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A REDUCTION APPROACH TO SILTING OBJECTS FOR DERIVED CATEGORIES OF HEREDITARY CATEGORIES

BY

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Abstract. Let $\mathcal{H}$ be a hereditary abelian category over a field k with finite-dimensional Hom and Ext spaces. It is proved that the bounded derived category $\mathcal{D}^b(\mathcal{H})$ has a silting object iff $\mathcal{H}$ has a tilting object iff $\mathcal{D}^b(\mathcal{H})$ has a simple-minded collection with acyclic Ext-quiver. Along the way, we obtain a new proof for the fact that every presilting object of $\mathcal{D}^b(\mathcal{H})$ is a partial silting object. We also consider the question of complements for pre-simple-minded collections. In contrast to presilting objects, a pre-simple-minded collection $\mathcal{R}$ of $\mathcal{D}^b(\mathcal{H})$ can be completed to a simple-minded collection iff the Ext-quiver of $\mathcal{R}$ is acyclic.

1. Introduction. Throughout this note, let k be a field. By a *hereditary abelian category*, we mean a hereditary abelian category over k with finite-dimensional Hom and Ext spaces.

Hereditary abelian categories with tilting objects and their bounded derived categories provide a framework for the classical tilting theory, which has been extensively studied since early eighties. The main examples of such categories are the category $\operatorname{mod} H$ of finitely generated right modules over a finite-dimensional hereditary k-algebra H and the category $\operatorname{coh} \mathbb{X}$ of coherent sheaves over an exceptional curve $\mathbb{X}$ in the sense of Lenzing [21]. A remarkable theorem of Happel and Reiten [13] shows that a connected hereditary abelian category with tilting object is either derived equivalent to $\operatorname{mod} H$ or to $\operatorname{coh} \mathbb{X}$.

Silting objects were first introduced in [19] as a generalization of tilting objects to parametrize bounded t-structures on derived categories of path algebras of Dynkin quivers. In recent years, the topic has obtained a lot of attention due to the work of Aihara and Iyama [3], in which a mutation theory for silting objects has been developed. Moreover, a reduction theorem has been proved, which establishes a correspondence between certain

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silting objects in a triangulated category $\mathcal{T}$ and silting objects in its Verdier quotient $\mathcal{T}/\mathcal{S}$ with respect to a thick subcategory $\mathcal{S}$. Various connections between silting objects and other topics in representation theory have been discovered, such as bounded t-structures, co-t-structures, torsion pairs and simple-minded collections etc. (cf. [20, 6] for instance).

The aim of this note is to study the bounded derived category $\mathcal{D}^b(\mathcal{H})$ of a hereditary abelian category $\mathcal{H}$ from the viewpoint of silting theory. It is known that there are triangulated categories which do not admit a silting object. Our first result is a characterization of the existence of silting objects in $\mathcal{D}^b(\mathcal{H})$.

THEOREM 1.1. *Let $\mathcal{H}$ be a hereditary abelian category. The following are equivalent:*

(1) *$\mathcal{H}$ has a tilting object;*
(2) *$\mathcal{D}^b(\mathcal{H})$ has a tilting object;*
(3) *$\mathcal{D}^b(\mathcal{H})$ has a silting object;*
(4) *$\mathcal{D}^b(\mathcal{H})$ has a simple-minded collection whose* Ext*-quiver is acyclic.*

The equivalence between (1) and (2) was proved in [14, Theorem 1.7] and our result yields a new proof for this fact. It is also known that if $\mathcal{D}^b(\mathcal{H})$ admits a 2-term silting object, then $\mathcal{H}$ admits a tilting object (cf. [7, Proposition 3.10]).

Let $\mathcal{T}$ be a Krull–Schmidt triangulated category with silting objects. One of open questions in silting theory is whether a presilting object in $\mathcal{T}$ can be completed to a silting object (cf. [6, Question 3.13] and [2, Question 2.14]). The following result gives a positive answer for the bounded derived category of a hereditary abelian category with tilting objects.

THEOREM 1.2. *Let $\mathcal{H}$ be a hereditary abelian category with tilting objects. Every presilting object of $\mathcal{D}^b(\mathcal{H})$ can be completed to a silting object.*

We remark that Theorem 1.2 is not new. In particular, Brüstle and Yang [6] have suggested a proof by the transitivity of the action of the braided group on exceptional sequences. In [22, 23], the result has been proved for the category $\mathcal{H} = \mathsf{mod}\, H$ for a finite-dimensional hereditary k-algebra H by different methods.

"Simple-minded collection" is a dual notion of "silting object". We consider the analogous question of complements for a pre-simple-minded collection in $\mathcal{D}^b(\mathcal{H})$. In contrast to presilting objects, there are pre-simple-minded collections which cannot be completed to simple-minded collections.

THEOREM 1.3. *Let $\mathcal{H}$ be a hereditary abelian category with tilting objects. A pre-simple-minded collection $\mathcal{X}$ of $\mathcal{D}^b(\mathcal{H})$ can be completed to a simple-minded collection if and only if the* Ext*-quiver of $\mathcal{X}$ is acyclic.*

Our proofs of Theorems 1.1–1.3 are inspired by the reduction approach of [10], where the Iyama–Yoshino reduction was applied to study the connectedness of the cluster-tilting graph of a hereditary abelian category. In the present paper, we apply silting reduction to investigate silting objects in the bounded derived category $\mathcal{D}^b(\mathcal{H})$ of a hereditary abelian category $\mathcal{H}$. A key observation is that the localization of $\mathcal{D}^b(\mathcal{H})$ with respect to the thick subcategory generated by an exceptional object is triangle equivalent to the bounded derived category of another hereditary abelian category (cf. Lemma 3.6 and Theorem 3.8).

The paper is organized as follows. In Section 2, we recall basic results for silting theory and simple-minded collections. In Section 3, we investigate the localization of the bounded derived category of a hereditary abelian category with respect to an exceptional object. In particular, Theorem 3.8 is proved. We present the proofs of Theorems 1.1–1.3 in Section 4.

Notation. Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}, \mathcal{Y}$ two full subcategories of $\mathcal{T}$.

