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DECOMPOSITION OF IDEMPOTENT 2-COCYCLES

BY

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Abstract. Let L be a finite Galois field extension of K with Galois group G. We decompose any idempotent 2-cocycle f using finite sequences of descending two-sided ideals of the corresponding weak crossed product algebra A_f. We specialize the results in case f is the corresponding idempotent 2-cocycle f_r for some semilinear map $r : G \to \Omega$, where Ω is a multiplicative monoid with minimum element.

1. Introduction. Let L be a finite Galois field extension of K with Galois group G. A function $f : G\times G \to L$ is called a *normalized weak* 2*-cocycle* (of G over L) if it satisfies the conditions $f(\sigma,\tau)f(\sigma\tau,\rho) = f^{\sigma}(\tau,\rho)f(\sigma,\tau\rho)$ for $\sigma,\tau,\rho \in G$ and $f(1,\sigma) = f(\sigma,1) = 1$ for $\sigma \in G$. Associated to a weak 2-cocycle f is a K-algebra A_f called the *weak crossed product algebra associated to* f, first introduced in [HLS83]. The K-algebra A_f is defined as an L-vector space $A_f = \sum_{\sigma\in G} Lx_\sigma$ having the symbols x_σ, $\sigma \in G$, as an L-basis and multiplication defined by the rules $x_\sigma l = l^{\sigma}x_\sigma$ and $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$ for $\sigma,\tau \in G$ and $l \in L$. The inertial group $H(f)$ (or H if is clear from the context) is defined as $H(f) = \{\sigma \in G : f(\sigma,\sigma^{-1}) \neq 0\}$. Then $A_f = \sum_{\sigma\in H(f)} Lx_\sigma + J_f$, where J_f (or J if f is clear from the context) is the Jacobson radical of A_f and the unique maximal two-sided ideal of A_f. By the notation $I \triangleleft A_f$ we always mean that I is a two-sided ideal of A_f. D. E. Haile [H82] gave the structure of the two-sided ideals of A_f. In particular, he proved that if $I \neq (0)$ is a two-sided ideal of A_f, then $I = \sum Lx_\sigma$, where the sum is taken over those $\sigma \in G$ such that $x_\sigma \in I$ [H82, Lemma 2.2]. Moreover, if f is an idempotent 2-cocycle, i.e. taking only the values 0 and 1, then every ideal I of A_f is of the form $I = \sum_{x_\sigma\in I} I_\sigma$, where I_σ is the ideal of A_f generated by x_σ [H82, Proposition 2.4].

We recall some results from [LT17]. We denote by $E^2(G,L)$ the set of idempotent 2-cocycles and by $E^2(G,L;H)$ the set of idempotent 2-cocycles

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with inertial group H. We set $G^* = G\backslash H$. To avoid trivialities we suppose that $G^* \neq \emptyset$ and so A_f is not a simple algebra. Let Ω be a multiplicative totally ordered monoid with minimum element 1 satisfying the relations $x < y \Rightarrow xz < yz$ and $zx < zy$ for all $x, y, z \in \Omega$. We denote by $\mathrm{Sl}(G)$ the set of functions $r : G \to \Omega$ satisfying the relations $r(1) = 1$ and $r(\sigma\tau) \leq r(\sigma)r(\tau)$ for every $\sigma, \tau \in G$. Let $M_r = \{\sigma \in G : r(\sigma) = 1\}$. It was shown in [LT17, Proposition 4.2 and Theorem 5.2] that M_r is a group and the function $f_r : G \times G \to L$ defined by the rule

$$f_r(\sigma, \tau) = \begin{cases} 1 & \text{if } r(\sigma\tau) = r(\sigma)r(\tau), \\ 0 & \text{if } r(\sigma\tau) < r(\sigma)r(\tau), \end{cases}$$

is an element of $E^2(G, L; M_r)$.

Let $t + 1$ be the nilpotency of J. To every f we associate a partition $\{N_i\}_{i=1}^t$ of G^* defined by $N_k(f) = \{\sigma \in G^* : x_\sigma \in J^k \setminus J^{k+1}\}$ for $1 \leq k \leq t$. We observe that [LT17, Lemma 3.1]

$$N_1(f) = \{\sigma \in G^* : f(\sigma_1, \sigma_2) = 0 \text{ for all } \sigma_1, \sigma_2 \in G^* \text{ with } \sigma_1\sigma_2 = \sigma\}.$$

An ordered set $(\sigma_1, \ldots, \sigma_k)$ of elements of $N_1(f)$ is called a *generator of $\sigma \in G^*$ with respect to f* if $x_\sigma = x_{\sigma_1} \ldots x_{\sigma_k}$. We denote by g_σ a generator of $\sigma \in G^*$ and by Γ_f the set of all generators of all elements of G^* with respect to f. So the elements of Γ_f are words with letters from $N_1(f)$. The product of two generators is defined by concatenation.

Pick any $f \in E^2(G, L; H)$. In order to find a function $s : G \to \Omega$, for some Ω, such that $f = f_s$ it was shown in [LT17, Proposition 6.10 and Theorem 6.12] that one has to start from a map $s' : N_1(f) \to \Omega \setminus \{1\}$ satisfying certain conditions and then extend it over the whole of G by

$$s(\sigma) = \begin{cases} \psi(g_\sigma), & g_\sigma \in \Gamma_f, \\ 1, & \sigma \in H, \end{cases}$$

where $\psi(g) = \prod_{\sigma \in g} s'(\sigma)$ for any generator g. The first condition is that for every $\rho \in G^*$ such that x_ρ is a left-right annihilator of J_f, all the generators g_ρ of ρ must map to the same value through ψ (i.e. $\psi(g_\rho) = \text{constant}$). Ignoring the elements whose classes lie exactly above H in the graph of f (in this case the equalities are trivial since from [LT17, Remark 6.4] every such element has exactly one generator), we see that the fewer the elements corresponding to maximal classes the fewer the equations that must be satisfied, and that the fewer the generators of each ρ the fewer the terms in each equation. The second condition is that for every such ρ, $\psi(g_\rho)$ must be minimum among all the words w_ρ with letters from $N_1(f)$. More formally, $\psi(g_\rho) < \psi(w_\rho)$ for all $w_\rho \notin \Gamma_f$. Finally, s' must be constant for all the elements $H\sigma H$, that is, $s'(h_1\sigma h_2) = s'(\sigma)$ for every $\sigma \in N_1(f)$ and $h_1, h_2 \in H$.

The problem we are interested in is which elements f of $E^2(G,L)$ afford the relation $f = f_s$ for some $s \in \mathrm{Sl}(G)$. In this article, as a first step, we decompose f in such a way that its constituents can take the form f_s. The decomposition that we are going to demonstrate is intended to simplify the system of equations implied by the first condition mentioned above [LT17, Proposition 6.10]. The main tool that will be used is a new construction of an idempotent 2-cocycle from an existing one by means of a finite sequence of descending two-sided ideals of A_f (Definition 2.2). We investigate the particular case of the family of ideals $\{J, I\}$ and examine the corresponding algebra. Using certain ideals we construct weak crossed product algebras whose graphs have a unique maximal class. We specialize the results for the case $f = f_r$ for some $r \in \mathrm{Sl}(G)$.

2. Idempotent 2-cocycles arising from a finite sequence of descending ideals. Let $f \in E^2(G,L)$, and let I be a two-sided ideal of A_f and H the inertial group of f. For $\sigma, \tau \in G$ and $h, h_1, h_2 \in H$ it is easy to prove that $f(h_1\sigma, \tau h_2) = f(\sigma, \tau)$ and $f(\sigma h, \tau) = f(\sigma, h\tau)$. Also, if $f(\sigma,\tau) = 1$ and $\sigma \in G^*$ or $\tau \in G^*$, then $\sigma\tau \notin H$. Moreover, a direct consequence of these formulas is that if $x_\sigma \in I$, then $x_{h_1\sigma h_2} \in I$ for every $h_1, h_2 \in H$. Also, if $x_{h_1\sigma h_2} \in I$ for some $h_1, h_2 \in H$, then $x_\sigma \in I$. Another useful observation that we will use frequently is that if $x_\tau \in I_\sigma$, then there exist $\rho_1, \rho_2 \in G$ such that $x_\tau = x_{\rho_1} x_\sigma x_{\rho_2}$.

