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Duality for double iterated outer L^p spaces

by

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Abstract. We study the double iterated outer L^p spaces, that is, the outer L^p spaces associated with three exponents and defined on sets endowed with a measure and two outer measures. We prove that in the case of finite sets, under certain relations between the outer measures, the double iterated outer L^p spaces are isomorphic to Banach spaces uniformly in the set, the measure, and the outer measures. We achieve this by showing the expected duality properties between them. We also provide counterexamples demonstrating that the uniformity does not hold in arbitrary settings on finite sets without further assumptions, at least in a certain range of exponents. We prove the isomorphism to Banach spaces and the duality properties between the double iterated outer L^p spaces also in the upper half 3-space infinite setting described by Uraltsev, going beyond the case of finite sets.

1. Introduction. The theory of L^p spaces for outer measures, or outer L^p spaces, was introduced by Do and Thiele [10] in the context of time-frequency analysis. It provides a framework to encode the boundedness of linear and multilinear operators satisfying certain symmetries in a two-step programme. The programme consists of a version of Hölder's inequality for outer L^p spaces together with the boundedness of certain embedding maps between classical and outer L^p spaces associated with wave packet decompositions. This scheme of proof turns out to be applicable not only in time-frequency analysis, see for example [1–3, 5, 6, 9, 14–16], but in other contexts too, see for example [7, 8, 10–13].

Although the theory of outer L^p spaces comes in a broad generality of settings, the outer L^p spaces used in [10] are specifically defined by quasi-norms reminiscent in nature of iterated Lebesgue norms. In particular, the two Lebesgue norms involved in the definition of outer L^p quasi-norms are associated with the two structures on a set provided by a measure and an outer measure. We recall that an *outer measure* μ on a set X is a monotone,

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subadditive function from $\mathcal{P}(X)$, the power set of X, to the extended positive half-line, attaining the value 0 on the empty set. Similarly, in [14] Uraltsev considered outer L^p spaces associated with three structures on a set, namely a measure and two outer measures, once again in the context of time-frequency analysis and in the spirit of the aforementioned two-step programme. Outer L^p spaces associated with three structures were used in [1–3, 6, 14–16].

As a matter of fact, one can define outer L^p spaces associated with arbitrary $n+1$ structures on a set, namely a measure and n outer measures. We refer to these spaces as *iterated outer L^p spaces*, and we provide a definition in detail. We start by recalling the classical product of L^p spaces on a set with a Cartesian product structure. Given a collection of couples (X_i, ω_i) of finite sets with strictly positive weights, we define recursively the product L^p quasi-norms for functions on their Cartesian product as follows. For any $n \in \mathbb{N}$, let

$$Y^n = \prod_{i=1}^{n} X_i,$$

where, for $n = 0$, the empty Cartesian product is meant to be $\{\emptyset\}$. Note that, for any $x \in X_n$, a function f on Y^n defines a function $f(\cdot, x)$ on Y^{n-1}. Given a collection of exponents $p_i \in (0, \infty]$, we define the classical product $\mathbb{L}_n$ quasi-norm of a function f on Y^n, where

$$\mathbb{L}_n = L^{p_n}_{\omega_n}(L^{p_{n-1}}_{\omega_{n-1}}(\dots L^{p_1}_{\omega_1})),$$

by the recursion

$$\|f(x)\|_{\mathbb{L}_0} = |f(x)|, \tag{1.1}$$

$$\|f\|_{\mathbb{L}_n} = \big\| \|f(\cdot, x)\|_{\mathbb{L}_{n-1}} \big\|_{L^{p_n}(X_n, \omega_n)}. \tag{1.2}$$

The theory of outer L^p spaces allows for a generalization of this definition to settings where the underlying set has no Cartesian product structure. For the purpose of this paper, we provide the definition of the iterated outer L^p quasi-norms in the form of a recursion analogous to that in (1.1), (1.2).

Let X be a finite set together with a collection of outer measures μ_i on it. To avoid cumbersome details, we make the harmless assumption that every μ_i is finite and strictly positive on every nonempty element of $\mathcal{P}(X)$. In fact, it is reasonable that subsets of X on which either of the outer measures is 0 or ∞ should contribute only trivially to the iterated outer L^p spaces on X, and we ignore them altogether. Throughout the paper, we avoid recalling this assumption, but the reader should always consider it implicitly stated whenever we refer to outer measures.

Given a collection of exponents $p_i \in (0, \infty]$, we define the iterated outer $\mathbf{L}_n$ quasi-norm of a function f on X, where

$$\mathbf{L}_n = L^{p_n}_{\mu_n}(\ell^{p_{n-1}}_{\mu_{n-1}}(\dots \ell^{p_1}_{\mu_1})),$$

by the recursion

$$\|f\|_{\mathbf{L}_0} = \sup_{x\in X} |f(x)|, \tag{1.3}$$

$$\mathbf{I}_n(f) = \sup_{\emptyset\neq A\subseteq X} \mu_n(A)^{-(p_{n-1})^{-1}} \|f1_A\|_{\mathbf{L}_{n-1}}, \tag{1.4}$$

$$\|f\|_{\mathbf{L}_n} = \begin{cases} \mathbf{I}_n(f) & \text{if } p_n = \infty, \\ \left[\int_0^\infty p_n \lambda^{p_n} \inf\{\mu_n(B)\colon \mathbf{I}_n(f1_{B^c}) \le \lambda\} \frac{\mathrm{d}\lambda}{\lambda}\right]^{1/p_n} & \text{if } p_n \neq \infty, \end{cases} \tag{1.5}$$

where $p_0 = \infty$, and the exponent ∞^{-1} is understood to be 0. We refer to the space defined by the quantity in (1.5) as the *iterated outer L^p space* $\mathbf{L}_n$ or $L^{p_n}_{\mu_n}(\ell^{p_{n-1}}_{\mu_{n-1}}(\dots \ell^{p_1}_{\mu_1}))$, where we define the argument of the supremum in (1.4) as

$$\ell^{p_{n-1}}_{\mu_{n-1}}(\dots \ell^{p_1}_{\mu_1})(f)(A) = \mu_n(A)^{-(p_{n-1})^{-1}} \|f1_A\|_{\mathbf{L}_{n-1}}, \tag{1.6}$$

and the infimum in (1.5) as

$$\mu_n(\ell^{p_{n-1}}_{\mu_{n-1}}(\dots \ell^{p_1}_{\mu_1})(f) > \lambda) = \inf\{\mu_n(B)\colon \mathbf{I}_n(f1_{B^c}) \le \lambda\}. \tag{1.7}$$

In the language of the L^p theory for outer measure spaces, the quantity in (1.6) defines a *size*, and that in (1.7) defines the *super level measure* of a function f at level λ with respect to the size.

If the outer measure μ_1 is a measure ω, then for every $p_1 \in (0,\infty]$,

$$\|f\|_{\mathbf{L}_1} = \|f\|_{L^{p_1}(X,\omega)},$$

hence we can begin the recursion in (1.3)–(1.5) from $\mathbf{L}_1$. In fact, the general case already had this form. The quasi-norm defined by the collections of outer measures μ_i and exponents p_i is the same one defined by the collections of outer measures $\widetilde{\mu}_i$ and exponents $\widetilde{p}_i$, where $\widetilde{\mu}_1$ is the counting measure, $\widetilde{p}_1 = \infty$, and $\widetilde{\mu}_{i+1} = \mu_i$, $\widetilde{p}_{i+1} = p_i$ for every $i \in \mathbb{N}$. Therefore, without loss of generality, we always assume that μ_1 is a measure ω associated with a finite and strictly positive weight that we denote by ω as well, with a slight abuse of notation. As before, throughout the paper, we avoid recalling this assumption, but the reader should always consider it implicitly stated whenever we refer to measures.

The classical product $\mathbb{L}_n$ quasi-norms defined in (1.2) are a special case of the iterated outer $\mathbf{L}_n$ ones defined in (1.5), with the same collection of exponents and the following collection of outer measures μ_i. For any $1 \le j \le n$, we define

$$Y^n_j = \prod_{i=j}^n X_i,$$

and we observe that the set Y^n has a canonical partition $\mathcal{Z}_j$, namely

$$\mathcal{Z}_j = \{Y_1^{j-1} \times z \colon z \in Y_j^n\},$$

where the set $Y_1^0 \times z$ is understood to be the singleton $\{z\}$. For every $A \subseteq Y^n$, let

$$\mu_i(A) = \inf_Z \Big\{ \sum_{z \in Z} \prod_{j=i}^{n} \omega_j(\pi_j(z)) \Big\}, \tag{1.8}$$

where π_j is the projection in the coordinate in X_j, and the infimum is taken over all subsets Z of Y_i^n such that A is covered by the elements of $\mathcal{Z}_i$ associated with Z.

The theory of classical product L^p spaces is well-developed: see for example [4]. In the range of exponents $p_i \in [1, \infty]$, the quantities defined in (1.2) are norms, and they satisfy the expected duality properties. On the other hand, the theory of outer L^p spaces is a theory of quasi-norms, mainly developed in [10] towards the real interpolation features of these quantities like Radon–Nikodym results, Hölder's inequality, and Marcinkiewicz interpolation, due to the aforementioned two-step programme.

However, as showed in [10], the iterated outer L^p spaces satisfy some properties analogous to those of the iterated classical ones; in particular, a one-direction "collapsing effect" and a version of Hölder's inequality up to a uniform constant, namely

$$\|f\|_{L^1(X,\omega)} \leq C \|f\|_{L^1_{\mu_n}(\ell^1_{\mu_{n-1}}(\dots \ell^1_\omega))}, \tag{1.9}$$

$$\sup_g \{ \|fg\|_{L^1_{\mu_n}(\dots \ell^1_\omega)} \colon \|g\|_{L^{p_n'}_{\mu_n}(\dots \ell^{p_1'}_\omega)} \leq 1 \} \leq C \|f\|_{L^{p_n}_{\mu_n}(\dots \ell^{p_1}_\omega)}, \tag{1.10}$$

where, for every $1 \leq i \leq n$, the exponent p_i' is the Hölder conjugate of p_i, satisfying

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1.$$

In [11], we studied the opposite inequalities in (1.9) and in (1.10) in the single iterated case, that is, when $n = 2$. We proved the equivalence in both cases up to constants depending on $p_i \in (1, \infty)$ but independent of the measure ω, the outer measure $\mu = \mu_2$, and the set X, as long as X is finite. These in turn imply the equivalence of the outer $L^{p_2}_\mu(\ell^{p_1}_\omega)$ quasi-norms to the norms defined by the supremum in (1.10). The endpoint cases $p_1 = \infty$ and $p_2 = 1$ exhibit a different behaviour, and we refer to [11] for more details.

In the present paper, we focus on the analogous opposite inequalities in (1.9) and in (1.10) in the double iterated case, that is, when $n = 3$. Already in this case, the study of the opposite inequalities becomes substantially more difficult due to the interplay between the subadditivity of the two outer measures and the exponents. We start by recalling the setting. Let X be a finite set, μ, ν outer measures, and ω a measure. Given three exponents $p, q, r \in (0, \infty]$, we define the double iterated outer L^p space $L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ through the quasi-norm in (1.5), with $\mu_1 = \omega$, $\mu_2 = \nu$, $\mu_3 = \mu$, and $p_1 = r$, $p_2 = q$, $p_3 = p$.

Before stating our main results, we introduce some auxiliary definitions. They depend on parameters $\Phi, K \geq 1$ that we will not recall every time.

Given a subset A of X, we say that a subset B of X is a *μ-parent set of A (with parameter Φ)* if $A \subseteq B$ and

$$\mu(B) \leq \Phi \mu(A). \tag{1.11}$$

A *μ-parent function* $\mathbf{B}$ *(with parameter Φ)* is then a monotone function from $\mathcal{P}(X)$ to itself, associating every subset A of X with a μ-parent set (with parameter Φ) $\mathbf{B}(A)$.

Moreover, given a collection $\mathcal{E}$ of subsets of X, we say that a function $\mathcal{C}$ from $\mathcal{P}(X)$ to the set of subcollections of pairwise disjoint elements in $\mathcal{E}$ is a *μ-covering function (with parameter Φ)* if the function $\mathbf{B}_{\mathcal{C}}$ from $\mathcal{P}(X)$ to itself defined by

$$\mathbf{B}_{\mathcal{C}}(A) = \bigcup_{E \in \mathcal{C}(A)} E$$

is a μ-parent function (with parameter Φ).

Next, we say that a collection $\mathcal{A}$ of pairwise disjoint subsets of X is *ν-Carathéodory (with parameter K)* if, for every subset U of X,

$$\sum_{A \in \mathcal{A}} \nu(U \cap A) \leq K \nu\Big(U \cap \bigcup_{A \in \mathcal{A}} A\Big). \tag{1.12}$$

Finally, we define two conditions for the quadruple $(X, \mu, \nu, \mathcal{C})$.

Condition 1.1 (Canopy). *We say that $(X, \mu, \nu, \mathcal{C})$ satisfies the* canopy condition (with parameters Φ, K) *if $\mathcal{C}$ is a μ-covering function (with parameter Φ), and for every ν-Carathéodory collection $\mathcal{A}$ (with parameter K), for every subset D of X disjoint from $\mathbf{B}_{\mathcal{C}}(\bigcup_{A \in \mathcal{A}} A)$, the collection $\mathcal{A} \cup \{D\}$ is still ν-Carathéodory (with parameter K).*

Condition 1.2 (Crop). *We say that $(X, \mu, \nu, \mathcal{C})$ satisfies the* crop condition (with parameters Φ, K) *if $\mathcal{C}$ is a μ-covering function (with parameter Φ), and for every collection $\mathcal{A}$ in $\mathcal{E}$, there exists a ν-Carathéodory subcollection $\mathcal{D}$ of $\mathcal{A}$ (with parameter K) such that, for every subset F of X disjoint from $\bigcup_{D \in \mathcal{D}} D$, we have*

$$\mathbf{B}_{\mathcal{C}}(F) = \mathbf{B}_{\widetilde{\mathcal{C}}}(F), \quad \textit{where} \quad \widetilde{\mathcal{C}}(F) = \mathcal{C}(F) \setminus \mathcal{A}.$$

We are now ready to state our main results.

THEOREM 1.3. *For all $q, r \in (0, \infty]$, $\Phi, K \geq 1$, there exist constants $C_1 = C_1(q, r, \Phi, K)$, $C_2 = C_2(q, r, \Phi, K)$ such that the following holds.*

Let X be a finite set, μ, ν outer measures, ω a measure, and $\mathcal{C}$ a μ-covering function such that $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 1.1. *Then, for every function $f \in L^q_\mu(\ell^q_\nu(\ell^r_\omega))$ on X,*

$$C_1^{-1}\|f\|_{L^q_\nu(\ell^r_\omega)} \leq \|f\|_{L^q_\mu(\ell^q_\nu(\ell^r_\omega))} \leq C_2\|f\|_{L^q_\nu(\ell^r_\omega)}. \tag{1.13}$$

If $q \leq r$ or $q = \infty$, the constant C_1 does not depend on Φ, K.

If $q \geq r$, the constant C_2 does not depend on Φ, K.

THEOREM 1.4. *For all $p, q \in (1, \infty)$, $r \in [q, \infty)$, $\Phi, K \geq 1$, there exists a constant $C = C(p, q, r, \Phi, K)$ such that the following holds.*

Let X be a finite set, μ, ν outer measures, ω a measure, and $\mathcal{C}$ a μ-covering function such that $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 1.1. *Then:*

(i) *For every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have*

$$C^{-1}\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leq \sup_{\|g\|_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))}=1} \|fg\|_{L^1(X,\omega)} \leq C\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}. \tag{1.14}$$

(ii) *For every collection $\{f_n : n \in \mathbb{N}\} \subseteq L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ of functions on X, we have*

$$\Big\|\sum_{n\in\mathbb{N}} f_n\Big\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leq C\sum_{n\in\mathbb{N}}\|f_n\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}. \tag{1.15}$$

For all $p, q \in (1, \infty), r \in (1, q]$, $\Phi, K \geq 1$, there exists a constant $C = C(p, q, r, \Phi, K)$ such that the analogous property holds for every finite set X, outer measures μ, ν, measure ω, and μ-covering function $\mathcal{C}$ such that $(X, \mu, \nu, \mathcal{C})$ satisfies the crop condition 1.2.

If $q = r$, the constant C does not depend on Φ, K.

The first result describes one instance of the "collapsing effect". When we have two consecutive outer L^p space structures associated with the same exponent, under certain conditions, the "exterior" one can be disregarded. We recall that, as a consequence of the "collapsing effect" in the single iterated case [11, Theorem 1.1(i)], we have

$$C^{-1}\|f\|_{L^p_\mu(\ell^r_\omega)} \leq \|f\|_{L^p_\mu(\ell^r_\nu(\ell^r_\omega))} \leq C\|f\|_{L^p_\mu(\ell^r_\omega)} \quad \text{for all } p, r \in (0, \infty],$$

where the constant $C = C(p, r)$ does not depend on Φ, K, and it is uniform in X, μ, ν, ω. Hence, the double iterated outer L^p spaces are reduced to single iterated ones. In particular, when $p = q = r \in (0, \infty]$, we have the full "collapsing effect"

$$C^{-1}\|f\|_{L^r(X,\omega)} \leq \|f\|_{L^r_\mu(\ell^r_\nu(\ell^r_\omega))} \leq C\|f\|_{L^r(X,\omega)}, \tag{1.16}$$

with constant $C = C(r)$ uniform in X, μ, ν, ω.

The second result yields the sharpness of outer Hölder's inequality. As a consequence, the iterated outer $L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ quasi-norm inherits from the $L^1(X,\omega)$-pairing a quasi-triangle inequality up to a constant uniform in the number of the summands, which is stated in the second property. Therefore, in the prescribed range of exponents, the double iterated outer L^p space is uniformly isomorphic to a Banach space with norm defined by the supremum in (1.14). Moreover, it is the Köthe dual space of the outer $L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))$ space, and we refer to [11] for an explanation of the use of the term "Köthe duality" in this context.

The main focus of both of the theorems is on the dependence of the constants in (1.13)–(1.15). A priori, for every fixed finite setting (X,μ,ν,ω) the constants are finite, but they depend on (X,μ,ν,ω). The theorems state that the constants depend on (X,μ,ν,ω) only through the parameters Φ, K associated with the canopy condition 1.1 or the crop condition 1.2. Moreover, we can exhibit counterexamples showing that, at least for the exponents p,q,r in a certain range, the constants cannot be chosen uniformly in Φ, K. We present the counterexamples in Subsection 3.4. It might be of interest to provide conditions weaker than the canopy condition 1.1 and the crop condition 1.2 that would still give control on the constants. However, this line of research is beyond the scope of the paper.

We also comment that the range of exponents p,q,r addressed by the aforementioned counterexamples points out a substantial difference between the cases of single iterated and double iterated outer L^p spaces. In the former case, there are no pathological behaviours of the outer L^p spaces in the range of exponents $(1,\infty)^2$. In the latter case, as we describe in Subsection 3.4, the range of exponents addressed by the counterexamples contains an open subset of $(1,\infty)^3$.

Finally, we mention the dichotomy between the cases $q > r$ and $q < r$ in the statement of the two theorems, in particular in view of the reduction to the single iterated outer L^p spaces in the case $q = r$. In Theorem 1.3 the dichotomy is in part explained by the counterexamples we exhibit in Subsection 3.4. It would be interesting to clarify whether in Theorem 1.4 the dichotomy is an intrinsic feature of the problem or it is just an artefact of the argument used in the proof. If the former case were true, it would be interesting to clarify how the dichotomy relates to the conditions guaranteeing a control on the constants.

Before moving on, we comment on the definition of ν-Carathéodory collections and the conditions we stated before the results. We start by observing that the Carathéodory measurability test with respect to an outer measure μ^* corresponds to checking that the couple $\{E, E^c\}$ is μ^*-Carathéodory with parameter 1. In particular, when ν is a measure, every collection of pairwise disjoint measurable subsets of X is ν-Carathéodory with parameter $K = 1$. This fact implies that, in the single iterated case, we can always deal with ν-Carathéodory collections, which come with desirable properties. In particular, for every set X, outer measure μ, measure ω, the quadruple $(X, \mu, \omega, \mathrm{Id})$ satisfies both the canopy condition 1.1 and the crop condition 1.2 with parameters $\Phi = K = 1$.

The extension of the results stated in Theorems 1.3 and 1.4 to infinite settings under reasonable assumptions should not be a surprise. However, this level of generality is beyond the scope of the paper. We concern ourselves only with two specific infinite settings, namely the one described by Uraltsev [14] and a slight variation of it, both defined on the upper half 3-space. Although not equivalent, these settings exhibit similar geometric properties. We focus mainly on the latter.

