty,$$

where $\varphi\in C_c^\infty(0,\infty)$ with $\operatorname{supp}\varphi\subset[1/4,1]$ and $\delta_aF(\cdot)=F(a\cdot)$, the operator $F(L)$ is of weak type $(1,1)$ and is bounded on $L^p(X)$ for all $1<p<\infty$. In addition,

$$\|F(L)\|_{L^1(X)\to L^{1,\infty}(X)} \le C_s\Big(\sup_{a>0}\|\varphi\delta_aF\|_{W_s^\infty}+|F(0)|\Big).$$

In this article we assume that L satisfies the following sub-Gaussian heat kernel estimate: the heat kernel $p_t(x,y)$ of the semigroup e^{-tL} satisfies

$$(\mathrm{UE}_m) \qquad |p_t(x,y)| \le \begin{cases} CV(x,\sqrt{t})^{-1}\exp\big(-c\frac{d(x,y)^2}{t}\big), & 0<t<1,\\ CV(x,\sqrt[m]{t})^{-1}\exp\big(-c\big(\frac{d(x,y)^m}{t}\big)^{\frac{1}{m-1}}\big), & t\ge 1,\end{cases}$$

where $m\ge 2$, for all x, $y\in X$. Several examples of operators satisfying sub-Gaussian heat kernel estimates are given in [GT01], [CCFR17, Section 5], [C15, Section 3], [HS01, Section 5]. These examples include Laplace–Beltrami operators on fractal Riemannian manifolds and quantum graphs.

A natural question is the boundedness of $F(L)$ if we replace the Gaussian heat kernel estimate (GE_m) of Theorem 1.1 by the sub-Gaussian estimate (UE_m). Following [LZ18, p. 1602], we introduce a new distance

$$\tilde d(x,y)=\max\{d(x,y),d(x,y)^{2/m}\}.$$

With this new distance, the sub-Gaussian estimate (UE_m) turns into the classical Gaussian heat kernel estimate of order m, that is,

$$(1.1) \qquad |p_t(x,y)| \le C\mu(\tilde B(x,\sqrt[m]{t}))^{-1}\exp\Big(-c\Big(\frac{\tilde d(x,y)^m}{t}\Big)^{\frac{1}{m-1}}\Big),$$

where $\tilde{B}(x,r) = \{y : \tilde{d}(x,y) < r\}$. The homogeneous dimension of the ambient space with respect to the distance $\tilde{d}$ is $mn/2$ (see [LZ18, (3.7)]). Hence, we can apply Theorem 1.1 to obtain a spectral multiplier theorem with smoothness index $s > Q/2 = mn/4$ of F.

The aim of this article is to establish a spectral multiplier theorem with the sub-Gaussian heat kernel estimate (UE_m), which requires the smoothness index $s > n/2$ of F, instead of $s > mn/4$. More precisely, we have the following result.

THEOREM 1.2. *Let (X, d, μ) be a metric measure space satisfying the doubling condition with homogeneous dimension n. Suppose that $s > n/2$ and L satisfies the sub-Gaussian heat kernel estimate* (UE_m). *Then for any Borel bounded function F such that*

$$\sup_{a>0} \|\varphi \delta_a F\|_{W_s^\infty} < \infty,$$

where $\varphi \in C_c^\infty(0,\infty)$ with $\operatorname{supp}\varphi \subset [1/4, 1]$, the operator $F(L)$ is of weak type $(1,1)$ and is bounded on $L^p(X)$ for all $1 < p < \infty$. In addition,

$$\|F(L)\|_{L^1(X)\to L^{1,\infty}(X)} \le C_s \Big(\sup_{a>0} \|\varphi \delta_a F\|_{W_s^\infty} + |F(0)|\Big).$$

We mention that in [DOS02, proof of Theorem 1.1], a crucial use is made of the Phragmén–Lindelöf theorem to extend the heat kernel estimate from the real line to the complex right half-plane. Our approach is based on the methods of Jensen–Nakamura [JN94, JN95] and D'Ancona–Nicola [DN16], using a commutator argument; see also [BDN20, COSY20, CDFLY, MM13]. Interestingly, in the whole proof, we do not use the semigroup structure of the heat semigroup. Crucial estimates are derived from the heat kernel estimate for a fixed time t and in the process we do not need the heat kernel estimate for other times; see Section 3 for details. So our approach is in line with (UE_m), which has different decay characteristics for $t \ge 1$ and $0 < t < 1$.

The structure of this paper is as follows. In Section 2, we describe some notations, assumptions and preliminary results. In Section 3, we use amalgam blocks methods and commutator techniques to obtain a crucial off-diagonal estimate for compact spectral multipliers. The proof of our main result, Theorem 1.2, will be given in Section 4.

2. Assumptions and preliminary results

2.1. Assumptions and notations. Throughout this paper, we assume that (X, d, μ) is a metric measure space, where μ is a σ-finite measure on X satisfying the doubling property: there exists a constant $C \ge 1$ such that

$$\text{(DP)} \qquad \mu(B(x,2r)) \le C\mu(B(x,r)), \qquad \forall x \in X,\ r > 0,$$

where $B(x,r)=\{y\in X: d(x,y)<r\}$ denotes the ball with center x and radius r. Note that (DP) implies that there exist constants $C>0$ and $n>0$ such that

$$\mu(B(x,\lambda r))\le C\lambda^n\mu(B(x,r)),\qquad \forall\lambda\ge 1,\ x\in X,\ r>0. \tag{2.1}$$

In Euclidean space with Lebesgue measure, n corresponds to the dimension of the space.

In addition, recall that the kernel $p_t(x,y)$ of the semigroup e^{-tL} $(t>0)$, generated by $-L$ satisfies the sub-Gaussian estimate

$$|p_t(x,y)|\le \begin{cases} CV(x,\sqrt{t})^{-1}\exp\big(-c\frac{d(x,y)^2}{t}\big), & 0<t<1,\\ CV(x,\sqrt[m]{t})^{-1}\exp\big(-c\big(\frac{d(x,y)^m}{t}\big)^{\frac{1}{m-1}}\big), & t\ge 1,\end{cases} \tag{UE$_m$}$$

where $m\ge 2$, for all $x,y\in X$.

