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Trotter–Kato product formula in symmetric F-normed ideals

by

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Abstract. We prove the Trotter–Kato product formula for arbitrary symmetric F-normed ideals which are closed with respect to logarithmic submajorization. This class of ideals include all symmetric quasi-Banach ideals.

1. Introduction. Let A and B be (possibly unbounded) non-negative self-adjoint operators on a separable Hilbert space H. Trotter [T59] proved that if the algebraic sum $C = A + B$ is essentially self-adjoint on $\mathrm{dom}(A) \cap \mathrm{dom}(B)$, then the Trotter product formula

$$\lim_{n\to\infty} (e^{-tA/n} e^{-tB/n})^n = e^{-tC}, \quad t \geq 0, \tag{1}$$

holds, with convergence in the strong operator topology. This formula extends the Lie product formula [RS80, Theorem VIII.29] which applies to bounded operators.

Later this formula was extended by Kato [K78] to more general form sums $C = A \dot{+} B$ of two non-negative self-adjoint operators A and B on $\mathrm{dom}(C) = \overline{\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})}$. Kato showed that

$$\lim_{n\to\infty} (e^{-tA/n} e^{-tB/n})^n = e^{-tC} P_0, \quad t \geq 0, \tag{2}$$

in the strong operator topology, where the convergence is uniform in $t \in [0, T]$ for any $0 < T < \infty$, and P_0 denotes the orthogonal projection from H onto $\mathrm{dom}(C)$. Furthermore, in [K74, K78], Kato proved that (2) is also true for a wider class of so-called Kato functions (see Definition 2.10). The formulas involving Kato functions are known as *Trotter–Kato product formulas*.

A particular point of interest has been to strengthen the convergence in (2). The first such result in operator norm topology for exponential func-

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tions was proved by Rogava [R93], assuming that the algebraic sum $C = A+B$ is self-adjoint on $\mathrm{dom}(C) = \mathrm{dom}(A) \subseteq \mathrm{dom}(B)$. He proved that there exists a constant Const > 0 such that

$$\|(e^{-tA/n}e^{-tB/n})^n - e^{-tC}\|_\infty \leq \frac{\mathrm{Const}\cdot\log(n)}{\sqrt{n}}, \quad n \geq 2,$$

uniformly in $t \in [0,T]$, $0 < T < \infty$, where $\|\cdot\|_\infty$ is the operator norm.

In [NZ99], Neidhardt and Zagrebnov considered the same problem replacing the strong operator convergence in the Trotter–Kato product formula by operator norm convergence. They proved the Trotter–Kato product formula for sums of two non-negative self-adjoint operators.

The first attempt of lifting strong operator convergence in the Trotter product formula to trace norm convergence was made by Zagrebnov [Z88]. He proved the Trotter product formula for some classes of Gibbs semigroups, the ones which belong to the trace class ideal. In [NZ90, NZa90], Neidhardt and Zagrebnov proved that the Trotter–Kato product formula holds in the trace class norm when A or B generates a self-adjoint Gibbs semigroup.

First results concerning some symmetric (quasi-)normed ideals were obtained by Hiai [H97], where the Trotter product formula was proved in the Schatten ideals $\mathcal{L}_p(H)$ for $0 < p < \infty$ assuming that $e^{-A} \in \mathcal{L}_p(H)$. Hiai conjectured that the Trotter–Kato product formula holds in any fully symmetric ideal of compact operators (see [H97, Problem 3.16]). Later, in [NZa99] (see also [Za19, Chapter 6]), Neidhardt and Zagrebnov confirmed this conjecture by showing that the Trotter–Kato product formula holds in the norm of any fully symmetric normed ideal $\mathcal{L}_\phi(H)$ of compact operators away from $t_0 > 0$ if the so-called Kac operator (the transfer matrix) $e^{-tB/2}e^{-tA}e^{-tB/2}$ belongs to this ideal for some $t = t_0$, i.e.,

$$\begin{aligned} &\lim_{n\to\infty}(e^{-tA/n}e^{-tB/n})^n = e^{-tC}P_0, \\ &\lim_{r\to\infty}(e^{-tB/2r}e^{-tA/r}e^{-tB/2r})^r = e^{-tC}P_0, \end{aligned} \tag{3}$$

where the limit is in the norm of the fully symmetric normed ideal. Moreover, they proved similar formulas for the whole class of Kato functions, and additionally that the functions $f(\cdot)$, $g(\cdot)$ and operators A, B are interchangeable in their respective formulas. Later, in [Z19], the convergence of the Trotter–Kato product formula was proved for the Dixmier ideal $\mathcal{M}_{1,\infty}(H)$.

In this paper, using the methods from [NZa99] and the notion of the closedness of an ideal with respect to logarithmic submajorization, we extend the results of Hiai [H97] and Neidhardt and Zagrebnov [NZa99] (see, e.g., Examples 2.4 and 3.8). Namely, we prove the Trotter–Kato product formula in an arbitrary symmetric F-normed ideal which is closed with respect to logarithmic submajorization (see Theorem 3.6). This class of ideals includes all symmetric quasi-Banach ideals, so that the Trotter–Kato product formula holds in any such ideal. We also consider error bounds for such formulas and show some examples similar to [NZa99, Theorem 5.1] (see Proposition 4.3 and Corollary 4.4).

The paper is organized as follows. In Section 2, we present necessary preliminaries and notions and in Section 3 we prove our main result, the convergence of the Trotter–Kato product formula in symmetric F-normed ideals. Finally, in Section 4 we investigate error bounds for such product formulas and present some examples where they can be computed directly.

2. Preliminaries. Let H be a separable Hilbert space and $\mathcal{L}(H)$ be the C^*-algebra of all bounded linear operators on H equipped with the uniform norm $\|\cdot\|_\infty$. Denote by $\mathcal{L}_\infty(H)$ the ideal of all compact operators on H and for any $X \in \mathcal{L}_\infty(H)$ denote by $\{s_j(X)\}_{j\geq 1}$ the singular values of the operator X, i.e. the eigenvalues $\{\lambda_j(|X|)\}_{j\geq 1}$ of the operator $|X| = (X^*X)^{1/2}$ arranged in decreasing order, and counting multiplicities. If $\{e_j\}_{j\geq 1}$ is an orthonormal basis in H, then, for any bounded sequence $a = \{a_j\}_{j\geq 1}$, we can define the diagonal operator $\operatorname{diag}(a) = \sum_{j\geq 1} a_j \langle \cdot, e_j\rangle e_j$ on H. In the following we use Const to denote a positive constant which may vary from line to line, and we denote the integer and fractional parts of a real number $r \in \mathbb{R}_+$ by $[r]$ and $\{r\}$, respectively. For bounded operators $\{X_n\}_{n\geq 1}$ and X, we denote by $\operatorname{s-lim}_{n\to\infty} X_n = X$ the convergence of the sequence $\{X_n\}_{n\geq 1}$ to an operator X in the strong operator topology.

2.1. Symmetric F-normed ideals and logarithmic submajorization. We recall the definition of F-norm.

Definition 2.1. Let Ω be a linear space over $\mathbb{C}$. A function $\|\cdot\|$ from Ω to $[0, \infty)$ is called an *F-norm* if for any $x, y \in \Omega$ the following conditions hold:

(i) $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$;
(ii) $\|\alpha x\| \leq \|x\|$ for any $\alpha \in \mathbb{C}$ such that $|\alpha| \leq 1$;
(iii) $\lim_{\alpha\to 0} \|\alpha x\| = 0$;
(iv) $\|x + y\| \leq \|x\| + \|y\|$.

The couple $(\Omega, \|\cdot\|)$ is called an *F-normed space*. Note that if $\|\cdot\|$ is an F-norm on Ω, then it induces a vector topology which is metrizable [KPR84, Chapter 1], and conversely, if Ω is metrizable then it can be equipped with an F-norm (see e.g. [K69, Chapter 3], [KPR84, Chapter 1]). In particular, any quasi-normed space is an F-normed space [KPR84, Chapter 1, Section 3].

Remark 2.2. To avoid confusion we do not use the notion of F-space for complete F-normed spaces, since the notion of F-space often refers to complete metrizable locally convex spaces. In this paper, we do not assume any local convexity.

We now define symmetric F-normed ideals.

DEFINITION 2.3. Let $\mathcal{I}(H)$ be a two-sided ideal in $\mathcal{L}(H)$ equipped with an F-norm $\|\cdot\|_{\mathcal{I}}$. We say that $\mathcal{I}(H)$ is a *symmetric F-normed ideal* if $X \in \mathcal{I}(H)$, $Y \in \mathcal{L}(H)$ and $s_j(Y) \le s_j(X)$ for any $j \in \mathbb{N}$ imply that $Y \in \mathcal{I}(H)$ and $\|Y\|_{\mathcal{I}} \le \|X\|_{\mathcal{I}}$. When $\|\cdot\|_{\mathcal{I}}$ is a (quasi-)norm, we say that $\mathcal{I}(H)$ is *symmetric (quasi-)normed ideal.*

EXAMPLE 2.4. (i) ([DSZ16, Section 4]) A simple example of a complete symmetric F-normed ideal is the trace class ideal $\mathcal{L}_1(H)$ equipped with the functional

$$\|X\|_{\log} := \sum_{k=1}^{\infty} \log(1 + s_k(X)), \quad X \in \mathcal{L}_1(H).$$

(ii) ([LSZ13, Example 1.2.6]) Another important example is the weak-l_p ideal $\mathcal{L}_{p,\infty}(H)$ for $0 < p < \infty$, defined as

$$\mathcal{L}_{p,\infty}(H) := \Big\{ X \in \mathcal{L}_\infty(H) : \sup_{k\ge 1} k^{1/p} s_k(X) < \infty \Big\}$$

equipped with

$$\|X\|_{p,\infty} := \sup_{k\ge 1} k^{1/p} s_k(X), \quad X \in \mathcal{L}_{p,\infty},$$

which is a symmetric Banach ideal for $1 < p < \infty$ (a symmetric quasi-Banach ideal for $0 < p \le 1$). It is well known that $\mathcal{L}_{p,\infty}(H)$ fails to be normable for $0 < p \le 1$ (see, for example, [P09, p. 210] and [H66, pp. 259–260]). Hence, the class of ideals considered in this paper contains some important examples beyond the results of [H97] and [NZa99].