- We always denote by [1] the suspension functor of $\mathcal{T}$ unless otherwise stated.
- Denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects Z which admit a triangle $X \to Z \to Y \to X[1]$, where $X \in \mathcal{X}$, $Y \in \mathcal{Y}$.
- For an integer l, set $\mathcal{X}[l] := \{X[l] \mid X \in \mathcal{X}\}$.
- Denote by $\mathcal{X}[\geq 0]$ (resp. $\mathcal{X}[> 0]$) the subcategory of $\mathcal{T}$ consisting of objects $X[l]$ for $X \in \mathcal{X}$ and $0 \leq l \in \mathbb{Z}$ (resp. $0 < l \in \mathbb{Z}$). Similarly, we define the subcategories $\mathcal{X}[\leq 0]$ and $\mathcal{X}[< 0]$.
- Let $\mathsf{add}\,\mathcal{X}$ be the smallest full subcategory of $\mathcal{T}$ which is closed under finite coproducts, summands, isomorphisms and containing $\mathcal{X}$. If $\mathcal{X}$ consists of a single object X, we simply denote it by $\mathsf{add}\,X$.
- Denote by $\mathsf{thick}(\mathcal{X})$ the thick subcategory of $\mathcal{T}$ containing $\mathcal{X}$.
- If $\mathcal{T}$ is Krull–Schmidt and $M \in \mathcal{T}$, denote by $|M|$ the number of pairwise non-isomorphic indecomposable direct summands of M.

2. Preliminaries

2.1. Perpendicular category and Verdier quotient. Let $\mathcal{T}$ be a Krull–Schmidt triangulated category and $\mathcal{M}$ a subcategory of $\mathcal{T}$. A morphism $f : M \to N$ is a *right $\mathcal{M}$-approximation* of $N \in \mathcal{T}$ if $M \in \mathcal{M}$ and $\mathsf{Hom}_{\mathcal{T}}(M', f)$ is surjective for any $M' \in \mathcal{M}$. The subcategory $\mathcal{M} \subset \mathcal{T}$ is *contravariantly finite* if every object in $\mathcal{T}$ has a right $\mathcal{M}$-approximation. Dually, we define a *left $\mathcal{M}$-approximation* and *covariantly finite subcategory.* We say that $\mathcal{M}$ is *functorially finite* if it is contravariantly finite and covariantly finite.

Define

$$\mathcal{M}^{\perp} := \{N \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(M, N) = 0 \text{ for all } M \in \mathcal{M}\},$$
$$^{\perp}\mathcal{M} := \{N \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(N, M) = 0 \text{ for all } M \in \mathcal{M}\}.$$

The subcategory $\mathcal{M}^{\perp}$ (resp. $^{\perp}\mathcal{M}$) is called the *right* (resp. *left*) *perpendicular* category of $\mathcal{M}$ in $\mathcal{T}$. If $\mathcal{M}$ is a triangulated subcategory of $\mathcal{T}$, then both $\mathcal{M}^{\perp}$ and $^{\perp}\mathcal{M}$ are triangulated subcategories of $\mathcal{T}$.

Recall that a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathcal{T}$ is a *torsion pair* of $\mathcal{T}$ if $\mathsf{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = 0$ and $\mathcal{X} * \mathcal{Y} = \mathcal{T}$. The following useful result is known as Wakamatsu's Lemma (cf. [3, Lemma 2.22]).

LEMMA 2.1. *Let $\mathcal{M}$ be a subcategory of $\mathcal{T}$ such that $\mathcal{M} * \mathcal{M} \subseteq \mathcal{M}$.*

(1) *If $\mathcal{M}$ is contravariantly finite, then $(\mathcal{M}, \mathcal{M}^{\perp})$ is a torsion pair of $\mathcal{T}$.*
(2) *If $\mathcal{M}$ is covariantly finite, then $(^{\perp}\mathcal{M}, \mathcal{M})$ is a torsion pair of $\mathcal{T}$.*

Let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$ and $\mathcal{T}/\mathcal{S}$ the Verdier quotient of $\mathcal{T}$ with respect to $\mathcal{S}$. Denote by $\mathbb{L} : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ the localization functor. We denote by $\iota : \mathcal{S}^{\perp} \hookrightarrow \mathcal{T}$ (resp. $\iota : {}^{\perp}\mathcal{S} \hookrightarrow \mathcal{T}$) the inclusion functor. The following well-known result identifies the Verdier quotient $\mathcal{T}/\mathcal{S}$ with certain subcategories of $\mathcal{T}$.

LEMMA 2.2. *Let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$.*

(1) *If $\mathcal{S}$ is contravariantly finite, then the composition $\mathbb{L} \circ \iota : \mathcal{S}^{\perp} \to \mathcal{T}/\mathcal{S}$ is an equivalence of triangulated categories.*
(2) *If $\mathcal{S}$ is covariantly finite, then the composition $\mathbb{L} \circ \iota : {}^{\perp}\mathcal{S} \to \mathcal{T}/\mathcal{S}$ is an equivalence of triangulated categories.*

Proof. It is straightforward to check that the functor $\mathscr{L} := \mathbb{L} \circ \iota$ induces an isomorphism

$$\mathsf{Hom}_{\mathcal{T}}(X, Y) \cong \mathsf{Hom}_{\mathcal{T}/\mathcal{S}}(\mathscr{L}(X), \mathscr{L}(Y))$$

for any $X, Y \in \mathcal{S}^{\perp}$ (resp. $^{\perp}\mathcal{S}$). On the other hand, the functor $\mathscr{L}$ is dense by Lemma 2.1. ∎

2.2. Silting theory. We follow [3]. For simplicity, we only consider Hom-finite Krull–Schmidt triangulated categories and silting objects.

Let $\mathcal{T}$ be a Hom-finite Krull–Schmidt triangulated category. An object M of $\mathcal{T}$ is a *presilting* object if $\mathsf{Hom}_{\mathcal{T}}(M, M[i]) = 0$ for all $i > 0$. A presilting object M is *silting* if $\mathsf{thick}(M) = \mathcal{T}$. A silting object M of $\mathcal{T}$ is a *tilting object* if $\mathsf{Hom}_{\mathcal{T}}(M, M[i]) = 0$ for $i \neq 0$. It is known that there exist Hom-finite Krull–Schmidt triangulated categories which do not admit silting objects.

Let $T = M \oplus \overline{T}$ be a basic silting object of $\mathcal{T}$ with indecomposable direct summand M. Consider the triangle

$$N \to T_M \xrightarrow{f_M} M \to N[1],$$

where f_M is a minimal right $\mathsf{add}\,\overline{T}$-approximation of M. According to [3, Theorem 2.31], $N \oplus \overline{T}$ is a silting object of $\mathcal{T}$, and $N \oplus \overline{T}$ is called the *right mutation* of T with respect to M. Dually, if we consider the triangle induced by a minimal left $\mathsf{add}\,\overline{T}$-approximation of M, we obtain the *left mutation* of T with respect to M.