We often just use B instead of $B(x,r)$. Given $\lambda>0$, we write λB for the λ-dilated ball, with the same center as B and radius λr. We denote by $V(x,r)$ the volume of $B(x,r)$. For $1\le p\le\infty$, we denote the L^p norm of a function f by $\|f\|_p$ and the scalar product of $L^2(X,\mathrm{d}\mu)$ by $\langle\cdot,\cdot\rangle$. If T is a bounded linear operator from $L^p(X,\mathrm{d}\mu)$ to $L^q(X,\mathrm{d}\mu)$ for $1\le p,q\le\infty$, we write $\|T\|_{p\to q}$ for the operator norm of T. Given a subset $E\subseteq X$, we denote the characteristic function of E by χ_E and set

$$P_Ef(x)=\chi_E(x)f(x).$$

Given a ball $B\subseteq X$, we define

$$U_0(B)=B,\qquad U_j(B)=2^jB\setminus 2^{j-1}B,\quad j\ge 1.$$

For a given function $F:\mathbb{R}\to\mathbb{C}$ and $R>0$, we define $\delta_RF:\mathbb{R}\to\mathbb{C}$ by putting $\delta_RF(x)=F(Rx)$. We also set

$$\rho(t):=\begin{cases} t^2, & 0<t<1,\\ t^m, & t\ge 1.\end{cases} \tag{2.2}$$

2.2. Preliminary results. Since X is separable, for every $r>0$ there exists a sequence $\{x_i\}_{i=0}^\infty\subset X$ such that $d(x_i,x_j)>r/2$ for $i\ne j$ and $\sup_{x\in X}\inf_i d(x_i,x)\le r/2$. Set

$$D:=\bigcup_{i\in\mathbb{N}}B(x_i,r/4).$$

We define the *amalgam block* $Q_i(r)$ by

$$Q_i(r):=B(x_i,r/4)\cup\Big(B(x_i,r/2)\setminus\Big(\bigcup_{j<i}B(x_j,r/2)\setminus D\Big)\Big).$$

Then $\{Q_i(r)\}_{i=0}^\infty$ is a countable partition of X with $Q_i(r)\cap Q_j(r)=\emptyset$ if $i\ne j$. Note that $B(x_i,r/4)\subseteq Q_i(r)\subseteq B(x_i,r/2)$, thus $\mu(Q_i(r))\approx\mu(B(x_i,r))$ by the doubling property (DP).

With the partition of X as above, for every $r > 0$, we define a family $\{\eta_{k,r}(x)\}$ of functions by

$$\eta_{k,r}(x) := d(x_k, x)/r, \qquad k \in \mathbb{N}.$$

Then we define multi-commutator operators:

DEFINITION 2.1. Let T be a bounded operator on $L^2(X)$. For every $r > 0$ we define multi-commutators $\mathrm{ad}^j_{k,r} : L^2(X) \to L^2(X)$ by

$$\begin{aligned} \mathrm{ad}^0_{k,r}(T) &:= \eta_{k,r}T - T\eta_{k,r}, && k \in \mathbb{N}, \\ \mathrm{ad}^j_{k,r}(T) &:= \mathrm{ad}^{j-1}_{k,r}(\eta_{k,r}T - T\eta_{k,r}), && j \in \mathbb{N}^+,\ k \in \mathbb{N}. \end{aligned}$$

The next lemma states that the L^2 boundedness of a multi-commutator of a Schrödinger group can be derived from an off-diagonal estimate.

LEMMA 2.2. *Assume that a self-adjoint operator A satisfies the following exponential off-diagonal estimate for some fixed $R > 0$:*

$$\begin{aligned} &\|P_{B(x,R^{-1/m})} A P_{B(y,R^{-1/m})}\|_{1\to 2} \\ &\qquad \le C V(y, R^{-1/m})^{-1/2} \exp\big(-c(\sqrt[m]{R}\, d(x,y))^{\frac{m}{m-1}}\big) \end{aligned}$$

with constants $C, c > 0$. Then for any $j \in \mathbb{N}$, there exists a constant $C_j > 0$ independent of R such that for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\|\mathrm{ad}^j_{k,R^{-1/m}}(e^{itA})\|_{2\to 2} \le C_j(1 + |t|)^j.$$

Proof. See [COSY20, (3.11)]; we omit the details. ∎

Next, we state the following result about the boundedness of singular integrals.

PROPOSITION 2.3. *Suppose that an operator L satisfies the sub-Gaussian heat kernel estimate* (UE$_m$). *Let T be a bounded linear operator from $L^2(X)$ to $L^2(X)$ and let $k_r(x, y)$ be the kernel of the composite operator $T(I - e^{-\rho(r)L})$, where $\rho(r)$ is given in* (2.2). *Assume that there exist constants $C, c > 0$ such that*

$$\int_{d(x,y)\ge cr} |k_r(x,y)|\, d\mu(x) \le C \tag{2.3}$$

for all $y \in X$. Then the operator T is of weak type $(1,1)$*, and there exists a constant $C_1 > 0$ such that*

$$\|T\|_{L^1(X)\to L^{1,\infty}(X)} \le C_1(C + \|T\|^2_{2\to 2}).$$

Proof. The proof can be obtained by adapting the argument of Duong–McIntosh [DM99, Theorem 1] or Hebisch [H90] to the sub-Gaussian heat kernel estimate (UE$_m$) as in [CCFR17, Theorem 1.2]; see also [LZ18, Proposition 3]. ∎

3. Off-diagonal estimates and kernel estimates

3.1. Off-diagonal estimates. Compared to the Gaussian heat kernel estimate, the sub-Gaussian heat kernel estimate is difficult to extend to the complex right half-plane. To overcome this difficulty, we will use commutator techniques and amalgam block methods presented in [COSY20] to get an off-diagonal estimate of a compactly supported multiplier. Instead of employing the estimate of $p_t(x,y)$ for all $t>0$, we just require the kernel estimate for some fixed t_0, which is related to the support of the compactly supported multiplier.