(iii) Now we present an example of a symmetric F-normed ideal which cannot be equipped with an equivalent quasi-norm. Let $\{t_k\}_{k\ge 1}$ be a sequence of positive numbers such that t_1 is an arbitrary positive number and t_n is a positive solution of the equation $t_n^2 + t_n = t_{n-1}^2$ for each $n \ge 2$. Define a function $M(t)$ on $[0, \infty)$ by

$$M(t) = \begin{cases} \frac{t^2(t^2+t)}{t_{2k}^2}, & t_{2k+1} \le t < t_{2k},\ k \ge 1, \\ t_{2k+1}^2, & t_{2k+2} \le t < t_{2k+1},\ k \ge 0, \\ t^2, & t \ge t_1, \end{cases}$$
$$M(0) = 0.$$

Then the Orlicz sequence space l^M, defined as

$$l^M = \Big\{ x = \{x_n\}_{n\ge 1} \in l_\infty : \sum_{n=1}^{\infty} M(x_n) < \infty \Big\}$$

equipped with

$$\|x\|_{l^M} = \inf \Big\{\varepsilon > 0 : \sum_{n=1}^{\infty} M(x_n/\varepsilon) < \varepsilon\Big\},$$

is a complete F-normed Calkin sequence space (for details, see [R59, Section 4], [MO58, Theorem 2] and [LSZ13, Section 2.4]). Moreover, as shown in [R59, Section 4] the function M does not satisfy condition (b) of [R59, Theorem 1]. Hence, by [R59, Theorem 1], the space l^M is not locally bounded, that is, it cannot be equipped with an equivalent quasi-norm (see e.g. [KPR84, Chapter I.3] or [K69, Chapter 3, Section 15.10]). Passing to operator ideals, [HLS17, Theorem 3.8] guarantees that the corresponding Orlicz ideal

$$\mathcal{L}^M := \big\{X \in \mathcal{L}_\infty(H) : s(X) := \{s_n(X)\}_{n\geq 1} \in l^M\big\}$$

equipped with

$$\|X\|_{\mathcal{L}^M} = \inf \Big\{\varepsilon > 0 : \sum_{n=1}^{\infty} M(s_n(X)/\varepsilon) < \varepsilon\Big\}, \quad X \in \mathcal{L}^M,$$

is a complete symmetric F-normed ideal which cannot be equipped with a quasi-norm (since otherwise the corresponding sequence space l^M would be quasi-normed too [HLS17, Theorem 3.9]).

For convenience, we assume that $\|U\|_{\mathcal{I}} = \|U\|_\infty$ for any rank-one operator from $\mathcal{L}(H)$. As in the case of symmetric normed ideals (see e.g. [GK69, Chapter III, Paragraph 2]) one can show that any F-norm is unitarily invariant and $\|X^*\|_{\mathcal{I}} = \|X\|_{\mathcal{I}}$. Note that since $s_j(XY) \leq \|X\|_\infty s_j(Y) = s_j(\|X\|_\infty Y)$ for all $j \in \mathbb{N}$, $X \in \mathcal{L}(H)$ and $Y \in \mathcal{I}(H)$, it follows that

$$\|XY\|_{\mathcal{I}} \leq \big\|\|X\|_\infty Y\big\|_{\mathcal{I}} \leq \lceil \|X\|_\infty \rceil \|Y\|_{\mathcal{I}}, \tag{4}$$

where the second inequality follows from the repeated use of the triangle inequality and $\lceil\cdot\rceil$ denotes the ceiling function.

For any $X, Y \in \mathcal{L}_\infty(H)$ we say that Y is *logarithmically submajorized* by X (denoted by $Y \prec\prec_{\log} X$) if

$$\prod_{j=1}^{k} s_j(Y) \leq \prod_{j=1}^{k} s_j(X), \quad \forall k \in \mathbb{N}.$$

Definition 2.5. A symmetric F-normed ideal $\mathcal{I}(H)$ is said to be *closed with respect to logarithmic submajorization* if

(i) $X \in \mathcal{I}(H)$, $Y \in \mathcal{L}(H)$ and $Y \prec\prec_{\log} X$ imply that $Y \in \mathcal{I}(H)$;
(ii) there is a constant $C_{\mathcal{I}} > 0$ such that $\|Y\|_{\mathcal{I}} \leq C_{\mathcal{I}} \|X\|_{\mathcal{I}}$ for any $X, Y \in \mathcal{I}(H)$ with $Y \prec\prec_{\log} X$.

We note that our definition of closedness with respect to logarithmic submajorization is stronger than that in [SZ14, Definition 6] and includes an inequality for the F-norm (ii) of Definition 2.5.

Remark 2.6. If $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ is a symmetric quasi-Banach ideal, then $\mathcal{I}(H)$ is closed with respect to logarithmic submajorization. Indeed, by [K98, Proposition 3.2], any symmetric quasi-Banach ideal is geometrically stable, i.e., if $X \in \mathcal{I}(H)$, then

$$\overline{\operatorname{diag}}(s_j(X)) \\ := \operatorname{diag}(s_1(X), (s_1(X)s_2(X))^{1/2}, \dots, (s_1(X)\cdots s_j(X))^{1/j}, \dots) \in \mathcal{I}(H),$$

and

$$\|\overline{\operatorname{diag}}(s_j(X))\|_{\mathcal{I}} \le \operatorname{Const} \|X\|_{\mathcal{I}} \tag{5}$$

for some constant Const > 0 depending only on the modulus of concavity of the quasi-norm. Furthermore, by [SZ14, Lemma 35], any geometrically stable ideal satisfies condition (i) of Definition 2.5. Since $s_j(Y) \le (\prod_{k=1}^{j} s_k(Y))^{1/j}$, $j \ge 1$, the symmetry of the quasi-norm $\|\cdot\|_{\mathcal{I}}$ and (5) imply that

$$\|Y\|_{\mathcal{I}} \le \|\overline{\operatorname{diag}}(s_j(Y))\|_{\mathcal{I}} \le \|\overline{\operatorname{diag}}(s_j(X))\|_{\mathcal{I}} \le \operatorname{Const} \|X\|_{\mathcal{I}}$$

provided that $Y \prec\prec_{\log} X$. Thus, condition (ii) of Definition 2.5 is satisfied too.

However, Remark 2.6 is no longer true for a general symmetric F-normed ideal (even when the ideal $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ is complete). To demonstrate this we present the following example.

Example 2.7. Consider the ideal of trace class operators

$$\mathcal{L}_1(H) = \Big\{ X \in \mathcal{L}_\infty(H) : \sum_{j=1}^{\infty} s_j(X) < \infty \Big\},$$

and the functional $\|\cdot\|_{\log\log}$ defined as

$$\|X\|_{\log\log} = \sum_{j=1}^{\infty} \log(1 + \log(1 + s_j(X))), \quad \forall X \in \mathcal{L}(H).$$

One can easily show that $\|\cdot\|_{\log\log}$ is finite on $\mathcal{L}_1(H)$. Futhermore, an argument similar to [DSZ16, Lemma 4.1] shows that $(\mathcal{L}_1(H), \|\cdot\|_{\log\log})$ is a symmetric F-normed ideal, which is complete. From the fact that any quasi-Banach ideal is closed with respect to logarithmic submajorization (see Remark 2.6) we know that $\mathcal{L}_1(H)$ satisfies condition (i) of Definition 2.5, i.e. $Y \prec\prec_{\log} X$, $X \in \mathcal{L}_1(H)$ implies $Y \in \mathcal{L}_1(H)$. For any $n \in \mathbb{N}$ consider the finite-rank operators

$$Y_n = \operatorname{diag}(\underbrace{e^n, \dots, e^n}_{n \text{ times}}, 0, \dots), \quad X_n = \operatorname{diag}(e^{n^2}, \underbrace{1, \dots, 1}_{n-1 \text{ times}}, 0, \dots).$$

It is easy to see that $Y_n \prec\prec_{\log} X_n$, $n \in \mathbb{N}$, and

$$\frac{\|Y_n\|_{\log\log}}{\|X_n\|_{\log\log}} = \frac{n\cdot\log(1+\log(1+e^n))}{\log(1+\log(1+e^{n^2}))+(n-1)\log(1+\log(2))} \to \infty, \quad n \to \infty.$$

Therefore, there is no constant $C_{\mathcal{I}} > 0$ such that $\|Y_n\|_{\log\log} \le C_{\mathcal{I}}\|X_n\|_{\log\log}$ for all $n \in \mathbb{N}$.

We recall two lemmas that we shall use.

LEMMA 2.8 ([NZa99, Lemma 2.5]). *Let X, X_0, Y, Y_0 be non-negative self-adjoint compact operators such that $X^r \le X_0$ and $Y^r \le Y_0$ for some $r \ge 1$. Then*

$$(Y^{1/2}XY^{1/2})^r \prec\prec_{\log} Y_0^{1/2}X_0Y_0^{1/2}. \tag{6}$$

LEMMA 2.9 ([H95, Theorem 2.3]). *Let X and Y be non-negative self-adjoint operators on a separable Hilbert space H and let $Z = X \dot{+} Y$ be their form sum. Then*

$$e^{-tZ} \prec\prec_{\log} (e^{-tY/2r}e^{-tX/r}e^{-tY/2r})^r, \quad t \ge 0.$$

2.2. Kato functions and the Trotter–Kato product formula. Next we recall the definition of Kato functions and dominated Kato functions.