Let $\mathcal{T}$ be a Hom-finite Krull–Schmidt triangulated category with a silting object T. It follows from [3, Theorem 2.27] that the Grothendieck group $\mathsf{G}_0(\mathcal{T})$ of $\mathcal{T}$ is a free abelian group of rank $|T|$. In particular, the images of the indecomposable direct summands of T in $\mathsf{G}_0(\mathcal{T})$ form a $\mathbb{Z}$-basis of $\mathsf{G}_0(\mathcal{T})$. As a consequence, each silting object of $\mathcal{T}$ has the same number of pairwise non-isomorphic indecomposable direct summands. A presilting object $M \in \mathcal{T}$ is a *partial silting* object if there is an object $N \in \mathcal{T}$ such that $M \oplus N$ is a silting object. It is an open question whether a presilting object in $\mathcal{T}$ is a partial silting object (cf. [6, Question 3.13]).

We denote by $\mathsf{silt}\,\mathcal{T}$ the set of isomorphism classes of basic silting objects of $\mathcal{T}$. The following reduction theorem plays a central role in our investigation.

THEOREM 2.3 ([3, Theorem 2.37]). *Let $\mathcal{T}$ be a Hom-finite Krull–Schmidt triangulated category, $\mathcal{S}$ a functorially finite thick subcategory of $\mathcal{T}$ and $\mathcal{T}/\mathcal{S}$ the Verdier quotient. Denote by $\mathbb{L} : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ the localization functor. For any $D \in \mathsf{silt}\,\mathcal{S}$, there is a bijective map*

$$\{T \in \mathsf{silt}\,\mathcal{T} \mid D \in \mathsf{add}\,T\} \to \mathsf{silt}\,\mathcal{T}/\mathcal{S}$$

given by $T \mapsto \mathbb{L}(T)$.

Let us recall the inverse map of the bijection following the proof of [3, Theorem 2.37]. Denote by

$$\mathcal{S}_D^{\leq 0} := \bigcup_{l \geq 0} \mathsf{add}\,D * \mathsf{add}\,D[1] * \cdots * \mathsf{add}\,D[l]$$

and $\mathcal{S}_D^{<0} := \mathcal{S}_D^{\leq 0}[1]$. It is known that $\mathcal{S}_D^{<0}$ is covariantly finite in $\mathcal{T}$. Since $\mathcal{S}$ is functorially finite, we may identify $\mathcal{T}/\mathcal{S}$ with $\mathcal{S}^{\perp}$. Let $N \in \mathcal{S}^{\perp}$ be a silting object of $\mathcal{S}^{\perp}$. Consider the triangle

$$S_N \to T_N \to N \xrightarrow{g} S_N[1],$$

where $S_N[1] \in \mathcal{S}_D^{<0}$ and g is a minimal left $\mathcal{S}_D^{<0}$-approximation. According to [3, proof of Theorem 2.37], $T_N \oplus D$ is a silting object of $\mathcal{T}$ such that $\mathbb{L}(T_N \oplus D) = \mathbb{L}(N)$.

2.3. Simple-minded collections. Let $\mathcal{T}$ be a Hom-finite k-linear triangulated category and $\mathcal{X} = \{X_1, \ldots, X_r\}$ a collection of objects. We call $\mathcal{X}$ a *pre-simple-minded collection* (= *pre-SMC*) if the following conditions hold for $i, j = 1, \ldots, r$:

- $\mathsf{Hom}_{\mathcal{T}}(X_i, X_j[m]) = 0$ for any $m < 0$;
- $\mathsf{End}_{\mathcal{T}}(X_i)$ is a division algebra and $\mathsf{Hom}_{\mathcal{T}}(X_i, X_j)$ vanishes for $i \neq j$.

In particular, every object in a pre-SMC is indecomposable. For a pre-SMC $\mathcal{X}$, its Ext-*quiver* $Q_{\mathcal{X}}$ is defined as follows:

- the vertices of $Q_{\mathcal{X}}$ are indexed by objects of $\mathcal{X}$;
- for $X_i, X_j \in \mathcal{X}$, there are $\frac{\dim_k \mathsf{Hom}_{\mathcal{T}}(X_i, X_j[1])}{\dim_k \mathsf{End}_{\mathcal{T}}(X_i)}$ arrows from X_i to X_j.

A pre-SMC $\mathcal{X}$ of $\mathcal{T}$ is a *simple-minded collection* (= SMC) (*cohomologically Schurian* in [4]) if $\mathsf{thick}(\mathcal{X}) = \mathcal{T}$. Similar to the case of silting objects, if $\mathcal{T}$ admits a SMC $\mathcal{X}$, then the Grothendieck group $\mathsf{G}_0(\mathcal{T})$ of $\mathcal{T}$ is a free abelian group of rank $|\mathcal{X}|$. We denote by $\mathsf{SMC}\,\mathcal{T}$ the set of isomorphism classes of SMCs of $\mathcal{T}$.

Let $\mathcal{R}$ be a pre-SMC of $\mathcal{T}$. Denote by $\mathsf{SMC}_{\mathcal{R}}\,\mathcal{T}$ the set of isomorphism classes of SMCs of $\mathcal{T}$ containing $\mathcal{R}$. Let $\mathcal{H}_{\mathcal{R}}$ be the smallest extension-closed subcategory of $\mathcal{T}$ containing $\mathcal{R}$. Define

$$\mathcal{Z} := \mathcal{R}[\geq 0]^{\perp} \cap {}^{\perp}\mathcal{R}[\leq 0].$$

The following reduction theorem for SMCs has been established in [17].

Theorem 2.4 ([17, Theorem 3.1]). *Assume that $\mathcal{H}_{\mathcal{R}}$ satisfies the following conditions:*

- *$\mathcal{H}_{\mathcal{R}}$ is contravariantly finite in $\mathcal{R}[> 0]^{\perp}$ and covariantly finite in ${}^{\perp}\mathcal{R}[< 0]$;*
- *for any $X \in \mathcal{T}$, we have $\mathsf{Hom}_{\mathcal{T}}(X, \mathcal{H}_{\mathcal{R}}[i]) = 0 = \mathsf{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{R}}, X[i])$ for $i \ll 0$.*

Then

(1) *the composition $\mathcal{Z} \hookrightarrow \mathcal{T} \to \mathcal{T}/\mathsf{thick}(\mathcal{R})$ is an additive equivalence $\mathcal{Z} \xrightarrow{\sim} \mathcal{T}/\mathsf{thick}(\mathcal{R})$;*
(2) *there is a bijection*

$$\mathsf{SMC}_{\mathcal{R}}\,\mathcal{T} \to \mathsf{SMC}\,\mathcal{T}/\mathsf{thick}(\mathcal{R})$$

sending $\mathcal{X} \in \mathsf{SMC}_{\mathcal{R}}\,\mathcal{T}$ to $\mathcal{X} \setminus \mathcal{R} \in \mathsf{SMC}\,\mathcal{T}/\mathsf{thick}(\mathcal{R})$.