We briefly recall the setting that we describe in detail in Subsection 4.3. Let X be the upper half 3-space $\mathbb{R} \times (0, \infty) \times \mathbb{R}$, and ω the measure induced on it by the Lebesgue measure $\mathrm{d}y\mathrm{d}t\mathrm{d}\eta$ on $\mathbb{R}^3$. On X, we define two outer measures by means of the following covering construction. Given a collection $\mathcal{S}$ of subsets of X and a pre-measure $\sigma \colon \mathcal{S} \to (0, \infty)$, we define the outer measure $\mu \colon \mathcal{P}(X) \to [0, \infty]$ on an arbitrary subset A of X by

$$\mu(A) = \inf \Big\{ \sum_{S \in \mathcal{S}'} \sigma(S) \colon \mathcal{S}' \subseteq \mathcal{S},\, A \subseteq \bigcup_{S \in \mathcal{S}'} S \Big\}. \tag{1.17}$$

First, for any dyadic interval $I \subseteq \mathbb{R}$, let $D(I)$ be the dyadic strip given by the Cartesian product between I, the interval $(0, |I|]$ and $\mathbb{R}$. Let $\mathcal{D}$ be the collection of all the dyadic strips, and for every $D(I) \in \mathcal{D}$ let σ be the length of the base I.

Second, for any couple of dyadic intervals $I, \widetilde{I} \subseteq \mathbb{R}$ with inverse lengths, let $T(I, \widetilde{I})$ be the dyadic tree given by the union of the Cartesian products between a dyadic interval $J \subseteq I$, the interval $(0, |J|]$, and the dyadic interval $\widetilde{J} \supseteq \widetilde{I}$ with inverse length of J. Let $\mathcal{T}$ be the collection of all the dyadic trees, and for every $T(I, \widetilde{I}) \in \mathcal{T}$ let τ be the length of the base I.

Now, let μ, ν be the outer measures on X associated with $(\mathcal{D}, \sigma)$, $(\mathcal{T}, \tau)$, respectively, as in (1.17). As we will see in Appendix A, for every dyadic strip D in $\mathcal{D}$ and every dyadic tree T in $\mathcal{T}$, we have

$$\mu(D) = \sigma(D), \qquad \nu(T) = \tau(T).$$

We define the double iterated outer L^p space $L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ in the upper half 3-space setting through the quasi-norm in (1.5) for ω-measurable functions. We use $\mu_1 = \omega$, $\mu_2 = \nu$, $\mu_3 = \mu$, and we restrict the supremum in $\mathbf{I}_1$ to the ω-measurable sets, that in $\mathbf{I}_2$ to the dyadic trees in $\mathcal{T}$, and that in $\mathbf{I}_3$ to the dyadic strips in $\mathcal{D}$.

In this setting, we have both the "collapsing effect" and the sharpness of outer Hölder's inequality described in the finite setting in the previous theorems.

THEOREM 1.5. *Let (X, μ, ν, ω) be the dyadic upper half 3-space setting just described, $p, q, r \in (0, \infty)$. There exists a constant $C = C(p, q, r)$ such that the following hold.*

(i) *For all $q, r \in (0, \infty)$ and every function $f \in L^q_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have*

$$C^{-1} \|f\|_{L^q_\nu(\ell^r_\omega)} \leq \|f\|_{L^q_\mu(\ell^q_\nu(\ell^r_\omega))} \leq C \|f\|_{L^q_\nu(\ell^r_\omega)}. \tag{1.18}$$

(ii) *For all $p, q, r \in (1, \infty)$ and every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have*

$$C^{-1} \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leq \sup_{\|g\|_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))}=1} \|fg\|_{L^1(X,\omega)} \leq C \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}. \tag{1.19}$$

(iii) *For all $p, q, r \in (1, \infty)$ and every collection $\{f_n : n \in \mathbb{N}\} \subseteq L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ of functions on X, we have*

$$\Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leq C \sum_{n \in \mathbb{N}} \|f_n\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}. \tag{1.20}$$

The result analogous to Theorem 1.5 holds true even in the upper half 3-space setting with arbitrary strips and trees originally considered in [14] that we describe in detail in Subsection 5.3.

We conclude by pointing out that the outer L^p spaces used by Uraltsev are different from those defined in (1.5). In [14], the innermost size, namely the quantity in (1.6) for $n = 2$, is not defined by a single Lebesgue norm with respect to the measure ω, but by a sum of an L^2 norm and an L^∞ norm, making it more complicated to treat. The first step in the study of these spaces would be to extend the results stated in Theorem 1.5 to the case $r = \infty$. This is likely to be achieved exploiting the geometric properties of the strips and trees in the upper half 3-space in the same fashion of the boxes in the upper half-space in [11]. The second step, requiring new considerations, would be to address the issue of the variable exponent Lebesgue norm.

Guide to the paper. In Section 2, we review some preliminaries about outer L^p quasi-norms and, more specifically, single iterated outer L^p ones from [11]. In Section 3, we prove Theorems 1.3 and 1.4. Moreover, at least in a certain range of exponents $p, q, r \in (0, \infty]$, we present counterexamples showing that the constants appearing in the statements of these theorems are not independent of the setting (X, μ, ν, ω), as discussed above in the Introduction. In Section 4, we describe some settings in which we define a μ-covering function satisfying the canopy condition 1.1 and the crop condition 1.2. In Section 5, we prove Theorem 1.5 in the dyadic upper half 3-space setting reducing the problem to an equivalent one in a finite setting via an approximation argument. The proof relies on the geometric properties of the outer measures and the approximation properties of functions in iterated outer L^p spaces that we will prove in Appendices A and B, respectively.

2. Preliminaries. In this section, we make some observations about the outer L^p quasi-norms. Moreover, we review the decomposition result for functions in a single iterated outer L^p space, which is the main ingredient in proving the results corresponding to [11, Theorems 1.3 and 1.4]. It provides a model for the decomposition in the case of double iterated outer L^p spaces that we perform in Section 3.

First, for every $p \in (0,\infty)$, we observe that we can replace the integral defining the outer L^p quasi-norm in (1.5) by a discrete version of it. For every $\Psi > 1$, we have

$$\|f\|^p_{L^p_\mu(S)} \sim_{\Psi,p} \sum_{k\in\mathbb{Z}} \Psi^{kp}\mu(S(f) > \Psi^k) \sim_{\Psi,p} \sum_{k\in\mathbb{Z}} \Psi^{kp} \sum_{l\geq k} \mu(S(f) > \Psi^l), \tag{2.1}$$

where S is a size of the form ℓ^r_ω or $\ell^q_\nu(\ell^r_\omega)$, and more generally an arbitrary size in the sense of [10]. The equivalences in (2.1) follow by the monotonicity of the super level measure, Fubini and the bounds on the geometric series.

Next, let X be a finite set, μ,ν outer measures, and ω a measure. Since μ,ν are finite and strictly positive on every nonempty subset of X, by outer Hölder's inequality [10, Proposition 3.4], we have

$$\begin{gathered} L^q_\nu(\ell^r_\omega) \subseteq L^\infty_\nu(\ell^r_\omega), \\ L^p_\mu(\ell^q_\nu(\ell^r_\omega)) \subseteq L^\infty_\mu(\ell^q_\nu(\ell^r_\omega)) \cap L^\infty_\mu(\ell^\infty_\nu(\ell^r_\omega)). \end{gathered} \tag{2.2}$$

Finally, we recall two results for single iterated outer L^p spaces already appearing, explicitly or implicitly, in [11, Proposition 2.1], with their proofs.

Lemma 2.1. *For all $r \in (0,\infty)$, $N \geq 1$, there exist constants $C = C(r,N)$, $c = c(N)$ such that the following holds.*

Let X be a set, ν an outer measure, and ω a measure. Let $f \in L^\infty_\nu(\ell^r_\omega)$ be a function on X, let $k \in \mathbb{Z}$ satisfy

$$\|f\|_{L^\infty_\nu(\ell^r_\omega)} \in (2^k, 2^{k+1}], \tag{2.3}$$

and let A be a subset of X such that

$$\|f1_A\|^r_{L^r(X,\omega)} > 2^{(k-N)r}\nu(A). \tag{2.4}$$

Then

$$\nu(A) \leq C\nu(\ell^r_\omega(f) > c2^k). \tag{2.5}$$

Proof. Let $\varepsilon > 0$. Let $V(c2^k,\varepsilon)$ be an optimal set associated with the super level measure $\nu(\ell^r_\omega(f) > c2^k)$ up to the multiplicative constant $1+\varepsilon$, i.e.

$$\|f 1_{V(c2^k,\varepsilon)^c}\|_{L^\infty_\nu(\ell^r_\omega)} \leq c2^k, \tag{2.6}$$

$$(1+\varepsilon)\nu(\ell^r_\omega(f) > c2^k) \geq \nu(V(c2^k,\varepsilon)), \tag{2.7}$$

where c will be fixed later. We have

$$\begin{aligned}\nu(V(c2^k,\varepsilon)) &\geq 2^{-(k+1)r}\|f 1_{V(c2^k,\varepsilon)} 1_A\|^r_{L^r(X,\omega)} \\ &\geq 2^{-(k+1)r}(\|f 1_A\|^r_{L^r(X,\omega)} - \|f 1_{A\setminus V(c2^k,\varepsilon)}\|^r_{L^r(X,\omega)}) \\ &\geq 2^{-(k+1)r}(2^{(k-N)r} - c^r 2^{kr})\nu(A),\end{aligned}$$

where we use the monotonicity of ν and (2.3) in the first inequality, the r-orthogonality of the classical L^r quasi-norms of functions supported on disjoint sets in the second, (2.4) and (2.6) in the third. By choosing $c = 2^{-N-1}$, and taking ε arbitrarily small, the previous chain of inequalities together with (2.7) yields (2.5). ∎

PROPOSITION 2.2. *For all $q, r \in (0,\infty)$, there exist constants $C = C(q,r)$, $c = c(q,r)$ such that the following holds.*

Let X be a finite set, ν an outer measure, ω a measure. For every function $f \in L^q_\nu(\ell^r_\omega)$ on X, there exists a collection $\{U_j : j \in \mathbb{Z}\}$ of pairwise disjoint subsets of X such that if we set

$$V_j = \bigcup_{l \geq j} U_l,$$

then, for every $j \in \mathbb{Z}$,

$$\ell^r_\omega(f 1_{V^c_{j+1}})(U_j) > 2^j \qquad \text{when } U_j \neq \emptyset, \tag{2.8}$$

$$\|f 1_{V^c_j}\|_{L^\infty_\nu(\ell^r_\omega)} \leq 2^j, \tag{2.9}$$

$$\nu(\ell^r_\omega(f) > 2^j) \leq \nu(V_j), \tag{2.10}$$

$$\nu(U_j) \leq C\nu(\ell^r_\omega(f) > c2^j). \tag{2.11}$$

In particular,

$$\|f\|^q_{L^q_\nu(\ell^r_\omega)} \sim_{r,q} \sum_{j\in\mathbb{Z}} 2^{jq}\nu(U_j) \sim_{r,q} \sum_{j\in\mathbb{Z}} 2^{jq} \sum_{l\geq j} \nu(U_l). \tag{2.12}$$

Proof. The first four statements and their proof appeared already in [11, Proposition 2.1]. The equivalences in (2.12) follow by (2.1), (2.10), the definition of V_j, (2.11), Fubini, and the bounds for the geometric series. ∎

Throughout the paper, we often use the observations made in this section without explicitly referring to them. For example, the reader should always have in mind the equivalences in (2.1) whenever we consider an outer L^p quasi-norm, and the list of properties (2.8)–(2.12) whenever we perform such a decomposition.

3. Equivalence with norms. In this section, we study the equivalence of double iterated outer L^p quasi-norms with norms uniformly in the finite setting.

First, for all $q,r \in (0,\infty)$, we study the q-orthogonality behaviour of the outer $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions supported on disjoint sets. Accordingly, we show decomposition results for functions in the double iterated outer L^p space with respect to a size of the form $\ell^q_\nu(\ell^r_\omega)$. We use them to prove Theorem 1.3.

Next, for all $p,q,r \in (1,\infty)$, we produce a function g for which we have a good lower bound on the $L^1(X,\omega)$-pairing with f and a good upper bound on the $L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))$ quasi-norm of g. We use it to prove Theorem 1.4.

We conclude the section with the promised class of counterexamples.

3.1. q-orthogonality of the $L^q_\nu(\ell^r_\omega)$ quasi-norm. We start with a result about the sub- and q-superorthogonality of the $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions supported on arbitrary disjoint sets according to the case distinction $q \geq r$ or $q \leq r$. We present counterexamples to the validity of the inequality in the opposite directions in both cases $q < r$ or $q > r$ in Subsection 3.4.

Lemma 3.1. *For all $q \in (0,\infty)$, $r \in (0,\infty]$, there exists a constant $C = C(q,r)$ such that the following holds.*

Let X be a finite set, ν an outer measure, ω a measure. Let $\mathcal{A}$ be a collection of pairwise disjoint subsets of X. Then, for every function f on X,

$$\sum_{A\in\mathcal{A}} \|f1_A\|^q_{L^q_\nu(\ell^r_\omega)} \leq C\|f1_B\|^q_{L^q_\nu(\ell^r_\omega)} \qquad \text{for } q \geq r, \tag{3.1}$$

$$\|f1_B\|^q_{L^q_\nu(\ell^r_\omega)} \leq C \sum_{A\in\mathcal{A}} \|f1_A\|^q_{L^q_\nu(\ell^r_\omega)} \qquad \text{for } q \leq r, \tag{3.2}$$

where $B = \bigcup_{A\in\mathcal{A}} A$.

Proof. Without loss of generality, assume $q = 1$. In fact, for $\frac{r}{q} \in (0,\infty]$, we have

$$\|f\|^q_{L^q_\nu(\ell^r_\omega)} = \|f^q\|_{L^1_\nu(\ell^{r/q}_\omega)}.$$

Case I: $q = 1$, $r = \infty$. We have

$$\nu(\ell^\infty_\omega(f) > \lambda) = \nu(\{x \in X : |f(x)| > \lambda\}). \tag{3.3}$$

Together with the subadditivity of ν, this yields

$$\nu(\ell^\infty_\omega(f1_B) > \lambda) \leq \sum_{A\in\mathcal{A}} \nu(\ell^\infty_\omega(f1_A) > \lambda).$$

By integrating in $(0,\infty)$ on both sides, we obtain (3.2).

CASE II: $q=1$, $r\in(0,1]$. We start with the following observation. Let $\mathcal{E}$ be a collection of pairwise disjoint nonempty subsets of X such that, for every $E\in\mathcal{E}$,

$$\ell^r_\omega(f)(E)\in(2^j,2^{j+1}]. \tag{3.4}$$

Together with the r-orthogonality of the classical L^r quasi-norms of functions supported on disjoint sets and the subadditivity of ν, this yields

$$\ell^r_\omega(f)\Big(\bigcup_{E\in\mathcal{E}}E\Big)>\Big(\nu\Big(\bigcup_{E\in\mathcal{E}}E\Big)^{-1}\sum_{E\in\mathcal{E}}2^{jr}\nu(E)\Big)^{1/r}\geq 2^j. \tag{3.5}$$

Next,

$$\begin{aligned}\nu\Big(\bigcup_{E\in\mathcal{E}}E\Big)^{-1}\sum_{E\in\mathcal{E}}\ell^r_\omega(f)(E)\nu(E)&\leq 2^{j+1}\nu\Big(\bigcup_{E\in\mathcal{E}}E\Big)^{-1}\sum_{E\in\mathcal{E}}\nu(E)\\&\leq 2^{j+1}\Big(\nu\Big(\bigcup_{E\in\mathcal{E}}E\Big)^{-1}\sum_{E\in\mathcal{E}}\nu(E)\Big)^{1/r}\\&\leq 2\Big(\nu\Big(\bigcup_{E\in\mathcal{E}}E\Big)^{-1}\sum_{E\in\mathcal{E}}\|f1_E\|^r_{L^r(X,\omega)}\Big)^{1/r}\\&\leq 2\ell^r_\omega(f)\Big(\bigcup_{E\in\mathcal{E}}E\Big),\end{aligned}$$

where we use the upper bound on $\ell^r_\omega(f)(E)$ in (3.4) for every $E\in\mathcal{E}$ in the first inequality, the subadditivity of ν and $r\leq 1$ in the second, the lower bound on $\ell^r_\omega(f)(E)$ in (3.4) for every $E\in\mathcal{E}$ in the third, and the r-orthogonality of the classical L^r quasi-norms of functions supported on disjoint sets in the fourth. The previous chain of inequalities yields

$$\sum_{E\in\mathcal{E}}\ell^r_\omega(f)(E)\nu(E)\leq 2\ell^r_\omega(f)\Big(\bigcup_{E\in\mathcal{E}}E\Big)\nu\Big(\bigcup_{E\in\mathcal{E}}E\Big). \tag{3.6}$$

Moreover, let $j\in\mathbb{Z}$ and, for every $k\in\mathbb{Z}_{\leq j}$, let $\mathcal{E}_k$ be a collection of pairwise disjoint subsets of X such that

$$\ell^r_\omega(f)(E)\in(2^k,2^{k+1}]\quad\text{for every nonempty } E\in\mathcal{E}_k,$$

and

$$\ell^r_\omega(f)\Big(\bigcup_{E\in\mathcal{E}_k}E\Big)\in(2^j,2^{j+1}]\quad\text{for every nonempty union }\textstyle\bigcup_{E\in\mathcal{E}_k}E.$$

By (3.6) applied twice, we have

$$\begin{aligned}\sum_{k\leq j}\sum_{E\in\mathcal{E}_k}\ell^r_\omega(f)(E)\nu(E)&\leq 2\sum_{k\leq j}\ell^r_\omega(f)\Big(\bigcup_{E\in\mathcal{E}_k}E\Big)\nu\Big(\bigcup_{E\in\mathcal{E}_k}E\Big)\\&\leq 4\ell^r_\omega(f)\Big(\bigcup_{k\leq j}\bigcup_{E\in\mathcal{E}_k}E\Big)\nu\Big(\bigcup_{k\leq j}\bigcup_{E\in\mathcal{E}_k}E\Big),\end{aligned} \tag{3.7}$$

where the sums run over nonempty subsets E and $\bigcup_{E\in\mathcal{E}_k}E$.

Now, let $\{A_j : j \in \mathbb{Z}\}$, $\{B_j : j \in \mathbb{Z}\}$ be the collections associated with the decomposition in Proposition 2.2 of the functions $f1_A$, $f1_B$, respectively. By (3.5) and (3.7), we can pass from the collection $\{A_j : A \in \mathcal{A}, j \in \mathbb{Z}\}$ of pairwise disjoint subsets of X to a collection $\mathcal{E} = \{E_l : l \in \mathbb{Z}\}$ of pairwise disjoint subsets of X with strictly fewer elements such that

$$\ell^r_\omega(f)(E_l) \in (2^l, 2^{l+1}] \quad \text{when } E_l \neq \emptyset, \tag{3.8}$$

$$\sum_{A \in \mathcal{A}} \sum_{j \in \mathbb{Z}} 2^j \nu(A_j) \leq C \sum_{l \in \mathbb{Z}} 2^l \nu(E_l). \tag{3.9}$$

By the monotonicity of ν, we have

$$\begin{aligned} \|f1_{E_l \cap (\bigcup_{k \geq l-1} B_k)^c}\|^r_{L^r(X,\omega)} &\leq 2^{(l-1)r} \nu\Big(E_l \cap \Big(\bigcup_{k \geq l-1} B_k\Big)^c\Big) \\ &\leq 2^{(l-1)r} \nu(E_l). \end{aligned}$$

Together with (3.8), this yields

$$\begin{aligned} \sum_{k \geq l-1} \|f1_{E_l \cap B_k}\|^r_{L^r(X,\omega)} &= \|f1_{E_l \cap \bigcup_{k \geq l-1} B_k}\|^r_{L^r(X,\omega)} \\ &= \|f1_{E_l}\|^r_{L^r(X,\omega)} - \|f1_{E_l \cap (\bigcup_{k \geq l-1} B_k)^c}\|^r_{L^r(X,\omega)} \\ &\geq c 2^{lr} \nu(E_l). \end{aligned} \tag{3.10}$$

Therefore,

$$\begin{aligned} \sum_{l \in \mathbb{Z}} 2^l \nu(E_l) &\leq C \sum_{l \in \mathbb{Z}} 2^{l(1-r)} \sum_{k \geq l-1} \|f1_{E_l \cap B_k}\|^r_{L^r(X,\omega)} \\ &\leq C \sum_{k \in \mathbb{Z}} 2^{k(1-r)} \sum_{l \leq k+1} \|f1_{E_l \cap B_k}\|^r_{L^r(X,\omega)} \\ &\leq C \sum_{k \in \mathbb{Z}} 2^{k(1-r)} \|f1_{B_k}\|^r_{L^r(X,\omega)} \leq C \sum_{k \in \mathbb{Z}} 2^k \nu(B_k), \end{aligned}$$

where we use (3.10) in the first inequality, $r \leq 1$ in the second, and the r-orthogonality of the classical L^r quasi-norms of functions supported on disjoint sets in the third. Together with (2.12) for the collections $\{A_j : j \in \mathbb{Z}\}$, $\{B_j : j \in \mathbb{Z}\}$, and (3.9), the previous chain of inequalities yields (3.1).