DEFINITION 2.10. [NZa99] A Borel measurable function f defined on $[0,\infty)$ is called a *Kato function* if

$$0 \le f(x) \le 1, \quad f(0) = 1, \quad f'(0) = -1.$$

A Kato function f is called *regular* if $0 \le \sup_{s\in[x,\infty)} f(s) < 1$ for $x > 0$ and

$$\lim_{x\to\infty} \frac{\sup_{0\le s\le x} sf(s)}{x} = 0.$$

For example, the exponential function $f(x) = e^{-x}$, $x \ge 0$, is a regular Kato function.

DEFINITION 2.11. [NZa99] Let $f^D : [0,\infty) \to [0,\infty)$ be a Borel measurable function. A Kato function f is said to be *dominated* by f^D if for any $x \ge 0$ and $0 < q \le 1$ one has

$$f(qx)^{1/q} \le f^D(x).$$

Let A and B be (possibly unbounded) non-negative self-adjoint operators on a separable Hilbert space H. Throughout this paper we denote by $C = A \dot{+} B$ the form sum of A and B which is well-defined on $H_0 = \overline{\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})}$. Denote by P_0 the orthogonal projection from H onto H_0. Let $E_C(\cdot)$ be the spectral measure of the operator C. Now, following [NZa99], we define the notion of convergence of the Trotter–Kato product formula.

Definition 2.12. Let operators A, B and their form sum C be as above. Let also f, g be Kato functions and $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ be a symmetric F-normed ideal of compact operators. Then we say that

(i) *the Trotter–Kato product formula for the family* $\{f(tA)g(tB)\}_{t\geq 0}$ *converges locally uniformly away from* $t_0 > 0$ in $\mathcal{I}(H)$ if for any bounded interval $[\tau_0, \tau] \subset (t_0, \infty)$ there is a natural number $n_0 \geq 1$ such that for any $t \in [\tau_0, \tau]$ and $n \geq n_0$,

$$e^{-tC} \in \mathcal{I}(H_0), \quad (f(tA/n)g(tB/n))^n \in \mathcal{I}(H),$$

and we have the convergence

$$\lim_{n\to\infty} \sup_{t\in[\tau_0,\tau]} \|(f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}} = 0.$$

(ii) *the Trotter–Kato product formula for the symmetrized family* $\{g(tB)^{1/2} f(tA)g(tB)^{1/2}\}_{t\geq 0}$ *converges locally uniformly away from* $t_0 > 0$ in $\mathcal{I}(H)$ if for any bounded interval $[\tau_0, \tau] \subset (t_0, \infty)$ there is $r_0 \geq 1$ such that for any $t \in [\tau_0, \tau]$ and $r \geq r_0$,

$$e^{-tC} \in \mathcal{I}(H_0), \quad (g(tB/r)^{1/2} f(tA/r)g(tB/r)^{1/2})^r \in \mathcal{I}(H),$$

and we have the convergence

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|(g(tB/r)^{1/2} f(tA/r)g(tB/r)^{1/2})^r - e^{-tC}P_0\|_{\mathcal{I}} = 0.$$

In a similar way, one can define the notion of convergence for the families $\{g(tB)f(tA)\}_{t\geq 0}$ and $\{f(tA)^{1/2}g(tB)f(tA)^{1/2}\}_{t\geq 0}$ by exchanging f with g and A with B. Moreover, if the convergence holds for all these families, then we say that the Trotter–Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$ *for all families generated by* f *and* g.

When $\|\cdot\|_{\mathcal{I}}$ is the operator norm $\|\cdot\|_\infty$ we say that the Trotter–Kato product formula converges locally uniformly away from $t_0 > 0$ *in the operator norm.*

3. The Trotter–Kato product formula in symmetric F-normed ideals. In this section we present the main result of this paper. Namely, we prove the convergence of the Trotter–Kato product formula in a symmetric F-normed ideal closed with respect to the logarithmic submajorization. Let f and g be Kato functions. Introduce operator-valued functions $F(t)$ and $G(t)$ by setting

$$\begin{aligned} F(t) &= g(tB)^{1/2} f(tA)g(tB)^{1/2}, \quad t \geq 0, \\ G(t) &= f(tA)^{1/2} g(tB)f(tA)^{1/2}, \quad t \geq 0. \end{aligned} \tag{7}$$

First we demonstrate that it is sufficient to consider the convergence of the Trotter–Kato product formula for just one of the families $\{F(t)\}_{t\geq 0}$, $\{G(t)\}_{t\geq 0}$, $\{f(tA)g(tB)\}_{t\geq 0}$ and $\{g(tB)f(tA)\}_{t\geq 0}$. We start with an extension of the lemma given in [NZa99, Lemma 2.6] to the case of symmetric F-normed ideals.

LEMMA 3.1. *Let $X \in \mathcal{I}(H)$, $Y \in \mathcal{L}_\infty(H)$ and $Z \in \mathcal{L}(H)$ be self-adjoint operators, where $\mathcal{I}(H)$ is a symmetric F-normed ideal in $\mathcal{L}(H)$. If $\{Z(t)\}_{t\geq 0}$ is a family of self-adjoint bounded operators such that* $\text{s-lim}_{t\to+0} Z(t) = Z$, *then*

$$\lim_{r\to\infty} \sup_{t\in[0,T]} \|(Z(t/r) - Z)YX\|_{\mathcal{I}} = \lim_{r\to\infty} \sup_{t\in[0,T]} \|XY(Z(t/r) - Z)\|_{\mathcal{I}} = 0$$

for any $T \in (0, \infty)$.

Proof. Fix $T \in (0, \infty)$ and $\varepsilon > 0$. It is sufficient to prove

$$\lim_{r\to\infty} \sup_{t\in[0,T]} \|(Z(t/r) - Z)YX\|_{\mathcal{I}} = 0,$$

since the other equality can be showed by taking adjoints. Since Y is a compact operator, it can be represented as $Y = Y_1 + Y_2$, where Y_1 is a finite-rank operator and $\|Y_2\|_\infty < \delta$ for a given $\delta > 0$. Then, by the triangle inequality,

$$\sup_{t\in[0,T]} \|(Z(t/r) - Z)YX\|_{\mathcal{I}} \leq \sup_{t\in[0,T]} \|(Z(t/r) - Z)Y_1X\|_{\mathcal{I}} + \sup_{t\in[0,T]} \|(Z(t/r) - Z)Y_2X\|_{\mathcal{I}}. \tag{8}$$

We consider the two terms on the right hand side of (8) separately. Writing $Y_1 = \sum_{k=1}^m \langle\cdot, \xi_k\rangle\eta_k$ for some $\{\xi_k\}_{k=1}^m, \{\eta_k\}_{k=1}^m \subset H$ and $m \in \mathbb{N}$, and using the triangle inequality repeatedly, we have

$$\begin{aligned}\sup_{t\in[0,T]} \|(Z(t/r) - Z)Y_1X\|_{\mathcal{I}} &\leq \sum_{k=1}^m \sup_{t\in[0,T]} \|(Z(t/r) - Z)(\langle\cdot, \xi_k\rangle\eta_k)X\|_{\mathcal{I}} \\ &= \sum_{k=1}^m \sup_{t\in[0,T]} \|(Z(t/r) - Z)(\langle\cdot, \xi_k\rangle\eta_k)X\|_\infty \\ &\leq \|X\|_\infty \sum_{k=1}^m \sup_{t\in[0,T]} \|(Z(t/r) - Z)\eta_k\|.\end{aligned}$$

Since $\text{s-lim}_{t\to\infty} Z(t) = Z$, there exist $r_k \in \mathbb{R}$, $k = 1, \ldots, m$, such that

$$\sup_{t\in[0,T]} \|(Z(t/r_k) - Z)\eta_k\| < \frac{\varepsilon}{2m\|X\|_\infty}.$$

Setting $R_1 = \max_{1\le k\le m} r_k$, for any $r \ge R_1$ we have

$$\sup_{t\in[0,T]} \|(Z(t/r) - Z)Y_1X\|_{\mathcal{I}} < \varepsilon/2. \tag{9}$$

Now we consider the second term on the right hand side of (8). Since $\text{s-lim}_{t\to+0} Z(t) = Z$, it follows that $\text{s-lim}_{r\to\infty} Z(t/r) = Z$ uniformly in $t \in [0,T]$. Therefore, there exists a constant $C > 0$ and a large enough $R_2 \in \mathbb{R}_+$ such that $\sup_{t\in[0,T]} \|Z(t/r) - Z\|_\infty \le C$ for any $r \ge R_2$. Then, for a given $\varepsilon > 0$, using axiom (iii) of the F-norm, we can choose $\delta > 0$ such that $\|C\delta X\|_{\mathcal{I}} < \varepsilon$. By the symmetry of the F-norm, (4) and the choice of the operator Y_2, we have

$$\begin{aligned}\sup_{t\in[0,T]} \|(Z(t/r) - Z)Y_2X\|_{\mathcal{I}} &\le \sup_{t\in[0,T]} \big\|\|(Z(t/r) - Z)\|_\infty \|Y_2\|_\infty X\big\|_{\mathcal{I}} \\ &\le \|C\delta X\|_{\mathcal{I}} < \varepsilon/2\end{aligned} \tag{10}$$

for any $r \ge R_2$. Therefore, since $\varepsilon > 0$ is arbitrary, combining (9) and (10) with $r \ge \max\{R_1, R_2\}$, we conclude the proof. ∎

Now we prove a result similar to [NZa99, Proposition 3.1] that establishes equivalence of the convergences of the Trotter–Kato product formula for different families considered in Definition 2.12. We recall that $\{F(t)\}_{t\ge0}$, $\{G(t)\}_{t\ge0}$ are defined in (7).