Let $\mathrm{Ann}(J)$ be the ideal of left-right annihilators of J. We state the following proposition which will be used frequently.

Proposition 2.1. *Let $f \in E^2(G,L;H)$ and $\sigma \in G^*$. Then:*

(i) $\{x_\tau \in I_\sigma : \tau \in G^*\} = \bigcup_{g_\sigma} \{x_\tau : \exists g_\tau \in \Gamma_f$ *and* $h_1, h_2 \in H$ *with* $h_1 g_\sigma h_2 \leq g_\tau\}$.

(ii) *If* $x_\sigma \in \mathrm{Ann}(J)$, *then* $x_{h_1\sigma h_2} \in \mathrm{Ann}(J)$ *for* $h_1, h_2 \in H$, *and also*

$$I_\sigma = \sum_{h_1,h_2 \in H} L x_{h_1\sigma h_2}.$$

Proof. (i) Let $x_\tau \in I_\sigma$ for $\tau \in G^*$, and $g_\sigma \in \Gamma_f$. There exist $\rho_1, \rho_2 \in G$ such that $x_\tau = x_{\rho_1} x_\sigma x_{\rho_2}$. If $\rho_1, \rho_2 \in H$, then $\rho_1 g_\sigma \rho_2 = g_{\rho_1\sigma\rho_2} = g_\tau \in \Gamma_f$ and the claim is true; similarly for the other possible cases, that is, $\rho_1 \in H$ and $\rho_2 \notin H$, $\rho_1 \notin H$ and $\rho_2 \in H$, or $\rho_1, \rho_2 \in G^*$.

For the opposite direction, suppose that there exist $g_\sigma, g_\tau \in \Gamma_f$ and $h_1, h_2 \in H$ with $h_1 g_\sigma h_2 \leq g_\tau$. If $g_\sigma = (\sigma_1, \ldots, \sigma_l)$ then $(h_1\sigma_1, \sigma_2, \ldots, \sigma_{l-1}, \sigma_l h_2) \leq g_\tau$. We remark that

$$g_\tau = \begin{cases} (\tau_1, \ldots, \tau_a, h_1\sigma_1, \sigma_2, \ldots, \sigma_{l-1}, \sigma_l h_2, \tau_{a+1}, \ldots, \tau_b), \text{ or} \\ (\tau_1, \ldots, \tau_a, h_1\sigma_1, \sigma_2, \ldots, \sigma_{l-1}, \sigma_l h_2), \text{ or} \\ (h_1\sigma_1, \sigma_2, \ldots, \sigma_{l-1}, \sigma_l h_2, \tau_1, \ldots, \tau_b), \text{ or} \\ (h_1\sigma_1, \sigma_2, \ldots, \sigma_{l-1}, \sigma_l h_2), \end{cases}$$

where $\sigma_i, \tau_j \in G^*$, $a, b \geq 1$. We examine the first case and similarly we work for the second and third. We set $g_{\rho_1} = (\tau_1, \dots, \tau_a)$ and $g_{\rho_2} = (\tau_{a+1}, \dots, \tau_b)$. Then $g_\tau = g_{\rho_1} g_{h_1\sigma h_2} g_{\rho_2} = g_{\rho_1} h_1 g_\sigma h_2 g_{\rho_2} = g_{\rho_1 h_1} g_\sigma g_{h_2\rho_2}$, hence $x_\tau \in I_\sigma$. For the fourth case $g_\tau = h_1 g_\sigma h_2$ and so $x_\tau = x_{h_1} x_\sigma x_{h_2}$. Therefore in any case $x_\tau \in I_\sigma$ and we conclude that in order to calculate the ideal I_σ we are restricted to the elements of G^* whose generators contain a generator of an element of the set $H\sigma H$.

(ii) Let $x_\sigma \in \mathrm{Ann}(J)$ and $h_1, h_2 \in H$. For any $\tau \in G^*$ it follows that $\tau h_1 \in G^*$ and $x_\tau x_{h_1\sigma h_2} = x_{\tau h_1} x_\sigma x_{h_2} = 0$. Similarly $x_{h_1\sigma h_2} x_\tau = 0$ and hence $x_{h_1\sigma h_2} \in \mathrm{Ann}(J)$. To prove the equality let $x_\tau \in I_\sigma$. From (i) there exist $g_\sigma, g_\tau \in \Gamma_f$ and $h_1, h_2 \in H$ such that $g_{h_1\sigma h_2} = h_1 g_\sigma h_2 \leq g_\tau$. Since $x_{h_1\sigma h_2} \in \mathrm{Ann}(J)$ and the generators of the annihilators have maximum length, it follows that $g_{h_1\sigma h_2} = g_\tau$ and so $\tau = h_1\sigma h_2$. The opposite direction is obvious. ∎

The following construction is the main tool which we use throughout this article. All ideals are assumed to be two-sided.

Definition 2.2. Let $f \in E^2(G, L; H)$ and $\mathbf{I} = \{I_i\}_{i=1}^k$, $k \geq 2$, be a finite sequence of descending ideals of A_f, that is, $I_1 \supseteq \cdots \supseteq I_k$. We set

$$f_{\mathbf{I}}(\sigma, \tau) = \begin{cases} 1, & \sigma \in H \text{ or } \tau \in H, \\ 1, & f(\sigma, \tau) = 1 \text{ and } x_\sigma, x_\tau, x_{\sigma\tau} \in I_i \setminus I_{i+1} \\ & \text{for some } i \in \{1, \dots, k-1\}, \\ 0, & \text{elsewhere}, \end{cases}$$

for $\sigma, \tau \in G$. Also, for $\sigma \in G^*$ such that $x_\sigma \in I_1$, let $s(\sigma) = \max\{a \in \mathbb{N}^* : x_\sigma \in I_a,\ 1 \leq a \leq k\}$.

Definition 2.3. In $E^2(G, L)$ we define the relation

$$f \leq g \iff \{(\sigma, \tau) \in G \times G : f(\sigma, \tau) = 1\} \subseteq \{(\sigma, \tau) \in G \times G : g(\sigma, \tau) = 1\},$$

which is a partial order.

Remark 2.4. Let $f_{\mathbf{I}}(\sigma, \tau) = 1$ for some $\sigma, \tau \in G$. From Definition 2.2 it follows that $f(\sigma, \tau) = 1$ or $\sigma \in H$ or $\tau \in H$. In any case $f(\sigma, \tau) = 1$ and so $f_{\mathbf{I}} \leq f$. Also, for $x_\sigma, x_\tau \in I_1$, if $s(\sigma) = k$ or $s(\tau) = k$ or $s(\sigma\tau) = k$, then $f_{\mathbf{I}}(\sigma, \tau) = 0$.

Our aim is to prove that $f_{\mathbf{I}}$ is an idempotent 2-cocycle. We need the following lemma.

Lemma 2.5. *Let $\mathbf{I} = \{I_i\}_{i=1}^k$, $k \geq 2$, be a finite sequence of descending ideals of A_f and $f_{\mathbf{I}}$ the function of Definition 2.2. Then:*

(i) *$s(h_1\sigma h_2) = s(\sigma)$ for $x_\sigma \in I_1$, $h_1, h_2 \in H$.*
(ii) *If $x_\sigma, x_\tau \in I_1$ and $f(\sigma, \tau) = 1$, then $s(\sigma) \leq s(\sigma\tau)$ and $s(\tau) \leq s(\sigma\tau)$.*

(iii) $f_{\mathbf{I}}(h_1\sigma, \tau h_2) = f_{\mathbf{I}}(\sigma, \tau)$ *for* $\sigma, \tau \in G$ *and* $h_1, h_2 \in H$.
(iv) $f_{\mathbf{I}}(\sigma h, \tau) = f_{\mathbf{I}}(\sigma, h\tau)$ *for* $\sigma, \tau \in G$ *and* $h \in H$.