We may regard $\mathcal{Z}$ as a triangulated category via the additive equivalence $\mathcal{Z} \xrightarrow{\sim} \mathcal{T}/\mathsf{thick}(\mathcal{R})$ in Theorem 2.4(1). Denote by $\langle 1 \rangle$ the suspension functor of $\mathcal{Z}$. Then for each object $Z \in \mathcal{Z}$, $Z\langle 1 \rangle$ is determined by the following triangle of $\mathcal{T}$:

$$R_Z \xrightarrow{f_Z} Z[1] \to Z\langle 1 \rangle \to R_Z[1],$$

where f_Z is a minimal right $\mathcal{H}_{\mathcal{R}}$-approximation of $Z[1]$ (cf. [17, Lemma 3.4]).

The inverse map of the bijection in Theorem 2.4(2) is constructed as follows. Let $\overline{\mathcal{X}}$ be a SMC of $\mathcal{T}/\mathsf{thick}(\mathcal{R})$. Denote by $\hat{\mathcal{X}} \subset \mathcal{Z}$ the preimage of $\overline{\mathcal{X}}$ via the equivalence in (1). Then $\hat{\mathcal{X}} \cup \mathcal{R}$ is a SMC of $\mathcal{T}$, which is the preimage of $\overline{\mathcal{X}}$.

3. Hereditary abelian categories with tilting objects

3.1. Hereditary abelian categories. Let $\mathcal{H}$ be a hereditary abelian category and $\mathcal{D}^b(\mathcal{H})$ the bounded derived category of $\mathcal{H}$. Recall that an object $M \in \mathcal{D}^b(\mathcal{H})$ is *rigid* if $\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(M, M[1]) = 0$. It is *exceptional* if it is rigid and indecomposable. The following fundamental result is due to Happel and Ringel [12].

Lemma 3.1. *Let E and F be indecomposable objects in $\mathcal{H}$ such that*
$$\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(F, E[1]) = 0.$$
Then any non-zero homomorphism $f : E \to F$ is a monomorphism or epimorphism. In particular, the endomorphism ring of an exceptional object is a division algebra.

Let $\mathcal{M}$ be a full subcategory of $\mathcal{D}^b(\mathcal{H})$ and $M \in \mathcal{M}$ an indecomposable object. A path in $\mathcal{M}$ from M to itself is a *cycle* in $\mathcal{M}$, that is, a sequence of non-zero non-isomorphism between indecomposable objects in $\mathcal{M}$ of the form
$$M = M_0 \xrightarrow{f_1} M_1 \to \cdots \xrightarrow{f_r} M_r = M.$$

The following is a consequence of Lemma 3.1 (cf. [9, Lemma 4.2] or [12, Corollary 4.2]).

Lemma 3.2. *Let T be an object in $\mathcal{D}^b(\mathcal{H})$ such that $\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, T[1]) = 0$. Then the subcategory $\mathsf{add}\, T$ has no cycle.*

3.2. Hereditary categories with tilting objects. Let $\mathcal{H}$ be a hereditary abelian category. A rigid object $T \in \mathcal{H}$ is a *tilting object* provided that for $X \in \mathcal{H}$ with $\mathsf{Hom}_{\mathcal{H}}(T, X) = 0 = \mathsf{Ext}^1_{\mathcal{H}}(T, X)$, we have $X = 0$.

Throughout this subsection, we always assume that $\mathcal{H}$ admits a tilting object. As a consequence, the Grothendieck group $\mathsf{G}_0(\mathcal{D}^b(\mathcal{H}))$ is a free abelian group of finite rank, denoted by $\mathsf{rank}\, \mathsf{G}_0(\mathcal{D}^b(\mathcal{H}))$. If $T \in \mathcal{H}$ is a tilting object, then $\mathsf{rank}\, \mathsf{G}_0(\mathcal{D}^b(\mathcal{H})) = |T|$. The existence of tilting objects also implies that $\mathcal{D}^b(\mathcal{H})$ admits almost split triangles and hence $\mathcal{H}$ has almost split sequences [15]. Denote by $\tau : \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\mathcal{H})$ the Auslander–Reiten (= AR) translation functor, which restricts to the AR translation $\tau : \mathcal{H} \to \mathcal{H}$. The following is a reformulation of [13, Proposition 1.2(a)].

Proposition 3.3. *Assume that $\mathcal{H}$ is connected. If $\mathcal{H}$ is not equivalent to $\mathsf{mod}\, H$ for a finite-dimensional hereditary k-algebra H, then $\mathcal{H}$ has neither non-zero projective objects nor non-zero injective objects. Consequently, the AR translation $\tau : \mathcal{H} \to \mathcal{H}$ is an equivalence.*

Proof. According to [13, Proposition 1.2(a)], $\mathcal{H}$ has no non-zero projective objects. Hence for each indecomposable $X \in \mathcal{H}$, we have $\tau X \in \mathcal{H}$. Let X be an indecomposable object of $\mathcal{H}$. Let $X \to E \to \tau^{-1} X \xrightarrow{h} X[1]$ be the almost split triangle starting at X. To show that X is not injective, it suffices to show that $\tau^{-1}X \in \mathcal{H}$. Since h is non-zero, we conclude that either $\tau^{-1}X \in \mathcal{H}$ or $\tau^{-1}X \in \mathcal{H}[1]$. Suppose that $\tau^{-1}X \in \mathcal{H}[1]$. Then $\tau^{-1}X[-1] \in \mathcal{H}$. We have $X[-1] = \tau(\tau^{-1}X[-1]) \in \mathcal{H}$, a contradiction. In particular, $\mathcal{H}$ has no non-zero injective objects. ∎

We also have the following result from [11, Lemma 3.7]; the proof there is valid for an arbitrary field.

LEMMA 3.4. *Let $\mathcal{H}$ be a hereditary abelian category with tilting object. Let $M \in \mathcal{H}$ be a rigid object such that $|M| = \operatorname{rank} G_0(\mathcal{D}^b(\mathcal{H}))$. Then M is a tilting object.*

A rigid object $M \in \mathcal{H}$ is a *partial tilting* if there is an object $N \in \mathcal{H}$ such that $M \oplus N$ is a tilting object. The following seems to be known for experts, which has been proved by Happel and Unger [16, Proposition 3.1] over an algebraically closed field. Here we sketch a proof for arbitrary fields by cluster-tilting theory; we refer to [8] for all unexplained terminology in cluster-tilting theory.