Proposition 3.2. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H, and $\mathcal{I}(H)$ be any symmetric F-normed ideal closed with respect to logarithmic submajorization. Let f and g be Kato functions. Set*

$$\begin{aligned} F_1 &= \{F(t)\}_{t\ge0}, & F_3 &= \{f(tA)g(tB)\}_{t\ge0},\\ F_2 &= \{G(t)\}_{t\ge0}, & F_4 &= \{g(tB)f(tA)\}_{t\ge0}.\end{aligned}$$

Then the following assertions (A_i) *are equivalent for* $i = 1, 2, 3, 4$*:*

(A_i) *The Trotter–Kato product formula for the family F_i converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$.*

Proof. First we show that (A_1) implies (A_2). Fix a bounded interval $[\tau_0, \tau] \subset (t_0, \infty)$. Choose an interval $[a, b] \subset (t_0, \infty)$ such that $[\tau_0, \tau] \subset (a, b)$. Assume $r \in \mathbb{R}_+$ is decomposed as $r = [r] + \{r\}$ into its integer and fractional parts. For any $t > 0$ we can write

$$G(t/r)^r = G(t/r)^{\{r\}} f(tA/r)^{1/2} g(tB/r)^{1/2} F(t/r)^{[r]-1} g(tB/r)^{1/2} f(tA/r)^{1/2}. \tag{11}$$

Note that $F(t/r)^{[r]-1} = F(\theta/([r] - 1))^{[r]-1}$, where $\theta = t([r] - 1)/r$. By (A_1), for the interval $[a, b]$ we can find $R_1 \in \mathbb{N}$ such that $F(t/r)^{[r]-1} = F(\theta/([r]-1))^{[r]-1} \in \mathcal{I}(H)$ for any $\theta \in [a, b]$ and $[r] \ge R_1$. Note that $\theta \in [a, b]$ is equivalent to $t \in [ra/([r] - 1), rb/([r] - 1)]$. However, since $[\tau_0, \tau] \subset (a, b)$

and $\frac{r}{[r]-1} \to 1$ as $r \to \infty$, it follows that there exists a large enough $R_2 \in \mathbb{R}_+$ such that

$$[\tau_0, \tau] \subset [ra/([r]-1), b] \subseteq [ra/([r]-1), rb/([r]-1)]$$

for any $r \geq R_2$. Therefore, if $r \geq \max\{R_1, R_2\}$, then $F(t/r)^{[r]-1} \in \mathcal{I}(H)$ for any $t \in [\tau_0, \tau]$ and $r \geq R_{\max}$. Thus, (11) implies that $G(t/r)^r \in \mathcal{I}(H)$ for any $t \in [\tau_0, \tau]$ and $r \geq \max\{R_1, R_2\}$. Similarly, since $e^{-\theta C} \in \mathcal{I}(H_0)$ for any $\theta \in [a, b]$, it follows that $e^{-tC} \in \mathcal{I}(H_0)$ for any $t \in [\tau_0, \tau]$.

It remains to show the convergence from Definition 2.12(ii). Note that, for any $t \geq 0$ and $r \in \mathbb{R}_+$,

$$\begin{aligned}
&G(t/r)^r - e^{-tC}P_0 \\
&\quad = G(t/r)^{\{r\}} f(tA/r)^{1/2} g(tB/r)^{1/2} [F(t/r)^{[r]-1} - e^{-tC}P_0] \\
&\qquad\qquad \cdot g(tB/r)^{1/2} f(tA/r)^{1/2} \\
&\qquad + G(t/r)^{\{r\}} f(tA/r)^{1/2} g(tB/r)^{1/2} e^{-tC} P_0 [g(tB/r)^{1/2} - I] f(tA/r)^{1/2} \\
&\qquad + G(t/r)^{\{r\}} f(tA/r)^{1/2} g(tB/r)^{1/2} e^{-tC} P_0 [f(tA/r)^{1/2} - I] \\
&\qquad + G(t/r)^{\{r\}} f(tA/r)^{1/2} [g(tB/r)^{1/2} - I] e^{-tC} P_0 \\
&\qquad + G(t/r)^{\{r\}} [f(tA/r)^{1/2} - I] e^{-tC} P_0 + [G(t/r)^{\{r\}} - I] e^{-tC} P_0.
\end{aligned}$$

Therefore, using the triangle inequality and (4), noting that

$$\left\| G\left(\frac{t}{r}\right)^{\{r\}} \right\|_\infty \leq 1, \quad \left\| f\left(\frac{tA}{r}\right)^{1/2} \right\|_\infty \leq 1 \quad \text{and} \quad \left\| g\left(\frac{tB}{r}\right)^{1/2} \right\|_\infty \leq 1$$

for any $t > 0$ and $r \in \mathbb{R}_+$, we have

$$\begin{aligned}
(12) \qquad & \sup_{t \in [\tau_0, \tau]} \| G(t/r)^r - e^{-tC} P_0 \|_{\mathcal{I}} \\
\leq & \sup_{t \in [\tau_0, \tau]} \| F(t/r)^{[r]-1} - e^{-t([r]-1)C/r} P_0 \|_{\mathcal{I}} \\
& + \sup_{t \in [\tau_0, \tau]} \| e^{-t([r]-1)C/r} P_0 - e^{-tC} P_0 \|_{\mathcal{I}} + \sup_{t \in [\tau_0, \tau]} \| e^{-tC} P_0 (I - g(tB/r)^{1/2}) \|_{\mathcal{I}} \\
& + \sup_{t \in [\tau_0, \tau]} \| e^{-tC} P_0 (I - f(tA/r)^{1/2}) \|_{\mathcal{I}} + \sup_{t \in [\tau_0, \tau]} \| (I - g(tB/r)^{1/2}) e^{-tC} P_0 \|_{\mathcal{I}} \\
& + \sup_{t \in [\tau_0, \tau]} \| (I - f(tA/r)^{1/2}) e^{-tC} P_0 \|_{\mathcal{I}} + \sup_{t \in [\tau_0, \tau]} \| (I - G(t/r)^{\{r\}}) e^{-tC} P_0 \|_{\mathcal{I}}.
\end{aligned}$$

Since $[\tau_0, \tau] \subset (a, b)$ there exists a small enough $\delta > 0$ such that $\tau_0 - \delta > a$ and $e^{-tC}P_0 = e^{-(\tau_0-\delta)C} e^{-(t-\tau_0+\delta)C} P_0$, where $e^{-(\tau_0-\delta)C} \in \mathcal{I}(H_0)$ and $e^{-(t-\tau_0+\delta)C} P_0 \in \mathcal{L}_\infty(H)$. Moreover, the spectral theorem implies that $\text{s-lim}_{t\to+0} f(tA)^{1/2} = I$, $\text{s-lim}_{t\to+0} g(tB)^{1/2} = I$, and additionally s-$\lim_{t\to+0} G(t) = I$. Therefore, by Lemma 3.1,

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|e^{-tC}P_0(I - g(tB/r)^{1/2})\|_{\mathcal{I}} = \lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|(I - g(tB/r)^{1/2})e^{-tC}P_0\|_{\mathcal{I}} = 0 \tag{13}$$

and

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|e^{-tC}P_0(I - f(tA/r)^{1/2})\|_{\mathcal{I}} = \lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|(I - f(tA/r)^{1/2})e^{-tC}P_0\|_{\mathcal{I}} = 0 \tag{14}$$

and

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|(I - G(t/r)^{\{r\}})e^{-tC}P_0\|_{\mathcal{I}} \leq \lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|(I - G(t/r))e^{-tC}P_0\|_{\mathcal{I}} = 0, \tag{15}$$

where the last inequality follows from the symmetry of the F-norm.

Therefore, using (13)–(15) and (12), we obtain

$$\begin{aligned}\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|G(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} &\leq \lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|F(t/r)^{[r]-1} - e^{-t([r]-1)C/r}P_0\|_{\mathcal{I}} \\ &+ \lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|e^{-t([r]-1)C/r}P_0 - e^{-tC}P_0\|_{\mathcal{I}}.\end{aligned} \tag{16}$$

We estimate the two terms on the right hand side of (16) separately. Since the Trotter–Kato product formula converges for the family $\{F(t)\}_{t\geq 0}$ and the interval $[a,b]$, we have

$$\lim_{r\to\infty} \sup_{\theta\in[a,b]} \|F(\theta/([r]-1))^{[r]-1} - e^{-\theta C}P_0\|_{\mathcal{I}} = 0, \tag{17}$$

which is equivalent to

$$\lim_{r\to\infty} \sup_{t([r]-1)/r\in[a,b]} \|F(t/r)^{[r]-1} - e^{-t([r]-1)C/r}P_0\|_{\mathcal{I}} = 0,$$

and hence

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|F(t/r)^{[r]-1} - e^{-t([r]-1)C/r}P_0\|_{\mathcal{I}} = 0, \tag{18}$$

since $[\tau_0,\tau] \subset [ra/([r]-1), rb/([r]-1)]$ for $r \geq \max\{R_1, R_2\}$. Hence, the first term on the right hand side of (16) is zero. For the second term, since $[\tau_0,\tau] \subset (a,b)$ and $([r]-1)/r \to 1$ as $r \to \infty$, we can find $\varepsilon > 0$ and $R_3 \in \mathbb{R}_+$ such that $t([r]-1)/r - \varepsilon \in [a,b]$ for any $t \in [\tau_0,\tau]$ and $r \geq R_3$. Hence, $e^{-t([r]-1)C/r}P_0 =$

$e^{-(t([r]-1)/r-\varepsilon)C}P_0e^{-\varepsilon C}P_0$, where $e^{-(t([r]-1)/r-\varepsilon)C}P_0 \in \mathcal{I}(H)$ and $e^{-\varepsilon C}P_0 \in \mathcal{L}_\infty(H)$. Therefore, by Lemma 3.1,

$$\begin{aligned}\lim_{r\to\infty}\sup_{t\in[\tau_0,\tau]} &\|e^{-t([r]-1)C/r}P_0 - e^{-tC}P_0\|_{\mathcal{I}} \\ &= \lim_{r\to\infty}\sup_{t\in[\tau_0,\tau]} \|e^{-(t([r]-1)/r-\varepsilon)C}P_0e^{-\varepsilon C}P_0(I - e^{-(t-t([r]-1)/r))C}P_0)\|_{\mathcal{I}} = 0,\end{aligned}$$

which together with (18) applied to (16) proves the desired convergence. Therefore, for an arbitrary interval $[\tau_0,\tau]$ we find a number $R_{\max} := \max\{R_1, R_2,\ R_3\}$ such that the Trotter–Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$ for the family $\{G(t)\}_{t\geq 0}$.