Case III: $q = 1$, $r \in [1, \infty)$. Let A_j, B_j be defined as before. Then

$$\begin{aligned} \sum_{j \in \mathbb{Z}} 2^j \nu(B_j) &\leq \sum_{j \in \mathbb{Z}} 2^{j(1-r)} \|f1_{B_j}\|^r_{L^r(X,\omega)} \\ &\leq \sum_{A \in \mathcal{A}} \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} 2^{j(1-r)} \|f1_{A_k \cap B_j}\|^r_{L^r(X,\omega)} \end{aligned}$$

$$\leq \sum_{A\in\mathcal{A}}\sum_{k\in\mathbb{Z}}\Big(2^{k(1-r)}\sum_{j\geq k}\|f1_{A_k\cap B_j}\|^r_{L^r(X,\omega)}+\sum_{j<k}2^{j(1-r)}\|f1_{A_k\cap B_j}\|^r_{L^r(X,\omega)}\Big)$$
$$\leq C\sum_{A\in\mathcal{A}}\sum_{k\in\mathbb{Z}}\Big(2^{k(1-r)}\|f1_{A_k}\|^r_{L^r(X,\omega)}+\sum_{j<k}2^{j(1-r)}2^{jr}\nu(A_k\cap B_j)\Big)$$
$$\leq C\sum_{A\in\mathcal{A}}\sum_{k\in\mathbb{Z}}\Big(2^{k}\nu(A_k)+\sum_{j<k}2^{j}\nu(A_k)\Big),$$

where we use the r-orthogonality of the classical L^r quasi-norms for functions with disjoint supports in the second and in the fourth inequalities, and $r\geq 1$ in the third. Together with (2.12) for the collections $\{A_j : j\in\mathbb{Z}\}$, $\{B_j : j\in\mathbb{Z}\}$, the previous chain of inequalities yields (3.2). ∎

We continue with a result about the full q-orthogonality of the $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions supported on disjoint sets forming a ν-Carathéodory collection.

Lemma 3.2. *For all $q\in(0,\infty)$, $r\in(0,\infty]$, $K\geq 1$, there exist constants $C_1=C_1(q,r,K)$, $C_2=C_2(q,r,K)$ such that the following holds.*

Let X be a set, ν an outer measure, ω a measure. Let $\mathcal{A}$ be a ν-Carathéodory collection of pairwise disjoint subsets of X. Then, for every function f on X,

$$C_1^{-1}\|f1_B\|^q_{L^q_\nu(\ell^r_\omega)}\leq\sum_{A\in\mathcal{A}}\|f1_A\|^q_{L^q_\nu(\ell^r_\omega)}\leq C_2\|f1_B\|^q_{L^q_\nu(\ell^r_\omega)},\tag{3.11}$$

where $B=\bigcup_{A\in\mathcal{A}}A$.

Proof. As before, without loss of generality, we assume $q=1$.

Expanding the definition of the outer $L^1_\nu(\ell^r_\omega)$ quasi-norms in (3.11), we have

$$\|f1_B\|_{L^1_\nu(\ell^r_\omega)}=\int_0^\infty\nu(\ell^r_\omega(f1_B)>\lambda)\,\mathrm{d}\lambda,$$
$$\sum_{A\in\mathcal{A}}\|f1_A\|_{L^1_\nu(\ell^r_\omega)}=\int_0^\infty\sum_{A\in\mathcal{A}}\nu(\ell^r_\omega(f1_A)>\lambda)\,\mathrm{d}\lambda.$$

To show the desired inequalities, it is enough to prove that there exist constants $c=c(r,K)$, $C=C(r,K)$ such that, for every $\lambda>0$,

$$\nu(\ell^r_\omega(f1_B)>c\lambda)\leq\sum_{A\in\mathcal{A}}\nu(\ell^r_\omega(f1_A)>\lambda)\leq C\nu(\ell^r_\omega(f1_B)>\lambda).\tag{3.12}$$

By integrating in $(0,\infty)$ on both sides, we obtain (3.11).

Case I: $r=\infty$. By the subadditivity of ν and the ν-Carathéodory condition (1.12),

$$\nu(\{x \in B\colon f(x) > \lambda\}) \le \sum_{A \in \mathcal{A}} \nu(\{x \in A\colon f(x) > \lambda\})$$
$$\le K\nu(\{x \in B\colon f(x) > \lambda\}).$$

Together with (3.3), this yields (3.12).

Case II: $r \in (0, \infty)$. We start with the first inequality in (3.12). Let $\varepsilon > 0$. For every $A \in \mathcal{A}$, let $V(A, \lambda, \varepsilon)$ be an optimal set associated with the super level measure $\nu(\ell^r_\omega(f 1_A) > \lambda)$ up to the multiplicative constant $1 + \varepsilon$, namely

$$\|f 1_A 1_{V(A,\lambda,\varepsilon)^c}\|_{L^\infty_\nu(\ell^r_\omega)} \le \lambda, \tag{3.13}$$
$$(1 + \varepsilon)\nu(\ell^r_\omega(f 1_A) > \lambda) \ge \nu(V(A, \lambda, \varepsilon)), \tag{3.14}$$

and set

$$V = \bigcup_{A \in \mathcal{A}} V(A, \lambda, \varepsilon).$$

For every $U \subseteq X$, we have

$$\big(\ell^r_\omega(f 1_B 1_{V^c})(U)\big)^r \le \nu(U)^{-1} \sum_{A \in \mathcal{A}} \|f 1_A 1_{V(A,\lambda,\varepsilon)^c} 1_U\|^r_{L^r(X,\omega)}$$
$$\le \nu(U)^{-1} \sum_{A \in \mathcal{A}} \lambda^r \nu(U \cap A) \le K\lambda^r,$$

where we use the r-orthogonality of the classical L^r quasi-norms of functions with disjoint support in the first inequality, (3.13) in the second, and the ν-Carathéodory condition (1.12) in the third. Together with the subadditivity of ν and (3.14), the previous chain of inequalities yields

$$\nu(\ell^r_\omega(f 1_B) > K^{1/r}\lambda) \le (1 + \varepsilon) \sum_{A \in \mathcal{A}} \nu(\ell^r_\omega(f 1_A) > \lambda).$$

By taking ε arbitrarily small, we obtain the first inequality in (3.12).

We turn to the second inequality in (3.12). Let $\varepsilon > 0$. Let $V(\lambda, \varepsilon)$ be an optimal set associated with the super level measure $\nu(\ell^r_\omega(f 1_B) > \lambda)$ up to the multiplicative constant $1 + \varepsilon$, namely

$$\|f 1_{V(\lambda,\varepsilon)^c}\|_{L^\infty_\nu(\ell^r_\omega)} \le \lambda, \tag{3.15}$$
$$(1 + \varepsilon)\nu(\ell^r_\omega(f 1_B) > \lambda) \ge \nu(V(\lambda, \varepsilon)). \tag{3.16}$$

For every $U \subseteq X$, we have

$$(\ell^r_\omega(f 1_A 1_{V(\lambda,\varepsilon)^c})(U))^r \le \nu(U)^{-1} \|f 1_B 1_{V(\lambda,\varepsilon)^c} 1_U\|^r_{L^r(X,\omega)} \le \lambda^r,$$

where we use the monotonicity of the classical L^r quasi-norms in the first inequality, and (3.15) in the second. Together with the ν-Carathéodory condition (1.12) and (3.16), the previous chain of inequalities yields

$$\sum_{A \in \mathcal{A}} \nu(\ell^r_\omega(f 1_A) > \lambda) \le \sum_{A \in \mathcal{A}} \nu(V(\lambda, \varepsilon) \cap A) \le (1 + \varepsilon)K\nu(\ell^r_\omega(f 1_B) > \lambda).$$

By taking ε arbitrarily small, we obtain the second inequality in (3.12). ∎

3.2. Decomposition for double iterated outer L^p spaces. We start with the result corresponding to Lemma 2.1 in the case of sizes given by single iterated outer L^p quasi-norms. The proof relies on the q-suborthogonality of the $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions with disjoint supports as stated in (3.1) or in the second inequality in (3.11). Therefore, according to the relation between the exponents q, r, we allow the constants to depend on the parameter associated with the ν-Carathéodory collection formed by the disjoint sets.

LEMMA 3.3. *For all $q \in (0,\infty)$, $r \in (0,\infty]$, $K \geq 1$, $N \geq 1$, there exist constants $C = C(q,r,K,N)$, $c = c(q,r,K,N)$ such that the following holds.*

Let X be a set, μ,ν outer measures, and ω a measure. Let $f \in L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ be a function on X, let $k \in \mathbb{Z}$ satisfy

$$\|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \in (2^k, 2^{k+1}], \tag{3.17}$$

and let $\mathcal{A}$ be a ν-Carathéodory collection of subsets of X such that

$$\|f 1_A\|^q_{L^q_\nu(\ell^r_\omega)} > 2^{(k-N)q}\mu(A) \quad \text{for every } A \in \mathcal{A}. \tag{3.18}$$

Then

$$\sum_{A\in\mathcal{A}} \mu(A) \leq C\mu(\ell^q_\nu(\ell^r_\omega)(f) > c2^k). \tag{3.19}$$

If $q \geq r$ and X is finite, the constants C, c do not depend on K.

Proof. CASE I: *arbitrary q, r.* Let $\varepsilon > 0$. Let $F(c2^k,\varepsilon)$ be an optimal set associated with the super level measure $\mu(\ell^q_\nu(\ell^r_\omega)(f) > c2^k)$ up to the multiplicative constant $1+\varepsilon$, namely

$$\|f 1_{F(c2^k,\varepsilon)^c}\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \leq c2^k, \tag{3.20}$$

$$(1+\varepsilon)\mu(\ell^q_\nu(\ell^r_\omega)(f) > c2^k) \geq \mu(F(c2^k,\varepsilon)), \tag{3.21}$$

where c will be fixed later. For $B = \bigcup_{A\in\mathcal{A}} A$, we have

$$\begin{aligned}
\mu\big(F(c2^k,\varepsilon)\big) &\geq 2^{-(k+1)q}\|f 1_{F(c2^k,\varepsilon)} 1_B\|^q_{L^q_\nu(\ell^r_\omega)} \\
&\geq C2^{-(k+1)q}\sum_{A\in\mathcal{A}} \|f 1_{F(c2^k,\varepsilon)} 1_A\|^q_{L^q_\nu(\ell^r_\omega)} \\
&\geq C2^{-(k+1)q}\sum_{A\in\mathcal{A}} (C_\Delta^{-1}\|f 1_A\|_{L^q_\nu(\ell^r_\omega)} - \|f 1_{A\setminus F(c2^k,\varepsilon)}\|_{L^q_\nu(\ell^r_\omega)})^q \\
&\geq C2^{-(k+1)q}\sum_{A\in\mathcal{A}} (C_\Delta^{-1}2^{k-N} - c2^k)^q\mu(A),
\end{aligned}$$

where we use the monotonicity of μ and (3.17) in the first inequality, Lemma 3.2 applied to the ν-Carathéodory collection $\mathcal{A}$ in the second, the quasi-triangle inequality for the outer L^p quasi-norm of two summands in the third, and (3.18) and (3.20) in the fourth. By choosing

$$c = (2^{N+1}C_\Delta)^{-1},$$

and taking ε arbitrarily small, the chain of inequalities together with (3.21) yields (3.19).

Case II: $q \geq r$. We use (3.1) from Lemma 3.1 applied to every collection $\mathcal{A}$ of pairwise disjoint subsets of X in place of Lemma 3.2. ■

We are now ready to prove a series of decomposition results for functions in the outer L^p space with respect to a size of the form $\ell^q_\nu(\ell^r_\omega)$. Although the statements, as well as the proofs, are similar, we provide them separately in order to highlight the differences. The proofs rely on the selection of disjoint subsets where the size achieves the levels Ψ^k for a certain $\Psi > 1$. The key ingredient in order to perform such a selection exhaustively at each step is the q-suborthogonality of the $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions supported on certain disjoint sets. Therefore, according to the relation between the exponents q, r, we require the canopy condition 1.1, and we allow the constants to depend on the associated parameters.

We start with a decomposition result in the full range of exponents under the assumption of the canopy condition 1.1.

Proposition 3.4. *For all* $p, q, r \in (0, \infty)$, $\Phi, K \geq 1$, *there exist constants* $C = C(p, q, r, \Phi, K)$, $c = c(p, q, r, \Phi, K)$ *such that the following holds.*

Let X *be a finite set,* μ, ν *outer measures,* ω *a measure, and* $\mathcal{C}$ *a* μ-*covering function such that* $(X, \mu, \nu, \mathcal{C})$ *satisfies the canopy condition* 1.1. *For every function* $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ *on* X, *there exists a collection* $\{E_k : k \in \mathbb{Z}\}$ *of pairwise disjoint subsets of* X *such that, if we set*

$$F_k = \mathbf{B}_{\mathcal{C}}\Big(\bigcup_{l \geq k} E_l\Big),$$

then for every $k \in \mathbb{Z}$,

$$\ell^q_\nu(\ell^r_\omega)(f 1_{F^c_{k+1}})(E_k) > c 2^k \quad \text{when } E_k \neq \emptyset, \tag{3.22}$$

$$\|f 1_{F^c_k}\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \leq 2^k, \tag{3.23}$$

$$\mu(\ell^q_\nu(\ell^r_\omega)(f) > 2^k) \leq \mu(F_k), \tag{3.24}$$

$$\mu(E_k) \leq C \mu(\ell^q_\nu(\ell^r_\omega)(f) > c 2^k). \tag{3.25}$$

In particular,

$$\|f\|^p_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \sim_{p,q,r,\Phi,K} \sum_{k \in \mathbb{Z}} 2^{kp} \mu(E_k) \sim_{p,q,r,\Phi,K} \sum_{k \in \mathbb{Z}} 2^{kp} \sum_{l \geq k} \mu(E_l). \tag{3.26}$$

Proof. By (2.2), we have $f \in L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$. We define the collection $\{E_k : k \in \mathbb{Z}\}$ by a backward recursion on $k \in \mathbb{Z}$. For k large enough such that

$$\|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \leq 2^k,$$

we set E_k to be empty. Now, we fix k and assume to have selected E_l for every $l > k$. In particular, F_{k+1} is already well-defined. If there exists no subset A of X disjoint from F_{k+1} such that

$$\ell^q_\nu(\ell^r_\omega)(f)(A) > 2^k, \tag{3.27}$$

then we set E_k to be empty, and continue the recursion with $k-1$.

If there exists a subset A of X disjoint from F_{k+1} satisfying (3.27), we define an auxiliary ν-Carathéodory collection $\{E_{k,n}\colon n \in \mathbb{N}_k\}$ of subsets of X by a forward recursion on $n \in \mathbb{N}_k$. The existence of A provides the starting point $E_{k,1}$ for the recursion. Now, we fix n, assume to have selected $E_{k,m}$ for every $m \in \mathbb{N}$, $m < n$ forming a ν-Carathéodory collection, and set

$$F_{k,n-1} = F_{k+1} \cup \mathbf{B}_{\mathcal{C}}\Big(\bigcup_{m<n} E_{k,m}\Big).$$

If there exists a subset A of X disjoint from $F_{k,n-1}$ satisfying (3.27), then we choose such a set A to be $E_{k,n}$. By the canopy condition 1.1, the collection $\{E_{k,m}\colon m \leq n\}$ is still ν-Carathéodory. If no A satisfying (3.27) exists, we set $\mathbb{N}_k$ to be $\{1, \ldots, n-1\}$, stop the forward recursion, set

$$E_k = \bigcup_{n\in\mathbb{N}_k} E_{k,n},$$

and continue the backward recursion with $k-1$.

By construction, we have (3.23), (3.24) for every $k \in \mathbb{Z}$. By construction and Lemma 3.2 applied to the ν-Carathéodory collection $\{E_{k,n}\colon n \in \mathbb{N}_k\}$, we have (3.22) for every nonempty E_k. To prove (3.25), we observe that for every k such that 2^k is greater than the $L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ quasi-norm of f, the statement is true. For every other k, the proof follows by construction and Lemma 3.3.

The equivalences in (3.26) follow by (3.24), the definition of F_k, (3.25), Fubini, and bounds for geometric series. ∎

Under the assumption $q \geq r$ on the exponents, we can drop the assumption of the canopy condition 1.1. Moreover, for every function f, the collection $\{E_k\colon k \in \mathbb{Z}\}$ produced by the decomposition forms a partition of the support of f.

Proposition 3.5. *For all $p, q \in (0,\infty)$ and $r \in (0,q]$, there exist constants $C = C(p,q,r)$, $c = c(p,q,r)$ such that the following holds.*

Let X be a finite set, μ, ν outer measures, and ω a measure. For every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, there exists a collection $\{E_k\colon k \in \mathbb{Z}\}$ of pairwise disjoint subsets of X forming a partition of the support of f such that the sets

$$F_k = \bigcup_{l\geq k} E_l$$

have the properties stated in (3.22)–(3.26).

Proof. The argument is analogous to that in the previous proof. The only difference is in the definition of E_k, for which we do not need a second forward recursion.

In fact, we fix k and assume to have selected E_l for every $l > k$. In particular, F_{k+1} is already well-defined. We set $\mathcal{E}_k$ to be the collection of nonempty subsets of X disjoint from F_{k+1} satisfying (3.27). If $\mathcal{E}_k$ is empty, we set E_k to be empty, and continue the recursion with $k-1$. If $\mathcal{E}_k$ is not empty, we choose a subcollection $\mathcal{E}'_k$ of $\mathcal{E}_k$ satisfying the following conditions. First, the elements of $\mathcal{E}'_k$ are pairwise disjoint. Moreover, every element of $\mathcal{E}_k$ intersects at least one element of $\mathcal{E}'_k$. We can fulfil these conditions in finitely many steps, due to the finiteness of X. In fact, if there exists an element of $\mathcal{E}_k$ pairwise disjoint from every element of $\mathcal{E}'_k$, we add it to $\mathcal{E}'_k$. Then we set E_k to be the union of the subsets of X in $\mathcal{E}'_k$, so that the subset F_k satisfies the property in (3.23) by construction. By (3.1) in Lemma 3.1 and the subadditivity of ν, the subset E_k satisfies the property in (3.22).

Because of the definition of F_k, the collection $\{E_k\colon k \in \mathbb{Z}\}$ forms a partition of the support of f. ∎

In fact, under the assumption of the canopy condition 1.1 on the setting, we can obtain a slightly different decomposition result improving that in Proposition 3.4 in the full range of exponents. The refinement we obtain is that we produce a partition of the support of the function f in terms of two ν-Carathéodory collections $\{\widetilde{E}^1_k\colon k \in \mathbb{Z}\}, \{\widetilde{E}^2_k\colon k \in \mathbb{Z}\}$. These collections are associated with $\{E_k\colon k \in \mathbb{Z}\}$, the collection of pairwise disjoint subsets of X we define by backward recursion according to the values of $\ell^q_\nu(\ell^r_\omega)$, and the collections are involved in an equivalence analogous to (3.26). The improvement over Proposition 3.4 is clarified by the following observations. First, $\{E_k\colon k \in \mathbb{Z}\}$ in Proposition 3.4 is a ν-Carathéodory collection, but in general it is not a partition of the support of f. Next, the collection $\{F_k \setminus F_{k+1}\colon k \in \mathbb{Z}\}$ in Proposition 3.4 is a partition of the support of f, but in general it is not a ν-Carathéodory collection. Obtaining a partition of the support of f in terms of ν-Carathéodory collections is important in order to prove Theorem 1.3. The minor price we have to pay to obtain the refinement described above is to change the levels from $\{2^k\colon k \in \mathbb{Z}\}$ to $\{\Psi^k\colon k \in \mathbb{Z}\}$, for a certain $\Psi > 1$ depending on the exponents and the parameters.

PROPOSITION 3.6. *For all $p, q, r \in (0,\infty)$, $\Phi, K \geq 1$, there exist constants $C = C(p,q,r,\Phi,K)$, $c = c(p,q,r,\Phi,K)$, $\Psi = \Psi(\Phi,p)$ such that the following holds.*

Let X be a set, μ, ν outer measures, ω a measure, and $\mathcal{C}$ a μ-covering function such that $(X,\mu,\nu,\mathcal{C})$ satisfies the canopy condition 1.1. *For every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, there exists a collection $\{E_k\colon k \in \mathbb{Z}\}$ of* *pairwise disjoint subsets of X such that the sets*

$$F_k = \mathbf{B}_{\mathcal{C}}(\mathbf{B}_{\mathcal{C}}(F_{k+1} \cup E_k))$$

have the properties stated in (3.22)–(3.25) *with* 2^k *replaced by* Ψ^k.