In the whole section, we fix $R>0$. We assume that the multiplier function H is supported in $[R/4,R]$ and that the nonnegative self-adjoint operator L satisfies the following heat kernel estimate for some fixed $R>0$ and $m\ge 2$:

$$|p_{R^{-1}}(x,y)| \le CV(x,R^{-1/m})^{-1}\exp\big(-c(\sqrt[m]{R}\,d(x,y))^{\frac{m}{m-1}}\big) \tag{3.1}$$

for all $x,y\in X$.

Proposition 3.1. *Let $s\in\mathbb{N}$ and $j\ge 4$. Suppose that the operator L satisfies estimate* (3.1) *for some fixed $R>0$ and $m\ge 2$. Then for any $\varepsilon>0$ and any ball B with radius $R^{-1/m}$, there exists a constant C_s such that for all Borel functions H with* $\operatorname{supp} H\subseteq[R/4,R]$,

$$\|P_{U_j(B)}H(L)P_B\|_{1\to 2}\le C_s 2^{-js}V(B)^{-1/2}\|\delta_R H\|_{W^2_{s+1/2+\varepsilon}}, \tag{3.2}$$

where C_s is a constant which depends on s only, and does not depend on B, R or m.

Proposition 3.1 is based on Lemmas 3.2–3.4 below.

Lemma 3.2. *Suppose that L satisfies estimate* (3.1) *for some fixed $R>0$ and $m\ge 2$. Let $1\le p\le q\le\infty$. The following off-diagonal estimate holds:*

$$\begin{aligned}\|P_{B(x,R^{-1/m})}e^{-L/R}P_{B(y,R^{-1/m})}\|_{p\to q}\\ \le CV(y,R^{-1/m})^{1/q-1/p}\exp\big(-c(Rd(x,y)^m)^{\frac{1}{m-1}}\big)\end{aligned} \tag{3.3}$$

for all $x,y\in X$.

Proof. From (3.1), it is straightforward to get (3.3); we leave the details to the reader. ∎

Lemma 3.3. *Let $K_{H(L)}(x,y)$ be the kernel of the operator $H(L)$. Then for any ball $B\subseteq X$,*

$$\|P_{U_j(B)}H(L)P_B\|_{1\to 2}=\sup_{y\in B}\Big(\int_{U_j(B)}|K_{H(L)}(x,y)|^2\,d\mu(x)\Big)^{1/2}. \tag{3.4}$$

Proof. From Minkowski's inequality, for all $f \in L^1(X)$ we have

$$\begin{aligned}\|P_{U_j(B)}H(L)P_B f\|_2 &= \Big(\int\limits_{U_j(B)} \Big|\int\limits_B f(y)K_{H(L)}(x,y)\,d\mu(y)\Big|^2\,d\mu(x)\Big)^{1/2}\\ &\le \int\limits_B \Big(\int\limits_{U_j(B)} |K_{H(L)}(x,y)|^2\,d\mu(x)\Big)^{1/2}|f(y)|\,d\mu(y)\\ &\le \sup_{y\in B}\Big(\int\limits_{U_j(B)} |K_{H(L)}(x,y)|^2\,d\mu(x)\Big)^{1/2}\|f\|_1.\end{aligned}$$

Hence,

$$\|P_{U_j(B)}H(L)P_B\|_{1\to 2} \le \sup_{y\in B}\Big(\int\limits_{U_j(B)} |K_{H(L)}(x,y)|^2\,d\mu(x)\Big)^{1/2}.$$

On the other hand, it follows from the Riesz representation theorem, duality and Hölder's inequality that

$$\begin{aligned}&\sup_{y\in B}\Big(\int\limits_{U_j(B)} |K_{H(L)}(x,y)|^2\,d\mu(x)\Big)^{1/2} = \sup_{y\in B}\sup_{\|h\|_{L^2(U_j(B))}=1} |\langle h, K_{H(L)}(\cdot,y)\rangle|\\ &= \sup_{y\in B}\sup_{\|h\|_{L^2(U_j(B))}=1} |\langle \bar h, K_{H(L)^*}(y,\cdot)\rangle| = \sup_{y\in B}\sup_{\|h\|_{L^2(U_j(B))}=1} |H(L)^*(\bar h)(y)|\\ &= \sup_{\|h\|_{L^2(U_j(B))}=1}\sup_{\|g\|_{L^1(B)}=1} |\langle H(L)^*(\bar h), g\rangle|\\ &= \sup_{\|h\|_{L^2(U_j(B))}=1}\sup_{\|g\|_{L^1(B)}=1} |\langle \bar h, H(L)(g)\rangle|\\ &\le \sup_{\|g\|_{L^1(B)}=1} \|H(L)(g)\|_{L^2(U_j(B))} \le \|P_{U_j(B)}H(L)P_B\|_{1\to 2}.\end{aligned}$$

Hence,

$$\sup_{y\in B}\Big(\int\limits_{U_j(B)} |K_{H(L)}(x,y)|^2\,d\mu(x)\Big)^{1/2} \le \|P_{U_j(B)}H(L)P_B\|_{1\to 2}.$$

Combining the above, we obtain the result. ■

LEMMA 3.4. *Suppose that L satisfies* (3.1) *for some fixed $R > 0$ and $m \ge 2$. Let $j \ge 4$. Then for any $s \in \mathbb{N}$ and any ball B with radius $R^{-1/m}$, there exists a constant C_s independent of B, R and m such that*

$$\|P_{U_j(B)}e^{i\xi A}AP_B\|_{1\to 2} \le C_s 2^{-js}(1+|\xi|)^s V(B)^{-1/2},$$

where $A = e^{-L/R}$.