Note that we have the equalities

$$\begin{aligned}(f(tA/n)g(tB/n))^n &= f(tA/n)^{1/2}G(t/n)^{n-1}f(tA/n)^{1/2}g(tB/n),\\ (g(tB/n)f(tA/n))^n &= g(tB/n)(f(tA/n)g(tB/n))^{n-1}f(tA/n)\end{aligned}$$

and

$$F(t/r)^r = F(t/r)^{\{r\}}g(tB/r)^{1/2}f(tA/r)(g(tB/r)f(tA/r))^{[r]-1}g(tB/r)^{1/2}.$$

Therefore, the proofs of implications $(\mathrm{A}_2)\Rightarrow(\mathrm{A}_3)$, $(\mathrm{A}_3)\Rightarrow(\mathrm{A}_4)$ and $(\mathrm{A}_4)\Rightarrow(\mathrm{A}_1)$ are similar. ■

Next we present a lifting result similar to [NZa99, Proposition 3.2] which infers the convergence of the Trotter–Kato product formula in symmetric F-normed ideals from operator-norm convergence. Since the convergence of the Trotter–Kato product formula in symmetric F-normed ideals holds for different families if it holds for one of these families, it is sufficient to show the convergence for the family $\{F(t)\}_{t\geq 0}$.

PROPOSITION 3.3. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H. Let $\mathcal{I}(H)$ be a symmetric F-normed ideal closed with respect to logarithmic submajorization and f, g be Kato functions. Let $t_0 > 0$. Assume that*

(i) *the Trotter–Kato product formula for $\{F(t)\}_{t\geq 0}$ converges locally uniformly away from zero in the operator norm;*
(ii) *for any bounded interval $[\tau_0,\tau] \subset (t_0,\infty)$ there exists $r_0 \geq 1$ such that $F(t/r)^r \in \mathcal{I}(H)$ for any $t \in [\tau_0,\tau]$ and $r \geq r_0$ and*

$$M([\tau_0,\tau]) := \sup_{r\geq r_0}\sup_{t\in[\tau_0,\tau]} \|F(t/r)^r\|_{\mathcal{I}} < \infty;$$

(iii) $e^{-tC} \in \mathcal{I}(H_0)$ *for $t > t_0$.*

Then the Trotter–Kato product formula for the family $\{F(t)\}_{t\geq 0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$.

Proof. Fix $[\tau_0,\tau]\subset(t_0,\infty)$ and $\varepsilon>0$. Then there exists $\alpha\in(0,1)$ such that $\tau_0':=\alpha\tau_0>t_0$. It is clear that $\alpha t\in[\tau_0',\tau]$ for any $t\in[\tau_0,\tau]$. Denoting by E_C the spectral measure of C, we can write

$$
\begin{aligned}
(19)\quad F(t/r)^r-e^{-tC}P_0 &= (F(t/r)^{(1-\alpha)r}-e^{-(1-\alpha)tC}P_0)F(t/r)^{\alpha r}\\
&\quad + e^{-(1-\alpha)tC}E_C([0,N))P_0(F(t/r)^{\alpha r}-e^{-\alpha tC}P_0)\\
&\quad + e^{-(1-\alpha)tC}E_C([N,\infty))P_0(F(t/r)^{\alpha r}-e^{-\alpha tC}P_0).
\end{aligned}
$$

Using the triangle inequality, we infer

$$
\begin{aligned}
(20)\qquad &\sup_{t\in[\tau_0,\tau]}\|F(t/r)^r-e^{-tC}P_0\|_{\mathcal{I}}\\
&\le \sup_{t\in[\tau_0,\tau]}\|(F(t/r)^{(1-\alpha)r}-e^{(1-\alpha)tC}P_0)F(t/r)^{\alpha r}\|_{\mathcal{I}}\\
&\quad+\sup_{t\in[\tau_0,\tau]}\|e^{-(1-\alpha)tC}E_C([0,N))P_0(F(t/r)^{\alpha r}-e^{-\alpha tC}P_0)\|_{\mathcal{I}}\\
&\quad+\sup_{t\in[\tau_0,\tau]}\|e^{-(1-\alpha)tC}E_C([N,\infty))P_0(F(t/r)^{\alpha r}-e^{-\alpha tC}P_0)\|_{\mathcal{I}}.
\end{aligned}
$$

We estimate each term on the right hand side separately. For any $t\ge 0$ we have $F(t/r)^{\alpha r}=F(\alpha t/\alpha r)^{\alpha r}$. Hence, by (ii) for the interval $[\tau_0',\tau]$, there exists $r_0\in\mathbb{R}_+$ such that $F(\alpha t/\alpha r)\in\mathcal{I}(H)$ for any $\alpha t\in[\tau_0',\tau]$, $r\ge r_0$ and

$$M([\tau_0',\tau]):=\sup_{\alpha t\in[\tau_0',\tau]}\sup_{r\ge r_0}\|F(\alpha t/\alpha r)^{\alpha r}\|_{\mathcal{I}}<\infty.$$

Since $\alpha t\in[\tau_0',\tau]$ for any $t\in[\tau_0,\tau]$, we have $F(t/r)^{\alpha r}\in\mathcal{I}(H)$ for any $t\in[\tau_0,\tau]$, $r\ge r_0$ and

$$(21)\qquad \|F(t/r)^{\alpha r}\|_{\mathcal{I}}\le M([\tau_0',\tau]).$$

We now estimate the third term on the right hand side of (20). Since $CE_C([N,\infty))\ge NE_C([N,\infty))$, we have

$$(22)\qquad \sup_{t\in[\tau_0,\tau]}\|e^{-(1-\alpha)tC}E_C([N,\infty))P_0\|_\infty\le e^{-(1-\alpha)\tau_0N}$$

for $N\ge1$. Therefore, by (4) and the triangle inequality,

$$
\begin{aligned}
(23)\qquad &\sup_{t\in[\tau_0,\tau]}\|e^{-(1-\alpha)tC}E_C([N,\infty))P_0(F(t/r)^{\alpha r}-e^{-\alpha tC}P_0)\|_{\mathcal{I}}\\
&\le \sup_{t\in[\tau_0,\tau]}\Big\|\sup_{t\in[\tau_0,\tau]}\|e^{-(1-\alpha)tC}E_C([N,\infty))P_0\|_\infty F(t/r)^{\alpha r}\Big\|_{\mathcal{I}}\\
&\quad+\sup_{t\in[\tau_0,\tau]}\Big\|\sup_{t\in[\tau_0,\tau]}\|e^{-(1-\alpha)tC}E_C([N,\infty))P_0\|_\infty e^{-\alpha tC}P_0\Big\|_{\mathcal{I}}\\
&\le \sup_{t\in[\tau_0,\tau]}\|e^{-(1-\alpha)\tau_0N}F(t/r)^{\alpha r}\|_{\mathcal{I}}+\sup_{t\in[\tau_0,\tau]}\|e^{-(1-\alpha)\tau_0N}e^{-\alpha\tau_0C}P_0\|_{\mathcal{I}}
\end{aligned}
$$

for $N \geq 1$. Note that the sequence $\{e^{-(1-\alpha)\tau_0 N}\}_{N\geq 1}$ converges to zero as $N \to \infty$. By (21) and the fact that $\|e^{-\alpha\tau_0 C}P_0\|_{\mathcal{I}} < \infty$, Definition 2.1(iii) implies that for any given $\varepsilon > 0$, there exists a large enough natural number $N_{\max}$ such that

$$\sup_{t\in[\tau_0,\tau]} \|e^{-(1-\alpha)\tau_0 N}F(t/r)^{\alpha r}\|_{\mathcal{I}} + \sup_{t\in[\tau_0,\tau]} \|e^{-(1-\alpha)\tau_0 N}e^{-\alpha\tau_0 C}P_0\|_{\mathcal{I}} < \varepsilon/3 \tag{24}$$

for any $r \geq r_0$ and $n \geq N_{\max}$; let us fix this $N_{\max}$.

To estimate the second term on the right hand side of (20), we first note that $s_j(e^{-t\beta C}) = s_j(e^{-tC})^\beta$ for any $j \geq 1$ and $\beta > 0$ implies that e^{-tC} is compact for any $t > 0$. Therefore, the spectrum of C is discrete with the only accumulation point at infinity. Hence, the projection $E_C([0,N))$ is a finite-rank operator for each $N = 1, 2, \ldots$, and in particular $E_C([0,N)) \in \mathcal{I}(H)$ for any $N \geq 1$. Since $F(t/r)^{\alpha r} \to e^{-\alpha tC}$ in the strong operator topology [K74, K78], Lemma 3.1 (with $X = Y = E_C([0,N))$ and $Z(t/r) = F(t/r)^{\alpha r}$, $Z = e^{-\alpha tC}$) implies that

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|E_C([0,N))P_0(F(t/r)^{\alpha r} - e^{-\alpha tC})\|_{\mathcal{I}} = 0.$$

Therefore, by (4), we can find a large enough $r_{\max} \geq r_0$ such that

$$\sup_{t\in[\tau_0,\tau]} \|e^{-(1-\alpha)tC}E_C([0,N_{\max}))P_0(F(t/r)^{\alpha r} - e^{-\alpha tC}P_0)\|_{\mathcal{I}} < \varepsilon/3 \tag{25}$$

for any $r \geq r_{\max}$.