Proof. (i) If $s(\sigma) = k$, the proof is immediate. Suppose that $1 \leq s(\sigma) \leq k-1$. Since $x_\sigma \in I_{s(\sigma)}$, we have $x_{h_1\sigma h_2} \in I_{s(\sigma)}$. If we had $x_{h_1\sigma h_2} \in I_{s(\sigma)+1}$, then $x_{h_1^{-1}} x_{h_1\sigma h_2} x_{h_2^{-1}} = x_\sigma \in I_{s(\sigma)+1}$, a contradiction. Hence $s(h_1\sigma h_2) = s(\sigma)$.

(ii) We have $x_{\sigma\tau} = x_\sigma x_\tau \in I_{s(\sigma)} x_\tau \subseteq I_{s(\sigma)}$ and so $s(\sigma\tau) \geq s(\sigma)$. Similarly $s(\sigma\tau) \geq s(\tau)$.

(iii) Let $\sigma, \tau \in G$. If $\sigma \in H$, then by definition, $f_{\mathbf{I}}(h_1\sigma, \tau h_2) = f_{\mathbf{I}}(\sigma, \tau) = 1$. Similarly if $\tau \in H$. Next let $\sigma, \tau \in G^*$. If $x_\sigma \notin I_1$, then $x_{h_1\sigma} \notin I_1$ and so $f_{\mathbf{I}}(h_1\sigma, \tau h_2) = f_{\mathbf{I}}(\sigma, \tau) = 0$. Similarly if $x_\tau \notin I_1$ or $x_{\sigma\tau} \notin I_1$. Next suppose that $x_\sigma, x_\tau, x_{\sigma\tau} \in I_1$. From (i), $s(h_1\sigma) = s(\sigma)$, $s(\tau h_2) = s(\tau)$ and $s(h_1\sigma\tau h_2) = s(\sigma\tau)$. If $s(\sigma) = k$ or $s(\tau) = k$ or $s(\sigma\tau) = k$, then $f_{\mathbf{I}}(h_1\sigma, \tau h_2) = f_{\mathbf{I}}(\sigma, \tau) = 0$. Suppose that $1 \leq s(\sigma), s(\tau), s(\sigma\tau) \leq k-1$. If $f(\sigma, \tau) = 1$ and $s(\sigma) = s(\tau) = s(\sigma\tau)$, then $s(h_1\sigma) = s(\tau h_2) = s(h_1\sigma\tau h_2)$. So $f(h_1\sigma, \tau h_2) = 1$ and by definition $f_{\mathbf{I}}(\sigma, \tau) = f_{\mathbf{I}}(h_1\sigma, \tau h_2) = 1$. If $f(\sigma, \tau) = 0$ or $s(\sigma), s(\tau), s(\sigma\tau)$ are not all equal, then $f_{\mathbf{I}}(\sigma, \tau) = f_{\mathbf{I}}(h_1\sigma, \tau h_2) = 0$.

(iv) The proof is similar to (iii) above with slight modifications. In particular, we need to examine cases for $s(\sigma)$, $s(\tau)$ and $s(\sigma h\tau)$. ∎

THEOREM 2.6. *Let* $f \in E^2(G, L; H)$. *For every finite descending sequence* $\mathbf{I} = \{I_i\}_{i=1}^k$ *of ideals of* A_f, *the function* $f_{\mathbf{I}}$ *of Definition* 2.2 *is an element of* $E^2(G, L; H)$.

Proof. First we prove that the 2-cocycle condition holds for $f_{\mathbf{I}}$. Let $\sigma, \tau, \rho \in G$. If $\sigma \in H$, then by definition $f_{\mathbf{I}}(\sigma, \tau) = f_{\mathbf{I}}(\sigma, \tau\rho) = 1$. Also, from Lemma 2.5(iii), $f_{\mathbf{I}}(\sigma\tau, \rho) = f_{\mathbf{I}}(\tau, \rho)$ and the 2-cocycle condition is true. Similarly if $\tau \in H$ or $\rho \in H$.

Next suppose that $\sigma, \tau, \rho \in G^*$. If $f(\sigma, \tau) = 0$, then $f_{\mathbf{I}}(\sigma, \tau) = 0$. Since f is an idempotent 2-cocycle, either $f(\tau, \rho)$ or $f(\sigma, \tau\rho)$ is zero, so either $f_{\mathbf{I}}(\tau, \rho)$ or $f_{\mathbf{I}}(\sigma, \tau\rho)$ is zero and the 2-cocycle condition is true. The same argument applies when $f(\sigma\tau, \rho) = 0$ or $f(\sigma, \tau\rho) = 0$ or $f(\tau, \rho) = 0$.

Suppose that $f(\sigma, \tau) = f(\sigma\tau, \rho) = f(\tau, \rho) = f(\sigma, \tau\rho) = 1$. If $x_\sigma \notin I_1$ or $x_\sigma \in I_k$, then $f_{\mathbf{I}}(\sigma, \tau) = f_{\mathbf{I}}(\sigma, \tau\rho) = 0$. Similarly in the following cases: $x_\tau \notin I_1$ or $x_\tau \in I_k$; $x_\rho \notin I_1$ or $x_\rho \in I_k$; $x_{\sigma\tau} \notin I_1$ or $x_{\sigma\tau} \in I_k$; $x_{\tau\rho} \notin I_1$ or $x_{\tau\rho} \in I_k$; and $x_{\sigma\tau\rho} \notin I_1$ or $x_{\sigma\tau\rho} \in I_k$. Therefore in all the above cases the 2-cocycle condition holds.

Finally, we suppose that $x_\sigma, x_\tau, x_\rho, x_{\sigma\tau}, x_{\tau\rho}, x_{\sigma\tau\rho} \in I_1 \setminus I_k$. We distinguish several cases for the natural numbers $s(\sigma), s(\tau), s(\rho)$. If $s(\sigma) < s(\tau)$, then $s(\sigma) < s(\tau\rho)$ (Lemma 2.5(ii)) and so $f_{\mathbf{I}}(\sigma, \tau) = f_{\mathbf{I}}(\sigma, \tau\rho) = 0$. Hence the 2-cocycle condition holds for $f_{\mathbf{I}}$. If $s(\rho) < s(\tau)$, then $s(\rho) < s(\sigma\tau)$ and so $f_{\mathbf{I}}(\sigma\tau, \rho) = f_{\mathbf{I}}(\tau, \rho) = 0$. If $s(\tau) \leq s(\sigma)$ and $s(\tau) \leq s(\rho)$, we distinguish the following cases:

- $s(\tau) < s(\sigma) \leq s(\rho)$. Then $f_{\mathbf{I}}(\sigma,\tau) = f_{\mathbf{I}}(\tau,\rho) = 0$. Similarly if $s(\tau) < s(\rho) \leq s(\sigma)$.
- $s(\tau) = s(\sigma) < s(\rho)$. Then $f_{\mathbf{I}}(\tau,\rho) = 0$. If $s(\tau) = s(\sigma) = s(\sigma\tau)$, then $f_{\mathbf{I}}(\sigma\tau,\rho) = 0$. If $s(\tau) = s(\sigma) < s(\sigma\tau)$, then $f_{\mathbf{I}}(\sigma,\tau) = 0$. Similarly if $s(\tau) = s(\rho) < s(\sigma)$.
- $s(\tau) = s(\sigma) = s(\rho)$. If $s(\sigma) < s(\sigma\tau\rho)$, then $f_{\mathbf{I}}(\sigma,\tau\rho) = f_{\mathbf{I}}(\sigma\tau,\rho) = 0$. If $s(\sigma) = s(\sigma\tau\rho)$, then $s(\sigma) \leq s(\sigma\tau) \leq s(\sigma\tau\rho)$ and so $s(\sigma\tau) = s(\sigma\tau\rho)$ and similarly $s(\tau\rho) = s(\sigma\tau\rho)$. Then $f_{\mathbf{I}}(\sigma,\tau) = f_{\mathbf{I}}(\sigma\tau,\rho) = f_{\mathbf{I}}(\tau,\rho) = f_{\mathbf{I}}(\sigma,\tau\rho) = 1$.

Therefore again in all the above cases the 2-cocycle condition holds. For the inertial group we see that if $\sigma \in H(f_{\mathbf{I}})$ then $f_{\mathbf{I}}(\sigma,\sigma^{-1}) = 1$. From Remark 2.4, $f(\sigma,\sigma^{-1}) = 1$ and so $\sigma \in H$. ∎

For any subgroup H of G we set

$$f_0(\sigma,\tau) = \begin{cases} 1 & \text{if } \sigma \in H \text{ or } \tau \in H, \\ 0 & \text{elsewhere,} \end{cases}$$

the *Waterhouse idempotent*. The following proposition will be used frequently. The appropriate 2-cocycle f_0 (and its inertial group) will be clear from the context.