In particular, the ν*-Carathéodory collections* $\{\widetilde{E}_k^1 \colon k \in \mathbb{Z}\}$, $\{\widetilde{E}_k^2 \colon k \in \mathbb{Z}\}$ *defined by*

$$\widetilde{E}_k^1 = \mathbf{B}_{\mathcal{C}}(F_{k+1} \cup E_k) \setminus F_{k+1}, \qquad \widetilde{E}_k^2 = F_k \setminus \mathbf{B}_{\mathcal{C}}(F_{k+1} \cup E_k) \tag{3.28}$$

form a partition of the support of f, *and*

$$\begin{aligned} \|f\|^p_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} &\sim_{p,q,r,\Phi,K} \sum_{k\in\mathbb{Z}} \Psi^{kp}\mu(E_k) \\ &\sim_{p,q,r,\Phi,K} \sum_{k\in\mathbb{Z}} \Psi^{kp}\big(\mu(\widetilde{E}_k^1) + \mu(\widetilde{E}_k^2)\big). \end{aligned} \tag{3.29}$$

Proof. The argument is analogous to that in the proof of Proposition 3.4. The only difference is that we replace the levels 2^k with Ψ^k, where

$$\Psi = \Phi^{3/p}.$$

In fact, we define E_k by a double recursion as before, and $\widetilde{E}_k^1, \widetilde{E}_k^2$ as in (3.28). Due to their definition, the collections $\{\widetilde{E}_k^1 \colon k \in \mathbb{Z}\}, \{\widetilde{E}_k^2 \colon k \in \mathbb{Z}\}$ are ν-Carathéodory and they form a partition of the support of f.

We now turn to the proof of (3.29). By the properties corresponding to (3.25) and (3.24) in this setting, and the definition of F_k, we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \Psi^{kp}\mu(E_k) &\le C\sum_{k\in\mathbb{Z}} \Psi^{kp}\mu(\ell^q_\nu(\ell^r_\omega)(f) > c\Psi^k) \le C\|f\|^p_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \\ &\le C\sum_{k\in\mathbb{Z}} \Psi^{kp}\mu(\ell^q_\nu(\ell^r_\omega)(f) > \Psi^k) \le C\sum_{k\in\mathbb{Z}} \Psi^{kp}\sum_{l\ge k}(\mu(\widetilde{E}_l^1) + \mu(\widetilde{E}_l^2)). \end{aligned}$$

Moreover, by (3.28), $\mathcal{C}$ being a μ-covering function, and the definition of Ψ,

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \Psi^{kp}\sum_{l\ge k}(\mu(\widetilde{E}_l^1) + \mu(\widetilde{E}_l^2)) &\le C\sum_{k\in\mathbb{Z}} \Psi^{kp}\sum_{l\ge k}\sum_{j\ge l}\Phi^{2(j-l)}\mu(E_j) \\ &\le C\sum_{k\in\mathbb{Z}} \Psi^{kp}\sum_{j\ge k}\Phi^{2(j-k)}\mu(E_j) \\ &\le C\sum_{k\in\mathbb{Z}}\sum_{j\ge k}\Phi^{k-j}\Psi^{jp}\mu(E_j) \le C\sum_{j\in\mathbb{Z}}\Psi^{jp}\mu(E_j). \ \blacksquare \end{aligned}$$

Proof of Theorem 1.3. The case $q = \infty$ follows by definition. Therefore, without loss of generality, we assume $q = 1$.

Case I: arbitrary $r \in (0, \infty]$. For a function $f \in L^1_\mu(\ell^1_\nu(\ell^r_\omega))$, let $\{E_k \colon k \in \mathbb{Z}\}$, $\{\widetilde{E}_k^1 \colon k \in \mathbb{Z}\}$, $\{\widetilde{E}_k^2 \colon k \in \mathbb{Z}\}$ be the collections of pairwise disjoint subsets of X as in Proposition 3.6. By (3.29), the property corresponding to (3.22), and Lemma 3.2, we have

$$\begin{aligned}\|f\|_{L^1_\mu(\ell^1_\nu(\ell^r_\omega))} &\le C\sum_{k\in\mathbb{Z}}\Psi^k\mu(E_k) \le C\sum_{k\in\mathbb{Z}}\|f1_{E_k}\|_{L^1_\nu(\ell^r_\omega)} \le C\Big\|\sum_{k\in\mathbb{Z}}f1_{E_k}\Big\|_{L^1_\nu(\ell^r_\omega)}\\ &\le C\|f\|_{L^1_\nu(\ell^r_\omega)}.\end{aligned}$$

Moreover, by the quasi-triangle inequality for the outer L^p quasi-norm of two summands, Lemma 3.2, the property corresponding to (3.23), and (3.29),

$$\begin{aligned}\|f\|_{L^1_\nu(\ell^r_\omega)} &\le C\Big(\Big\|\sum_{k\in\mathbb{Z}}f1_{\widetilde{E}^1_k}\Big\|_{L^1_\nu(\ell^r_\omega)} + \Big\|\sum_{k\in\mathbb{Z}}f1_{\widetilde{E}^2_k}\Big\|_{L^1_\nu(\ell^r_\omega)}\Big)\\ &\le C\Big(\sum_{k\in\mathbb{Z}}\|f1_{\widetilde{E}^1_k}\|_{L^1_\nu(\ell^r_\omega)} + \sum_{k\in\mathbb{Z}}\|f1_{\widetilde{E}^2_k}\|_{L^1_\nu(\ell^r_\omega)}\Big)\\ &\le C\sum_{k\in\mathbb{Z}}\Psi^k(\mu(\widetilde{E}^1_k)+\mu(\widetilde{E}^2_k)) \le C\|f\|_{L^1_\mu(\ell^1_\nu(\ell^r_\omega))}.\end{aligned}$$

Case II: $q \ge r$. For $f \in L^1_\mu(\ell^1_\nu(\ell^r_\omega))$, let $\{E_k : k \in \mathbb{Z}\}$ be the collection of pairwise disjoint subsets of X as in Proposition 3.5. By the properties corresponding to (3.26) and (3.22), and (3.1) in Lemma 3.1, we have

$$\begin{aligned}\|f\|_{L^1_\mu(\ell^1_\nu(\ell^r_\omega))} &\le C\sum_{k\in\mathbb{Z}}2^k\mu(E_k) \le C\sum_{k\in\mathbb{Z}}\|f1_{E_k}\|_{L^1_\nu(\ell^r_\omega)} \le C\|\sum_{k\in\mathbb{Z}}f1_{E_k}\|_{L^1_\nu(\ell^r_\omega)}\\ &\le C\|f\|_{L^1_\nu(\ell^r_\omega)}.\end{aligned}$$

Case III: $q \le r$. For $f \in L^1_\mu(\ell^1_\nu(\ell^r_\omega))$, let $\{A_k : k \in \mathbb{Z}\}$ be the collection of optimal sets associated with the super level measures $\mu(\ell^1_\nu(\ell^r_\omega)(f) > 2^k)$, namely

$$\|f1_{A_k^c}\|_{L^\infty_\mu(\ell^1_\nu(\ell^r_\omega))} \le 2^k, \tag{3.30}$$

$$\mu(\ell^1_\nu(\ell^r_\omega)(f) > 2^k) = \mu(A_k). \tag{3.31}$$

By (3.2) in Lemma 3.1, (3.30), the monotonicity of μ, and (3.31), we have

$$\begin{aligned}\|f\|_{L^1_\nu(\ell^r_\omega)} &\le C\sum_{k\in\mathbb{Z}}\|f1_{A_k\setminus A_{k+1}}\|_{L^1_\nu(\ell^r_\omega)} \le C\sum_{k\in\mathbb{Z}}2^{k+1}\mu(A_k\setminus A_{k+1})\\ &\le C\sum_{k\in\mathbb{Z}}2^k\mu(A_k) \le C\|f\|_{L^1_\mu(\ell^1_\nu(\ell^r_\omega))}. \ \blacksquare\end{aligned}$$

3.3. Dualizing function candidate. We start by recalling the setting. Let $p, q, r \in (1,\infty)$ and $\Phi, K \ge 1$. Let X be a finite set, μ, ν outer measures, ω a measure, and $\mathcal{C}$ a μ-covering function. For $q < r$, we assume $(X,\mu,\nu,\mathcal{C})$ satisfies the canopy condition 1.1. For $q > r$, we assume $(X,\mu,\nu,\mathcal{C})$ satisfies the crop condition 1.2.

When $q = r$, the double iterated outer L^p quasi-norm is isomorphic to a single iterated one, and the results stated in Theorem 1.4 correspond to [11, properties (ii), (iii) of Theorem 1.1].

When $q \neq r$, for a function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we provide the candidate dualizing function g on X. We distinguish two cases.

CASE 1: $q > r$. Let $\{E_k : k \in \mathbb{Z}\}$ be the collection of pairwise disjoint subsets of X associated with the function f and the size $\ell^q_\nu(\ell^r_\omega)$ as in Proposition 3.5.

CASE 2: $q < r$. Let $\{E_k : k \in \mathbb{Z}\}$ be the collection of pairwise disjoint subsets of X associated with the function f and the size $\ell^q_\nu(\ell^r_\omega)$ as in Proposition 3.4.

In both cases, let $\{U^k_j : j \in \mathbb{Z}\}$ be the collection of pairwise disjoint subsets of E_k associated with the function $f 1_{E_k}$ and the size ℓ^r_ω as in Proposition 2.2. We define

$$f_{k,j}(x) = f(x) 1_{U^k_j}(x), \qquad f_k(x) = \sum_{j \in \mathbb{Z}} f_{k,j}(x) = f(x) \sum_{j \in \mathbb{Z}} 1_{U^k_j}(x).$$

When $q > r$, let

$$M = 2 + \left\lfloor \frac{\log_2 K}{r} \right\rfloor,$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. For

$$\mathcal{F}^k_j = \{F \in \mathcal{E} : \ell^r_\omega(f_{k,j})(F) \leq 2^{j-M}\},$$

let $\mathcal{G}^k_j$ be its ν-Carathéodory subcollection as in the crop condition 1.2; set

$$\widetilde{U}^k_j = U^k_j \setminus \bigcup_{G \in \mathcal{G}^k_j} G.$$

We set

$$W^k_j = \begin{cases} \widetilde{U}^k_j & \text{for } q > r, \\ U^k_j & \text{for } q < r, \end{cases}$$

and we define

$$\begin{aligned} g_{k,j}(x) &= f(x)^{r-1} 1_{W^k_j}(x), \\ g_k(x) &= \sum_{j \in \mathbb{Z}} 2^{j(q-r)} g_{k,j}(x) = f(x)^{r-1} \sum_{j \in \mathbb{Z}} 2^{j(q-r)} 1_{W^k_j}(x), \\ g(x) &= \sum_{k \in \mathbb{Z}} 2^{k(p-q)} g_k(x) = f(x)^{r-1} \sum_{k \in \mathbb{Z}} 2^{k(p-q)} \sum_{j \in \mathbb{Z}} 2^{j(q-r)} 1_{W^k_j}(x). \end{aligned} \tag{3.32}$$

LEMMA 3.7. *Let $p, q, r \in (1, \infty)$ with $q \neq r$, and $\Phi, K \geq 1$. There exists a constant $c = c(r, K)$ such that, for every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have*

$$\|f^r_{k,j} 1_{W^k_j}\|_{L^1(X,\omega)} \geq c 2^{jr} \nu(U^k_j). \tag{3.33}$$

Proof. Case I: $q > r$. We have

$$\begin{aligned}\|f^r_{k,j} 1_{W^k_j}\|_{L^1(X,\omega)} &\geq \|f^r_{k,j}\|_{L^1(X,\omega)} - \sum_{G \in \mathcal{G}^k_j} \|f^r_{k,j} 1_G\|_{L^1(X,\omega)} \\ &\geq 2^{jr}\nu(U^k_j) - \sum_{G \in \mathcal{G}^k_j} 2^{(j-M)r}\nu(U^k_j \cap G) \\ &\geq 2^{jr}\nu(U^k_j) - K2^{(j-M)r}\nu(U^k_j) \geq c2^{jr}\nu(U^k_j),\end{aligned}$$

where we use (2.8) and the control on the size ℓ^r_ω defining the elements of $\mathcal{F}^k_j$ in the second inequality, the ν-Carathéodory condition (1.12) for the collection $\mathcal{G}^k_j$ in the third, and the definition of M in the fourth.

Case II: $q < r$. The desired inequality follows by (2.8). ∎

The definition of g guarantees the following good lower bound on the classical L^1 norm of fg, and good upper bound on the outer $L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))$ quasi-norm of g.

Lemma 3.8. *Let $p, q, r \in (1, \infty)$ with $q \neq r$, and $\Phi, K \geq 1$. There exists a constant $c = c(p, q, r, \Phi, K)$ such that, for every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, and for g defined by* (3.32),

$$\|fg\|_{L^1(X,\omega)} \geq c\|f\|^p_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

Proof. By (3.33) and (2.12),

$$\begin{aligned}\|fg\|_{L^1(X,\omega)} &= \sum_{k \in \mathbb{Z}} 2^{k(p-q)} \sum_{j \in \mathbb{Z}} 2^{j(q-r)} \|f^r_{k,j} 1_{W^k_j}\|_{L^1(X,\omega)} \\ &\geq c \sum_{k \in \mathbb{Z}} 2^{k(p-q)} \sum_{j \in \mathbb{Z}} 2^{jq}\nu(U^k_j) \\ &\geq c \sum_{k \in \mathbb{Z}} 2^{k(p-q)} \|f_k\|^q_{L^q_\nu(\ell^r_\omega)}.\end{aligned}$$

For $q < r$, by (3.22) and (3.26),

$$\sum_{k \in \mathbb{Z}} 2^{k(p-q)} \|f_k\|^q_{L^q_\nu(\ell^r_\omega)} \geq c \sum_{k \in \mathbb{Z}} 2^{kp}\mu(E_k) \geq c\|f\|^p_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

For $q > r$, the properties in Proposition 3.5 corresponding to (3.22) and (3.26) yield the analogous chain of inequalities. ∎

Lemma 3.9. *Let $p, q, r \in (1, \infty)$ with $q \neq r$, and $\Phi, K \geq 1$. There exists a constant $C = C(p, q, r, \Phi, K)$ such that, for every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, for g defined by* (3.32),

$$\|g\|^{p'}_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} \leq C\|f\|^p_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}. \tag{3.34}$$

Proof. Case I: $q > r$. Let $\widetilde{k}, j$ be fixed. For all $F \subseteq X$ and $U \subseteq F$, we have

$$\begin{aligned}\ell^{r'}_\omega(g_{\widetilde{k}} 1_F 1_{(V^{\widetilde{k}}_j)^c})(U) &\le \sum_{\widetilde{j}<j} 2^{\widetilde{j}(q-r)} (\nu(U)^{-1} \| g_{\widetilde{k},\widetilde{j}} 1_{U \setminus V^{\widetilde{k}}_{\widetilde{j}+1}} \|^{r'}_{L^{r'}(X,\omega)})^{1/r'} \\ &\le \sum_{\widetilde{j}<j} 2^{\widetilde{j}(q-r)} (\nu(U)^{-1} \| f_{\widetilde{k},\widetilde{j}} 1_{U \setminus V^{\widetilde{k}}_{\widetilde{j}+1}} \|^{r}_{L^{r}(X,\omega)})^{1/r'} \le c 2^{j(q-1)},\end{aligned}$$

where we use the triangle inequality for the classical $L^{r'}$ norm in the first inequality, and (2.9) in the third. The previous chain of inequalities yields

$$\nu(\ell^{r'}_\omega(g_{\widetilde{k}} 1_F) > c 2^{j(q-1)}) \le \sum_{\widetilde{j} \ge j} \nu(W^{\widetilde{k}}_{\widetilde{j}} \cap F). \tag{3.35}$$

Moreover, for every fixed $\widetilde{j} \in \mathbb{Z}$, for $E = \mathbf{B}_\mathcal{C}(F)$, we have

$$\nu(W^{\widetilde{k}}_{\widetilde{j}} \cap F) \le C \nu(\ell^r_\omega(f_{\widetilde{k}} 1_E) > \widetilde{c} 2^{\widetilde{j}}). \tag{3.36}$$

In fact, we have two cases:

- If $W^{\widetilde{k}}_{\widetilde{j}} \cap F = \emptyset$, the left hand side in (3.36) is 0, and the inequality holds.
- If $W^{\widetilde{k}}_{\widetilde{j}} \cap F \neq \emptyset$, by the crop condition 1.2, we see that $E' = \mathbf{B}_\mathcal{C}(W^{\widetilde{k}}_{\widetilde{j}} \cap F) \subseteq E$ is covered by a collection of disjoint subsets that are not in $\mathcal{F}^{\widetilde{k}}_{\widetilde{j}}$, so that
$$\ell^r_\omega(f_{\widetilde{k},\widetilde{j}} 1_E)(U^{\widetilde{k}}_{\widetilde{j}} \cap E') \ge \widetilde{c} 2^{\widetilde{j}},$$
hence, by Lemma 2.1, we obtain (3.36).

Therefore, by (3.35) and (3.36),

$$\begin{aligned}\| g_{\widetilde{k}} 1_F \|^{q'}_{L^{q'}_\nu(\ell^{r'}_\omega)} &\le C \sum_{j \in \mathbb{Z}} 2^{jq} \nu(\ell^{r'}_\omega(g_{\widetilde{k}} 1_F) > c 2^{j(q-1)}) \\ &\le C \sum_{j \in \mathbb{Z}} 2^{jq} \sum_{\widetilde{j} \ge j} \nu(\ell^r_\omega(f_{\widetilde{k}} 1_E) > \widetilde{c} 2^{\widetilde{j}}) \le C \| f_{\widetilde{k}} 1_E \|^q_{L^q_\nu(\ell^r_\omega)}.\end{aligned} \tag{3.37}$$

Hence,

$$\begin{aligned}\ell^{q'}_\nu(\ell^{r'}_\omega)(g 1_{F^c_k})(F) &\le C \sum_{\widetilde{k}<k} 2^{\widetilde{k}(p-q)} (\mu(F)^{-1} \| g_{\widetilde{k}} 1_F \|^{q'}_{L^{q'}_\nu(\ell^{r'}_\omega)})^{1/q'} \\ &\le C \sum_{\widetilde{k}<k} 2^{\widetilde{k}(p-q)} (\mu(F)^{-1} \| f_{\widetilde{k}} 1_E \|_{L^q_\nu(\ell^r_\omega)})^{1/q'} \le C 2^{k(p-1)},\end{aligned}$$

where we use the quasi-triangle inequality for the outer $L^{q'}_\nu(\ell^{r'}_\omega)$ quasi-norm proved in [11] in the first inequality, (3.37) in the second, and the property in Proposition 3.5 corresponding to (3.23), and (1.11), in the third. The previous chain of inequalities yields

$$\mu(\ell^{q'}_{\nu}(\ell^{r'}_{\omega})(g) > C2^{k(p-1)}) \leq \mu(F_k) \leq \widetilde{C} \sum_{\widetilde{k} \geq k} \mu(E_{\widetilde{k}}).$$

Together with the property in Proposition 3.5 corresponding to (3.26), this leads to

$$\begin{aligned} \|g\|^{p'}_{L^{p'}_{\mu}(\ell^{q'}_{\nu}(\ell^{r'}_{\omega}))} &\leq \widetilde{C} \sum_{k \in \mathbb{Z}} 2^{kp} \mu(\ell^{q'}_{\nu}(\ell^{r'}_{\omega})(g) > C2^{k(p-1)}) \leq \widetilde{C} \sum_{k \in \mathbb{Z}} 2^{kp} \sum_{\widetilde{k} \geq k} \mu(E_{\widetilde{k}}) \\ &\leq \widetilde{C} \|f\|^{p}_{L^{p}_{\mu}(\ell^{q}_{\nu}(\ell^{r}_{\omega}))}. \end{aligned}$$

Case II: $q < r$. Let $\widetilde{k}$ be fixed. It is enough to prove that, for every $F \subseteq X$, we have

$$\|g_{\widetilde{k}} 1_F\|^{q'}_{L^{q'}_{\nu}(\ell^{r'}_{\omega})} \leq C \|f_{\widetilde{k}} 1_F\|^{q}_{L^{q}_{\nu}(\ell^{r}_{\omega})}. \tag{3.38}$$

The desired inequality in (3.34) then follows as in the previous case.