Proof. For all $f \in L^1(X)$, we first note that

$$\begin{aligned}\|P_{U_j(B)} e^{i\xi A} A P_B f\|_2 &\le \|P_{U_j(B)} e^{i\xi A} P_{(2^{j-3}B)^c} A P_B f\|_2 \\ &\quad + \|P_{U_j(B)} e^{i\xi A} P_{2^{j-3}B} A P_B f\|_2 \\ &:= \mathrm{I} + \mathrm{II}.\end{aligned}$$

For I, it follows from the L^2 boundedness of $e^{i\xi A}$ and Minkowski's inequality that

$$\begin{aligned}\mathrm{I} &\le \|P_{U_j(B)} e^{i\xi A}\|_{2\to 2} \|P_{(2^{j-3}B)^c} A P_B f\|_2 \le \|P_{(2^{j-3}B)^c} A P_B f\|_2 \\ &= \Big(\int_{(2^{j-3}B)^c} \Big| \int_B p_{R^{-1}}(x,z) f(z)\, d\mu(z) \Big|^2 d\mu(x) \Big)^{1/2} \\ &\le \int_B \Big(\int_{(2^{j-3}B)^c} |p_{R^{-1}}(x,z)|^2\, d\mu(x) \Big)^{1/2} |f(z)|\, d\mu(z).\end{aligned}$$

To continue the estimate of I, we first note that for any $x \in (2^{j-3}B)^c$ and $z \in B$,

$$d(x,z) \ge d(x,x_B) - d(x_B,z) \ge 2^{j-4} R^{-1/m},$$

where x_B is the center of B.

From estimates (3.1) and (2.1), we can easily deduce that there exist positive constants C and c such that

$$|p_{R^{-1}}(x,z)| \le C V(z, R^{-1/m})^{-1} \exp\big(-c(\sqrt[m]{R}\, d(x,z))^{\frac{m}{m-1}}\big)$$

for all $x, z \in X$. Therefore, for any $j \ge 4$,

$$\begin{aligned}&\int_{(2^{j-3}B)^c} |p_{R^{-1}}(x,z)|^2\, d\mu(x) \\ &\le \int_{(2^{j-3}B)^c} \big| V(z,R^{-1/m})^{-1} \exp\big(-c(\sqrt[m]{R}\, d(x,z))^{\frac{m}{m-1}}\big)\big|^2\, d\mu(x) \\ &\le C \exp(-16c2^j) V(z,R^{-1/m})^{-2} \int_X \exp\big(-c(\sqrt[m]{R}\, d(x,z))^{\frac{m}{m-1}}\big)\, d\mu(x) \\ &\le C \exp(-16c2^j) V(z,R^{-1/m})^{-1},\end{aligned} \tag{3.5}$$

where we also use the inequality

$$\exp\big(-c(\sqrt[m]{R}\, d(x,z))^{\frac{m}{m-1}}\big) \le \exp\big(-c2^{(j-4)\frac{m}{m-1}}\big) \le C\exp(-16c2^j).$$

Hence, from estimate (3.5) and the doubling property (DP), we obtain

$$\begin{aligned}\mathrm{I} &\le C\exp(-8c2^j) \int_B |f(z)| V(z,R^{-1/m})^{-1/2}\, d\mu(z) \\ &\le C\exp(-8c2^j) V(B)^{-1/2} \|f\|_1.\end{aligned} \tag{3.6}$$

For II, we set $r = R^{-1/m}$ and use the amalgam blocks $\{Q_k(r)\}$ to cover $2^{j-3}B$, and then use Minkowski's inequality to obtain

$$\begin{aligned} \text{II} &= \Big\| P_{U_j(B)} e^{i\xi A} \sum_{k\colon Q_k(r)\subseteq 2^{j-2}B} P_{Q_k(r)} P_{2^{j-3}B} A P_B f \Big\|_2 \\ &\le \sum_{k\colon Q_k(r)\subseteq 2^{j-2}B} \|P_{U_j(B)} e^{i\xi A} P_{Q_k(r)} P_{2^{j-3}B} A P_B f\|_2 \\ &= \sum_{k\colon Q_k(r)\subseteq 2^{j-2}B} \|\eta_{k,r}^{-s}\eta_{k,r}^{s} P_{U_j(B)} e^{i\xi A} P_{2^{j-3}B\cap Q_k(r)} A P_B f\|_2 \\ &\le 4^s 2^{-js} \sum_{k\colon Q_k(r)\subseteq 2^{j-2}B} \|\eta_{k,r}^{s} P_{U_j(B)} e^{i\xi A} P_{2^{j-3}B\cap Q_k(r)} A P_B f\|_2 \\ &\le 4^s 2^{-js} \sum_{k} \|\eta_{k,r}^{s} e^{i\xi A} P_{Q_k(r)}\|_{2\to 2} \|P_{Q_k(r)} A P_B f\|_2, \end{aligned} \tag{3.7}$$

where in the second inequality, we use the fact that $x \in U_j(B)$ and $Q_k(r) \subseteq 2^{j-2}B$ to find that $d(x_k, x) \ge 2^{j-2}r$ and

$$\eta_{k,r}^{-s} = (d(x_k, x)/r)^{-s} \le 2^{-(j-2)s}.$$

To continue, we need to estimate $\|\eta_{k,r}^{s} e^{i\xi A} P_{Q_k(r)}\|_{2\to 2}$ and $\sum_k \|P_{Q_k(r)} A P_B f\|_2$.

Estimate of $\|\eta_{k,r}^{s} e^{i\xi A} P_{Q_k(r)}\|_{2\to 2}$. For commutators, we have the following formula (see for example [JN95, Lemma 3.1]):

$$\eta_{k,r}^{s} T = \sum_{l=0}^{s} \Gamma(s,l)\, \mathrm{ad}_{k,r}^{l}(T)\, \eta_{k,r}^{s-l}, \tag{3.8}$$

where $\Gamma(b,0) = 1$ for $b \in \mathbb{N}$ and $\Gamma(b,l)$ is defined inductively by $\Gamma(b,l+1) = \sum_{i=l}^{b-1} \Gamma(i,l)$ for $b \ge 1$ and $1 \le l \le b-1$. Applying (3.8) and Lemma 2.2, we further find that

$$\begin{aligned} \|\eta_{k,r}^{s} e^{i\xi A} P_{Q_k(r)}\|_{2\to 2} &\le \sum_{l=0}^{s} \Gamma(s,l) \|\mathrm{ad}_{k,r}^{l}(e^{i\xi A}) \eta_{k,r}^{s-l} P_{Q_k(r)}\|_{2\to 2} \\ &\le \sum_{l=0}^{s} \Gamma(s,l) \|\mathrm{ad}_{k,r}^{l}(e^{i\xi A})\|_{2\to 2} \|\eta_{k,r}^{s-l} P_{Q_k(r)}\|_{2\to 2} \\ &\le \Big(\sum_{l=0}^{s} \Gamma(s,l)\Big) C_1 (1+|\xi|)^s, \end{aligned} \tag{3.9}$$

where we also use the fact that $\|\eta_{k,r}^{s-l} P_{Q_k(r)}\|_{2\to 2} \le 1$ for all $0 \le l \le s$.