PROPOSITION 3.5. *Each rigid object of $\mathcal{H}$ is a partial tilting object.*

Proof. Without loss of generality, we may assume that $\mathcal{H}$ is connected. Let M be a rigid object of $\mathcal{H}$. If $\mathcal{H} = \operatorname{mod} H$ for a finite-dimensional hereditary k-algebra H, it is well-known that M is a partial tilting module by classical tilting theory.

Assume that $\mathcal{H}$ is not equivalent to $\operatorname{mod} H$ for a finite-dimensional hereditary k-algebra H. By Proposition 3.3, $\mathcal{H}$ has neither non-zero projective objects nor non-zero injective objects. Let $\mathcal{C}(\mathcal{H}) := \mathcal{D}^b(\mathcal{H})/\tau^{-1} \circ [1]$ be the cluster category of $\mathcal{H}$, i.e. the orbit category of $\mathcal{D}^b(\mathcal{H})$ under the equivalence $\tau^{-1} \circ [1]$ (cf. [18]). The cluster category $\mathcal{C}(\mathcal{H})$ admits a canonical triangle structure such that the projection functor $\pi : \mathcal{D}^b(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ is a triangle functor. The projection functor π induces a bijection between the set of isomorphism classes of objects of $\mathcal{H}$ and the set of isomorphism classes of objects of $\mathcal{C}(\mathcal{H})$ (cf. [8, Section 3]). Moreover, an object $X \in \mathcal{H}$ is rigid if and only if $\pi(X)$ is rigid in $\mathcal{C}(\mathcal{H})$ (cf. [8, 24]).

Let T be a tilting object of $\mathcal{H}$. Then $\pi(T)$ is a cluster-tilting object of $\mathcal{C}(\mathcal{H})$ (cf. [5]). Since M is rigid, $\pi(M)$ is rigid, which can be completed to a cluster-tilting object of $\mathcal{C}(\mathcal{H})$ by [1, Theorem 4.1], i.e. there is $N \in \mathcal{H}$ such that $\pi(M) \oplus \pi(N)$ is a cluster-tilting object of $\mathcal{C}(\mathcal{H})$. As a consequence, $M \oplus N$ is rigid. According to [1, Corollary 4.5], $|M \oplus N| = |\pi(M) \oplus \pi(N)| = |T|$. We conclude that $M \oplus N$ is a tilting object of $\mathcal{H}$ by Lemma 3.4. ∎

3.3. Perpendicular category and localization. Let $\mathcal{H}$ be a hereditary abelian category and $E \in \mathcal{H}$ an exceptional object. Define

$$E^{\perp} := \{X \in \mathcal{H} \mid \operatorname{Hom}_{\mathcal{H}}(E, X) = 0 = \operatorname{Ext}^1_{\mathcal{H}}(E, X)\},$$

$$^{\perp}E := \{X \in \mathcal{H} \mid \operatorname{Hom}_{\mathcal{H}}(X, E) = 0 = \operatorname{Ext}^1_{\mathcal{H}}(X, E)\}.$$

It is straightforward to check that $E^{\perp}$ and ${}^{\perp}E$ are hereditary abelian subcategories of $\mathcal{H}$.

LEMMA 3.6. *Let $\mathcal{H}$ be a hereditary abelian category and $E \in \mathcal{H}$ an exceptional object. We have*

$$\mathcal{D}^b(E^{\perp}) \cong \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(E) \quad \textit{and} \quad \mathcal{D}^b({}^{\perp}E) \cong \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(E).$$

Proof. Since E is exceptional, according to Lemma 3.1, the indecomposable objects of $\mathsf{thick}(E)$ are precisely $E[i]$, $i \in \mathbb{Z}$. It follows that $\mathsf{thick}(E)$ is a functorially finite subcategory of $\mathcal{D}^b(\mathcal{H})$. By Lemma 2.2, we have $\mathsf{thick}(E)^{\perp} \cong \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(E)$. Consider the inclusion functor $\iota : E^{\perp} \hookrightarrow \mathcal{H}$, which induces a fully faithful triangle functor $\hat{\iota} : \mathcal{D}^b(E^{\perp}) \hookrightarrow \mathcal{D}^b(\mathcal{H})$. It is straightforward to check that the image of $\hat{\iota}$ coincides with $\mathsf{thick}(E)^{\perp}$. Consequently, $\mathcal{D}^b(E^{\perp}) \cong \mathsf{thick}(E)^{\perp} \cong \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(E)$. Similarly, one can prove $\mathcal{D}^b({}^{\perp}E) \cong \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(E)$. ■

LEMMA 3.7. *Let $\mathcal{H}$ be a hereditary abelian category and $E \in \mathcal{H}$ an exceptional object. Assume that $E^{\perp}$ has a tilting object M. Then $M \oplus E[1]$ is a silting object of $\mathcal{D}^b(\mathcal{H})$. Moreover, the right mutation of $M \oplus E[1]$ with respect to $E[1]$ is a tilting object of $\mathcal{H}$.*

Proof. Since $M \in E^{\perp}$ is a tilting object and $\mathcal{H}$ is hereditary, we have

$$\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(M \oplus E[1], M[i] \oplus E[i+1]) = 0 \quad \text{for all } i > 0.$$

Recall that $\mathsf{thick}(E)$ is a functorially finite subcategory of $\mathcal{D}^b(\mathcal{H})$. For any $X \in \mathcal{D}^b(\mathcal{H})$, consider the triangle

$$E_X \xrightarrow{f_X} X \to Z \to E_X[1],$$

where f_X is a minimal right $\mathsf{thick}(E)$-approximation of X. According to Lemma 2.1, there is a triangle

$$X_E \xrightarrow{f} X \to Y \to X_E[1],$$

where $X_E \in \mathsf{thick}(E)$ and $Y \in \mathsf{thick}(E)^{\perp}$. Notice that f is also a right $\mathsf{thick}(E)$-approximation of X. Consequently, we obtain the commutative diagram of triangles

$$\begin{array}{ccccccc}
E_X & \xrightarrow{f_X} & X & \longrightarrow & Z & \longrightarrow & E_X[1] \\
\downarrow{\scriptstyle \exists g_1} & & \| & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle g_1[1]} \\
X_E & \xrightarrow{f} & X & \longrightarrow & Y & \longrightarrow & X_E[1] \\
\downarrow{\scriptstyle \exists g_2} & & \| & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle g_2[1]} \\
E_X & \xrightarrow{f_X} & X & \longrightarrow & Z & \longrightarrow & E_X[1]
\end{array}$$

Since f_X is minimal, $g_2 \circ g_1$ is an isomorphism and hence $h_2 \circ h_1$ is an isomorphism. Consequently, Z is a direct summand of Y, which implies that $Z \in \mathsf{thick}(E)^{\perp}$. As in the proof of Lemma 3.6, we may identify $\mathcal{D}^b(E^{\perp})$

with $\mathsf{thick}(E)^{\perp}$. By the assumption that M is a tilting object of $E^{\perp}$, we conclude that $\mathsf{thick}(M) = \mathsf{thick}(E)^{\perp}$. Consequently, $X \in \mathsf{thick}(M \oplus E[1])$. Hence $\mathsf{thick}(M \oplus E[1]) = \mathcal{D}^b(\mathcal{H})$. In particular, $M \oplus E[1]$ is a silting object of $\mathcal{D}^b(\mathcal{H})$.