Let j be fixed. Let $V(2^j)$ be an optimal set associated with the super level measure $\nu(\ell^r_{\omega}(f_{\widetilde{k}} 1_F) > 2^j)$, namely

$$\|f_{\widetilde{k}} 1_F 1_{V(2^j)^c}\|_{L^{\infty}_{\nu}(\ell^r_{\omega})} \leq 2^j, \tag{3.39}$$

$$\nu(\ell^r_{\omega}(f_{\widetilde{k}} 1_F) > 2^j) = \nu(V(2^j)). \tag{3.40}$$

For every $U \subseteq F$, we have

$$\begin{aligned} \ell^{r'}_{\omega}(g_{\widetilde{k}} 1_F 1_{V(2^j)^c})(U) \leq{}& \sum_{\widetilde{j} < j} 2^{\widetilde{j}(q-r)} (\nu(U)^{-1} \|g_{\widetilde{k},\widetilde{j}} 1_{U \setminus V^{\widetilde{k}}_{\widetilde{j}+1}}\|^{r'}_{L^{r'}(X,\omega)})^{1/r'} \\ &+ (\nu(U)^{-1} \Big\| \sum_{\widetilde{j} \geq j} 2^{\widetilde{j}(q-r)} g_{\widetilde{k},\widetilde{j}} 1_F 1_{U \setminus V(2^j)} \Big\|^{r'}_{L^{r'}(X,\omega)})^{1/r'} \\ \leq{}& \sum_{\widetilde{j} < j} 2^{\widetilde{j}(q-r)} (\nu(U)^{-1} \|f_{\widetilde{k},\widetilde{j}} 1_{U \setminus V^{\widetilde{k}}_{\widetilde{j}+1}}\|^{r}_{L^{r}(X,\omega)})^{1/r'} \\ &+ 2^{j(q-r)} (\nu(U)^{-1} \Big\| \sum_{\widetilde{j} \geq j} f_{\widetilde{k},\widetilde{j}} 1_F 1_{U \setminus V(2^j)} \Big\|^{r'}_{L^{r'}(X,\omega)})^{1/r'} \\ \leq{}& c 2^{j(q-1)}, \end{aligned}$$

where we use the triangle inequality for the classical $L^{r'}$ norm in the first inequality, the condition $q < r$ in the second, and (3.23) and (3.39) in the third. Together with (3.40), the previous chain of inequalities yields, for every $j \in \mathbb{Z}$,

$$\nu(\ell^{r'}_{\omega}(g_{\widetilde{k}} 1_F) > c 2^{j(q-1)}) \leq \nu(\ell^r_{\omega}(f_{\widetilde{k}} 1_F) > 2^j).$$

The inequality in (3.38) follows upon multiplying by 2^{jq} and summing in $j \in \mathbb{Z}$ on both sides. ∎

Proof of Theorem 1.4. When $q = r$, the double iterated outer L^p quasi-norm is isomorphic to a single iterated one, and the proof corresponds to the one of properties (ii), (iii) of [11, Theorem 1.1].

When $q \neq r$, we proceed as follows.

(i) By (1.16), the $L^1(X, \omega)$-pairing of two functions f, g is equivalent to the outer $L^1_\mu(\ell^1_\nu(\ell^1_\omega))$ quasi-norm of the product fg. The second inequality in (1.14) is then given by outer Hölder's inequality [10, Proposition 3.4]. The first inequality in (1.14) is a corollary of Lemmas 3.8 and 3.9 for $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$.

(ii) The inequality in (1.15) is a corollary of the triangle inequality for the $L^1(X, \omega)$ norm and property (i). ∎

3.4. Counterexamples. For every $m \in \mathbb{N}$, we introduce the finite setting

$$\begin{aligned} X_m &= \{x_i : 1 \leq i \leq m\}, && \\ \omega_m(A) &= \mu_m(A) = |A| && \text{for every } A \subseteq X_m, \\ \nu_m(A) &= 1 && \text{for every } \emptyset \neq A \subseteq X_m, \\ f_i &= 1_{x_i} && \text{for every } 1 \leq i \leq m, \\ f &= 1_{X_m}. && \end{aligned}$$

In particular, the collection of singletons $\{\{x_i\} : 1 \leq i \leq m\}$ satisfies the ν_m-Carathéodory condition with parameter $K_m \geq m$.

First, we observe that, for every exponent $r \in (0, \infty]$, for every function g, for every nonempty subset A of X_m,

$$\ell^r_{\omega_m}(g)(A) = \|g 1_A\|_{L^r(X_m, \omega_m)}.$$

Therefore, for every exponent $r \in (0, \infty]$, for every function g,

$$\nu_m(\ell^r_{\omega_m}(g) > \lambda) = \begin{cases} \nu_m(X_m) = 1 & \text{for } \lambda \in [0, \|g\|_{L^\infty_{\nu_m}(\ell^r_{\omega_m})}), \\ \nu_m(\emptyset) = 0 & \text{for } \lambda \in [\|g\|_{L^\infty_{\nu_m}(\ell^r_{\omega_m})}, \infty), \end{cases}$$

where, here and later as well, for every level λ, we provide a subset of X_m realizing the infimum in the definition of the super level measure in (1.7).

Hence, for all exponents $q, r \in (0, \infty]$,

$$\|g\|_{L^q_{\nu_m}(\ell^r_{\omega_m})} = \|g\|_{L^\infty_{\nu_m}(\ell^r_{\omega_m})} = \|g\|_{L^r(X_m, \omega_m)}.$$

In particular, for every exponent $r \in (0, \infty]$,

$$\sum_{i=1}^m \|f_i\|_{L^1_{\nu_m}(\ell^r_{\omega_m})} = \sum_{i=1}^m 1 = m, \qquad \Big\| \sum_{i=1}^m f_i \Big\|_{L^1_{\nu_m}(\ell^r_{\omega_m})} = \|f\|_{L^1_{\nu_m}(\ell^r_{\omega_m})} = m^{1/r}.$$

When $r \in (0, \infty]$, $r \neq 1$, one of the constants C_1, C_2 of q-super- or sub-orthogonality in (3.11) blows up as m grows to infinity.

Next, we observe that, for all exponents $q, r \in (0, \infty]$, for every function g, for every nonempty subset A of X_m,

$$\ell^q_{\nu_m}(\ell^r_{\omega_m})(g)(A) = \mu_m(A)^{-1/q}\|g1_A\|_{L^q_{\nu_m}(\ell^r_{\omega_m})} = |A|^{-1/q}\|g1_A\|_{L^r(X_m,\omega_m)},$$

hence, for every exponent $r \in [1, \infty]$, for every strict subset B of X_m,

$$\|f1_{B^c}\|_{L^\infty_{\mu_m}(\ell^1_{\nu_m}(\ell^r_{\omega_m}))} = 1 = \ell^1_{\nu_m}(\ell^r_{\omega_m})(f1_{B^c})(\{x_i\}) \quad \text{for every } x_i \notin B.$$

Therefore, for every exponent $r \in [1, \infty]$,

$$\mu_m(\ell^1_{\nu_m}(\ell^r_{\omega_m})(f) > \lambda) = \begin{cases} \mu_m(X_m) = m & \text{for } \lambda \in [0, 1), \\ \mu_m(\emptyset) = 0 & \text{for } \lambda \in [1, \infty). \end{cases}$$

In particular, for every $r \in [1, \infty]$,

$$\|f\|_{L^1_{\mu_m}(\ell^1_{\nu_m}(\ell^r_{\omega_m}))} = m.$$

When $r \in (1, \infty]$, the constant C_2 of the "collapsing effect" in (1.13) blows up as m grows to infinity.

Finally, we observe that, for all exponents $q \in (1, \infty)$, $r \in (1, q]$, and every strict subset B of X_m,

$$\|f1_{B^c}\|_{L^\infty_{\mu_m}(\ell^q_{\nu_m}(\ell^r_{\omega_m}))} = |X_m \setminus B|^\alpha = \ell^q_{\nu_m}(\ell^r_{\omega_m})(f1_{B^c})(B^c),$$

where $\alpha = \alpha(r, q) = \frac{1}{r} - \frac{1}{q}$. Therefore, for all $q \in (1, \infty)$, $r \in (1, q]$, we have, for $1 \leq i \leq m$,

$$\mu_m(\ell^q_{\nu_m}(\ell^r_{\omega_m})(f) > \lambda) = \begin{cases} \mu_m(X_m^{m-i+1}) = m - i + 1 & \text{for } \lambda \in [(i-1)^\alpha, i^\alpha), \\ \mu_m(\emptyset) = 0 & \text{for } \lambda \in [m^\alpha, \infty), \end{cases}$$

where X^j_m is an arbitrary subset of X_m of cardinality j.

In particular, for all $p, q \in (1, \infty)$, $r \in (1, q]$, there exists a constant $c = c(p, q, r)$ such that, for every $m \in \mathbb{N}$ large enough,

$$\sum_{i=1}^m \|f_i\|_{L^p_{\mu_m}(\ell^q_{\nu_m}(\ell^r_{\omega_m}))} = \sum_{i=1}^m 1 = m,$$
$$\Big\|\sum_{i=1}^m f_i\Big\|_{L^p_{\mu_m}(\ell^q_{\nu_m}(\ell^r_{\omega_m}))} = \|f\|_{L^p_{\mu_m}(\ell^q_{\nu_m}(\ell^r_{\omega_m}))} \geq cm^{1/p-1/q+1/r}.$$

Therefore, the constants of the sharpness of outer Hölder's inequality in (1.14) and the triangle inequality in (1.15) blow up as m grows to infinity when

$$p, q, r \in (1, \infty), \qquad \frac{1}{p} - \frac{1}{q} + \frac{1}{r} > 1.$$

Now, for every $m \in \mathbb{N}$, we slightly modify the previous finite setting:

$$\begin{aligned} X_m &= \{x_i \colon 1 \leq i \leq m\}, && \\ \omega_m(A) &= |A| && \text{for every } A \subseteq X_m, \\ \nu_m(A) &= 1 && \text{for every } A \subseteq X_m, \\ \sigma_m(\{x_i\}) &= 2^{\beta(i-1)} && \text{for every } 1 \leq i \leq m, \\ f &= 1_{X_m}, && \end{aligned}$$

where $\beta = \beta(r) = 2/r$, and we let μ_m be the measure generated via (1.17) from σ_m. As in the previous setting, the collection $\{\{x_i\} \colon 1 \leq i \leq m\}$ of singletons satisfies the ν_m-Carathéodory condition with parameter $K_m \geq m$.

As in the previous setting, for all exponents $q, r \in (0, \infty]$, every function g, and every nonempty subset A of X_m,

$$\ell^q_{\nu_m}(\ell^r_{\omega_m})(g)(A) = \mu_m(A)^{-1/q} \|g 1_A\|_{L^q_{\nu_m}(\ell^r_{\omega_m})} = \mu_m(A)^{-1/q} \|g 1_A\|_{L^r(X_m, \omega_m)},$$

hence, for every $r \in (0, 1]$ and every $B \subsetneq X_m$,

$$\|f 1_{B^c}\|_{L^\infty_{\mu_m}(\ell^1_{\nu_m}(\ell^r_{\omega_m}))} = 2^{-\beta(j-1)} = \ell^1_{\nu_m}(\ell^r_{\omega_m})(f 1_{B^c})(\{x_j\}),$$

where $j = \min\{i \colon 1 \leq i \leq m,\, x_i \notin B\}$. Therefore, for every $r \in (0, 1]$ we have, for $1 \leq j < m$,

$$\mu_m(\ell^1_{\nu_m}(\ell^r_{\omega_m})(f) > \lambda) = \begin{cases} \mu_m(X_m) = \displaystyle\sum_{i=1}^m 2^{\beta(i-1)} & \text{for } \lambda \in [0, 2^{-\beta(m-1)}), \\ \mu_m(X^j_m) = \displaystyle\sum_{i=1}^j 2^{\beta(i-1)} & \text{for } \lambda \in [2^{-\beta j}, 2^{-\beta(j-1)}), \\ \mu_m(\emptyset) = 0 & \text{for } \lambda \in [1, \infty), \end{cases}$$

where $X^j_m = \{x_i \colon 1 \leq i \leq j\} \subseteq X_m$.

In particular, for every $r \in (0, 1]$, there exists a constant $C = C(r)$ such that

$$\|f\|_{L^1_{\nu_m}(\ell^r_{\omega_m})} = m^{1/r}, \qquad \|f\|_{L^1_{\mu_m}(\ell^1_{\nu_m}(\ell^r_{\omega_m}))} \leq Cm.$$

When $r \in (0, 1)$, the constant C_1 of the "collapsing effect" in (1.13) blows up as m grows to infinity.

4. Examples. In this section we present three settings in which we provide a μ-covering function $\mathcal{C}$ satisfying the canopy condition 1.1 and the crop condition 1.2.

4.1. Finite set with three measures. Let X be a finite set, and μ, ν, ω be three measures on it. The function $\mathcal{C}$ defined by

$$\mathcal{E} = \{\{x\} \colon x \in X\}, \qquad \mathcal{C}(A) = \{\{x\} \colon x \in A\},$$

is a μ-covering function with parameter $\Phi = 1$. The canopy and the crop conditions with parameters $\Phi = K = 1$ are satisfied because every collection of pairwise disjoint subsets of X is ν-Carathéodory with parameter $K = 1$, since ν is a measure, and we use the very definition of $\mathcal{C}$. The same conditions are satisfied by

$$\mathcal{E}' = \mathcal{P}(X), \quad \mathcal{C}'(A) = A.$$

4.2. Cartesian product of three finite sets with measures. Let X_1, X_2, X_3 be finite sets with measures $\omega_1, \omega_2, \omega_3$. Let μ, ν, ω be the outer measures μ_1, μ_2, μ_3 defined on X as in (1.8). The function $\mathcal{C}$ defined by

$$\mathcal{E} = \{X_1 \times X_2 \times \{z\} \colon z \in X_3\}, \quad \mathcal{C}(A) = \{X_1 \times X_2 \times \{z\} \colon z \in \pi_3(A)\},$$

where π_3 is the projection in X_3, is a μ-covering function with parameter $\Phi = 1$. The canopy and the crop conditions with parameters $\Phi = K = 1$ are satisfied because every collection of disjoint subsets of X of the form $X_1 \times X_2 \times Z$ is ν-Carathéodory with parameter $K = 1$, since on these sets, ν behaves like the measure $\omega_2 \otimes \omega_3$, and we use the very definition of $\mathcal{C}$. The same conditions are satisfied by

$$\mathcal{E}' = \{X_1 \times X_2 \times Z \colon Z \in \mathcal{P}(X_3)\}, \quad \mathcal{C}'(A) = X_1 \times X_2 \times \pi_3(A).$$

4.3. Upper half 3-space with dyadic strips and trees. Let X be the upper half 3-space with the measure induced by the Lebesgue measure on $\mathbb{R}^3$,

$$X = \mathbb{R}^3_+ = \mathbb{R}^2_+ \times \mathbb{R} = \mathbb{R} \times (0, \infty) \times \mathbb{R}, \quad \mathrm{d}\omega(y, t, \eta) = \mathrm{d}y \, \mathrm{d}t \, \mathrm{d}\eta. \tag{4.1}$$

To define the outer measures, we start by recalling the set $\mathcal{I}$ of dyadic intervals in $\mathbb{R}$,

$$I(m, l) = (2^l m, 2^l (m+1)], \quad \mathcal{I} = \{I(m, l) \colon m, l \in \mathbb{Z}\}.$$

Moreover, for all $m, l, n \in \mathbb{Z}$, we define the dyadic upper half-tile $H(m, l, n)$ by

$$H(m, l, n) = I(m, l) \times (2^{l-1}, 2^l] \times I(n, -l). \tag{4.2}$$

Now, let μ be the outer measure generated by the pre-measure σ on $\mathcal{D}$, the collection of dyadic strips, as in (1.17):

$$\begin{aligned} &D(m, l) = D\big(I(m, l)\big) = \bigcup_{l' \leq l} \bigcup_{m' = 2^{l-l'} m}^{2^{l-l'}(m+1)-1} \bigcup_{n' \in \mathbb{Z}} H(m', l', n'), \\ &\mathcal{D} = \{D(m, l) \colon m, l \in \mathbb{Z}\} = \{D(I) \colon I \in \mathcal{I}\}, \\ &\sigma\big(D(m, l)\big) = |I(m, l)| = 2^l \quad \text{for all } m, l \in \mathbb{Z}. \end{aligned} \tag{4.3}$$

Analogously, let ν be the outer measure generated by the pre-measure τ

on $\mathcal{T}$, the collection of dyadic trees, as in (1.17):

$$(4.4)\quad \begin{aligned} &T(m,l,n) = T(I(m,l), I(n,-l)) = \bigcup_{l' \le l} \bigcup_{m'=2^{l-l'}m}^{2^{l-l'}(m+1)-1} H(m', l', N(n,l')), \\ &\mathcal{T} = \{T(m,l,n) \colon m,l,n \in \mathbb{Z}\} = \{T(I,\widetilde{I}) \colon I, \widetilde{I} \in \mathcal{I},\ |I|\,|\widetilde{I}| = 1\}, \\ &\tau(T(m,l,n)) = |I(m,l)| = 2^l \quad \text{for all } m,l,n \in \mathbb{Z}, \end{aligned}$$

where $N(n,l')$ is defined by the condition

$$(4.5)\qquad I(n,-l) \subseteq I(N(n,l'), -l').$$

From now on, we assume all the strips and trees in this subsection are dyadic, and we do not repeat it.

Next, for every $L \in \mathbb{Z}$, we define

$$(4.6)\qquad Y_L = \mathbb{R} \times (0, 2^L] \times \mathbb{R}.$$

On Y_L, we have the measure ω_L and the outer measures μ_L, ν_L induced by ω, μ, ν. In particular, the outer measures μ_L, ν_L are equivalently generated as in (1.17) by the pre-measures σ, τ upon restricting the collections of dyadic strips and trees to those contained in Y_L, namely

$$\mathcal{D}_L = \{D(m,l) \colon m,l \in \mathbb{Z},\ l \le L\}, \qquad \mathcal{T}_L = \{T(m,l,n) \colon m,l,n \in \mathbb{Z},\ l \le L\}.$$

Moreover, we drop the subscript L in all the notation, as the definitions are consistent with the inclusions $Y_{L_1} \subseteq Y_{L_2}$ for $L_1 \le L_2$.

To define the function $\mathcal{C}$ and check that it satisfies the required conditions, we recall some properties of the geometry of dyadic strips and trees, introduce some auxiliary functions and formulate their properties. We postpone the proofs to Appendix A.

To make the notation more compact, we introduce a new symbol for the union of the elements of a collection of subsets of X,

$$\mathcal{L} \colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X), \qquad \mathcal{L}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A.$$

We start with two observations about the geometry of the intersections between strips, and between a strip and a tree.

Lemma 4.1. *Given two strips D_1, D_2 in $\mathcal{D}$, their intersection is again a strip in $\mathcal{D}$, possibly empty. If it is nonempty, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.*

Lemma 4.2. *Given a strip D in $\mathcal{D}$ and a tree T in $\mathcal{T}$, their intersection is again a tree T' in $\mathcal{T}$, possibly empty.*

Next, we make some observations about the behaviour of the outer measures μ, ν on strips, trees, their unions and their intersections.

Lemma 4.3. *For every strip D in $\mathcal{D}$ and for every tree T in $\mathcal{T}$,*

$$\mu(D) = \sigma(D) = |\pi(D)|, \tag{4.7}$$

$$\nu(T) = \tau(T) = |\pi(T)|, \tag{4.8}$$

where π is the projection on the first coordinate.

Moreover, for every tree T in $\mathcal{T}$,

$$\nu(T) = |\pi(T)| = |\pi(D(T))| = \mu(D(T)), \tag{4.9}$$

where $D(T)$ is the strip in $\mathcal{D}$ containing T defined by

$$D(T) = \pi(T) \times (0, |\pi(T)|] \times \mathbb{R}.$$

Lemma 4.4. *For every collection $\mathcal{D}_1$ of pairwise disjoint strips in $\mathcal{D}$,*

$$\mu(\mathcal{L}(\mathcal{D}_1)) = \sum_{D_1 \in \mathcal{D}_1} \mu(D_1) = \sum_{D_1 \in \mathcal{D}_1} |\pi(D_1)|. \tag{4.10}$$

Analogously, for every collection $\mathcal{T}_1$ of pairwise disjoint trees in $\mathcal{T}$,

$$\nu(\mathcal{L}(\mathcal{T}_1)) = \sum_{T_1 \in \mathcal{T}_1} \nu(T_1) = \sum_{T_1 \in \mathcal{T}_1} |\pi(T_1)|. \tag{4.11}$$

Moreover, for every collection $\mathcal{D}_1$ of pairwise disjoint strips in $\mathcal{D}$, and every tree T in $\mathcal{T}$,

$$\nu(T \cap \mathcal{L}(\mathcal{D}_1)) = \sum_{D_1 \in \mathcal{D}_1} \nu(T \cap D_1). \tag{4.12}$$

Finally, we introduce the auxiliary functions. First, we define

$$\mathcal{Q} \colon \mathcal{P}(X) \to \mathcal{P}(\mathcal{D}), \qquad \mathcal{Q}(A) = \{E \colon E \in \mathcal{D},\, E_+ \cap A \neq \emptyset\},$$

where E_+ is the upper half of the strip E,

$$E_+ = \{(x, s, \xi) \in E \colon s > \sigma(E)/2\}.$$

It satisfies the following properties:

$$A \subseteq \mathcal{L}(\mathcal{Q}(A)), \tag{4.13}$$

$$A_1 \subseteq A_2 \implies \mathcal{L}(\mathcal{Q}(A_1)) \subseteq \mathcal{L}(\mathcal{Q}(A_2)), \tag{4.14}$$

$$\mu(\mathcal{L}(\mathcal{Q}(A))) = \mu(A). \tag{4.15}$$

Next, we define

$$\mathcal{N} \colon \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D}), \qquad \mathcal{N}(\mathcal{D}_1) = \{E \colon E \in \mathcal{D},\, |\pi(E) \cap \pi(\mathcal{L}(\mathcal{D}_1))| \geq |\pi(E)|/2\}.$$

It associates a collection $\mathcal{D}_1$ of strips to the collection of strips whose associated space interval is at least half-covered by the space intervals associated with the elements of $\mathcal{D}_1$. It satisfies the following properties:

$$\mathcal{L}(\mathcal{D}_1) \subseteq \mathcal{L}(\mathcal{N}(\mathcal{D}_1)), \tag{4.16}$$

$$\mathcal{L}(\mathcal{D}_1) \subseteq \mathcal{L}(\mathcal{D}_2) \implies \mathcal{L}(\mathcal{N}(\mathcal{D}_1)) \subseteq \mathcal{L}(\mathcal{N}(\mathcal{D}_2)), \tag{4.17}$$

$$\mu(\mathcal{L}(\mathcal{N}(\mathcal{D}_1))) \leq 2\mu(\mathcal{L}(\mathcal{D}_1)). \tag{4.18}$$

Finally, we define

$$\mathcal{M}\colon \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D}),$$
$$\mathcal{M}(\mathcal{D}_1) = \{E\colon E \in \mathcal{D}_1,\ \forall D_1 \in \mathcal{D}_1 \setminus \{E\} \text{ we have } E \not\subseteq D_1\}.$$

It associates a collection $\mathcal{D}_1$ of strips to the subcollection of maximal elements with respect to inclusion. In particular, it is well-defined because, for every $L \in \mathbb{Z}$, the space Y_L is bounded in the second variable. In fact, by Lemma 4.1, the function $\mathcal{M}$ maps into the subset of collections of pairwise disjoint strips in $\mathcal{D}$. Moreover, it satisfies the following properties:

$$\mathcal{L}(\mathcal{D}_1) = \mathcal{L}(\mathcal{M}(\mathcal{D}_1)), \tag{4.19}$$

$$\mathcal{L}(\mathcal{D}_1) \subseteq \mathcal{L}(\mathcal{D}_2) \implies \mathcal{L}(\mathcal{M}(\mathcal{D}_1)) \subseteq \mathcal{L}(\mathcal{M}(\mathcal{D}_2)), \tag{4.20}$$

$$\mu(\mathcal{L}(\mathcal{D}_1)) = \mu(\mathcal{L}(\mathcal{M}(\mathcal{D}_1))) = \sum_{E \in \mathcal{M}(\mathcal{D}_1)} \mu(E). \tag{4.21}$$

We define the function $\mathcal{C}\colon \mathcal{P}(X) \to \dot{\mathcal{P}}(\mathcal{E})$ by

$$\mathcal{E} = \mathcal{D}, \qquad \mathcal{C}(A) = \mathcal{M}(\mathcal{N}(\mathcal{Q}(A))),$$

where $\dot{\mathcal{P}}(\mathcal{E})$ stands for the set of subcollections of pairwise disjoint elements in $\mathcal{E}$.