PROPOSITION 2.7. *$f_{\mathbf{I}} = f_0$ if and only if $I_a^2 \subseteq I_{a+1}$ for every $a \in \{1,\ldots,k-1\}$.*

Proof. We set $H = H(f_{\mathbf{I}}) = H(f_0)$ and $G^* = G\backslash H$. First, suppose that $I_a^2 \subseteq I_{a+1}$ for every $a \in \{1,\ldots,k-1\}$, and let $\sigma,\tau \in G^*$. Let $f(\sigma,\tau) = 1$. If $x_\sigma \notin I_1$ or $x_\sigma \in I_k$ or $x_\tau \notin I_1$ or $x_\tau \in I_k$ or $x_{\sigma\tau} \notin I_1$ or $x_{\sigma\tau} \in I_k$, then $f_{\mathbf{I}}(\sigma,\tau) = 0$ in these cases.

Next suppose that $x_\sigma, x_\tau, x_{\sigma\tau} \in I_1 \setminus I_k$. If the natural numbers $s(\sigma), s(\tau)$, $s(\sigma\tau)$ are not all equal, then by definition $f_{\mathbf{I}}(\sigma,\tau) = 0$. If $s(\sigma) = s(\tau) = s(\sigma\tau) = a$ for some $1 \leq a \leq k-1$, then $x_{\sigma\tau} = x_\sigma x_\tau \in I_{s(\sigma)} I_{s(\tau)} = I_a^2$. From the assumption it follows that $x_{\sigma\tau} \in I_{a+1} = I_{s(\sigma\tau)+1}$, a contradiction, which implies that this case is impossible. Therefore in any case $f_{\mathbf{I}}(\sigma,\tau) = 0$ for $\sigma,\tau \in G^*$.

Next suppose that $f_{\mathbf{I}} = f_0$. Let $a \in \{1,\ldots,k-1\}$. If $I_a^2 \neq 0$ and $x_\sigma \in I_a^2$, then there exist $x_{\sigma_1}, x_{\sigma_2} \in I_a$ such that $f(\sigma_1,\sigma_2) = 1$ with $\sigma_1\sigma_2 = \sigma$. If $x_\sigma \notin I_{a+1}$, then $s(\sigma) \leq a$. Since from Lemma 2.5(ii) we have $a \leq s(\sigma_1) \leq s(\sigma) \leq a$, it follows that $s(\sigma_1) = s(\sigma) = a$ and similarly $s(\sigma_2) = s(\sigma) = a$. But then $f_{\mathbf{I}}(\sigma_1,\sigma_2) = 1$, contrary to assumption. So $x_\sigma \in I_{a+1}$. ∎

We will need a new operation.

DEFINITION 2.8. For $\{f_i\}_{i=1}^k$ a finite family of elements of $E^2(G,L)$ we define

$$\bigvee_{i=1}^{k} f_i(\sigma,\tau) = f_1(\sigma,\tau) \vee \cdots \vee f_k(\sigma,\tau)$$
$$= \begin{cases} 0 & \text{if } f_i(\sigma,\tau) = 0,\ \forall i = 1,\dots,k, \\ 1 & \text{if } \exists i \in \{1,\dots,k\} \text{ such that } f_i(\sigma,\tau) = 1. \end{cases}$$

The set $E^2(G,L)$ in general is not closed under the operation $\vee$. We will encounter some instance where the result of the operation is indeed an idempotent 2-cocycle. We remark that inside $E^2(G,L;H)$ we have $f \vee f_0 = f_0 \vee f = f$ for H the inertial group of f_0.

LEMMA 2.9. *Let $f \in E^2(G,L;H)$ and $\{I_i\}_{i=1}^k$, $k \geq 3$, be a finite descending sequence of ideals of A_f. Then for $a \in \{2,\dots,k-1\}$ we get $f_{\{I_1,\dots,I_a,\dots,I_k\}} = f_{\{I_1,\dots,I_a\}} \vee f_{\{I_a,\dots,I_k\}}$.*

Proof. We set $f' = f_{\{I_1,\dots,I_a,\dots,I_k\}}$, $f_1 = f_{\{I_1,\dots,I_a\}}$ and $f_2 = f_{\{I_a,\dots,I_k\}}$. First let $\sigma,\tau \in G^*$ be such that $f'(\sigma,\tau) = 1$. Then $f(\sigma,\tau) = 1$ and $x_\sigma, x_\tau, x_{\sigma\tau} \in I_b \setminus I_{b+1}$ for some $1 \leq b \leq k-1$. If $1 \leq b \leq a-1$, then $f_1(\sigma,\tau) = 1$. If $a \leq b \leq k-1$, then $f_2(\sigma,\tau) = 1$.

For the opposite direction, let $\sigma,\tau \in G$ be such that $f_1(\sigma,\tau) \vee f_2(\sigma,\tau) = 1$. If $\sigma,\tau \in G^*$, then $f(\sigma,\tau) = 1$ and $x_\sigma, x_\tau, x_{\sigma\tau} \in I_i \setminus I_{i+1}$ for some $1 \leq i \leq a-1$, or $x_\sigma, x_\tau, x_{\sigma\tau} \in I_i \setminus I_{i+1}$ for some $a \leq i \leq k-1$. Therefore in any case $f'(\sigma,\tau) = 1$. ■

By repeated application of Lemma 2.9 we obtain

COROLLARY 2.10. *Let $\{I_i\}_{i=1}^k$, $k \geq 2$, be a finite descending sequence of ideals of A_f. Then*

$$f_{\{I_1,\dots,I_k\}} = \bigvee_{i=1}^{k-1} f_{\{I_i,I_{i+1}\}}. \ ■$$

The following proposition establishes a decomposition formula for an idempotent 2-cocycle. We will need the fact that if $x_\sigma \in \sum_{i=1}^k I_i$, then $x_\sigma \in I_i$ for some $i \in \{1,\dots,k\}$. Also, if $x_\sigma \notin \sum_{i=1}^k I_i$, then $x_\sigma \notin I_i$ for every $i \in \{1,\dots,k\}$.

PROPOSITION 2.11. *Let I and $I_1,\dots,I_k$, $k \geq 2$, be ideals of A_f such that $I_i \subseteq I$ for every $i \in \{1,\dots,k\}$. The following identities hold:*

(i) $f_{\{I,\sum_{i=1}^k I_i\}} = \prod_{i=1}^k f_{\{I,I_i\}}$,
(ii) $f_{\{I,\bigcap_{i=1}^k I_i\}} = \bigvee_{i=1}^k f_{\{I,I_i\}}$.

Proof. We set $P = \sum_{i=1}^k I_i \lhd A_f$. Then from Theorem 2.6 we get $H(f_{\{I,P\}}) = H(f_{\{I,I_i\}}) = H(f) = H$ for every i. We set $G^* = G \setminus H$. Let $\sigma,\tau \in G^*$. If $f(\sigma,\tau) = 0$, then from Remark 2.4 we have $f_{\{I,P\}}(\sigma,\tau) = f_{\{I,I_i\}}(\sigma,\tau) = 0$ for every i. Next suppose that $f(\sigma,\tau) = 1$. If $x_\sigma \notin I$ or $x_\tau \notin I$ or $x_{\sigma\tau} \notin I$, then by definition $f_{\{I,P\}}(\sigma,\tau) = f_{\{I,I_i\}}(\sigma,\tau) = 0$ for every i. If $x_\sigma, x_\tau, x_{\sigma\tau} \in I$, then:

- If $x_{\sigma\tau} \notin P$, then $x_\sigma, x_\tau \notin P$. Also $x_{\sigma\tau} \notin I_i$ for every i, and so $x_\sigma, x_\tau \notin I_i$ for every i. It follows that $f_{\{I,P\}}(\sigma,\tau) = f_{\{I,I_i\}}(\sigma,\tau) = 1$ for every i.
- If $x_{\sigma\tau} \in P$, then $f_{\{I,P\}}(\sigma,\tau) = 0$. Also there exists $a \in \{1,\dots,k\}$ such that $x_{\sigma\tau} \in I_a$. Then $f_{\{I,I_a\}}(\sigma,\tau) = 0$ and so $\prod_{i=1}^k f_{\{I,I_i\}}(\sigma,\tau) = 0$.