Let $M' \xrightarrow{f_{E[1]}} E[1]$ be a minimal right $\mathsf{add}\, M$-approximation of $E[1]$, which fits into the triangle

$$E \to N \to M' \xrightarrow{f_{E[1]}} E[1].$$

It follows from [3, Theorem 2.31] that $M \oplus N$ is a silting object of $\mathcal{D}^b(\mathcal{H})$. Since $\mathcal{H}$ is an extension-closed subcategory of $\mathcal{D}^b(\mathcal{H})$, we conclude that $N \in \mathcal{H}$. It is straightforward to show that $N \oplus M$ is a tilting object of $\mathcal{H}$. Indeed, since $N \oplus M$ is a silting object, we have $\mathsf{Ext}^1_{\mathcal{H}}(N \oplus M, N \oplus M) = 0$ and $\mathsf{thick}(N \oplus M) = \mathcal{D}^b(\mathcal{H})$. Let $L \in \mathcal{H}$ be such that $\mathsf{Hom}_{\mathcal{H}}(N \oplus M, L) = 0 = \mathsf{Ext}^1_{\mathcal{H}}(N \oplus M, L)$. It is clear that $\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(N \oplus M, L[i]) = 0$ for all $i \in \mathbb{Z}$. Consequently, for any object $Y \in \mathsf{thick}(N \oplus M) = \mathcal{D}^b(\mathcal{H})$, we have $\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(Y, L) = 0$. In particular, $\mathsf{Hom}_{\mathcal{H}}(L, L) = 0$, which implies $L = 0$. This completes the proof. ∎

The following result plays a fundamental role in our reduction approach to silting objects in $\mathcal{D}^b(\mathcal{H})$.

THEOREM 3.8. *Let $\mathcal{H}$ be a hereditary abelian category with tilting objects and $E \in \mathcal{H}$ an exceptional object. There is a hereditary abelian category $\mathcal{H}'$ with tilting objects such that $\mathcal{D}^b(\mathcal{H}')$ is triangle equivalent to $\mathcal{D}^b(\mathcal{H})/\mathsf{thick}(E)$. Moreover,* $\mathsf{rank}\, G_0(\mathcal{D}^b(\mathcal{H})) = \mathsf{rank}\, G_0(\mathcal{D}^b(\mathcal{H}')) + 1$.

Proof. Without loss of generality, we may assume that $\mathcal{H}$ is connected. By [13, Theorem 3.5], $\mathcal{H}$ is derived equivalent either to the category $\mathsf{mod}\, H$ of a finite-dimensional hereditary k-algebra H, or to the category $\mathsf{coh}\, \mathbb{X}$ of coherent sheaves over an exceptional curve $\mathbb{X}$ in the sense of Lenzing [21]. Since we are working with derived categories, we may further assume that $\mathcal{H} = \mathsf{mod}\, H$ or $\mathcal{H} = \mathsf{coh}\, \mathbb{X}$.

Let us consider the case $\mathcal{H} = \mathsf{coh}\, \mathbb{X}$. By Proposition 3.5, there is an object $M \in \mathcal{H}$ such that $E \oplus M$ is a tilting object of $\mathcal{H}$. Applying [13, Proposition 1.4], we conclude that $\mathcal{H}' := E^{\perp}$ is a connected hereditary abelian category with tilting object. By Lemma 3.6, $\mathcal{D}^b(\mathcal{H}') \cong \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(E)$. Let N be a basic tilting object of $\mathcal{H}' = E^{\perp}$. Lemma 3.7 shows that $N \oplus E[1]$ is a basic silting object of $\mathcal{D}^b(\mathcal{H})$. Consequently,

$$\mathsf{rank}\, G_0(\mathcal{D}^b(\mathcal{H})) = \mathsf{rank}\, G_0(\mathcal{D}^b(\mathcal{H}')) + 1.$$

Now we turn to the case $\mathcal{H} = \mathsf{mod}\, H$. We may assume that E is not projective. There is an H-module M such that $E \oplus M$ is a tilting object of $\mathcal{H}$ and M is a tilting object of $E^{\perp}$. In particular, $E^{\perp}$ is a hereditary abelian category with tilting object M, and the result follows from Lemma 3.6. ∎

4. Proofs of the main results

4.1. Proof of Theorem 1.1. (1)⇒(2)⇒(3) are obvious.

(3)⇒(1): Since $\mathcal{D}^b(\mathcal{H})$ admits silting objects, $\mathsf{G}_0(\mathcal{D}^b(\mathcal{H}))$ is a free abelian group of finite rank. We argue by induction on $n := \operatorname{rank} \mathsf{G}_0(\mathcal{D}^b(\mathcal{H}))$. The statement is clear for $n = 1$. Indeed, let M be a silting object of $\mathcal{D}^b(\mathcal{H})$ which is indecomposable. We may assume that $M \in \mathcal{H}$. Consequently, M is a tilting object of $\mathcal{H}$ by Lemma 3.4.

Now assume that the statement is true for $n < n_0$. Let $\mathcal{H}$ be a hereditary abelian category with $\operatorname{rank} \mathsf{G}_0(\mathcal{D}^b(\mathcal{H})) = n_0$. Let $T = E \oplus \overline{T}$ be a basic silting object of $\mathcal{D}^b(\mathcal{H})$, where E is indecomposable. Again, we may assume that $E \in \mathcal{H}$. In particular, E is an exceptional object. By Lemma 3.6, we have $\mathcal{D}^b(\mathcal{H})/\mathsf{thick}(E) \cong \mathcal{D}^b(E^{\perp})$. Clearly, E is a silting object of $\mathsf{thick}(E)$ and $\mathsf{thick}(E)$ is functorially finite in $\mathcal{D}^b(\mathcal{H})$. Denote by $\mathbb{L} : \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(E)$ the localization functor. By Theorem 2.3, $\mathbb{L}(\overline{T})$ is a silting object of $\mathcal{D}^b(\mathcal{H})/\mathsf{thick}(E) \cong \mathcal{D}^b(E^{\perp})$. Consequently, $\operatorname{rank} \mathsf{G}_0(\mathcal{D}^b(E^{\perp})) = n_0 - 1$. By induction, $E^{\perp}$ has a tilting object. We conclude that $\mathcal{H}$ has a tilting object by Lemma 3.7.