Note that the estimate of $\|\eta_{k,r}^{s} e^{i\xi A} P_{Q_k(r)}\|_{2\to 2}$ is independent of k. To continue (3.7), it remains to sum $\|P_{Q_k(r)} A P_B f\|_2$ over k.

Estimate of $\sum_k \|P_{Q_k(r)} A P_B f\|_2$. It follows from $X = \bigcup_i Q_i(r)$ with $Q_i(r) \cap Q_j(r) = \emptyset$ for $i \ne j$ and Lemma 3.2 that

$$\begin{aligned}
(3.10)\qquad \sum_k \|P_{Q_k(r)}AP_Bf\|_2 &= \sum_k \Big\| P_{Q_k(r)}A\sum_i P_{B\cap Q_i(r)}f\Big\|_2\\
&\le \sum_k\sum_i \|P_{Q_k(r)}AP_{B\cap Q_i(r)}f\|_2\\
&\le \sum_i\sum_k \|P_{Q_k(r)}AP_{B\cap Q_i(r)}\|_{1\to 2}\|P_{B\cap Q_i(r)}f\|_1\\
&\le C\sum_{i,k}\exp\bigg(-c\Big(\frac{d(x_k,x_i)}{r}\Big)^{\frac{m}{m-1}}\bigg)V(Q_i(r))^{-1/2}\|P_{B\cap Q_i(r)}f\|_1\\
&\le C\sum_i\sum_k\exp\bigg(-c\Big(\frac{d(x_k,x_i)}{r}\Big)^{\frac{m}{m-1}}\bigg)V(B)^{-1/2}\|P_{B\cap Q_i(r)}f\|_1.
\end{aligned}$$

Using estimate (2.1) and the fact that $d(x_i,x_k)>r/2$ for $i\neq j$, we find that

$$\begin{aligned}
(3.11)\qquad &\sum_k \exp\bigg(-c\Big(\frac{d(x_k,x_i)}{r}\Big)^{\frac{m}{m-1}}\bigg)\\
&\le \sum_{l=0}^{\infty}\sum_{k:\,2^{l-1}r<d(x_i,x_k)\le 2^l r}\exp\bigg(-c\Big(\frac{d(x_k,x_i)}{r}\Big)^{\frac{m}{m-1}}\bigg)\\
&\le \sum_{l=0}^{\infty}\#\big(\{k: 2^{l-1}r<d(x_i,x_k)\le 2^l r\}\big)\exp(-c2^{l-1})\\
&\le \sum_{l=0}^{\infty}\sup_{\{k:\,2^{l-1}r<d(x_i,x_k)\le 2^l r\}}\frac{V(x_k,2^l r)}{V(x_k,r)}\exp(-c2^{l-1})\\
&\le C\sum_{l=0}^{\infty}2^{ln}\exp(-c2^{l-1})\le C,
\end{aligned}$$

where # denotes the number of elements.

Hence, estimates (3.10), (3.11) and the disjoint partition $\{Q_i(r)\}_{i=0}^{\infty}$ of X imply that

$$\sum_k \|P_{Q_k(r)}AP_Bf\|_2\le C\sum_i V(B)^{-1/2}\|P_{B\cap Q_i(r)}f\|_1 = CV(B)^{-1/2}\|P_Bf\|_1,$$

which, together with estimates (3.7) and (3.9), implies that

$$(3.12)\qquad \mathrm{II}\le 4^s\Big(\sum_{l=0}^{s}\Gamma(s,l)\Big)C_1 2^{-js}(1+|\xi|)^s V(B)^{-1/2}\|P_Bf\|_1.$$

Combining the estimates (3.6) and (3.12), we conclude that

$$\|P_{U_j(B)}e^{i\xi A}AP_B\|_{1\to 2}\le C_s2^{-js}(1+|\xi|)^sV(B)^{-1/2},$$

where $C_s=4^s(\sum_{l=0}^s\Gamma(s,l))C_1$.

This completes the proof of Lemma 3.4. ∎

Proof of Proposition 3.1. Inspired by [COSY20] and [CDFLY], we write $G(t) = H(-R\log t)t^{-1}$. Then $H(L) = G(A)A$, where $A = e^{-L/R}$. Using the inverse Fourier transform we get

$$H(L) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{G}(\xi)e^{i\xi A}A\,d\xi.$$

In combination with Lemma 3.4, for any $\varepsilon > 0$,

$$\begin{aligned}\|P_{U_j(B)}H(L)P_B\|_{1\to2} &\le \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|P_{U_j(B)}e^{i\xi A}AP_B\|_{1\to2}|\hat{G}(\xi)|\,d\xi \\ &\le C_s 2^{-js}V(B)^{-1/2}\int_{-\infty}^{+\infty} |\hat{G}(\xi)|(1+|\xi|)^s\,d\xi \\ &\le C_s 2^{-js}V(B)^{-1/2}\|G\|_{W^2_{s+1/2+\varepsilon}}.\end{aligned}$$

Noting that H is supported in $[R/4, R]$, we have

$$\|G\|_{W^2_{s+1/2+\varepsilon}} \le \|\delta_R H\|_{W^2_{s+1/2+\varepsilon}}.$$

This completes the proof. ∎

3.2. Related kernel estimates. Note that in Proposition 3.1, we require $s \in \mathbb{N}$. In this subsection, we extend the range of s from $\mathbb{N}$ to $\mathbb{R} > 0$ and improve the off-diagonal estimates for multipliers with compact support in Proposition 3.1 from $W^2_{s+1/2+\varepsilon}$ to $W^\infty_{s+\varepsilon}$.