Therefore we get the first statement. For the second we set $P = \bigcap_{i=1}^k I_i \lhd A_f$. The proof is identical in all cases except when $f(\sigma,\tau) = 1$ and $x_\sigma, x_\tau, x_{\sigma\tau} \in I$. In this case, if $x_{\sigma\tau} \notin P$, then $f_{\{I,P\}}(\sigma,\tau) = f_{\{I,I_a\}}(\sigma,\tau) = 1$. If $x_{\sigma\tau} \in P$, then $f_{\{I,P\}}(\sigma,\tau) = f_{\{I,I_i\}}(\sigma,\tau) = 0$ for every i. ∎

3. The crossed product algebra A_{f_I}

3.1. The idempotent 2-cocycle $f_{\{J,I\}}$. The special case of Theorem 2.6 for the family of ideals $\{I_1 = J,\ I_2 = I\}$ for some ideal I of A_f is of interest so we restate it as a separate proposition.

PROPOSITION 3.1. *Let $f \in E^2(G,L;H)$ and $I \lhd A_f$. Then the function defined by*

$$f_I(\sigma,\tau) = f_{\{J,I\}}(\sigma,\tau) = \begin{cases} 1 & \text{if } \sigma \in H \text{ or } \tau \in H, \\ 1 & \text{if } f(\sigma,\tau) = 1,\ x_{\sigma\tau} \notin I,\ \sigma \notin H,\ \tau \notin H, \\ 0 & \text{elsewhere} \end{cases}$$

is an element of $E^2(G,L;H)$.

Despite the new notation, f_0 will still denote the Waterhouse idempotent and not the idempotent 2-cocycle $f_{\{J,0\}}$ corresponding to the zero ideal (which is equal to f). The following proposition calculates the set $N_1(f_I)$ explicitly.

PROPOSITION 3.2. *If $I \lhd A_f$, then $N_1(f_I) = N_1(f) \cup \{\sigma \in G^* : x_\sigma \in I\}$.*

Proof. Let H be the inertial group of both f and f_I as in Proposition 3.1. We set $G^* = G \setminus H$. First we prove that $N_1(f) \subseteq N_1(f_I)$. For this pick any $\sigma \in N_1(f)$. Suppose there exist $\tau,\rho \in G^*$ with $\tau\rho = \sigma$ such that $f_I(\tau,\rho) = 1$. From Remark 2.4 it follows that $f(\tau,\rho) = 1$, which is impossible. So $f_I(\tau,\rho) = 0$ for all $\tau,\rho \in G^*$ with $\tau\rho = \sigma$, and hence $\sigma \in N_1(f_I)$.

Next we prove that $\{\sigma \in G^* : x_\sigma \in I\} \subseteq N_1(f_I)$. For this choose $\sigma \in G^*$ such that $x_\sigma \in I$ and $\tau,\rho \in G^*$ with $\tau\rho = \sigma$. If we had $f_I(\tau,\rho) = 1$, then $f(\tau,\rho) = 1$ and $x_\tau x_\rho = x_\sigma \in I$. So by definition $f_I(\tau,\rho) = 0$, a contradiction.

For the converse, let $\sigma \in N_1(f_I)$. If there exist $\tau,\rho \in G^*$ such that $\tau\rho = \sigma$ and $f(\tau,\rho) = 1$, then $x_\tau x_\rho = x_\sigma \in I$ (since $x_\sigma \notin I$ would imply $f_I(\tau,\rho) = 1$, a contradiction). If $f(\tau,\rho) = 0$ for all $\tau,\rho \in G^*$ such that $\tau\rho = \sigma$, then $\sigma \in N_1(f)$. ∎

Let $x_\sigma \in \mathrm{Ann}(J)$. We call $\sigma \in G^*$ a *trivial annihilator* of f if $\sigma \in N_1(f)$. Otherwise σ is a *non-trivial annihilator* of f. All annihilators correspond to maximal elements in both graphs of f. Intuitively, the classes of trivial annihilators lie above H in both graphs of f. We recall that every ordered part of a generator is also a generator [LT17, Proposition 6.3]. If $f(\sigma,\tau) = 1$ for some $\sigma, \tau \in G^*$, then $g_\sigma g_\tau = g_{\sigma\tau}$. We set $A_{f_I} = \sum_{\sigma\in G} Ly_\sigma$, where $\{y_\sigma : \sigma \in G\}$ is an L-basis of A_{f_I}.

LEMMA 3.3. *If $I \lhd A_f$ and $x_\rho \in I$, then ρ is a trivial annihilator of f_I.*

Proof. From Proposition 3.2 we know that $\rho \in N_1(f_I)$. The only generator of ρ with respect to f_I is (ρ) (see [LT17, Remark 6.4]). It remains to prove that y_ρ is an annihilator of J_{f_I}. Suppose it is not. We extend (ρ) to a generator (with respect to f_I) of an annihilator, say $g_\tau = (\rho_1, \ldots, \rho, \ldots, \rho_k) \in \Gamma_{f_I}$, $k \geq 1$ [or $(\rho, \rho_1, \ldots, \rho_k)$ or $(\rho_1, \ldots, \rho_k, \rho)$]. Since $y_{\rho_1} \ldots y_\rho \ldots y_{\rho_k} = y_\tau$, from Remark 2.4 it follows that $x_{\rho_1} \ldots x_\rho \ldots x_{\rho_k} = x_\tau$. But $x_\rho \in I$, so $x_\tau \in I$. From Proposition 3.2 it would follow that $\tau \in N_1(f_I)$ and so the only generator of τ with respect to f_I would be (τ), a contradiction. ∎

THEOREM 3.4. *Let $I \lhd A_f$.*

(i) *Let $a : G \to L$ be the function defined by*

$$a(\sigma) = \begin{cases} 1 & \text{if } x_\sigma \notin I, \\ 0 & \text{if } x_\sigma \in I. \end{cases}$$

Then the function $\phi : A_f \to A_{f_I}$ defined by $\phi(x_\sigma) = a(\sigma)y_\sigma$ and extended by linearity is a K-algebra homomorphism with $\ker(\phi) = I$.

(ii) *The function $\psi : A_{f_I} \to A_f/I$ defined by*

$$\psi\Big(\sum_{\sigma\in G} l_\sigma y_\sigma\Big) = \sum_{\sigma\in G} l_\sigma \overline{x}_\sigma,$$

where $\overline{x}_\sigma = x_\sigma + I$, is a K-algebra epimorphism with kernel $\sum_{x_\sigma\in I} Ly_\sigma$,

(iii) *$A_{f_I} \cong (A_f/I) \oplus I$ as L-modules.*

Proof. (i) If $I = 0$, then $f_I = f$, $a(\sigma) = 1$ for $\sigma \in G$ and the statement is true. So suppose that $I \neq 0$. First we prove that for $\sigma, \tau \in G$,

$$f(\sigma,\tau)a(\sigma\tau) = a(\sigma)a(\tau)f_I(\sigma,\tau).$$

If $\sigma \in H$, then $f(\sigma,\tau) = f_I(\sigma,\tau) = 1$ and $a(\sigma) = 1$. In this case $a(\sigma\tau) = a(\tau)$, since both $x_{\sigma\tau} = x_\sigma x_\tau, x_\tau$ are elements of I or both are not. So the equality holds. Similarly if $\tau \in H$. Next suppose that $\sigma, \tau \in G^*$. Suppose that $f(\sigma,\tau) = 1$. If $x_{\sigma\tau} = x_\sigma x_\tau \notin I$, then $x_\sigma, x_\tau \notin I$, $a(\sigma) = a(\tau) = a(\sigma\tau) = 1$ and the equality is true. If $x_{\sigma\tau} \in I$, then $a(\sigma\tau) = 0$, $f_I(\sigma,\tau) = 0$ and again the equality is true. But $\phi(x_\sigma x_\tau) = f(\sigma,\tau)a(\sigma\tau)y_{\sigma\tau}$ and $\phi(x_\sigma)\phi(x_\tau) = a(\sigma)\sigma(a(\tau))f_I(\sigma,\tau)y_{\sigma\tau}$ so ϕ is K-algebra homomorphism.