(1)⇒(4): Let T be a basic tilting object of $\mathcal{H}$. Denote by $A = \mathsf{End}_{\mathcal{H}}(T)$ the endomorphism algebra of T. Denote by $S_1, \ldots, S_n$ the pairwise non-isomorphic simple A-modules. We have an equivalence of triangulated categories $\mathbb{F} : \mathcal{D}^b(\mathsf{mod}\, A) \to \mathcal{D}^b(\mathcal{H})$. Clearly, $\mathbb{F}(S_1), \ldots, \mathbb{F}(S_n)$ is a SMC of $\mathcal{D}^b(\mathcal{H})$. Moreover, its Ext-quiver is acyclic by Lemma 3.2.

(4)⇒(1): Since $\mathcal{D}^b(\mathcal{H})$ admits a SMC, $\mathsf{G}_0(\mathcal{D}^b(\mathcal{H}))$ is a free abelian group of finite rank. We use induction on $n := \operatorname{rank} \mathsf{G}_0(\mathcal{D}^b(\mathcal{H}))$. If $n = 1$, let $\mathcal{X}$ be a SMC of $\mathcal{D}^b(\mathcal{H})$ which consists of exactly one object S. Since the Ext-quiver of S is acyclic, we conclude that S is exceptional. We may assume that $S \in \mathcal{H}$. As a consequence, S is a tilting object of $\mathcal{H}$ by Lemma 3.4.

Assume that the statement is true for $n < n_0$. Let $\mathcal{H}$ be a hereditary abelian category with $\operatorname{rank} \mathsf{G}_0(\mathcal{D}^b(\mathcal{H})) = n_0$ and $\mathcal{X} = \{X_1, \ldots, X_{n_0}\}$ a SMC of $\mathcal{D}^b(\mathcal{H})$ with acyclic Ext-quiver. Since the Ext-quiver of $\mathcal{X}$ is acyclic, $X_1, \ldots, X_{n_0}$ are exceptional. Furthermore, we may renumber them so that

$$\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(X_i, X_j[1]) = 0 \quad \text{whenever } i > j.$$

We may assume that $X_1 \in \mathcal{H}$. Denote by $\mathcal{H}_{X_1}$ the smallest extension-closed subcategory of $\mathcal{D}^b(\mathcal{H})$ containing X_1. Since X_1 is exceptional, we have $\mathcal{H}_{X_1} = \mathsf{add}\, X_1$. Consequently, $\mathcal{H}_{X_1}$ satisfies the conditions of Theorem 2.4. Let

$$\mathcal{Z} := X_1[\geq 0]^{\perp} \cap {}^{\perp}X_1[\leq 0].$$

Since $\mathcal{X}$ is a SMC, we deduce that $X_2, \ldots, X_{n_0} \in \mathcal{Z}$. By Theorem 2.4, $\{X_2, \ldots, X_{n_0}\}$ is a SMC of $\mathcal{Z} \cong \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(X_1)$. According to Lemma 3.6, $\mathcal{D}^b(\mathcal{H})/\mathsf{thick}(X_1) \cong \mathcal{D}^b(X_1^{\perp})$. Hence $\operatorname{rank} \mathsf{G}_0(\mathcal{D}^b(X_1^{\perp})) = n_0 - 1$.

Recall that $\langle 1\rangle$ is the suspension functor of $\mathcal{Z}$. For every X_i with $2 \leq i \leq n_0$, we have a triangle $R_{X_i} \xrightarrow{f_i} X_i[1] \to X_i\langle 1\rangle \to R_{X_i}[1]$ in $\mathcal{D}^b(\mathcal{H})$, where f_i is a minimal right $\mathsf{add}\, X_1$-approximation of $X_i[1]$. Applying $\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(X_j, -)$ to the above triangle, we obtain a long exact sequence

$$\cdots \to \mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(X_j, X_i[1]) \to \mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(X_j, X_i\langle 1\rangle) \to \mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(X_j, R_{X_i}[1]) \to \cdots .$$

Consequently, for any $1 < i \leq j$, we obtain

$$\mathsf{Hom}_{\mathcal{Z}}(X_j, X_i\langle 1\rangle) = \mathsf{Hom}_{\mathcal{D}^b(H)}(X_j, X_i\langle 1\rangle) = 0.$$

In particular, the Ext-quiver of the SMC $\{X_2, \dots, X_{n_0}\}$ of $\mathcal{Z} \cong \mathcal{D}^b(X_1^{\perp})$ is acyclic. By induction, $X_1^{\perp}$ has a tilting object. We conclude that $\mathcal{H}$ has a tilting object by Lemma 3.7.

4.2. Proof of Theorem 1.2. We prove this statement by induction on $n := \mathsf{rank}\, \mathsf{G}_0(\mathcal{D}^b(\mathcal{H}))$. The statement is clear for $n = 1$. Indeed, $\mathcal{H} = \mathsf{add}\, E$ in this case, where E is an exceptional object of $\mathcal{H}$. Moreover, E is a tilting object of $\mathcal{H}$. Hence every indecomposable presilting object of $\mathcal{D}^b(\mathcal{H})$ is a silting object.

Assume that the statement is true for $n < n_0$. Let $\mathcal{H}$ be a hereditary abelian category with tilting objects such that $\mathsf{rank}\, \mathsf{G}_0(\mathcal{D}^b(\mathcal{H})) = n_0$. Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic presilting object of $\mathcal{D}^b(\mathcal{H})$ with indecomposable direct summands $T_1, \dots, T_r$. We may assume that the $T_i \in \mathcal{H}[t_i]$ are such that

- $t_1 \leq t_2 \leq \cdots \leq t_r = 0$;
- $\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(T_i, T_j) = 0$ whenever $i > j$ by Lemma 3.2.