Lastly, since the Trotter–Kato product formula converges locally uniformly away from zero in the operator norm, we have

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|F(t/r)^{(1-\alpha)r} - e^{(1-\alpha)tC}P_0\| = 0.$$

Therefore, by (21), the symmetry of the F-norm and Definition 2.1(iii), there exists $R \geq r_{\max}$ such that

$$\begin{aligned} &\sup_{t\in[\tau,\tau_0]} \|(F(t/r)^{(1-\alpha)r} - e^{(1-\alpha)tC}P_0)F(t/r)^{\alpha r}\|_{\mathcal{I}} \\ &\quad \leq \sup_{t\in[\tau,\tau_0]} \left\| \sup_{t\in[\tau,\tau_0]} \|F(t/r)^{(1-\alpha)r} - e^{(1-\alpha)tC}P_0\|_\infty F(t/r)^{\alpha r} \right\|_{\mathcal{I}} < \varepsilon/3 \end{aligned} \tag{26}$$

for any $r \geq R$.

Finally, combining (24)–(26) and the representation (20), for any bounded interval $[\tau_0, \tau]$ and $\varepsilon > 0$ we can find $R \in \mathbb{R}_+$ such that

$$\sup_{t\in[\tau_0,\tau]} \|F(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} < \varepsilon$$

for $r \geq R$. This proves that the Trotter–Kato product formula for the family $\{F(t)\}_{t\geq 0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$. ∎

In Lemmas 3.4 and 3.5 below, we verify that conditions (ii) and (iii) of Proposition 3.3 are satisfied.

Lemma 3.4. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and let $f^D, g^D : [0,\infty) \to [0,\infty)$ be bounded Borel measurable functions such that $F^D(t_0) = g^D(t_0B)^{1/2} f^D(t_0A) g^D(t_0B)^{1/2} \in \mathcal{I}(H)$ for some $t_0 > 0$, where $\mathcal{I}(H)$ is a symmetric F-normed ideal closed with respect to logarithmic submajorization. If Kato functions f and g are dominated by f^D and g^D, then $F(t/r)^r \in \mathcal{I}(H)$ and*

$$\|F(t/r)^r\|_{\mathcal{I}} \leq C_{\mathcal{I}} \cdot \|F^D(t_0)\|_{\mathcal{I}} \tag{27}$$

for all $t_0 \leq t \leq rt_0$, $r \geq 1$ and the constant $C_{\mathcal{I}} > 0$ from Definition 2.5.

Proof. Let $X = f(tA/r)$, $Y = g(tB/r)$ and $X_0 = f^D(t_0A)$, $Y_0 = g^D(t_0B)$. We have

$$X^r = f(tA/r)^r \leq f(tA/r)^{rt_0/t} \leq f^D(t_0A) = X_0,$$
$$Y^r = g(tB/r)^r \leq g(tB/r)^{rt_0/t} \leq g^D(t_0B) = Y_0,$$

for $t_0 \leq t \leq rt_0$ and $r \geq 1$. Therefore, by Lemma 2.8 we have

$$F(t/r)^r = (Y^{1/2}XY^{1/2})^r \prec\prec_{\log} Y_0^{1/2}X_0Y_0^{1/2} = F^D(t_0),$$

for $t_0 \leq t \leq rt_0$ and $r \geq 1$. Since $\mathcal{I}(H)$ is closed with respect to logarithmic submajorization, it follows that $F(t/r)^r \in \mathcal{I}(H)$ and

$$\|F(t/r)^r\|_{\mathcal{I}} \leq C_{\mathcal{I}} \cdot \|F^D(t_0)\|_{\mathcal{I}}. \blacksquare$$

Next we show that under the assumption of Lemma 3.4 one has $e^{-tC} \in \mathcal{I}(H_0)$ for $t > t_0$.

Lemma 3.5. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and let $f^D, g^D : [0,\infty) \to [0,\infty)$ be bounded Borel measurable functions such that $F^D(t_0) = g^D(t_0B)^{1/2} f^D(t_0A) g^D(t_0B)^{1/2} \in \mathcal{I}(H)$ for some $t_0 > 0$, where $\mathcal{I}(H)$ is a symmetric F-normed ideal closed with respect to logarithmic submajorization. If Kato functions f and g are dominated by f^D and g^D, then $e^{-tC} \in \mathcal{I}(H_0)$ and*

$$\|e^{-tC}\|_{\mathcal{I}} \leq C_{\mathcal{I}} \|F^D(t_0)\|_{\mathcal{I}} \tag{28}$$

for $t \geq t_0$ and a constant $C_{\mathcal{I}} > 0$ from Definition 2.5.

Proof. Note that, for $x \geq 0$, we have

$$e^{-x} = \lim_{r\to\infty} f(x/r)^r \leq f^D(x), \qquad e^{-x} = \lim_{r\to\infty} g(x/r)^r \leq g^D(x).$$

Hence, the assumption of the lemma implies that $f_0(x) = g_0(x) = e^{-x}$ are dominated by f^D and g^D, respectively. Lemma 3.4 guarantees that $(e^{-tB/2r}e^{-tA/r}e^{-tB/2r})^r \in \mathcal{I}(H)$ for $t_0 \leq t \leq rt_0$. By Lemma 2.9, we have

$$e^{-tC}P_0 \prec\prec_{\log} (e^{-tB/2r}e^{-tA/r}e^{-tB/2r})^r \in \mathcal{I}(H), \quad r \geq 1, \tag{29}$$

for $t_0 \le t \le rt_0$. Therefore, by the assumption on the ideal $\mathcal{I}(H)$ and (29), we find that $e^{-tC}P_0 \in \mathcal{I}(H)$ and (28) holds. ∎

Finally, a combination of the previous lemmas and propositions yields the main result of the present paper.

THEOREM 3.6. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and $\mathcal{I}(H)$ be a symmetric F-normed ideal closed with respect to logarithmic submajorization. Let $f^D : [0,\infty) \to [0,\infty)$ be a Borel measurable functions such that $f^D(t_0A) \in \mathcal{I}(H)$ for some $t_0 > 0$. If g is any Kato function and f is any regular Kato function dominated by f^D, then the Trotter–Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$ for all families generated by f and g.*

Proof. A similar argument to the proof of Lemma 3.5 implies that e^{-x} is dominated by f^D. Thus, $e^{-t_0A} \in \mathcal{I}(H)$ provided that $f^D(t_0A) \in \mathcal{I}(H)$. In particular, $(I + t_0A)^{-1} \in \mathcal{L}_\infty(H)$. Then, since f is regular, the Trotter–Kato product formula converges locally uniformly away from zero in the operator norm for the family $\{F(t)\}_{t\ge0}$ [NZ99, Theorem 3.7].

Setting $g^D(x) \equiv 1$, $x \ge 0$, it easily follows that $F^D(t_0) \in \mathcal{I}(H)$. Notice that any Kato function g is dominated by $g^D(\cdot) \equiv 1$. Hence, by Lemmas 3.4 and 3.5, for any bounded interval $[\tau_0,\tau] \subset (t_0,\infty)$ there exists $r_0 \ge 1$ such that $e^{-tC} \in \mathcal{I}(H_0)$ and $F(t/r)^r \in \mathcal{I}(H)$ for any $t \in [\tau_0,\tau]$ and $r \ge r_0$ such that

$$\sup_{t\in[\tau_0,\tau]} \sup_{r\ge r_0} \|F(t/r)^r\|_{\mathcal{I}} \le \|F^D(t_0)\|_{\mathcal{I}} < \infty.$$

Thus, all assumptions of Proposition 3.3 are satisfied, and the Trotter–Kato product formula for $\{F(t)\}_{t\ge0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$. Finally, by Proposition 3.2, we have the convergence of the Trotter–Kato product formula in the symmetric F-normed ideal $\mathcal{I}(H)$ for all other families generated by f and g. ∎

As a corollary of Theorem 3.6, we have analogous result for symmetric quasi-Banach ideals.

COROLLARY 3.7. *Let $\mathcal{I}(H)$ be a symmetric quasi-Banach ideal and let operators A, B and functions f, g, f^D be as in Theorem 3.6. If $f^D(t_0A) \in \mathcal{I}(H)$ for some $t_0 > 0$, then the Trotter–Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$ for all families generated by f and g.*

We present a concrete example of a symmetric quasi-Banach ideal and non-commuting operators A and B of Corollary 3.7, which is not covered by [H97] and [NZa99].

EXAMPLE 3.8. Let $I = \mathcal{L}_{1,\infty}$ be the weak-l_1 ideal from Example 2.4(ii), equipped with its natural quasi-norm

$$\|X\|_{1,\infty} = \sup_{j\geq 1} j s_j(X).$$

Let $D : W^{1,2}(\mathbb{T}) \to L^2(\mathbb{T})$ be a self-adjoint operator given by $Df = \frac{1}{i}\frac{df}{dt}$, $f \in W^{1,2}(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. It is known that $(1+|D|)^{-1} \in \mathcal{L}_{1,\infty}$. Indeed, for $k \geq 1$,

$$s_1((1+|D|)^{-1}) = 1, \quad s_{2k}((1+|D|)^{-1}) = s_{2k+1}((1+|D|)^{-1}) = \frac{1}{k+1}.$$

We set $A = \log(1+|D|)$. Hence, $e^{-A} = (1+|D|)^{-1} \in \mathcal{L}_{1,\infty}$. Moreover, $(1+|D|)^{-1}$ generates the principal ideal $\mathcal{L}_{1,\infty}$, and therefore e^{-A} cannot belong to any ideal which is strictly smaller than $\mathcal{L}_{1,\infty}$.

We claim that no operator B with non-trivial continuous spectrum commutes with A. Indeed, otherwise it commutes with all spectral projections of A. Let $p_n = \chi_{\{n\}}(|D|)$ be a spectral projection of A. It follows that

$$B = \sum_{n\in\mathbb{Z}_+} Bp_n = \sum_{n\in\mathbb{Z}_+} p_n B p_n,$$

where the sums are taken in the strong operator topology. Each of the operators $p_n B p_n$ is of finite rank; hence its spectrum consists of eigenvalues. Since the operators $\{p_n B p_n\}_{n\in\mathbb{Z}_+}$ are pairwise orthogonal, the spectrum of B consists of eigenvalues as well, contrary to the assumption. In particular, if we take $B = M_h$, $h \in L_\infty(\mathbb{T})$, then the operators A and B do not commute.