For the kernel of ϕ we see that if $x = \sum l_\sigma x_\sigma \in I$ for $\sigma \in G$ with $l_\sigma \neq 0$ for every σ, then $x_\sigma \in I$ and so $a(\sigma) = 0$. Hence $\phi(x) = \sum l_\sigma a(\sigma)y_\sigma = 0$ and so $x \in \ker(\phi)$. Also, if $x = \sum l_\sigma x_\sigma \in \ker(\phi)$ with $l_\sigma \neq 0$ for every σ, then $\phi(x) = \sum l_\sigma a(\sigma)y_\sigma = 0$, from which $a(\sigma) = 0$ for every σ, and so $x_\sigma \in I$ for every σ. It follows that $x \in I$.

(ii) ψ is obviously an L-module homomorphism. Let $y_\sigma, y_\tau \in A_{f_I}$. Then $\psi(y_\sigma y_\tau) = f_I(\sigma,\tau)x_{\sigma\tau} + I$ and $\psi(y_\sigma)\psi(y_\tau) = \overline{x}_\sigma\overline{x}_\tau = f(\sigma,\tau)x_{\sigma\tau} + I$. If $f_I(\sigma,\tau) = 1$, then $f(\sigma,\tau) = 1$ and so $\psi(y_\sigma y_\tau) = \psi(y_\sigma)\psi(y_\tau) = x_{\sigma\tau} + I$. Suppose next that $f_I(\sigma,\tau) = 0$. If $x_{\sigma\tau} \in I$, then $\psi(y_\sigma y_\tau) = \psi(y_\sigma)\psi(y_\tau) = 0$. If $x_{\sigma\tau} \notin I$, then by the definition of f_I we must have $f(\sigma,\tau) = 0$ and so again $\psi(y_\sigma y_\tau) = \psi(y_\sigma)\psi(y_\tau) = 0$. Finally,

$$\begin{aligned}\ker(\psi) &= \Big\{\sum_{\sigma\in G} l_\sigma y_\sigma : \sum_{\sigma\in G} l_\sigma \overline{x}_\sigma = 0\Big\}\\ &= \Big\{\sum_{\sigma\in G} l_\sigma y_\sigma : l_\sigma = 0 \text{ or } x_\sigma \in I\Big\} = \sum_{x_\sigma\in I} L y_\sigma.\end{aligned}$$

(iii) Consider the function $i : A_f \to A_{f_I}$ defined by

$$i\Big(\sum_{\sigma\in G} l_\sigma x_\sigma\Big) = \sum_{\sigma\in G} l_\sigma y_\sigma,$$

which is obviously an L-module isomorphism. Then the sequence

$$0 \to I \xrightarrow{i} A_{f_I} \underset{\theta}{\overset{\psi}{\rightleftarrows}} A_f/I \to 0$$

is exact, ψ of (ii) is surjective and $\ker(\psi) = \sum_{x_\sigma\in I} L y_\sigma = i(I)$. Let $\theta : A_f/I \to A_{f_I}$ be the identity homomorphism when A_f/I is viewed as a subalgebra of A_{f_I} through ϕ of (i). Then, for any $\sigma \in G$, $\psi\circ\theta(\overline{x}_\sigma) = \psi(a(\sigma)y_\sigma) = a(\sigma)\overline{x}_\sigma = \overline{x}_\sigma$. So $\psi\circ\theta$ is the identity map of A_f/I, and the above sequence is split. The result follows. ∎

REMARK 3.5. Let $\phi \in \mathrm{End}_K(A_f)$. From Theorem 3.4, by selecting $I = \ker(\phi)$ it follows that $\phi(A_f) \cong A_f/\ker(\phi) = A_f/\ker(\phi) \cong \phi(A_f)$. So $\phi(A_f)$ can be viewed as a subalgebra of $A_{f_{\ker(\phi)}}$, hence $A_{f_{\ker(\phi)}} = \phi(A_f) \oplus \ker(\phi)$.

3.2. Ideals of A_{f_I}. The idempotent 2-cocycle of Definition 2.2 involves a finite family of ideals. It turns out that to study it, we can focus only on two types of idempotent 2-cocycles, namely $f_{\{J,I\}}$ and $f_{\{I,0\}}$. The next proposition is a key ingredient to this claim. Let $\{I_i\}_{i=1}^k$ be a finite descending sequence of ideals of A_f. Then $\{I_i/I_k\}_{i=1}^k$ is a finite descending sequence of ideals of A_f/I_k. We set $A_k = A_{f_{I_k}} = \sum_{\sigma\in G} Ly_\sigma$. Let $\psi_k : A_k \to A_f/I_k$ be the epimorphism of Theorem 3.4(ii).

PROPOSITION 3.6. *In the above notation,*

$$f_{\{I_1,\dots,I_k\}} = (f_{I_k})_{\{P_1,\dots,P_{k-1},P_k=\ker(\psi_k)\}},$$

where $\{P_i\}_{i=1}^k$ *is a finite descending sequence of ideals of* A_k *such that* $\psi_k(P_i) = I_i/I_k$ *for* $1 \le i \le k$.

Proof. Let $A_f = \sum_{\sigma\in G} Lx_\sigma$. We set

$$f_1 = f_{\{I_1,\dots,I_k\}}, \qquad f_2 = (f_{I_k})_{\{P_1,\dots,P_{k-1},P_k=\ker(\psi_k)\}}$$

and $H = H(f) = H(f_1) = H(f_{I_k}) = H(f_2)$. Let $\sigma,\tau \in G^*$ be such that $f_2(\sigma,\tau) = 1$. We have $\sigma\tau \in G^*$. Then $f_{I_k}(\sigma,\tau) = 1$ and $y_\sigma, y_\tau, y_{\sigma\tau} \in P_i \setminus P_{i+1}$ for some $1 \le i \le k-1$. So $f(\sigma,\tau) = 1$ and $x_\sigma, x_\tau, x_{\sigma\tau} \in J \setminus I_k$ and $\psi_k(y_\sigma), \psi_k(y_\tau), \psi_k(y_{\sigma\tau}) \in (I_i/I_k) \setminus (I_{i+1}/I_k)$ for some $1 \le i \le k-1$. Then $f(\sigma,\tau) = 1$ and $x_\sigma, x_\tau, x_{\sigma\tau} \notin I_k$ and $x_\sigma + I_k, x_\tau + I_k, x_{\sigma\tau} + I_k \in (I_i/I_k)\setminus(I_{i+1}/I_k)$ for some $1 \le i \le k-1$. So $f(\sigma,\tau) = 1$ and $x_\sigma, x_\tau, x_{\sigma\tau} \in I_i \setminus I_{i+1}$ for some $1 \le i \le k-1$. It follows that $f_1(\sigma,\tau) = 1$. For the opposite direction we note that $\psi_k^{-1}(x_\sigma + I_k) = \{y_\sigma + x : x \in \ker(\psi_k)\}$ and we follow the arguments backwards. ■

LEMMA 3.7. *Let* $I_1 \lhd A_f$ *and* $I_2 = \sum I_\sigma$ *for some* $x_\sigma \in I_1$ *such that* σ *is a trivial annihilator of* f. *Then* $f_{\{I_1,I_2\}} = f_{\{I_1,0\}}$.