Set $\overline{T} := T_1 \oplus \cdots \oplus T_{r-1}$. Clearly, $\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(T_r[i], \overline{T}) = 0$ for all $i \in \mathbb{Z}$. In particular, $\overline{T} \in \mathsf{thick}(T_r)^{\perp}$. Recall that $\mathsf{thick}(T_r)^{\perp} \cong \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(T_r)$ by Lemma 3.6. As a consequence, $\overline{T}$ is a presilting object of $\mathcal{D}^b(\mathcal{H})/\mathsf{thick}(T_r)$. On the other hand, by Theorem 3.8, $\mathcal{D}^b(\mathcal{H})/\mathsf{thick}(T_r)$ is triangle equivalent to $\mathcal{D}^b(\mathcal{H}')$ for a hereditary abelian category $\mathcal{H}'$ with tilting objects such that $\mathsf{rank}\, \mathsf{G}_0(\mathcal{D}^b(\mathcal{H}')) = n_0 - 1$. By induction, every presilting object of $\mathcal{D}^b(\mathcal{H}')$ is a partial silting object. Consequently, there is an $N \in \mathsf{thick}(T_r)^{\perp}$ such that $\overline{T} \oplus N$ is a silting object of $\mathsf{thick}(T_r)^{\perp} \cong \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(T_r)$. By Theorem 2.3, there is a silting object M of $\mathcal{T}$ such that $T_r \in \mathsf{add}\, M$ and $\mathbb{L}(M) = \mathbb{L}(\overline{T} \oplus N)$, where $\mathbb{L} : \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(T_r)$ is the localization functor. It remains to show that $\overline{T} \in \mathsf{add}\, M$.

Set $\mathcal{S}_{T_r}^{\leq 0} = \bigcup_{l \geq 0} \mathsf{add}\, T_r * \mathsf{add}\, T_r[1] * \cdots * \mathsf{add}\, T_r[l]$ and $\mathcal{S}_{T_r}^{< 0} = \mathcal{S}_{T_r}^{\leq 0}[1]$. Since $T = \overline{T} \oplus T_r$ is a presilting object, $\mathsf{Hom}_{\mathcal{T}}(\overline{T}, X) = 0$ for any $X \in \mathcal{S}_{T_r}^{< 0}$. By the construction of M (cf. Section 2.2), $\overline{T}$ is a direct summand of M. In particular, T is a partial silting object.

4.3. Proof of Theorem 1.3. Let T be a basic tilting object of $\mathcal{H}$. Set $\Lambda = \mathsf{End}_{\mathcal{H}}(T)$, which has global dimension at most 2. It is well-known that $\mathcal{D}^b(\mathsf{mod}\,\Lambda) \cong \mathcal{D}^b(\mathcal{H})$. By [20, Theorem 6.1], there is a one-to-one correspondence between $\mathsf{silt}\,\mathcal{D}^b(\mathcal{H})$ and $\mathsf{SMC}\,\mathcal{D}^b(\mathcal{H})$. Let P be a basic silting object of $\mathcal{D}^b(\mathcal{H})$ and $A := \mathsf{End}_{\mathcal{D}^b(\mathcal{H})}(P)$. Again by [20, Theorem 6.1], P determines a bounded t-structure of $\mathcal{D}^b(\mathcal{H})$ whose heart is equivalent to $\mathsf{mod}\,A$ (cf. also [9, Section 2.4]). Via this equivalence, the simple A-modules form a SMC of $\mathcal{D}^b(\mathcal{H})$ corresponding to P. According to Lemma 3.2, the Ext-quiver of an arbitrary SMC of $\mathcal{D}^b(\mathcal{H})$ is acyclic. In particular, this implies the "only if" part.

It remains to prove the "if" part. We use induction on $n := \mathsf{rank}\,\mathsf{G}_0(\mathcal{D}^b(H))$. The case $n = 1$ is clear. Assume that the assertion is true for $n < n_0$. Let $\mathcal{H}$ be a hereditary abelian category with tilting objects such that $\mathsf{rank}\,\mathsf{G}_0(\mathcal{D}^b(\mathcal{H})) = n_0$. Let $\mathcal{X} = \{X_1, \ldots, X_r\}$ be a pre-SMC of $\mathcal{D}^b(\mathcal{H})$ whose Ext-quiver is acyclic. We may assume that $X_1 \in \mathcal{H}$ and

$$\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(X_i, X_j[1]) = 0 \quad \text{whenever } i \geq j.$$

Set $\mathcal{R} := \{X_1\}$, which is a pre-SMC of $\mathcal{D}^b(\mathcal{H})$. Denote by $\mathcal{H}_{\mathcal{R}}$ the smallest extension-closed subcategory of $\mathcal{D}^b(\mathcal{H})$ containing $\mathcal{R}$. Since X_1 is exceptional, we have $\mathcal{H}_{\mathcal{R}} = \mathsf{add}\,X_1$. Consequently, $\mathcal{H}_{\mathcal{R}}$ satisfies the conditions of Theorem 2.4 and there is a bijection between $\mathsf{SCM}_{\mathcal{R}}\,\mathcal{D}^b(\mathcal{H})$ and $\mathsf{SCM}\,\mathcal{D}^b(\mathcal{H})/\mathsf{thick}(X_1)$. Set $\mathcal{Z} := \mathcal{R}[\geq 0]^{\perp} \cap {}^{\perp}\mathcal{R}[\leq 0]$. By definition of pre-SMC, clearly $X_2, \ldots, X_r \in \mathcal{Z}$. Recall that $\langle 1 \rangle$ is the suspension functor of $\mathcal{Z}$. By definition of $\langle 1 \rangle$ (cf. Section 2.3), for each object $X \in \mathcal{Z}$ and $n > 0$, we have $X\langle n \rangle \in X[n] * \mathsf{add}\,X_1[n] * \cdots * \mathsf{add}\,X_1[1]$. As a consequence, we obtain $\mathsf{Hom}_{\mathcal{Z}}(X_i\langle n \rangle, X_j) = 0$ for $2 \leq i, j \leq r$ and $n > 0$. In particular, $\{X_2, \ldots, X_r\}$ is a pre-SMC of $\mathcal{Z}$. Let $\mathbb{L} : \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\mathcal{H})/\mathsf{thick}(X_1)$ be the localization functor. Theorem 2.4(1) shows that $\{\mathbb{L}(X_2), \ldots, \mathbb{L}(X_r)\}$ is a pre-SMC of $\mathcal{D}^b(\mathcal{H})/\mathsf{thick}(X_1)$. Similar to the proof of (4)⇒(1) in Theorem 1.1, one can show that the Ext-quiver of $\{\mathbb{L}(X_2), \ldots, \mathbb{L}(X_r)\}$ is acyclic. According to Theorem 3.8, $\mathcal{D}^b(\mathcal{H})/\mathsf{thick}(X_1) \cong \mathcal{D}^b(\mathcal{H}')$, where $\mathcal{H}'$ is a hereditary abelian category with tilting objects such that $\mathsf{rank}\,\mathsf{G}_0(\mathcal{D}^b(\mathcal{H}')) = n_0 - 1$. By induction, $\{\mathbb{L}(X_2), \ldots, \mathbb{L}(X_r)\}$ can be completed to a SMC of $\mathcal{D}^b(\mathcal{H})/\mathsf{thick}(X_1)$. Now the result follows from Theorem 2.4(2).

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