Let the function f^D from Corollary 3.7 be the exponential function e^{-x}, $x \geq 0$. Then $f^D(A) = e^{-A} = (1+|D|)^{-1} \in \mathcal{L}_{1,\infty}$ for $t_0 = 1$. Therefore, by Corollary 3.7, the Trotter–Kato product formula converges locally uniformly away from $t_0 = 1$ in $\mathcal{L}_{1,\infty}$ for all families generated by the exponential function.

Now we want to specify a particular subclass of Kato functions to present a necessary and sufficient condition for the convergence of the Trotter–Kato product formula in $\mathcal{I}(H)$ similar to [NZa99, Theorem 4.10]. A Kato function f is said to be *self-dominated* if for any $x \geq 0$ and $0 < q \leq 1$, one has $f(qx)^{1/q} \leq f(x)$.

THEOREM 3.9. *Let A and B be non-negative self-adjoint operators on the separable Hilbert space H and let $\mathcal{I}(H)$ be a symmetric F-normed ideal closed with respect to logarithmic submajorization. Let f and g be self-dominated Kato functions which additionally satisfy*

$$\sup_{x>0} \frac{xf(x)}{1-f(x)} < \infty, \quad \sup_{x>0} \frac{xg(x)}{1-g(x)} < \infty. \tag{30}$$

There exists $t_0 > 0$ such that the Trotter–Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$ for all families generated by f and g if and only if there exist $s_0 > 0$ and $p \in \mathbb{Z}_+$ such that $F(s_0)^p \in \mathcal{I}(H)$.

Proof. The necessity follows from Definition 2.12.

Assume that there exist $s_0 > 0$ and $p \in \mathbb{Z}_+$ such that $F(s_0)^p \in \mathcal{I}(H) \subset \mathcal{L}_\infty(H)$. Then $F(s_0)$ is a compact operator and a similar argument to [NZa99, Theorem 4.6] implies that $(I + A)^{-1}(I + B)^{-1} \in \mathcal{L}_\infty(H)$. Hence, by [NZ99, Theorem 5.3], the Trotter–Kato product formula converges locally uniformly away from zero in the operator norm. Finally, the sufficiency follows from Proposition 3.3 together with Lemmas 3.4 and 3.5, on taking $f^D(x) \equiv f(x)$ and $g^D(x) \equiv g(x)$ for $x \geq 0$ and $t_0 = ps_0$. ∎

4. Error bound for the Trotter–Kato product formula in a symmetric F-normed ideal $\mathcal{I}(H)$. In this section we determine error bounds for the Trotter–Kato product formulas from Section 3 and give some examples where they can be computed directly. As before, we assume that A and B are non-negative self-adjoint operators on a separable Hilbert space H with the form sum $C = A \dot{+} B$ and $\mathcal{I}(H)$ is a symmetric F-normed ideal closed with respect to logarithmic submajorization. Let f and g be Kato functions and

$$F(t) := g(tB)^{1/2} f(tA) g(tB)^{1/2}, \quad t \geq 0,$$
$$G(t) := f(tA)^{1/2} g(tB) f(tA)^{1/2}, \quad t \geq 0.$$

We first define the notion of an error bound as in [NZa99]. Assume that $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ (or $\varepsilon : \mathbb{N} \to \mathbb{R}_+$) is a function such that $\lim_{x\to\infty} \varepsilon(x) = 0$.

Definition 4.1. A function $\varepsilon_{\mathcal{I}} : \mathbb{R}_+ \to \mathbb{R}_+$ is called an *error bound* for the Trotter–Kato product formula for the family $\{F(t)\}_{t\geq 0}$ away from $t_0 > 0$ in $\mathcal{I}(H)$ if for any bounded interval $[a, b] \subset (t_0, \infty)$ there exists $1 \leq r_0 \in \mathbb{R}_+$ such that

$$F(t/r)^r - e^{-tC} P_0 \in \mathcal{I}(H) \quad \text{and} \quad \|F(t/r)^r - e^{-tC} P_0\|_{\mathcal{I}} \leq \text{Const} \cdot \varepsilon(r)$$

for any $t \in [a, b]$, $r \geq r_0$ and some constant $\text{Const} > 0$.

In a similar way, one can define the notion of an error bound for the families $\{G(t)\}_{t\geq 0}$, $\{f(tA)g(tB)\}_{t\geq 0}$ and $\{g(tB)f(tA)\}_{t\geq 0}$, assuming that $r \in \mathbb{N}$ in the last two cases. When $\|\cdot\|_{\mathcal{I}}$ is the operator norm $\|\cdot\|_\infty$, we simply write $\varepsilon(\cdot)$ instead of $\varepsilon_{\mathcal{I}}(\cdot)$, and call it an error bound for the Trotter–Kato product formula away from $t_0 > 0$ in the operator norm.

First we prove an auxiliary lemma.

Lemma 4.2. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and $\mathcal{I}(H)$ be a symmetric F-normed ideal. Let $f^D, g^D : \mathbb{R}_+ \to \mathbb{R}_+$ be Borel measurable functions such that $F^D(t_0) :=$*

$g^D(t_0B)^{1/2}f^D(t_0A)g^D(t_0B)^{1/2} \in \mathcal{I}(H)$ for some $t_0 > 0$. Then $G^D(t_0) := f^D(t_0A)^{1/2}g^D(t_0B)f^D(t_0A)^{1/2} \in \mathcal{I}(H)$ and

$$s_j(G^D(t_0)) \le s_j(F^D(t_0)), \quad j \ge 1. \tag{31}$$

Proof. Let

$$E(t) = f^D(tA)^{1/2}g^D(tB)^{1/2}, \quad t \ge 0.$$

Note that $|E(t)| = F^D(t)^{1/2}$ and $|E(t)^*| = G^D(t)^{1/2}$. Therefore, denoting by $U(t)$ the partial isometry in the polar decomposition

$$E(t) = U(t)F^D(t)^{1/2}, \quad t \ge 0,$$

we conclude that

$$G^D(t_0) = E(t_0)E(t_0)^* = U(t_0)F^D(t_0)U(t_0)^* \in \mathcal{I}(H),$$

which also implies (31). ∎

Now we present the following result which is helpful to compute error bounds for the Trotter–Kato product formula in some special symmetric F-normed ideals. The proof is similar to [NZa99, Theorem 5.1].

Proposition 4.3. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and $\mathcal{I}(H)$ be a symmetric F-normed ideal closed with respect to logarithmic submajorization. Let $f^D, g^D : \mathbb{R}_+ \to \mathbb{R}_+$ be Borel measurable functions such that $F^D(t_0) = g^D(t_0B)^{1/2}f^D(t_0A)g(t_0B)^{1/2} \in \mathcal{I}(H)$ for some $t_0 > 0$. Assume that f and g are Kato functions dominated by f^D and g^D, respectively. Then:*

(i) *For families $\{F(t)\}_{t\ge0}$ and $\{G(t)\}_{t\ge0}$, we have*

$$\|F(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \le 2C_{\mathcal{I}}\big\|\|F(t/r)^{r/2} - e^{-tC/2}P_0\|_\infty F_D(t_0)\big\|_{\mathcal{I}},$$
$$\|G(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \le 2C_{\mathcal{I}}\big\|\|G(t/r)^{r/2} - e^{-tC/2}P_0\|_\infty F_D(t_0)\big\|_{\mathcal{I}}$$

for $2t_0 \le t \le rt_0$ and $r \ge 2$.

(ii) *For families $\{f(tA)g(tB)\}_{t\ge0}$ and $\{g(tB)f(tA)\}_{t\ge0}$, we have*

$$\begin{aligned}&\|(f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}}\\ &\quad\le C_{\mathcal{I}}\big\|\|(f(tA/n)g(tB/n))^k - e^{-ktC/n}P_0\|_\infty F^D(t_0)\big\|_{\mathcal{I}}\\ &\quad\quad + C_{\mathcal{I}}\big\|\|(f(tA/n)g(tB/n))^m - e^{-mtC/n}P_0\|_\infty F^D(t_0)\big\|_{\mathcal{I}},\end{aligned}$$

$$\begin{aligned}&\|(g(tB/n)f(tA/n))^n - e^{-tC}P_0\|_{\mathcal{I}}\\ &\quad\le C_{\mathcal{I}}\big\|\|(g(tB/n)f(tA/n))^k - e^{-ktC/n}P_0\|_\infty F^D(t_0)\big\|_{\mathcal{I}}\\ &\quad\quad + C_{\mathcal{I}}\big\|\|(g(tB/n)f(tA/n))^m - e^{-mtC/n}P_0\|_\infty F^D(t_0)\big\|_{\mathcal{I}}\end{aligned}$$

for $t_0 \le (m-1)t/n \le (m-1)t_0$, $m \ge 2$ and $kt/n \ge t_0$, where $k := [n/2]$ and $m := [(n+1)/2]$ with $n \ge 3$.

Proof. (i) For $r \in \mathbb{R}_+$, we write

$$(32)\quad \begin{aligned} &F(t/r)^r - e^{-tC}P_0 \\ &= (F(t/r)^{r/2} - e^{-tC/2}P_0)F(t/r)^{r/2} + e^{-tC/2}P_0(F(t/r)^{r/2} - e^{-tC/2}P_0). \end{aligned}$$

By (4), we have

$$(33)\quad \begin{aligned} \|F(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \le {}& \big\| \|F(t/r)^{r/2} - e^{-tC/2}P_0\|_\infty F(t/r)^{r/2} \big\|_{\mathcal{I}} \\ &+ \big\| \|F(t/r)^{r/2} - e^{-tC/2}P_0\|_\infty e^{-tC/2}P_0 \big\|_{\mathcal{I}}. \end{aligned}$$

Since $F^D(t_0) \in \mathcal{I}(H)$, Lemmas 3.4 and 3.5 imply

$$\begin{aligned} &F(t/r)^{r/2} \prec\prec_{\log} F^D(t_0) && \text{for } 2t_0 \le t \le rt_0,\ r \ge 2, \text{ and} \\ &e^{-tC/2}P_0 \prec\prec_{\log} F^D(t_0) && \text{for } t \ge 2t_0. \end{aligned}$$

Since $\mathcal{I}(H)$ is closed with respect to logarithmic submajorization, it follows that

$$\|F(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \le 2C_{\mathcal{I}} \big\| \|F(t/r)^{r/2} - e^{-tC/2}P_0\|_\infty F^D(t_0) \big\|_{\mathcal{I}}$$

for $2t_0 \le t \le rt_0$.