Proof. From the identity $f_{\{I_1,I_2+0\}} = f_{\{I_1,I_2\}} f_{\{I_1,0\}}$ of Proposition 2.11(i) it follows that $f_{\{I_1,I_2\}} \le f_{\{I_1,0\}}$. Next let $\sigma,\tau \in G^*$ be such that $f_{\{I_1,0\}}(\sigma,\tau) = 1$. Then $f(\sigma,\tau) = 1$ and $x_\sigma, x_\tau, x_{\sigma\tau} \in I_1$. It could not be $x_\sigma \in I_2$, since x_σ is not an annihilator of J. Similarly $x_\tau \notin I_2$. Finally, since $\sigma\tau \notin N_1(f)$ [LT17, Lemma 3.1], it follows that $x_{\sigma\tau} \notin I_2$. So $f(\sigma,\tau) = 1$ and $x_\sigma, x_\tau, x_{\sigma\tau} \in I_1 \setminus I_2$. From the definition $f_{\{I_1,I_2\}}(\sigma,\tau) = 1$ and so $f_{\{I_1,0\}} \le f_{\{I_1,I_2\}}$. ■

REMARK 3.8. Using Corollary 2.10 we can decompose any $f_{\mathbf{I}} \in E^2(G,L)$ into idempotent 2-cocycles of the type $f_{\{I',I\}}$, which in turn, from Proposition 3.6, take the form $(f_I)_{\{P,ker(\psi)\}}$, where $\psi : A_{f_I} \to A_f/I$ and P is an ideal of A_{f_I} containing $\ker(\psi)$ such that $\psi(P) = I'/I$. From Lemma 3.3 we know that if $x_\sigma \in I$, then σ is a trivial annihilator of f_I. From Theorem 3.4(ii) and Lemma 3.7 we have $(f_I)_{\{P,\ker(\psi)\}} = (f_I)_{\{P,0\}}$. In essence, the only idempotent 2-cocycles that are needed are of the types $f_{\{J,I\}}$ and $f_{\{I,0\}}$.

Another useful fact derived from Proposition 3.6 is that repeated applications of Proposition 3.1 can be avoided.

PROPOSITION 3.9. *Let* $f \in E^2(G,L)$, $I \lhd A_f$, $P \lhd A_{f_I}$ *and* $\psi : A_{f_I} \to A_f/I$ *be the algebra epimorphism of Theorem* 3.4(ii). *Then* $(f_I)_P = f_{I_1}$, *where* I_1 *is an ideal of* A_f *such that* $I_1/I = \psi(P)$.

Proof. We note that J_{f_I} is the unique maximal ideal in A_{f_I}, and so by the lattice isomorphism theorem we have $\psi(J_{f_I}) = J_f/I$, which is the unique maximal ideal in A_f/I.

First we prove the claim for P containing the kernel of ψ. Since $\psi(P)$ is an ideal of A_f/I, there exists an ideal I_1 of A_f containing I such that $\psi(P) = I_1/I$. Since I_1 is nilpotent, let k be the smallest positive integer such that $I_1^{2^k} \subseteq I$. Suppose that $k \geq 2$. Then $\{I_1^{2^i}/I\}_{i=1}^{k-1}$ is a finite descending sequence of ideals of A_f/I. Let $\{P_i\}_{i=1}^{k-1}$ be a finite descending sequence of ideals of A_{f_I} containing $\ker(\psi)$ such that $\psi(P_i) = I_1^{2^i}/I$ for $i = 1, \dots, k-1$. We set $P_0 = P$. Since $(I'/I)^2 \subseteq I'^2/I$ for any ideal $I' \supsetneq I$, for every $i \in \{0, \dots, k-2\}$ we have

$$\psi(P_i^2) \subseteq \psi(P_i)^2 \subseteq (I_1^{2^i}/I)^2 \subseteq (I_1^{2^i})^2/I = I_1^{2^{i+1}}/I = \psi(P_{i+1}).$$

It follows that $P_i^2 \subseteq P_{i+1}$. For $i = k-1$ we have $\psi(P_{k-1}^2) \subseteq (I_1^{2^{k-1}}/I)^2 = 0$ and so $P_{k-1}^2 \subseteq \ker(\psi)$. Then from Propositions 3.6 and 2.7 we have

$$\begin{aligned} f_{\{J_f, I_1, I_1^2, \dots, I_1^{2^{k-1}}, I\}} &= (f_I)_{\{J_{f_I}, P_0, P_1, \dots, P_{k-1}, \ker(\psi)\}} \\ &= (f_I)_{\{J_{f_I}, P_0\}} \vee (f_I)_{\{P_0, P_1\}} \vee \cdots \vee (f_I)_{\{P_{k-1}, \ker(\psi)\}} \\ &= (f_I)_{\{J_{f_I}, P\}} = (f_I)_P. \end{aligned}$$

But $f_{\{J_f, I_1, I_1^2, \dots, I_1^{2^{k-1}}, I\}} = f_{\{J_f, I_1\}} \vee f_{\{I_1, I_1^2\}} \vee \cdots \vee f_{\{I_1^{2^{k-1}}, I\}} = f_{I_1}$ as claimed. The cases for $k = 0$ $[I_1 = I$ and $P = \ker(\psi)]$ and $k = 1$ $[I_1^2 \subseteq I]$ are handled accordingly by omitting the irrelevant terms.

Now suppose that P does not contain $\ker(\psi)$. Then for the ideal $P' = P + \ker(\psi)$, taking into consideration that $(f_I)_{\ker(\psi)} = (f_I)_{\{J_{f_I}, 0\}} = f_I$ as in Remark 3.8, from the first part of the proof and Proposition 2.11(i) we have $(f_I)_P = (f_I)_P (f_I)_{\ker(\psi)} = (f_I)_{P'} = f_{I_1}$ where I_1 is an ideal of A_f such that $I_1/I = \psi(P') = \psi(P)$. ■

4. Decomposition of idempotent 2-cocycles using ideals. Proposition 2.11(ii) together with the next proposition, whose proof is immediate, imply that for every $f \in E^2(G, L; H)$ the subset $\{f_I : I \lhd A_f\}$ is a monoid with respect to $\vee$ with unit element $f_0 = f_{\{J,J\}}$ and zero element $f = f_{\{J,0\}}$.

PROPOSITION 4.1. *Let $\{I_i\}_{i=1}^k$ be a finite family of two-sided ideals of A_f. If $\bigcap_{i=1}^k I_i = \{0\}$, then*

$$f = \bigvee_{i=1}^k f_{I_i}. \ ■$$

Proposition 4.1 leads to the following result.

PROPOSITION 4.2. *Let $I \lhd A_f$. Then $f_I = f$ if and only if $I = 0$ or $I = \sum I_\sigma$ for some trivial annihilators $\sigma \in G^*$ of f.*

Proof. First suppose that $f_I = f$. Since $H(f_I) = H(f)$, from the definition it follows that $x_{\sigma\tau} \notin I$ for every $\sigma, \tau \notin H$ such that $f(\sigma, \tau) = 1$. Suppose that $I \neq 0$ and let $x_\rho \in I$ for $\rho \in G^*$. We notice that if $\rho \notin N_1(f)$, then there exist $\sigma, \tau \in G^*$ such that $\rho = \sigma\tau$ with $f(\sigma, \tau) = 1$. But $x_\rho = x_{\sigma\tau} \in I_\rho \subseteq I$ and so $f_I(\sigma, \tau) = 0$, contrary to the assumption that $f_I = f$. So $\rho \in N_1(f)$. If ρ is not an annihilator of f, then there exists $\tau \in G^*$ such that $f(\rho, \tau) = 1$ or $f(\tau, \rho) = 1$. Consider the first case (the second is handled similarly). We have $x_{\rho\tau} = x_\rho x_\tau \in I$ and so $f_I(\rho, \tau) = 0$, again contrary to assumption. It follows that ρ is a trivial annihilator of f for every $x_\rho \in I$. Since $I = \sum_{x_\rho \in I} I_\rho$, we are done. The opposite direction is a direct consequence of Lemma 3.7 and the definition of f_I. ∎

REMARK 4.3. The relation $\sigma \sim \tau \Leftrightarrow \sigma \in H\tau H$ is an equivalence relation on G. We denote by $[\sigma]$ the class of $\sigma \in G$. Let A be the subset of G such that for every $\sigma \in A$, x_σ is an annihilator of J. Then A is not empty. If $\sigma \in A$, then from Proposition 2.1 we have $H\sigma H \subseteq A$. So $A = \biguplus_{i=1}^{k} H\sigma_i H$, where $A' = \{\sigma_1, \ldots, \sigma_k\}$ is a complete set of representatives of the classes of A.

PROPOSITION 4.4. *Let $f \in E^2(G, L)$. For every $\rho \in G^*$ such that $\rho \notin N_1(f)$, there exists an ideal I of A_f such that $[\rho]$ is the unique class of non-trivial annihilators of f_I, with respect to the equivalence relation of Remark* 4.3.