We now prove that $\mathcal{C}$ is a μ-covering function and that the setting $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 1.1 and the crop condition 1.2.

LEMMA 4.5. *The function $\mathcal{C}$ is a μ-covering function for every choice of the parameter $\Phi \geq 2$.*

Proof. We recall that

$$\mathbf{B}_{\mathcal{C}}(A) = \mathcal{L}(\mathcal{M}(\mathcal{N}(\mathcal{Q}(A)))).$$

By (4.13), (4.16) and (4.19),

$$A \subseteq \mathbf{B}_{\mathcal{C}}(A).$$

By (4.14), (4.17) and (4.20),

$$A_1 \subseteq A_2 \implies \mathbf{B}_{\mathcal{C}}(A_1) \subseteq \mathbf{B}_{\mathcal{C}}(A_2).$$

Moreover, by (4.21), (4.18) and (4.15),

$$\mu\big(\mathbf{B}_{\mathcal{C}}(A)\big) \leq 2\mu(A). \ \blacksquare$$

LEMMA 4.6. *The setting $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition* 1.1 *for every choice of parameters Φ, $K \geq 2$.*

Proof. Let $\mathcal{A}$ be a ν-Carathéodory collection of subsets of X with parameter K, and $\widetilde{D}$ a subset of X disjoint from $\mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A}))$. We claim that the collection $\mathcal{A} \cup \{\widetilde{D}\}$ is still ν-Carathéodory with the same parameter K. In particular, we want to prove that for every subset U of X,

$$\sum_{A\in\mathcal{A}} \nu(U\cap A)+\nu(U\cap\widetilde{D})\le K\nu(U). \tag{4.22}$$

Without loss of generality, we assume $U\cap\widetilde{D}\neq\emptyset$, otherwise the inequality follows by the ν-Carathéodory property for the collection $\mathcal{A}$. In particular, we have $\widetilde{D}\neq\emptyset$.

First, we prove (4.22) under some additional assumptions on $\widetilde{D}$ and U. After that, we obtain the general case in a series of generalization steps.

Step 1. Let $\widetilde{D}$ be a nonempty set of the form

$$D\setminus\mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A})), \tag{4.23}$$

where D is a strip in $\mathcal{D}$, and $\mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A}))\subsetneq D$. We claim that, for every tree T in $\mathcal{T}$,

$$\sum_{A\in\mathcal{A}} \nu(T\cap A)+\nu(T\cap D)\le K\nu(T). \tag{4.24}$$

The version of (4.22) for the particular choices of T and $\widetilde{D}$ follows by the monotonicity of ν.

Without loss of generality, we assume T to be contained in D. The result for an arbitrary tree T follows by that for $T\cap D$, which by Lemma 4.2 is a tree as well, and the monotonicity of ν.

For every tree T contained in D with nonempty intersection with $\widetilde{D}$, we have

$$D(T)\notin\mathcal{N}(\mathcal{Q}(\mathcal{L}(\mathcal{A}))).$$

Together with (4.9), this yields

$$\nu(T)=|\pi(D(T))|\ge 2\big|\pi\big(D(T)\cap\mathcal{L}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))\big)\big|.$$

By (4.19) and the disjointness of the elements of the collection $\mathcal{M}(\mathcal{D}_1)$ for every $\mathcal{D}_1\subseteq\mathcal{D}$, we have

$$\begin{aligned}|\pi(D(T)\cap\mathcal{L}(\mathcal{Q}(\mathcal{L}(\mathcal{A}))))| &= \big|\pi\big(D(T)\cap\mathcal{L}(\mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A}))))\big)\big|\\ &= \sum_{E\in\mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))} |\pi(D(T)\cap E)|.\end{aligned}$$

By the monotonicity of the Lebesgue measure, Lemma 4.2, and (4.9), we have

$$\begin{aligned}\sum_{E\in\mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))} |\pi(D(T)\cap E)| &\ge \sum_{E\in\mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))} |\pi(T\cap E)|\\ &\ge \sum_{E\in\mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))} \nu(T\cap E).\end{aligned}$$

By (4.12) and the monotonicity of ν,

$$\sum_{E\in\mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))} \nu(T\cap E) \geq \nu\big(T\cap\mathcal{L}(\mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A}))))\big) \geq \nu(T\cap\mathcal{L}(\mathcal{A})).$$

Together with the condition $K\geq 2$ and the ν-Carathéodory property for the collection $\mathcal{A}$, the previous chains of inequalities yield

$$\begin{aligned} K\nu(T) &\geq \nu(T\cap D) + 2(K-1)\nu(T\cap\mathcal{L}(\mathcal{A})) \geq \nu(T\cap D) + K\nu(T\cap\mathcal{L}(\mathcal{A})) \\ &\geq \nu(T\cap D) + \sum_{A\in\mathcal{A}}\nu(T\cap A). \end{aligned}$$

STEP 2. Let $\widetilde{D}$ be a nonempty set of the form

$$\widetilde{D} = \bigcup_{D'\in\mathcal{D}'} \widetilde{D}' = \bigcup_{D'\in\mathcal{D}'} (D'\setminus \mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A}))),$$

where $\mathcal{D}'$ is a collection of pairwise disjoint strips. We claim that, for every tree T in $\mathcal{T}$, we have (4.22) for the particular choices of T and $\widetilde{D}$.

By definition, for every strip $\mathcal{D}'$,

$$D'\not\subseteq \mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A})).$$

Therefore, by Lemma 4.1,

$$\mathcal{C}(\mathcal{L}(\mathcal{A})) = \mathcal{C}_1 \cup \bigcup_{D'\in\mathcal{D}'} \mathcal{C}_{D'},$$

where the elements of $\mathcal{C}_1$ are disjoint from $\mathcal{L}(\mathcal{D}')$, while, for every D' in $\mathcal{D}'$, the elements of $\mathcal{C}_{D'}$ are contained in D'. In particular,

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_1 \cup \bigcup_{D'\in\mathcal{D}'} \mathcal{A}_{D'} \\ &= \{A\colon A\in\mathcal{A},\, A\subseteq\mathcal{L}(\mathcal{C}_1)\} \cup \bigcup_{D'\in\mathcal{D}'} \{A\colon A\in\mathcal{A},\, A\subseteq\mathcal{L}(\mathcal{C}_{D'})\}. \end{aligned}$$

Then

$$\begin{aligned} K\nu(T) &\geq K\nu\Big(T\cap\Big(\mathcal{L}(\mathcal{C}(\mathcal{L}(\mathcal{A})))\cup\bigcup_{D'\in\mathcal{D}'} D'\Big)\Big) \\ &\geq K\nu(T\cap\mathcal{L}(\mathcal{C}_1)) + K\sum_{D'\in\mathcal{D}'}\nu(T\cap D') \\ &\geq \sum_{A\in\mathcal{A}_1}\nu(T\cap A) + \sum_{D'\in\mathcal{D}'}\Big(\sum_{A\in\mathcal{A}_{D'}}\nu(T\cap A) + \nu(T\cap D')\Big) \\ &\geq \sum_{A\in\mathcal{A}}\nu(T\cap A) + \nu(T\cap\mathcal{L}(\mathcal{D}')) \\ &\geq \sum_{A\in\mathcal{A}}\nu(T\cap A) + \nu(T\cap\widetilde{D}), \end{aligned} \tag{4.25}$$

where we use the monotonicity of ν in the first and in the fifth inequalities, (4.12) in the second, the ν-Carathéodory property for the collection $\{A\colon A \in \mathcal{A},\ A \subseteq \mathcal{L}(\mathcal{C}_1)\}$ and (4.24) for each D' in $\mathcal{D}'$ in the third, Fubini and (4.12) in the fourth.

STEP 3. Let $\widetilde{D}$ be an arbitrary nonempty set disjoint from $\mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A}))$. We claim that, for every tree T in $\mathcal{T}$, we have (4.22).

For $\mathcal{D}' = \mathcal{M}(\mathcal{Q}(\widetilde{D}))$, we define

$$\widetilde{D}_1 = \bigcup_{D' \in \mathcal{D}'} \big(D' \setminus \mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A}))\big).$$

By (4.25) and the monotonicity of ν,

$$K\nu(T) \geq \sum_{A \in \mathcal{A}} \nu(T \cap A) + \nu(T \cap \widetilde{D}_1) \geq \sum_{A \in \mathcal{A}} \nu(T \cap A) + \nu(T \cap \widetilde{D}). \tag{4.26}$$

STEP 4. Let $\widetilde{D}$ be an arbitrary nonempty set disjoint from $\mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A}))$. We claim that, for every $U \subseteq X$, we have (4.22).

In fact, there exists a collection $\mathcal{T}' \subseteq \mathcal{T}$ covering U ν-optimally, namely

$$U \subseteq \bigcup_{T \in \mathcal{T}'} T, \tag{4.27}$$

$$\sum_{T \in \mathcal{T}'} \tau(T) = \nu(U). \tag{4.28}$$

By (4.26) for every tree T in $\mathcal{T}'$, the subadditivity of ν, and (4.27),

$$\begin{aligned} K \sum_{T \in \mathcal{T}'} \nu(T) &\geq \sum_{T \in \mathcal{T}'} \Big(\sum_{A \in \mathcal{A}} \nu(T \cap A) + \nu(T \cap \widetilde{D}) \Big) \\ &\geq \sum_{A \in \mathcal{A}} \sum_{T \in \mathcal{T}'} \nu(T \cap A) + \sum_{T \in \mathcal{T}'} \nu(T \cap \widetilde{D}) \\ &\geq \sum_{A \in \mathcal{A}} \nu(U \cap A) + \nu(U \cap \widetilde{D}). \end{aligned}$$

Together with (4.28), this yields (4.22). ∎

LEMMA 4.7. *The setting* $(X, \mu, \nu, \mathcal{C})$ *satisfies the crop condition* 1.2 *for every choice of parameters* $\Phi \geq 2$, $K \geq 1$.

Proof. For every collection $\mathcal{A}$ of strips in $\mathcal{D}$, let $\mathcal{B} = \mathcal{M}(\mathcal{A})$. The subcollection $\mathcal{B}$ is ν-Carathéodory with parameter $K = 1$. Moreover, for every subset F of X disjoint from $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{A})$, we have

$$\mathcal{C}(F) \cap \mathcal{A} = \mathcal{Q}(F) \cap \mathcal{A} = \emptyset,$$

and this yields $\mathbf{B}_{\mathcal{C}}(F) = \mathbf{B}_{\widetilde{\mathcal{C}}}(F)$. ∎

5. Double iterated outer L^p spaces on the upper half 3-space. In this section we prove Theorem 1.5 in the dyadic upper half 3-space setting described in (4.1), (4.3) and (4.4), reducing the problem to an equivalent one in a finite setting via an approximation argument.

We start by stating some auxiliary results about the approximation of functions in outer L^p spaces. We use them to prove the approximation of functions in outer L^p spaces on the upper half 3-space X by functions with support in X_J for a certain $J \in \mathbb{N}$, where

$$X_J = (-2^J J, 2^J J] \times (2^{-J}, 2^J] \times (-2^J J, 2^J J]. \tag{5.1}$$

On X_J, we have the measure ω_J and the outer measures μ_J, ν_J induced by ω, μ, ν. In particular, this setting inherits the definition of the function $\mathcal{C}$ on Y_J, for Y_J defined in (4.6), and its properties (Lemmas 4.5–4.7).

Next, for any $J \in \mathbb{N}$, we introduce a finite setting X'_J and exhibit a map between functions on X_J and on X'_J preserving the double iterated outer L^p quasi-norms. We use Theorems 1.3, 1.4 in the finite settings to prove Theorem 1.5.

Finally, we conclude the section with some observations about the result analogous to Theorem 1.5 for double iterated outer L^p spaces in the upper half 3-space setting where the outer measures are defined by arbitrary strips and trees originally considered in [14].

5.1. Approximation results. First, we state a result about the approximation of functions in $L^p_\mu(S)$ by functions in $L^p_\mu(S) \cap L^\infty_\mu(S)$, for a size S of the form ℓ^r_ω or $\ell^q_\nu(\ell^r_\omega)$, and more generally an arbitrary size in the definition in [10].

LEMMA 5.1. *For every $p \in (0,\infty)$, there exists a constant $C = C(p)$ such that the following holds.*

Let X be a set, μ an outer measure, and S a size. For every $f \in L^p_\mu(S)$, there exists a subset A of X such that $f 1_A$ is in $L^p_\mu(S) \cap L^\infty_\mu(S)$ and

$$\|f\|_{L^p_\mu(S)} \le C \|f 1_A\|_{L^p_\mu(S)}.$$

Next, we state a result about the behaviour of the super level measures for single iterated outer L^p spaces for increasing cut-offs of a function in a general setting.

LEMMA 5.2 (Monotonic convergence I). *For every $r \in (0,\infty)$, there exist constants $C = C(r)$, $c = c(r)$ such that the following holds.*

Let X be a set, ν an outer measure, and ω a measure. Let $\{X_J : J \in \mathbb{N}\}$ be an increasing sequence of subsets of X such that

$$X = \bigcup_{J \in \mathbb{N}} X_J,$$

and let $f \in L^\infty_\nu(\ell^r_\omega)$ *be a function on* X*. Then, for every* $k \in \mathbb{Z}$*, there exists* $J = J(r, f, k) \in \mathbb{N}$ *such that*

$$\nu(\ell^r_\omega(f) > 2^k) \le C \sum_{l \ge k} \nu(\ell^r_\omega(f 1_{X_J}) > c2^l).$$

Finally, we state a result about the behaviour of the super level measures for double iterated outer L^p spaces for increasing cut-offs of a function in the dyadic upper half 3-space setting.

Lemma 5.3 (Monotonic convergence II). *For all* $q, r \in (0, \infty)$*, there exist constants* $C = C(q, r)$*,* $c = c(q, r)$ *such that the following holds.*

Let $f \in L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ *be a function on* $X = \mathbb{R} \times (0, \infty) \times \mathbb{R}$*, and let* $\{X_J : J \in \mathbb{N}\}$ *be the increasing sequence of subsets of* X *defined in* (5.1)*. Then, for every* $k \in \mathbb{Z}$*, there exists* $J = J(q, r, f, k) \in \mathbb{Z}$ *such that*

$$\mu(\ell^q_\nu(\ell^r_\omega)(f) > 2^k) \le C \sum_{l \ge k} \mu(\ell^q_\nu(\ell^r_\omega)(f 1_{X_J}) > c2^l).$$

The previous three results will be proved in Appendix B. We use them to show the following results about the approximation of functions in $L^q_\nu(\ell^r_\omega)$ and $L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ by functions with support in X_j for a certain $j \in \mathbb{N}$.

Lemma 5.4. *For all* $q, r \in (0, \infty)$*, there exists a constant* $C = C(q, r)$ *such that the following holds.*

For every $f \in L^q_\nu(\ell^r_\omega)$*, there exists* $J = J(q, r, f) \in \mathbb{N}$ *such that*

$$\|f 1_{X_J}\|_{L^q_\nu(\ell^r_\omega)} \le \|f\|_{L^q_\nu(\ell^r_\omega)} \le C \|f 1_{X_J}\|_{L^q_\nu(\ell^r_\omega)}.$$

Proof. The first inequality follows by the monotonicity of the outer L^p quasi-norms.

To prove the second inequality, by Lemma 5.1, we assume f is in $L^q_\nu(\ell^r_\omega) \cap L^\infty_\nu(\ell^r_\omega)$. Next, we observe that there exists $K = K(q, r, f) \in \mathbb{N}$ such that

$$\|f\|^q_{L^q_\nu(\ell^r_\omega)} \le C \sum_{k \in \mathbb{Z}} 2^{kq} \nu(\ell^r_\omega(f) > 2^k) \le C \sum_{k \in [-K, K]} 2^{kq} \nu(\ell^r_\omega(f) > 2^k).$$

By Lemma 5.2, for every $k \in [-K, K]$, there exists a $\widetilde{J} = \widetilde{J}(r, f, k) \in \mathbb{N}$ such that

$$\nu(\ell^r_\omega(f) > 2^k) \le C \sum_{l \ge k} \nu(\ell^r_\omega(f 1_{X_{\widetilde{J}}}) > c2^l).$$

By taking $J = \max_{k \in [-K, K]} \widetilde{J}(k, f, r)$, the previous inequalities yield

$$\|f\|^q_{L^q_\nu(\ell^r_\omega)} \le C \sum_{k \in [-K, K]} 2^{kq} \sum_{l \ge k} \nu(\ell^r_\omega(f 1_{X_J}) > c2^l) \le C \|f 1_{X_J}\|^q_{L^q_\nu(\ell^r_\omega)}. \blacksquare$$

Lemma 5.5. *For all* $p, q, r \in (0, \infty)$*, there exists a constant* $C = C(p, q, r)$ *such that the following holds.*

For every $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$, there exists $J = J(p,q,r,f) \in \mathbb{N}$ such that

$$\|f1_{X_J}\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leq \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leq C\|f1_{X_J}\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

Proof. One uses the same argument as in the previous proof, with Lemma 5.2 replaced by Lemma 5.3. ∎

5.2. Equivalence with finite settings. We introduce the following finite setting:

$$X' = \mathbb{Z}^3,$$
$$\omega'(m,l,n) = 1,$$
$$D'(m,l) = \{(m',l',n') : m' \in [2^{l-l'}m, 2^{l-l'}(m+1)),\, l' \leq l,\, n' \in \mathbb{Z}\},$$
$$\mathcal{D}' = \{D'(m,l) : m,l \in \mathbb{Z}\},$$
$$\sigma'(D'(m,l)) = 2^l \quad \text{for all } m,l \in \mathbb{Z},$$
$$T'(m,l,n) = \{(m',l',n') : m' \in [2^{l-l'}m, 2^{l-l'}(m+1)),\, l' \leq l,\, n' = N(n,l')\},$$
$$\mathcal{T}' = \{T'(m,l,n) : m,l,n \in \mathbb{Z}\},$$
$$\tau'(T'(m,l,n)) = 2^l \quad \text{for all } m,l,n \in \mathbb{Z},$$

where $N(n,l')$ is defined by (4.5), and μ',ν' are defined by σ',τ' as in (1.17). Moreover, for every $J \in \mathbb{N}$, we define

$$X'_J = \{(m,l,n) \in X' : \\ l \in (-J,J],\, m \in [-J2^{J-l}, J2^{J-l}),\, n \in [-J2^{J+l}, J2^{J+l})\},$$

On X_J, we have the measure ω'_J and the outer measures μ'_J,ν'_J induced by ω',μ',ν'. In fact, the outer measure μ'_J is equivalently generated by the pre-measure σ'_J on $\mathcal{D}'_J$ as in (1.17):

$$D'_J(m,l) = D'(m,l) \cap X'_J,$$
$$\mathcal{D}'_J = \{D'_J(m,l) : m,l \in \mathbb{Z},\, D'_J(m,l) \neq \emptyset\},$$
$$\sigma_J(D'_J(m,l)) = 2^l \quad \text{for all } m,l \in \mathbb{Z},\, D'_J(m,l) \neq \emptyset,$$

and the outer measure ν'_J by the pre-measure τ'_J on $\mathcal{T}'_J$ as in (1.17):

$$T'_J(m,l,n) = T'(m,l,n) \cap X'_J,$$
$$\mathcal{T}'_J = \{T'_J(m,l,n) : m,l,n \in \mathbb{Z},\, T'_J(m,l,n) \neq \emptyset\},$$
$$\tau'_J(T'_J(m,l,n)) = 2^l \quad \text{for all } m,l,n \in \mathbb{Z},\, T'_J(m,l,n) \neq \emptyset.$$

The setting on X'_J inherits the definition of the function $\mathcal{C}$ on X_J and its properties (Lemmas 4.5–4.7) via the map associating every triple $(m,l,n) \in X'$ to $H(m,l,n)$, the pairwise disjoint subsets of X defined in (4.2).