For the family $\{G(t)\}_{t\ge 0}$ we have a similar estimate via decomposition similar to (32) with an additional reference to Lemma 4.2.

(ii) We prove the result for the family $\{f(tA)g(tB)\}_{t\ge 0}$; the argument for $\{g(tB)f(tA)\}_{t\ge 0}$ is similar and is omitted. Let $n \in \mathbb{N}$ and write $n = k + m$ with

$$k := \left[\frac{n}{2}\right], \quad m := \left[\frac{n+1}{2}\right], \quad n \ge 3.$$

We have

$$(34)\quad \begin{aligned} &(f(tA/n)g(tB/n))^n - e^{-tC}P_0 \\ &\qquad = ((f(tA/n)g(tB/n))^k - e^{-ktC/n}P_0)(f(tA/n)g(tB/n))^m \\ &\qquad\quad + e^{-ktC/n}P_0((f(tA/n)g(tB/n))^m - e^{-mtC/n}P_0). \end{aligned}$$

Therefore, the triangle inequality and (4) imply

$$(35)\quad \begin{aligned} &\|(f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}} \\ &\quad \le \big\| \|(f(tA/n)g(tB/n))^k - e^{-ktC/n}P_0\|_\infty (f(tA/n)g(tB/n))^m \big\|_{\mathcal{I}} \\ &\quad\quad + \big\| e^{-ktC/n}P_0 \|(f(tA/n)g(tB/n))^m - e^{-mtC/n}P_0\|_\infty \big\|_{\mathcal{I}}. \end{aligned}$$

We consider the two terms on the right hand side of (35) separately. Note that

$$(f(tA/n)g(tB/n))^m = f(tA/n)g(tB/n)^{1/2}F(t/n)^{m-1}g(tB/n)^{1/2}.$$

By Lemma 3.4, $F(t/n)^{m-1} \in \mathcal{I}(H)$ when $t_0 \le (m-1)t/n \le (m-1)t_0$ and $m-1 \ge 1$. Therefore, $(f(tA/n)g(tB/n))^m \in \mathcal{I}(H)$ and

$$(f(tA/n)g(tB/n))^m \prec\prec_{\log} F(t/n)^{m-1} \prec\prec_{\log} F^D(t_0) \tag{36}$$

for $t_0 \le (m-1)t/n \le (m-1)t_0$ and $m-1 \ge 1$. Lemma 3.5 implies that $e^{-ktC/n}P_0 \in \mathcal{I}(H)$ and

$$e^{-ktC/n}P_0 \prec\prec_{\log} F^D(t_0) \tag{37}$$

for $kt/n \ge t_0$. Hence, by (35)–(37) together with the fact that $\mathcal{I}(H)$ is closed with respect to logarithmic submajorization, we have

$$\begin{aligned} &\|(f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}} \\ &\quad \le C_{\mathcal{I}}\big\|\|(f(tA/n)g(tB/n))^k - e^{-ktC/n}P_0\|_\infty F^D(t_0)\big\|_{\mathcal{I}} \\ &\quad\quad + C_{\mathcal{I}}\big\|\|(f(tA/n)g(tB/n))^m - e^{-mtC/n}P_0\|_\infty F^D(t_0)\big\|_{\mathcal{I}} \end{aligned} \tag{38}$$

for $t_0 \le (m-1)t/n \le (m-1)t_0$, $m \ge 2$ and $kt/n \ge t_0$. ∎

Now we present some examples similar to [NZa99, Theorem 5.1], where the error bounds can be computed directly. Recall that the notion of an error bound in the operator norm is defined as in Definition 4.1.

EXAMPLE 4.4. (i) Let $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ be a symmetric F-normed ideal closed with respect to logarithmic submajorization and let f, g, f^D and g^D be as in Proposition 4.3 such that $F^D(t_0) \in \mathcal{I}(H)$ for some $t_0 > 0$, and

$$\|\alpha F^D(t_0)\|_{\mathcal{I}} = O(\alpha^q), \quad \alpha \to 0. \tag{39}$$

Let $\varepsilon(r)$, $r \ge 1$, be an error bound for the Trotter–Kato product formula in the operator norm away from $t_0 > 0$. Then, by Proposition 4.3(i),

$$\|F(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \le 2C_{\mathcal{I}}\big\|\|F(t/r)^{r/2} - e^{-tC/2}P_0\|_\infty F_D(t_0)\big\|_{\mathcal{I}}.$$

Hence, the symmetry of the F-norm and (39) imply that

$$\|F(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \le \mathrm{Const} \cdot \|\varepsilon(r/2)F_D(t_0)\|_{\mathcal{I}} \le \mathrm{Const} \cdot \varepsilon(r/2)^q$$

for some constant Const > 0. Hence, the function $\varepsilon_{\mathcal{I}}(r) = \varepsilon(r/2)^q, r \ge 2$, is an error bound for the Trotter–Kato product formula for the family $\{F(t)\}_{t\ge 0}$ away from $2t_0$ in $\mathcal{I}(H)$. Similarly, by the second inequality of Proposition 4.3(i), we have an analogous error bound for $\{G(t)\}_{t\ge 0}$ away from $2t_0$ in $\mathcal{I}(H)$. For the family $\{f(tA)g(tB)\}_{t\ge 0}$, using Proposition 4.3(ii) and arguments above, we have

$$\begin{aligned} &\|(f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}} \\ &\quad \le \mathrm{Const} \cdot \bigg(\bigg\|\varepsilon\bigg(\bigg[\frac{n}{2}\bigg]\bigg)F^D(t_0)\bigg\|_{\mathcal{I}} + \bigg\|\varepsilon\bigg(\bigg[\frac{n+1}{2}\bigg]\bigg)F^D(t_0)\bigg\|_{\mathcal{I}}\bigg), \end{aligned}$$

which together with (39) implies

$$\|(f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}} \\ \leq \mathrm{Const} \cdot \left(\varepsilon\left(\left[\frac{n}{2}\right]\right)^q + \varepsilon\left(\left[\frac{n+1}{2}\right]\right)^q\right)$$

for some constant Const > 0. Therefore, $\varepsilon_{\mathcal{I}}(n) = \varepsilon\big(\big[\frac{n}{2}\big]\big)^q + \varepsilon\big(\big[\frac{n+1}{2}\big]\big)^q$, $n \geq 3$, is an error bound for the Trotter–Kato product formula for the family $\{f(tA)g(tB)\}_{t\geq 0}$ away from $2t_0$ in $\mathcal{I}(H)$. In a similar way, by the second inequality of Proposition 4.3(ii), we have an analogous error bound for $\{g(tB)f(tA)\}_{t\geq 0}$.

(ii) Let $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ be a symmetric quasi-normed ideal and $\varepsilon(r)$, $r \in \mathbb{R}_+$, be an error bound for the Trotter–Kato product formula in the operator norm away from $t_0 > 0$. Then, under the assumptions of Proposition 4.3, the arguments similar to (4.4) and the homogeneity of the quasi-norm imply that

$$\varepsilon_{\mathcal{I}}(r) = \varepsilon(r/2), \quad 2 \leq r \in \mathbb{R}_+,$$

and

$$\varepsilon_{\mathcal{I}}(n) = \varepsilon\left(\left[\frac{n}{2}\right]\right) + \varepsilon\left(\left[\frac{n+1}{2}\right]\right), \quad 3 \leq n \in \mathbb{N},$$

are the error bounds locally away from $2t_0$ in $\mathcal{I}(H)$ for the families $\{F(t)\}_{t\geq 0}$, $\{G(t)\}_{t\geq 0}$ and $\{f(tA)g(tB)\}_{t\geq 0}$, $\{g(tB)f(tA)\}_{t\geq 0}$, respectively.

(iii) Let $\mathcal{L}_1(H)$ be the trace class ideal equipped with the functional $\|X\|_{\log} = \sum_{k\geq 1} \log(1 + s_k(X))$, $X \in \mathcal{L}_\infty(H)$. By [DSZ16], $(\mathcal{L}_1(H), \|\cdot\|_{\log})$ is a complete symmetric F-normed ideal. For $X \in \mathcal{L}_1(H)$ and $\alpha \in [0,1]$, we have

$$\|\alpha X\|_{\log} \leq \|\alpha X\|_1 \leq \alpha(1 + \|X\|_\infty)\|X\|_{\log}, \tag{40}$$

where $\|\cdot\|_1$ is the trace class norm. Let $\varepsilon(r)$, $r \in \mathbb{R}_+$, be an error bound for the Trotter–Kato product formula in the operator norm away from $t_0 > 0$. Then, under the assumptions of Proposition 4.3, (40) and the arguments similar to (4.4) imply that the error bounds are $\varepsilon_{\mathcal{I}}(r) = \varepsilon(r/2)$, $2 \leq r \in \mathbb{R}_+$, and $\varepsilon_{\mathcal{I}}(n) = \varepsilon\big(\big[\frac{n}{2}\big]\big) + \varepsilon\big(\big[\frac{n+1}{2}\big]\big)$, $3 \leq n \in \mathbb{N}$, for families $\{F(t)\}_{t\geq 0}$, $\{G(t)\}_{t\geq 0}$ and $\{f(tA)g(tB)\}_{t\geq 0}$, $\{g(tB)f(tA)\}_{t\geq 0}$, respectively.

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