First, we have the following result.

Lemma 3.5. *Suppose that L satisfies* (3.1) *for some fixed $R > 0$ and $m \ge 2$. Then there exists a constant $C > 0$ independent of R such that for all Borel functions H with* $\operatorname{supp} H \subseteq [R/4, R]$,

$$\|K_{H(L)}(\cdot, y)\|_2 \le CV(y, R^{-1/m})^{-1/2}\|H\|_\infty \quad \text{for all } y \in X.$$

Proof. See [DOS02, Lemma 2.2]. ∎

Lemma 3.6. *Let $s > 0$. Suppose that L satisfies* (3.1) *for some fixed $R > 0$ and $m \ge 2$. Then for any $\varepsilon > 0$, there exists a constant $C = C(s, \varepsilon) > 0$ independent of R such that for all Borel functions H with* $\operatorname{supp} H \subseteq [R/4, R]$,

$$\Big(\int_X |K_{H(L)}(x,y)|^2(1 + \sqrt[m]{R}\,d(x,y))^{2s}\,d\mu(x)\Big)^{1/2} \le CV(y, R^{-1/m})^{-1/2}\|\delta_R H\|_{W^\infty_{s+\varepsilon}} \quad \text{for all } y \in X.$$

Proof. Let $B := B(y, R^{-1/m})$. We decompose X into dyadic rings $\{U_j(B)\}_{j\ge4}$ and the ball $8B$. By Minkowski's inequality,

$$\Big(\int_X |K_{H(L)}(x,y)|^2(1+\sqrt[m]{R}\,d(x,y))^{2s}\,d\mu(x)\Big)^{1/2}$$
$$\le \Big(\int_{8B} |K_{H(L)}(x,y)|^2(1+\sqrt[m]{R}\,d(x,y))^{2s}\,d\mu(x)\Big)^{1/2}$$
$$+\sum_{j\ge 4}\Big(\int_{U_j(B)} |K_{H(L)}(x,y)|^2(1+\sqrt[m]{R}\,d(x,y))^{2s}\,d\mu(x)\Big)^{1/2}.$$

For the diagonal part, we note that $x\in 8B(y,R^{-1/m})$ implies the inequality $(1+\sqrt[m]{R}\,d(x,y))^{2s}\le 9^{2s}$. It follows from Lemma 3.5 that

$$\Big(\int_{8B} |K_{H(L)}(x,y)|^2(1+\sqrt[m]{R}\,d(x,y))^{2s}\,d\mu(x)\Big)^{1/2}$$
$$\le 9^s\Big(\int_{8B} |K_{H(L)}(x,y)|^2\,d\mu(x)\Big)^{1/2}$$
$$\le C9^sV(B)^{-1/2}\|\delta_R H\|_{L^\infty}\le C_sV(B)^{-1/2}\|\delta_R H\|_{W^2_{1/2+\varepsilon}},$$

where in the last inequality we use the imbedding $W^2_{1/2+\varepsilon}\subset L^\infty$.

For the off-diagonal part, we first note that for any $x\in U_j(B)$, $d(x,y)\le 2^jR^{-1/m}$, and so

$$(1+\sqrt[m]{R}\,d(x,y))^{2s}\le(1+2^j)^s. \tag{3.13}$$

From estimate (3.13), Lemma 3.3 and Proposition 3.1, it follows that

$$\Big(\int_{U_j(B)} |K_{H(L)}(x,y)|^2(1+\sqrt[m]{R}\,d(x,y))^{2s}\,d\mu(x)\Big)^{1/2} \tag{3.14}$$
$$\le(1+2^j)^s\Big(\int_{U_j(B)} |K_{H(L)}(x,y)|^2\,d\mu(x)\Big)^{1/2}$$
$$\le 2^s2^{js}\sup_{z\in B}\Big(\int_{U_j(B)} |K_{H(L)}(x,z)|^2\,d\mu(x)\Big)^{1/2}$$
$$\le 2^s2^{js}\|P_{U_j(B)}H(L)P_B\|_{1\to 2}$$
$$\le 2^sC(\tilde{s})2^{-j(\tilde{s}-s)}V(B)^{-1/2}\|\delta_R H\|_{W^2_{\tilde{s}+1/2+\varepsilon}},$$

where $\tilde{s}\in\mathbb{N}$ and $s<\tilde{s}\le s+1$. By the relation between s and $\tilde{s}$, we have

$$\|\delta_R H\|_{W^2_{\tilde{s}+1/2+\varepsilon}}\le\|\delta_R H\|_{W^2_{s+3/2+\varepsilon}}. \tag{3.15}$$

Inserting (3.15) into (3.14) and summing over j we obtain

$$\sum_{j\ge 4}\Big(\int_{U_j(B)} |K_{H(L)}(x,y)|^2(1+\sqrt[m]{R}\,d(x,y))^{2s}\,d\mu(x)\Big)^{1/2}$$
$$\le 2^sC(\tilde{s})V(B)^{-1/2}\|\delta_R H\|_{W^2_{s+3/2+\varepsilon}}.$$

Gathering the above estimates, we obtain

$$\Big(\int_X |K_{H(L)}(x,y)|^2(1+\sqrt[m]{R}\,d(x,y))^{2s}\,d\mu(x)\Big)^{1/2} \le \frac{\tilde{C}(s)\|\delta_R H\|_{W^2_{s+3/2+\varepsilon}}}{V(B)^{1/2}},$$

where $\tilde{C}(s) = 2^s C(\tilde{s}) + C_s$.

Meanwhile,

$$\|\delta_R H\|_{W^2_{s+3/2+\varepsilon}} \le \|\delta_R H\|_{W^\infty_{s+3/2+\varepsilon}},$$

since $\operatorname{supp}\delta_R H \subseteq [1/4,1]$.