Moreover, every function f on X that is in $L^r_{\mathrm{loc}}(X,\omega)$ for some $r \in (0,\infty]$ defines a function $F(f,r)$ on X' by

$$F(f,r)(m,l,n) = \|f 1_{H(m,l,n)}\|_{L^r(X,\omega)}.$$

For every fixed $r \in (0,\infty]$, the map between functions on X and on X' just described preserves the iterated outer L^p quasi-norms.

LEMMA 5.6. *Let $p,q,r \in (0,\infty)$. For every f supported in X_J for any $J \in \mathbb{N}$, we have*

$$\|f\|_{L^q_\nu(\ell^r_\omega)} = \|F(f,r)\|_{L^q_{\nu'}(\ell^r_{\omega'})}, \qquad \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} = \|F(f,r)\|_{L^p_{\mu'}(\ell^q_{\nu'}(\ell^r_{\omega'}))}.$$

Proof. Let $J \in \mathbb{N}$ be fixed, and assume that f is supported in X_J.

We start by observing that $F(f,r)$ is supported in X'_J. Moreover, in both cases, we can restrict to considering only the elements of $\mathcal{D}_J, \mathcal{T}_J$ and $\mathcal{D}'_J, \mathcal{T}'_J$, since

$$\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} = \|f\|_{L^p_{\mu_J}(\ell^q_{\nu_J}(\ell^r_{\omega_J}))},$$
$$\|F(f,r)\|_{L^p_{\mu'}(\ell^q_{\nu'}(\ell^r_{\omega'}))} = \|F(f,r)\|_{L^p_{\mu'_J}(\ell^q_{\nu'_J}(\ell^r_{\omega'_J}))}.$$

In particular, for any $U \in \mathcal{T}_J$, we have $U = T_J(m,l,n)$, and we define $U' \in \mathcal{T}'_J$ by $U' = T'_J(m,l,n)$, hence

$$\nu_J(U) = \tau_J(U) = \tau'_J(U') = \nu'_J(U'). \tag{5.2}$$

Now, for any two collections $\mathcal{U}_1, \mathcal{U}_2$ of elements in $\mathcal{T}_J$, we define, for $i = 1,2$,

$$U_i = \mathcal{L}(\mathcal{U}_i), \qquad U'_i = \mathcal{L}(\mathcal{U}'_i),$$

and we have

$$F(f 1_{U_1 \setminus U_2}, r) = F(f,r) 1_{U'_1 \setminus U'_2}. \tag{5.3}$$

Next, by the definition of $F(f,r)$,

$$\|f\|_{L^r(X_J,\omega_J)} = \|F(f,r)\|_{L^r(X'_J,\omega'_J)}. \tag{5.4}$$

Therefore, for any $U \in \mathcal{T}_J$,

$$\|f 1_U\|_{L^r(X_J,\omega_J)} = \|F(f 1_U, r)\|_{L^r(X'_J,\omega'_J)} = \|F(f,r) 1_{U'}\|_{L^r(X'_J,\omega'_J)}, \tag{5.5}$$

where we use (5.4) in the first equality, and (5.3) in the second. Moreover, for any $A \subseteq X_J$, there exists a finite subcollection $\mathcal{U}$ of $\mathcal{T}_J$ such that $A \subseteq \mathcal{L}(\mathcal{U})$ and

$$\nu_J(A) = \sum_{U \in \mathcal{U}} \tau_J(U) = \sum_{U \in \mathcal{U}} \nu_J(U). \tag{5.6}$$

In particular,

$$\begin{aligned}(5.7)\qquad \nu_J(A)^{-1}\|f1_A\|^r_{L^r(X_J,\omega_J)} &\le \nu_J(A)^{-1}\sum_{U\in\mathcal{U}}\|f1_U\|^r_{L^r(X_J,\omega_J)}\\ &\le \nu_J(A)^{-1}\max_{V\in\mathcal{U}}\nu_J(V)^{-1}\|f1_V\|^r_{L^r(X_J,\omega_J)}\sum_{U\in\mathcal{U}}\nu_J(U)\\ &\le \max_{V\in\mathcal{U}}\nu_J(V)^{-1}\|f1_V\|^r_{L^r(X_J,\omega_J)},\end{aligned}$$

where we use the monotonicity and the r-orthogonality of the classical L^r quasi-norm in the first inequality, Hölder's inequality in the second, and (5.6) in the third. The analogous properties hold true for any F supported in X'_J.

Therefore, for any $\lambda > 0$, we have, for $F = F(f,r)$,

$$\begin{aligned}&\nu_J(\ell^r_{\omega_J}(f) > \lambda)\\ &= \inf\{\nu_J(A)\colon A\subseteq X_J,\ \sup\{\nu_J(B)^{-1/r}\|f1_B1_{A^c}\|_{L^r(X_J,\omega_J)}\colon B\subseteq X_J\}\le\lambda\}\\ &= \inf\{\nu_J(\mathcal{L}(\mathcal{U}))\colon \mathcal{U}\subseteq\mathcal{T}_J,\\ &\qquad \sup\{\nu_J(V)^{-1/r}\|f1_V1_{\mathcal{L}(\mathcal{U})^c}\|_{L^r(X_J,\omega_J)}\colon V\in\mathcal{T}_J\}\le\lambda\}\\ &= \inf\{\nu'_J(\mathcal{L}(\mathcal{U}'))\colon \mathcal{U}'\subseteq\mathcal{T}'_J,\\ &\qquad \sup\{\nu'_J(V')^{-1/r}\|F1_{V'}1_{\mathcal{L}(\mathcal{U}')^c}\|_{L^r(X'_J,\omega'_J)}\colon V'\in\mathcal{T}'_J\}\le\lambda\}\\ &= \inf\{\nu'_J(A')\colon A'\subseteq X'_J,\\ &\qquad \sup\{\nu'_J(B')^{-1/r}\|F1_{B'}1_{(A')^c}\|_{L^r(X'_J,\omega'_J)}\colon B'\subseteq X'_J\}\le\lambda\}\\ &= \nu'_J(\ell^r_{\omega'_J}(F) > \lambda),\end{aligned}$$

where we use (5.6) and (5.7) in the second equality, (5.2) and (5.5) in the third, and the analogues of (5.6) and (5.7) in the fourth. Hence

$$\|f\|_{L^q_{\nu_J}(\ell^r_{\omega_J})} = \|F(f,r)\|_{L^q_{\nu'_J}(\ell^r_{\omega'_J})}.$$

Applying an analogous argument to the "exterior" level of definition of the double iterated outer L^p space, we obtain

$$\|f\|_{L^p_{\mu_J}(\ell^q_{\nu_J}(\ell^r_{\omega_J}))} = \|F(f,r)\|_{L^p_{\mu'_J}(\ell^q_{\nu'_J}(\ell^r_{\omega'_J}))}. \ \blacksquare$$

Proof of Theorem 1.5. Let $p,q,r\in(0,\infty]$. By Lemmas 5.4 and 5.5, for every $f\in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$, there exists $J = J(f,p,q,r)\in\mathbb{N}$ such that

$$(5.8)\qquad \begin{aligned}\|f1_{X_J}\|_{L^q_\nu(\ell^r_\omega)} &\le \|f\|_{L^q_\nu(\ell^r_\omega)} \le C\|f1_{X_J}\|_{L^q_\nu(\ell^r_\omega)},\\ \|f1_{X_J}\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} &\le \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \le C\|f1_{X_J}\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))},\end{aligned}$$

where C is independent of f and J. By Lemma 5.6,

$$\|f1_{X_J}\|_{L^q_\nu(\ell^r_\omega)} = \|F(f1_{X_J},r)\|_{L^q_{\nu'}(\ell^r_{\omega'})} = \|F(f,r)1_{X'_J}\|_{L^q_{\nu'_J}(\ell^r_{\omega'_J})},$$

$$\begin{aligned}\|f1_{X_J}\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} &= \|F(f1_{X_J},r)\|_{L^p_{\mu'}(\ell^q_{\nu'}(\ell^r_{\omega'}))} \\ &= \|F(f,r)1_{X'_J}\|_{L^p_{\mu'_J}(\ell^q_{\nu'_J}(\ell^r_{\omega'_J}))}.\end{aligned} \tag{5.9}$$

(i) Let $q, r \in (0,\infty)$. By Theorem 1.3,

$$\begin{aligned}C^{-1}\|F(f,r)1_{X'_J}\|_{L^q_{\nu'_J}(\ell^r_{\omega'_J})} &\le \|F(f,r)1_{X'_J}\|_{L^q_{\mu'_J}(\ell^q_{\nu'_J}(\ell^r_{\omega'_J}))} \\ &\le C\|F(f,r)1_{X'_J}\|_{L^q_{\nu'_J}(\ell^r_{\omega'_J})},\end{aligned}$$

where C is independent of f and J. Together with (5.8) and (5.9), the previous chain of inequalities yields (1.18).

(ii) Let $p, q, r \in (1,\infty)$. By Theorem 1.4, for every $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$, there exists a function G on X'_J with unitary outer $L^{p'}_{\mu'_J}(\ell^{q'}_{\nu'_J}(\ell^{r'}_{\omega'_J}))$ quasi-norm such that

$$\begin{aligned}C^{-1}\|F(f,r)1_{X'_J}\|_{L^p_{\mu'_J}(\ell^p_{\nu'_J}(\ell^r_{\omega'_J}))} &\le \|F(f,r)1_{X'_J}G\|_{L^1(X'_J,\omega'_J)} \\ &\le C\|F(f,r)1_{X'_J}\|_{L^p_{\mu'_J}(\ell^p_{\nu'_J}(\ell^r_{\omega'_J}))},\end{aligned} \tag{5.10}$$

where C is independent of f and J. We define a function g on X by

$$\begin{aligned}&g(x,s,\xi) \\ &\quad = |f(x,s,\xi)|^{r-1} \sum_{m,l,n\in\mathbb{Z}} F(f,r)(m,l,n)^{1-r} G(m,l,n) 1_{H(m,l,n)}(x,s,\xi).\end{aligned}$$

By construction, $F(g,r') = G$. Together with Lemma 5.6, this yields

$$\|g\|_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} = \|G\|_{L^{p'}_{\mu'}(\ell^{q'}_{\nu'}(\ell^{r'}_{\omega'}))} = \|G\|_{L^{p'}_{\mu'_J}(\ell^{q'}_{\nu'_J}(\ell^{r'}_{\omega'_J}))} = 1.$$

Moreover, by construction,

$$\begin{aligned}\|fg\|_{L^1_\omega} &= \|F(f,r)G\|_{L^1(X'_J,\omega'_J)} = \|F(f,r)G\|_{L^1(X'_J,\omega'_J)} \\ &= \|F(f,r)1_{X'_J}G\|_{L^1(X'_J,\omega'_J)}.\end{aligned}$$

Together with (5.8)–(5.10), the last two chains of equalities yield (1.19).

(iii) The inequality in (1.20) is a corollary of the triangle inequality for the $L^1(X,\omega)$ norm and property (ii). ∎

5.3. Upper half 3-space with arbitrary strips and trees. We turn to the case of double iterated outer L^p spaces on the upper half 3-space setting where the outer measures are defined by arbitrary strips and trees. In particular, let

$$
\begin{aligned}
&X = \mathbb{R}^3_+ = \mathbb{R}^2_+ \times \mathbb{R} = \mathbb{R} \times (0,\infty) \times \mathbb{R},\\
&\mathrm{d}\omega(y,t,\eta) = \mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\eta,\\
&\widetilde{D}(x,s) = \{(y,t,\eta)\colon y \in x+(0,s],\ t \in (0,s],\ \eta \in \mathbb{R}\},\\
&\widetilde{\mathcal{D}} = \{\widetilde{D}(x,s)\colon x \in \mathbb{R},\ s \in (0,\infty)\},\\
&\widetilde{\sigma}(\widetilde{D}(x,s)) = s \quad \text{for all } x \in \mathbb{R},\ s \in (0,\infty),\\
&\widetilde{T}(x,s,\xi) = \{(y,t,\eta)\colon y \in x+(0,s],\ t \in (0,s],\ \eta \in \xi + (-t^{-1},t^{-1}]\},\\
&\widetilde{\mathcal{T}} = \{\widetilde{T}(x,s,\xi)\colon x \in \mathbb{R},\ s \in (0,\infty),\ \xi \in \mathbb{R}\},\\
&\widetilde{\tau}(\widetilde{T}(x,s,\xi)) = s \quad \text{for all } x \in \mathbb{R},\ s \in (0,\infty),\ \xi \in \mathbb{R},
\end{aligned}
\tag{5.11}
$$

where $\widetilde{\mu}, \widetilde{\nu}$ are defined by $\widetilde{\sigma}, \widetilde{\tau}$ as in (1.17).

On the one hand, the outer measures generated by dyadic strips and arbitrary ones are equivalent and we can substitute the outer measure $\widetilde{\mu}$ with μ. In particular, we have $\mathcal{D} \subseteq \widetilde{\mathcal{D}}$, and every element of $\widetilde{\mathcal{D}}$ is covered by at most two elements of $\mathcal{D}$ with comparable pre-measure.

On the other hand, the outer measures generated by dyadic trees and arbitrary ones are not equivalent. In fact, while for every dyadic tree T in $\mathcal{T}$ we have

$$\widetilde{\nu}(T) \leq \nu(T),$$

for every tree $\widetilde{T}$ in $\widetilde{\mathcal{T}}$ we have instead

$$\nu(\widetilde{T}) = \infty; \tag{5.12}$$

we postpone the proof to Appendix A. Therefore, we cannot trivially deduce the result of Theorem 1.5 in the setting described in (5.11) from Theorem 1.5 itself.

However, a reduction of the problem to an equivalent one in a finite setting via an approximation argument analogous to that described in the previous subsections still yields the desired result. We briefly comment on some additional observations, providing guidance to the readers interested in a complete proof.

First, the outer measure $\widetilde{\nu}$ is equivalent to $\widetilde{\nu}_d$, the outer measure defined as in (1.17) by the pre-measure $\widetilde{\tau}$ restricting the collection $\widetilde{\mathcal{T}}$ of trees to those associated with dyadic intervals, namely

$$\widetilde{\mathcal{T}}_d = \{\widetilde{T}(2^l m, 2^l, 2^{-l} n)\colon m,l,n \in \mathbb{Z}\} \subseteq \widetilde{\mathcal{T}}.$$

The geometry of the elements of $\mathcal{D}, \widetilde{\mathcal{T}}_d$ and their intersections is analogous to that of the elements of $\mathcal{D}, \mathcal{T}$. Therefore, for every function f in a double iterated outer L^p space in the setting $(X, \mu, \widetilde{\nu}_d, \omega)$, we can pass to a cut-off $f 1_{X_J}$ approximating the double iterated outer L^p quasi-norm of f, for X_J defined in (5.1).

Next, for every fixed $J \in \mathbb{N}$, we consider the outer measure $\widetilde{\nu}_{d,J}$ induced on Y_J by $\widetilde{\nu}_d$, where Y_J is defined in (4.6). We observe that $\widetilde{\nu}_{d,J}$ is equivalent to the outer measure generated as in (1.17) by the pre-measure $\widetilde{\tau}$ restricting the collection $\widetilde{\mathcal{T}}_d$ of trees to those contained in Y_J, namely

$$\widetilde{\mathcal{T}}_{d,J} = \{\widetilde{T}(2^l m, 2^l, 2^{-l} n) \colon m, l, n \in \mathbb{Z},\ l \leq J\} \subseteq \widetilde{\mathcal{T}}_d.$$

In the setting $(Y_J, \mu_J, \widetilde{\mathcal{T}}_{d,J}, \omega_J)$, we can state definitions and prove results based on the geometry of the elements of $\mathcal{D}_J, \widetilde{\mathcal{T}}_{d,J}$ analogous to those in Section 4. Therefore, for every $J \in \mathbb{N}$, we can define a μ_J-covering function $\widetilde{\mathcal{C}}$ satisfying the canopy condition 1.1 and the crop condition 1.2. In particular, this definition is inherited by $X_J \subseteq Y_J$.

Next, for every fixed $J \in \mathbb{N}$, we observe that the elements of $\mathcal{D}_J, \widetilde{\mathcal{T}}_{d,J}$ with nonempty intersection with X_J are finitely many. Therefore, we can introduce a finite setting with a point for every intersection and the induced measure and outer measures. In particular, we deduce the result corresponding to Theorem 1.5 via an argument analogous to that of the previous subsection.

A. Geometry of the dyadic upper half 3-space setting. In this appendix, we present the postponed proofs of the results involving the geometry of the dyadic strips and trees in the upper half 3-space stated in Section 4, and in (5.12) in Section 5.

We start by recalling that every dyadic strip D in $\mathcal{D}$ is determined by a dyadic interval I_D in $\mathcal{I}$, and has the form

$$D = I_D \times (0, |I_D|] \times \mathbb{R} = \pi(D) \times (0, |\pi(D)|] \times \mathbb{R}, \tag{A.1}$$

and every dyadic tree T in $\mathcal{T}$ is determined by two dyadic intervals $I_T, \widetilde{I}_T$ in $\mathcal{I}$ such that $|I_T|\,|\widetilde{I}_T| = 1$ and has the form

$$T = \bigcup_{J \in \mathcal{I},\, J \subseteq I_T} J \times (0, |J|] \times \widetilde{J}(T, J) = \bigcup_{J \in \mathcal{I},\, J \subseteq \pi(T)} J \times (0, |J|] \times \widetilde{J}(T, J), \tag{A.2}$$

where the dyadic interval $\widetilde{J}(T, J)$ in $\mathcal{I}$ is defined by the conditions

$$|\widetilde{J}(T, J)| = |J|^{-1}, \qquad \widetilde{I}_T = \widetilde{J}\big(T, \pi(T)\big) \subseteq \widetilde{J}(T, J).$$

Proof of Lemma 4.1. If $D_1 \cap D_2$ is empty, the statement is trivially satisfied. Therefore, we assume that the strips D_1, D_2 have a nonempty intersection. Hence the dyadic intervals $\pi(D_1), \pi(D_2)$ have a nonempty intersection as well. Therefore, either $\pi(D_1) \subseteq \pi(D_2)$ or $\pi(D_2) \subseteq \pi(D_1)$. Without loss of generality, we can restrict to the first case, the second being analogous. We have $|\pi(D_1)| \leq |\pi(D_2)|$, hence by (A.1), $D_1 \subseteq D_2$. ∎

Proof of Lemma 4.2. If $D \cap T$ is empty, the statement is trivial. Therefore, we assume that the strip D and the tree T intersect. Hence the dyadic intervals $\pi(D), \pi(T)$ intersect as well. Therefore, either $\pi(D) \subseteq \pi(T)$ or

$\pi(T) \subseteq \pi(D)$. In the first case, we have $|\pi(D)| \leq |\pi(T)|$, hence by (A.1) and (A.2),

$$D \cap T = T\big(\pi(D), \widetilde{J}(T, \pi(D))\big).$$

In the second case, $|\pi(T)| \leq |\pi(D)|$, hence by (A.1) and (A.2), $D \cap T = T$. ■

Proof of Lemma 4.3. Let D be a strip in $\mathcal{D}$. Then

$$\mu(D) = \inf\Big\{ \sum_{D_1 \in \mathcal{D}_1} \sigma(D_1) \colon \mathcal{D}_1 \subseteq \mathcal{D},\ D \subseteq \mathcal{L}(\mathcal{D}_1) \Big\}.$$

Therefore, clearly $\mu(D) \leq \sigma(D)$. To prove the opposite inequality, we observe that for every covering $\mathcal{D}_1$ of D by means of strips in $\mathcal{D}$, there exists a strip E in $\mathcal{D}_1$ such that

$$(x_D, |\pi(D)|, 0) \in E,$$

where x_D is the middle point of the dyadic interval $\pi(D)$. In particular, this implies

$$\sigma(E) \geq |\pi(D)|.$$

Therefore,

$$\sum_{D_1 \in \mathcal{D}_1} \sigma(D_1) \geq \sigma(D).$$

By taking the infimum over all possible coverings of D, we obtain (4.7).