As a consequence,

$$\Big(\int_X |K_{H(L)}(x,y)|^2(1+\sqrt[m]{R}\,d(x,y))^{2s}\,d\mu(x)\Big)^{1/2} \le \frac{\tilde{C}(s)\|\delta_R H\|_{W^\infty_{s+3/2+\varepsilon}}}{V(B)^{1/2}}. \tag{3.16}$$

From estimate (3.16), we see that in the kernel estimate of $K_{H(L)}(x,y)$, the required order of differentiability of the function $\delta_R H$ is $3/2$ greater than that of Lemma 3.6. To get rid of this additional $3/2$, we use the interpolation argument as in [DOS02, Lemma 4.3]. Define the linear operator $K_{y,R} : L^\infty([1/4,1]) \to L^2(X,d\mu)$ by

$$K_{y,R}(H)(x) := K_{\delta_{1/R}H}(x,y).$$

Let $d\mu_{y,s,R}(x) = (1+\sqrt[m]{R}\,d(x,y))^s\,d\mu(x)$ and let $L^2_{y,s,R}$ denote the weighted space $L^2(X, d\mu_{y,s,R})$. Then by Lemma 3.5,

$$\|K_{y,R}(H)\|_{L^2} \le CV(y,R^{-1/m})^{-1/2}\|H\|_{L^\infty([1/4,1])},$$

and by estimate (3.16),

$$\|K_{y,R}(H)\|_{L^2_{y,s,R}} \le \tilde{C}(s)V(y,R^{-1/m})^{-1/2}\|H\|_{W^\infty_{s+3/2+\varepsilon}([1/4,1])}.$$

It follows from interpolation that for all $\theta \in (0,1)$

$$\|K_{y,R}(H)\|_{L^2_{y,\theta s,R}} \le C(s,\varepsilon)V(y,R^{-1/m})^{-1/2}\|H\|_{W^\infty_{\theta s+3/2\theta+\theta\varepsilon}}.$$

Let $s' = s\theta$ and take θ small enough; we find that

$$\|K_{y,R}(H)\|_{L^2_{y,s',R}} \le C(s',\varepsilon')V(y,R^{-1/m})^{-1/2}\|H\|_{W^\infty_{s'+\varepsilon'}},$$

where $\varepsilon' = 3/2\theta + \theta\varepsilon$. Hence, we conclude that

$$\Big(\int_X |K_{H(L)}(x,y)|^2(1+\sqrt[m]{R}\,d(x,y))^{2s}\,d\mu(x)\Big)^{1/2} \le C(s,\varepsilon)V(y,R^{-1/m})^{-1/2}\|\delta_R H\|_{W^\infty_{s+\varepsilon}}.$$

The proof of Lemma 3.6 is complete. ∎

PROPOSITION 3.7. *Let $s > n/2$. Suppose that the operator L satisfies estimate* (3.1) *for some fixed $R > 0$ and $m \geq 2$. Then for any $\varepsilon > 0$, there exists a constant $C(s, \varepsilon) > 0$ independent of R such that for all Borel functions H with* $\operatorname{supp} H \subseteq [R/4, R]$,

$$\int_{d(x,y)\geq 2r} |K_{H(L)}(x,y)|\, d\mu(x) \leq C(s,\varepsilon)(1+\sqrt[m]{R}\, r)^{-s+n/2} \|\delta_R H\|_{W^\infty_{s+\varepsilon}}.$$

Proof. It is proved in [DOS02, Lemma 4.4] that for any $s > n/2$,

$$\Big(\int_{d(x,y)\geq 2r} (1+\sqrt[m]{R}\, d(x,y))^{-2s}\, d\mu(x)\Big)^{1/2} \leq CV(y, R^{-1/m})^{1/2}(1+\sqrt[m]{R}\, r)^{-s+n/2}.$$

Then combining with Lemma 3.6, we find that

$$\begin{aligned}\int_{d(x,y)\geq 2r} |K_{H(L)}(x,y)|\, d\mu(x) &\leq \Big(\int_X |K_{H(L)}(x,y)|^2 (1+\sqrt[m]{R}\, d(x,y))^{2s}\, d\mu(x)\Big)^{1/2} \\ &\quad \times \Big(\int_{d(x,y)\geq 2r} (1+\sqrt[m]{R}\, d(x,y))^{-2s}\, d\mu(x)\Big)^{1/2} \\ &\leq C(s,\varepsilon)(1+\sqrt[m]{R}\, r)^{-s+n/2}\|\delta_R H\|_{W^\infty_{s+\varepsilon}}. \qquad \blacksquare\end{aligned}$$

4. Proof of Theorem 1.2. Recall that $\rho(r) = r^2$ if $0 < r < 1$, and $\rho(r) = r^m$ if $r \geq 1$, by (2.2). Let $K_{F(L)(I-e^{-\rho(r)L})}(x,y)$ denote the kernel of the composite operator $F(L)(I - e^{-\rho(r)L})$. Note that $F(L)$ is a bounded operator on $L^2(X)$. To obtain the weak type $(1,1)$ of $F(L)$, from Proposition 2.3 it remains to show that there exists a constant $C > 0$ such that the kernel of $F(L)(I - e^{-\rho(r)L})$ satisfies

$$\int_{d(x,y)\geq 2r} |K_{F(L)(I-e^{-\rho(r)L})}(x,y)| d\mu(x) \leq C \quad \text{for all } y \in X. \tag{4.1}$$

Let us prove (4.1). To do it, we pick up a function $\varphi \in C_c^\infty(\mathbb{R})$ supported in $[1/4, 1]$ and satisfying $\sum_{k\in\mathbb{Z}} \varphi(2^{-k}t) = 1$ for all $t > 0$. Let $H_k(t) := (\delta_{2^{-k}}\varphi F)(t)(1 - e^{-\rho(r)t})$, and so

$$F(L)(I - e^{-\rho(r)L}) = \sum_{k\in\mathbb{Z}} H_k(L). \tag{4.2}$$

Obviously, for each $k \in \mathbb{Z}$, H_k is supported in $[2^k/4, 2^k]$. Let $K_{H_k(L)}(x,y)$ denote the kernel of $H_k(L)$. Noting that $s > n/2$, we take $s' > n/2$ and ε in Proposition 3.7 small enough such that $s' + \varepsilon < s$.