The statement for a tree T in $\mathcal{T}$ in (4.8) follows by an analogous argument considering the point $(x_T, |\pi(T)|, \xi_T)$, where x_T is the middle point of $\pi(T)$, and ξ_T is the middle point of $\widetilde{J}(T, \pi(T))$.

The statement in (4.9) follows by the definition of $D(T)$, (4.7), and (4.8). ■

Proof of Lemma 4.4. Let $\mathcal{D}_1$ be a collection of pairwise disjoint strips in $\mathcal{D}$. Then

$$\mu(\mathcal{L}(\mathcal{D}_1)) \leq \sum_{D_1 \in \mathcal{D}_1} \mu(D_1)$$

by the subadditivity of μ. To prove the opposite inequality, we consider a covering $\mathcal{D}_2$ of $\mathcal{L}(\mathcal{D}_1)$. Without loss of generality, we assume that E in $\mathcal{D}_2$ is strictly contained in any element of D_1, otherwise it would be useless in the covering. Therefore, $E \not\subseteq \mathcal{L}(\mathcal{D}_1)$, and, by Lemma 4.1,

$$\mathcal{D}_1 = \mathcal{D}_{1,E} \cup \widetilde{\mathcal{D}}_1,$$

where every element of $\mathcal{D}_{1,E}$ is contained in E, and every element of the other collection is disjoint from E. In particular,

$$\mathcal{L}(\mathcal{D}_{1,E}) \subseteq E. \tag{A.3}$$

As a consequence,

$$\sigma(E) = |\pi(E)| \geq |\pi\big(\mathcal{L}(\mathcal{D}_{1,E})\big)| = \sum_{D_1 \in \mathcal{D}_{1,E}} |\pi(D_1)| = \sum_{D_1 \in \mathcal{D}_{1,E}} \mu(D_1),$$

where we use (4.7) in the first and in the third equalities, (A.3) and the monotonicity of π and the Lebesgue measure in the inequality, and the distributivity of the projection over set union and the additivity of the Lebesgue measure on the disjoint intervals in $\pi(\mathcal{D}_1)$ in the second equality. Together with the observation that for every element D_1 of $\mathcal{D}_1$ there exists at least one E in $\mathcal{D}_2$ such that $D_1 \in \mathcal{D}_{1,E}$, we obtain

$$\sum_{E\in\mathcal{D}_2} \sigma(E) \geq \sum_{E\in\mathcal{D}_2} \sum_{D_1\in\mathcal{D}_{1,E}} \mu(D_1) \geq \sum_{D_1\in\mathcal{D}_1} \mu(D_1).$$

By taking the infimum over all possible coverings of $\mathcal{L}(\mathcal{D}_1)$, we obtain (4.10).

The statement for a collection $\mathcal{T}_1$ of pairwise disjoint trees in (4.11) follows by an analogous argument. The additional observation is that the collection $\mathcal{T}$ of trees splits as

$$\mathcal{T} = \mathcal{T}_+ \cup \mathcal{T}_-,$$

where the elements of $\mathcal{T}_+$ are all contained in $\mathbb{R}\times(0,\infty)\times(0,\infty)$, while the elements of $\mathcal{T}_-$ are all contained in $\mathbb{R}\times(0,\infty)\times(-\infty,0]$. In particular, every element of the first family is disjoint from every element of the second one.

The statement in (4.12) follows by Lemma 4.2 and (4.11). ∎

Proof of (4.13), (4.14). Let A be a subset of X. For every point (x,s,ξ) in A, there exist $l \in \mathbb{Z}$ such that $s \in (2^{l-1}, 2^l]$, and $m \in \mathbb{Z}$ such that $x \in I(m,l)$. Hence,

$$(x,s,\xi) \in D(m,l)_+,$$

proving (4.13).

Next, let $A_1, A_2 \subseteq X$ be such that $A_1 \subseteq A_2$. By the definition of $\mathcal{Q}$, we have $\mathcal{Q}(A_1) \subseteq \mathcal{Q}(A_2)$. Taking the union of the elements of the collection in both cases, we obtain the desired inclusion, proving (4.14). ∎

Proof of (4.16), (4.17). Let $\mathcal{D}_1$ be a collection of strips. By the definition of $\mathcal{N}$, we have $\mathcal{D}_1 \subseteq \mathcal{N}(\mathcal{D}_1)$. Taking the union of the elements of the collection in both cases, we obtain the desired inclusion, proving (4.16).

Next, let $\mathcal{D}_1, \mathcal{D}_2$ be two collections of strips such that $\mathcal{L}(\mathcal{D}_1) \subseteq \mathcal{L}(\mathcal{D}_2)$. In particular, $\pi(\mathcal{L}(\mathcal{D}_1)) \subseteq \pi(\mathcal{L}(\mathcal{D}_2))$. By the definition of $\mathcal{N}$, we have $\mathcal{N}(\mathcal{D}_1) \subseteq \mathcal{N}(\mathcal{D}_1)$. Taking the union of the elements of the collection in both cases, we obtain the desired inclusion, proving (4.17). ∎

Proof of (4.19), (4.20). Let $\mathcal{D}_1$ be a collection of strips. As $\mathcal{M}(\mathcal{D}_1) \subseteq \mathcal{D}_1$, we have $\mathcal{L}(\mathcal{M}(\mathcal{D}_1)) \subseteq \mathcal{L}(\mathcal{D}_1)$.

To prove the reverse inclusion, we observe that for every strip D' in $\mathcal{D}_1 \setminus \mathcal{M}(\mathcal{D}_1)$, there exists a finite collection of strips in $\mathcal{D}$ strictly containing D'. In particular, there exists a maximal one in $\mathcal{D}_1$, which then belongs to $\mathcal{M}(\mathcal{D}_1)$

and is unique by definition. Taking the union of the elements of the collection in both cases, we obtain the desired inclusion, proving (4.19).

The monotonicity property in (4.20) follows trivially. ■

Proof of (4.21), (4.18), (4.15). The equalities in (4.21) follow by (4.19) and (4.10).

Now, we turn to the proof of (4.18). By (4.19),

$$\mathcal{N} \circ \mathcal{M} = \mathcal{N},$$

hence

$$\mu(\mathcal{L}(\mathcal{N}(\mathcal{D}_1))) = \mu(\mathcal{L}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))).$$

By (4.19) and (4.10),

$$\mu(\mathcal{L}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))) = \mu(\mathcal{L}(\mathcal{M}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1))))) = \sum_{E \in \mathcal{M}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))} |\pi(E)|,$$

$$\mu(\mathcal{L}(\mathcal{D}_1)) = \mu(\mathcal{L}(\mathcal{M}(\mathcal{D}_1))) = \sum_{E \in \mathcal{M}(\mathcal{D}_1)} |\pi(E)|.$$

By the disjointness of the elements in $\mathcal{M}(\mathcal{D}_1)$ and Lemma 4.1, we can partition the collection $\mathcal{M}(\mathcal{D}_1)$ into pairwise disjoint subcollections $\mathcal{M}(\mathcal{D}_1)_E$, one for each element $E \in \mathcal{M}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))$, so that

$$\mathcal{L}(\mathcal{M}(\mathcal{D}_1)_E) \subseteq E.$$

By the definition of $\mathcal{N}$,

$$\sum_{E \in \mathcal{M}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))} |\pi(E)| \leq 2 \sum_{E \in \mathcal{M}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))} \sum_{F \in \mathcal{M}(\mathcal{D}_1)_E} |\pi(F)| \leq 2 \sum_{F \in \mathcal{M}(\mathcal{D}_1)} |\pi(F)|.$$

Together with the previous equalities, this yields (4.18).

Finally, we turn to the proof of (4.15). The inequality

$$\mu(A) \leq \mu(\mathcal{L}(\mathcal{Q}(A)))$$

follows by (4.13) and the monotonicity of μ. The inequality

$$\mu(\mathcal{L}(\mathcal{Q}(A))) = \mu(\mathcal{L}(\mathcal{M}(\mathcal{Q}(A)))) \leq \mu(A)$$

follows by an argument analogous to the one used to prove (4.10) upon observing that for every E in $\mathcal{M}(\mathcal{Q}(A))$, the intersection of E_+ and A is nonempty. ■

Proof of (5.12). Without loss of generality, we assume that the tree $\widetilde{T} \in \widetilde{\mathcal{T}}$ is of the form

$$\widetilde{T}(0,1,1) = \{(y,t,\eta) \colon y \in (0,1],\ t \in (0,1],\ \eta \in 1 + (-t^{-1}, t^{-1}]\}.$$

Next, let $\widetilde{T}_0$ be the subset of $\widetilde{T}$ defined by

$$\widetilde{T}_0 = \widetilde{T}(0,1,1) \cap (0,1] \times (0,1] \times (0,\infty).$$

Due to the monotonicity of ν, it is enough to show that

$$\nu(\widetilde{T}_0) = \infty.$$

Now, let $\mathcal{U}_0 \subseteq \mathcal{T}$ be a covering of $\widetilde{T}_0$ by dyadic trees. For every $l \in \mathbb{N}$, let V_l be the subset of $\widetilde{T}_0$ defined by

$$V_l = (0,1] \times (2^{-l-1}, 2^{-l}] \times (2^l, 2^l + 1],$$

and let $\mathcal{U}_0(l)$ be the subcollection of $\mathcal{U}_0$ defined by its dyadic tree with nonempty intersection with V_l. In particular,

$$V_l \subseteq \mathcal{L}(\mathcal{U}_0(l)),$$

and, for every $l' \in \mathbb{N}$, $l' \neq l$, for every $U \in \mathcal{U}_0(l)$, we claim that

$$U \cap V_{l'} = \emptyset.$$

In particular, the dyadic tree U has the form $T(m, -j, n(l,j))$, where $j \in \mathbb{Z}$, $j \leq l$, $m \in \mathbb{Z}$, $0 \leq m < 2^j$, and $n(l,j) \in \mathbb{Z}$ is defined by the condition

$$I(n(l,j), j) \subseteq I(1,l).$$

If $j > l'$, we have

$$U \subseteq \mathbb{R} \times (0, 2^{l'-1}] \times \mathbb{R}, \qquad V_{l'} \subseteq \mathbb{R} \times (2^{l'-1}, 2^{l'}] \times \mathbb{R},$$

yielding the desired disjointness.

If $j < l'$, we distinguish two cases.

Case I: $l < l'$. We have

$$I(n(l,j), j) \subseteq I(1,l) \subseteq I(0,l'), \qquad (2^{l'}, 2^{l'} + 1] \subseteq I(1,l'),$$

yielding the desired disjointness.

Case II: $l > l'$. We have

$$I(n(l,j), j) \subseteq I(1,l), \qquad (2^{l'}, 2^{l'} + 1] \subseteq I(1,l') \subseteq I(0,l),$$

yielding the desired disjointness.

Therefore, the subcollections $\mathcal{U}_0(l)$ are pairwise disjoint, and

$$\sum_{T \in \mathcal{U}_0} \tau(T) \geq \sum_{l \in \mathbb{N}} \sum_{T \in \mathcal{U}_0(l)} \tau(T) \geq \sum_{l \in \mathbb{N}} \nu(V_l).$$

It is enough to observe that, for every $l \in \mathbb{N}$,

$$\nu(V_l) = 1.$$

In fact, for every covering $\mathcal{V}_l$ of V_l by dyadic trees in $\mathcal{T}$,

$$\pi(V_l) \subseteq \pi\Big(\bigcup_{V \in \mathcal{V}_l} V\Big) \subseteq \bigcup_{V \in \mathcal{V}_l} \pi(V),$$

hence

$$1 = |\pi(V_l)| \leq \sum_{V \in \mathcal{V}_l} |\pi(V)| = \sum_{V \in \mathcal{V}_l} \tau(V). \blacksquare$$

B. Approximation for outer L^p spaces. In this appendix, we present the postponed proofs of the approximation results stated in Section 5.

Proof of Lemma 5.1. We have

$$\|f\|^p_{L^p_\mu(S)} \leq C \sum_{k \in \mathbb{Z}} 2^{kp} \mu(S(f) > 2^k).$$

In particular, there exists $k_0 \in \mathbb{N}$ such that, for every $\widetilde{k} \in \mathbb{N}$, $\widetilde{k} \geq k_0$,

$$\|f\|^p_{L^p_\mu(S)} \leq C \sum_{k \leq \widetilde{k}} 2^{kp} \mu(S(f) > 2^k). \tag{B.1}$$

If $\mu(S(f) > 2^{k_0}) = 0$, then $f \in L^\infty_\mu(S)$, and we can take $A = X$.

Otherwise, we claim that there exists $k_1 \in \mathbb{N}$, $k_1 > k_0$, such that

$$\mu(S(f) > 2^{k_1 - 1}) > 2^p \mu(S(f) > 2^{k_1}). \tag{B.2}$$

If not, for every $k \in \mathbb{N}$, $k > k_0$, we would have

$$2^{kp} \mu(S(f) > 2^k) \geq 2^{k_0 p} \mu(S(f) > 2^{k_0}) > 0,$$

yielding the contradiction

$$\|f\|^p_{L^p_\mu(S)} \geq C \sum_{k=k_0+1}^{\infty} 2^{kp} \mu(S(f) > 2^k) \geq C \sum_{k=k_0+1}^{\infty} 2^{k_0 p} \mu(S(f) > 2^{k_0}) = \infty.$$

Now, let B be an optimal set associated with $\mu(\ell^r(f) > 2^{k_1})$ up to a factor $2^{-1}(1+2^p)$, namely

$$\|f 1_{B^c}\|_{L^\infty_\mu(S)} \leq 2^{k_1}, \tag{B.3}$$

$$\mu(S(f) > 2^{k_1}) \leq \mu(B) \leq \frac{1+2^p}{2} \mu(S(f) > 2^{k_1}), \tag{B.4}$$

and define $A = B^c$, so that $f 1_A \in L^\infty_\mu(S)$.

We claim that for every $k \in \mathbb{N}$, $k < k_1$,

$$\mu(S(f 1_A) > 2^k) \geq \frac{1-2^{-p}}{2} \mu(S(f) > 2^k). \tag{B.5}$$

If not, there would exist $\widetilde{k} \in \mathbb{N}$, $\widetilde{k} < k_1$, such that

$$\mu(S(f 1_A) > 2^{\widetilde{k}}) < \frac{1-2^{-p}}{2} \mu(S(f) > 2^{\widetilde{k}}),$$

yielding the contradiction

$$\begin{aligned}
\mu(S(f) > 2^{\widetilde{k}}) &\leq \mu(S(f 1_A) > 2^{\widetilde{k}}) + \mu(B) \\
&< \frac{1-2^{-p}}{2} \mu(S(f) > 2^{\widetilde{k}}) + \frac{1+2^p}{2} 2^{-p} \mu(S(f) > 2^{k_1 - 1}) \\
&\leq \mu(S(f) > 2^{\widetilde{k}}),
\end{aligned}$$

where we use (B.3) and the subadditivity of μ in the first inequality, (B.4) and (B.2) in the second, and the monotonicity of the super level measure $\mu(S(f) > \lambda)$ in λ in the third.

Therefore, by (B.1) and (B.5),

$$\begin{aligned}\|f\|^p_{L^p_\mu(S)} &\le C \sum_{k<k_1} 2^{kp} \mu(S(f) > 2^k) \\ &\le C \sum_{k<k_1} 2^{kp} \mu(S(f1_A) > 2^k) \\ &\le C \|f1_A\|^p_{L^p_\mu(S)}. \quad \blacksquare\end{aligned}$$

Proof of Lemma 5.2. Without loss of generality, upon normalization of f, we assume that

$$1 < \|f\|_{L^\infty_\nu(\ell^r_\omega)} \le 2.$$

For every $k \in \mathbb{Z}$, $k > 0$, the super level measure of f associated with the level 2^k is zero, and the desired inequality is trivially satisfied.

For $k \le 0$, we prove the desired inequality by induction. In particular, we prove that there exist constants $C = C(r)$, $c = c(r)$, and a bounded sequence $\{C_k : C_k < C,\ k \in \mathbb{Z},\ k \le 0\}$ such that

$$\nu(\ell^r_\omega(f) > 2^k) \le C_k \sum_{l \ge k} \nu(\ell^r_\omega(f1_{X_j}) > c2^l).$$

CASE I: $k = 0$. By the r-orthogonality of the classical L^r quasi-norm on sets with disjoint supports, there exists a set B_0 such that

$$\ell^r_\omega(f)(B_0) > 1, \tag{B.6}$$

$$\nu(\ell^r_\omega(f) > 1) \le \nu(B_0). \tag{B.7}$$

By the monotonicity of the classical L^r quasi-norm and (B.6), there exists $j \in \mathbb{N}$ such that

$$\ell^r_\omega(f1_{X_j})(B_0) > 1.$$

Since

$$\|f1_{X_j}\|_{L^\infty_\nu(\ell^r_\omega)} \le \|f\|_{L^\infty_\nu(\ell^r_\omega)} \le 2,$$

by Lemma 2.1 we obtain

$$\nu(B_0) \le C_0 \nu(\ell^r_\omega(f1_{X_j}) > c).$$

Together with (B.7), this yields the desired inequality.

CASE II: $k < 0$. We assume that there exists $j = j(r, f, k+1) \in \mathbb{N}$ with

$$\nu(\ell^r_\omega(f) > 2^{k+1}) \le C_{k+1} \sum_{l \ge k+1} \nu(\ell^r_\omega(f1_{X_j}) > c2^l). \tag{B.8}$$

Now, for every $\varepsilon > 0$, there exists a set A_{k+1} such that

$$\|f 1_{A^c_{k+1}}\|_{L^\infty_\nu(\ell^r_\omega)} \le 2^{k+1}, \tag{B.9}$$

$$\nu(\ell^r_\omega(f) > 2^{k+1}) \le \nu(A_{k+1}) \le (1+\varepsilon)\nu(\ell^r_\omega(f) > 2^{k+1}). \tag{B.10}$$

We will fix ε later. In particular,

$$\nu(\ell^r_\omega(f) > 2^k) \le \nu(A_{k+1}) + \nu(\ell^r_\omega(f 1_{A^c_{k+1}}) > 2^k). \tag{B.11}$$

If

$$\|f 1_{A^c_{k+1}}\|_{L^\infty_\nu(\ell^r_\omega)} \le 2^k,$$

we obtain

$$\nu(\ell^r_\omega(f) > 2^k) \le \nu(A_{k+1}) \le (1+\varepsilon) C_{k+1} \sum_{l \ge k+1} \nu(\ell^r_\omega(f 1_{X_j}) > c2^l).$$

Otherwise,

$$2^k < \|f 1_{A^c_{k+1}}\|_{L^\infty_\nu(\ell^r_\omega)} \le 2^{k+1}.$$

Applying to $f 1_{A^c_{k+1}}$ an argument analogous to that of the previous case, we obtain $j = j(r, f, k) \in \mathbb{N}$, without loss of generality greater than $j(r, f, k+1)$, such that

$$\nu(\ell^r_\omega(f 1_{A^c_{k+1}}) > 2^k) \le C_0 \nu(\ell^r_\omega(f 1_{A^c_{k+1}} 1_{X_j}) > c2^k) \le C_0 \nu(\ell^r_\omega(f 1_{X_j}) > c2^k).$$

Together with (B.11), (B.10), and (B.8), the previous chain of inequalities yields

$$\nu(\ell^r_\omega(f) > 2^k) \le (1+\varepsilon) C_{k+1} \sum_{l \ge k+1} \nu(\ell^r_\omega(f 1_{X_j}) > c2^l) + C_0 \nu(\ell^r_\omega(f 1_{X_j}) > c2^k).$$

By choosing $\varepsilon = \varepsilon(k) = 2^{2^k} - 1$ and defining $C_k = 2^{1-2^k} C_0, C = 2C_0$, we obtain the desired inequality. ■

Proof of Lemma 5.3. The proof is analogous to that of Lemma 5.2 upon the following observation. Without loss of generality, it is enough to proceed in the case

$$1 < \|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \le 2.$$

Therefore, for every dyadic strip $E \in \mathcal{D}$, we have $f 1_E \in L^q_\nu(\ell^r_\omega)$. Moreover, there exists a collection of maximal dyadic strips $\{E_n \colon E_n \in \mathcal{D},\ n \in \mathbb{N}\}$ such that

$$\ell^q_\nu(\ell^r_\omega)(f)(E_n) > 1, \qquad \mu(\ell^q_\nu(\ell^r_\omega)(f) > 1) \le \sum_{n \in \mathbb{N}} \mu(E_n).$$

In particular, there exists a finite subcollection such that

$$\mu(\ell^q_\nu(\ell^r_\omega)(f) > 1) \le 2 \sum_{n=1}^{N} \mu(E_n).$$

Since the dyadic strips are maximal, they are disjoint, hence, by Lemma 4.4, they are ν-Carathéodory with parameter 1.

Now we apply an argument analogous to that used to prove Lemma 5.2 with the monotonicity of the classical L^r quasi-norms replaced by Lemma 5.4, and Lemma 2.1 replaced by Lemma 3.3. ∎

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