If $0 < 2^k \le 1$, we apply Proposition 3.7 and the heat kernel estimate (UE_m) for $t \ge 1$ to obtain

$$\int_{d(x,y)\ge 2r} |K_{H_k(L)}(x,y)|\, d\mu(x) \le C(s',\varepsilon)(1+2^{k/m}r)^{-s'+n/2}\|\delta_{2^k}H_k\|_{W^\infty_{s'+\varepsilon}} \le C(s',\varepsilon)(1+2^{k/m}r)^{-s'+n/2}\|\delta_{2^k}H_k\|_{W^\infty_s}. \tag{4.3}$$

For $2^k > 1$, we apply Proposition 3.7 and the heat kernel estimate (UE_m) for $0 < t < 1$ to obtain

$$\int_{d(x,y)\ge 2r} |K_{H_k(L)}(x,y)|\, d\mu(x) \le C(s',\varepsilon)(1+2^{k/2}r)^{-s'+n/2}\|\delta_{2^k}H_k\|_{W^\infty_s}. \tag{4.4}$$

Now we let $\tilde{\varphi} \in C_c^\infty([1/8,2])$ with $\tilde{\varphi} \equiv 1$ on the support of φ. We find that

$$\begin{aligned}\|\delta_{2^k}H_k\|_{W^\infty_s} &= \|\varphi(t)F(2^kt)(1-e^{-\rho(r)2^kt})\|_{W^\infty_s}\\ &\le \|\varphi(t)F(2^kt)\|_{W^\infty_s}\|\tilde{\varphi}(t)(1-e^{-\rho(r)2^kt})\|_{C^{[s]+1}([1/8,2])}\\ &\le C\sup_{a>0}\|\varphi\delta_aF\|_{W^\infty_s}\min\{1,2^k\rho(r)\}.\end{aligned} \tag{4.5}$$

From (4.2)–(4.5), we obtain

$$\text{LHS of (4.1)} \le C(s',\varepsilon)\sup_{a>0}\|\varphi\delta_aF\|_{W^\infty_s}\cdot E(m,r), \tag{4.6}$$

where

$$\begin{aligned}E(m,r) = &\sum_{k:\,0<2^k\le 1}(1+2^{k/m}r)^{-s'+n/2}\min\{1,2^k\rho(r)\}\\ &+\sum_{k:\,2^k>1}(1+2^{k/2}r)^{-s'+n/2}\min\{1,2^k\rho(r)\}.\end{aligned}$$

To estimate $E(m,r)$, we note that for $m \ge 2$,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}(1+2^{k/m}r)^{-s'+n/2}(\min\{1,2^{k/m}r\})^m\\ &\qquad\le \sum_{k:\,2^{k/m}r\le 1}(2^{k/m}r)^m + \sum_{k:\,2^{k/m}r\ge 1}(2^{k/m}r)^{-s'+n/2} \le C.\end{aligned}$$

Consider the following two cases: $r \ge 1$ and $0 < r < 1$.

CASE 1: $r \geq 1$. In this case, $\rho(r) = r^m$. Since $s' > n/2$, we have

$$\begin{aligned} E(m,r) \leq & \sum_{k\colon 0<2^k\leq 1} (1+2^{k/m}r)^{-s'+n/2}(\min\{1, 2^{k/m}r\})^m \\ & + r^{-s'+n/2} \sum_{k\colon 2^k>1} 2^{\frac{k}{2}(-s'+n/2)} \\ & \leq C + Cr^{-s'+n/2} \leq C. \end{aligned}$$

CASE 2: $0 < r < 1$. In this case, $\rho(r) = r^2$. Since $s' > n/2$, we have

$$\begin{aligned} E(m,r) \leq & \sum_{k\colon 0<2^k\leq 1} (2^{k/2}r)^2 + \sum_{k\colon 2^k>1} (1+2^{k/2}r)^{-s'+n/2}(\min\{1, 2^{k/2}r\})^2 \\ & \leq Cr^2 + C \leq C. \end{aligned}$$

From Cases 1 and 2 above, we know that $E(m,r) \leq C$ with $C > 0$ independent of $r > 0$ and $y \in X$. Therefore, it follows from (4.6) that

$$\int_{d(x,y)\geq 2r} |K_{F(L)(I-e^{-\rho(r)L})}(x,y)|\, d\mu(x) \leq C_s \sup_{a>0} \|\varphi\delta_a F\|_{W_s^\infty},$$

for all $y \in X$. Hence, the desired estimate (4.1) follows readily. From Proposition 2.3, we conclude that $F(L)$ is of weak type $(1,1)$ and

$$\|F(L)\|_{L^1\to L^{1,\infty}} \leq C_s\Big(\sup_{a>0} \|\varphi\delta_a F\|_{W_s^\infty} + \|F\|_\infty^2\Big).$$

By interpolation and duality, $F(L)$ is bounded on $L^p(X)$ $(1 < p < \infty)$. The proof of Theorem 1.2 is complete. ∎

As an application, we consider the imaginary powers $L^{i\beta}$, $\beta \in \mathbb{R}$, to obtain the following corollary.

COROLLARY 4.1. *Suppose that L satisfies the sub-Gaussian heat kernel estimate* (UE$_m$). *Then $L^{i\beta}$, $\beta \in \mathbb{R}$, is bounded from $L^1(X)$ to $L^{1,\infty}(X)$, and is bounded on $L^p(X)$ for $1 < p < \infty$. In addition, for any $\varepsilon > 0$, there exists a constant $C_{n,\varepsilon} > 0$ such that*

$$\begin{aligned} \|L^{i\beta}\|_{L^1\to L^{1,\infty}} & \leq C_{n,\varepsilon}\Big(\sup_{a>0} \|\varphi(t)(at)^{i\beta}\|_{W^\infty_{n/2+\varepsilon}} + 1\Big) \\ & \leq C_{n,\varepsilon}(1+|\beta|)^{n/2+\varepsilon}. \end{aligned}$$

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