Proof. Let $I = \sum_{x_\rho \notin I_\sigma} I_\sigma$. We remark that $x_\rho \notin I$. First we prove that ρ is an annihilator of f_I. For this, let $\tau \in G^*$. If $f(\rho, \tau) = 0$, then $f_I(\rho, \tau) = 0$ (Remark 2.4). Suppose that $f(\rho, \tau) = 1$. Then $x_\rho \notin I_{\rho\tau}$ and so $x_{\rho\tau} \in I$. From the definition of f_I it follows that $f_I(\rho, \tau) = 0$. Similarly we prove that $f_I(\tau, \rho) = 0$ for every $\tau \in G^*$. Since by the assumption $\rho \notin N_1(f)$ and also $x_\rho \notin I$, from Proposition 3.2 it follows that $\rho \notin N_1(f_I)$ and so ρ is a non-trivial annihilator of f_I.

Finally we prove that $[\rho]$ is the unique class of non-trivial annihilators of f_I. Suppose there exists $\sigma \in G^*$ such that σ is a non-trivial annihilator of f_I with $[\rho] \neq [\sigma]$. If we had $x_\sigma \notin I$, then $x_\rho \in I_\sigma$. So there exist $\sigma_1, \sigma_2 \in G$ such that $x_{\sigma_1} x_\sigma x_{\sigma_2} = x_\rho$. If both σ_1, σ_2 are in H, then $[\rho] = [\sigma]$ contrary to assumption. So at least one of the two is an element of G^*, say σ_1. Then $f(\sigma_1, \sigma) = 1$ and $x_{\sigma_1\sigma} \notin I$ (otherwise, $x_\rho \in I$, impossible). So $f_I(\sigma_1, \sigma) = 1$, contrary to the assumption that σ is an annihilator of f_I. If we had $x_\sigma \in I$, then σ would be a trivial annihilator of f_I, again a contradiction. ∎

Aljouiee [A05] studied weak crossed product algebras whose graphs have a unique maximal element, i.e. have no trivial annihilators, and a unique class of non-trivial annihilators. In particular, he showed that such an algebra is Frobenius [A05, Theorem 1.6]. In the next theorem we give a procedure to decompose any idempotent 2-cocyle to idempotent 2-cocycles having a unique class of non-trivial annihilators.

THEOREM 4.5. *Let $f \in E^2(G,L)$, $f \neq f_0$. Then f has a unique class of non-trivial annihilators or there exist ideals $\{I_i\}_{i=1}^k$ such that $f = \bigvee_{i=1}^k f_{I_i}$, $f_0 < f_{I_i} < f$ and each f_{I_i} has a unique class of non-trivial annihilators.*

Proof. Let $A = \{\rho_1, \dots, \rho_k\}$ for some $k \geq 1$ be a complete set of representatives of the classes of elements of G^* such that $x_{\rho_i} \in J^2$ ($A \neq \emptyset$ since $f \neq f_0$). If $k = 1$, then $[\rho_1]$ is the unique class of non-trivial annihilators. So suppose that $k \geq 2$. We set $I_i = \sum_{x_{\rho_i} \notin I_\sigma} I_\sigma$. From Proposition 4.4, f_{I_i} has $[\rho_i]$ as the unique class of non-trivial annihilators for every $i = 1, \dots, k$. If $\bigcap_{i=1}^k I_i = \{0\}$, then by Proposition 4.1, $f = \bigvee_{i=1}^k f_{I_i}$. If $\bigcap_{i=1}^k I_i \neq \{0\}$, then let $x_\tau \in \bigcap_{i=1}^k I_i$. If $x_\tau \in J^2$, then $\tau \in [\rho_j]$ for some $j \in \{1, \dots, k\}$ and so $\tau = h_1 \rho_j h_2$, $h_1, h_2 \in H$. Since $x_{\rho_j} \notin I_j$, we have $x_{h_1} x_{\rho_j} x_{h_2} = x_\tau \notin I_j$, which contradicts the assumption that $x_\tau \in \bigcap_{i=1}^k I_i$. Hence $x_\tau \notin J^2$ and so $\tau \in N_1(f)$. If τ is not an annihilator of f, then there exists $\rho \in G^*$ such that $f(\tau, \rho) = 1$ [or $f(\rho, \tau) = 1$]. Then $x_{\tau\rho} \in J^2$ and so $x_\tau \notin I_a$ for some a. But this contradicts the choice of x_τ. It follows that for every $x_\tau \in \bigcap_{i=1}^k I_i$, τ is a trivial annihilator of f. From Propositions 2.11 and 4.2 we have

$$\bigvee_{i=1}^k f_{I_i} = f_{\bigcap_{i=1}^k I_i} = f_{\sum I_\tau} = f.$$

We now prove that $f_0 < f_{I_i} < f$ for every $i = 1, \dots, k$. We know that $f_I \leq f$, for any ideal I. First suppose that ρ_i is a non-trivial annihilator of f. Since the class $[\rho_i]$ is not unique, let ρ be another non-trivial annihilator such that $\rho \notin [\rho_i]$. Then from Proposition 2.1(ii) we get $x_{\rho_i} \notin I_\rho$ and so $x_\rho \in I_i$. Since $\rho \notin N_1(f)$, it follows that there exist $\sigma_1, \sigma_2 \in G^*$ such that $f(\sigma_1, \sigma_2) = 1$ with $\sigma_1 \sigma_2 = \rho$. But then $f_{I_i}(\sigma_1, \sigma_2) = 0$, which proves that $f_{I_i} < f$. If ρ_i is not an annihilator where $\rho_i \in N_k(f)$ for some $k \geq 2$, then there exists $\tau \in G^*$ such that $f(\rho_i, \tau) = 1$ [or $f(\tau, \rho_i) = 1$]. Then $x_{\rho_i} \notin I_{\rho_i \tau}$ and so $x_{\rho_i \tau} \in I_i$. Hence $f_{I_i}(\rho_i, \tau) = 0$, and so again $f_{I_i} < f$.

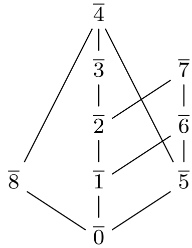
Finally, since $x_{\rho_i} \notin I_i$ and $x_{\rho_i} \in J^2$, it follows that $J^2 \subsetneq I_i$ and so, from Proposition 2.7, we have $f_{I_i} > f_0$. ∎

EXAMPLE 4.6. Let $G = \mathbb{Z}/9\mathbb{Z}$ and $r = \{0,1,2,3,4,1,2,3,3\} \in \mathrm{Sl}(G)$ with set of generators

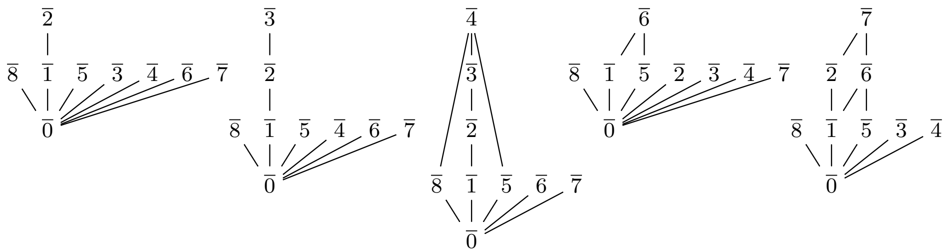
$$\begin{aligned} \Gamma_{f_r} = \big\{ &\{(\bar{1})\}, \{(\bar{1},\bar{1})\}, \{(\bar{1},\bar{1},\bar{1})\}, \{(\bar{5},\bar{8}), (\bar{8},\bar{5}), (\bar{1},\bar{1},\bar{1},\bar{1})\}, \{(\bar{5})\}, \\ &\{(\bar{1},\bar{5}), (\bar{5},\bar{1})\}, \{(\bar{1},\bar{1},\bar{5}), (\bar{1},\bar{5},\bar{1}), (\bar{5},\bar{1},\bar{1})\}, \{(\bar{8})\} \big\}. \end{aligned}$$

The corresponding table of values and the graph of f_r are

$$\begin{vmatrix} 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&0&1&1&1&0&0\\ 1&1&1&0&0&1&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&1\\ 1&1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&0&0&0 \end{vmatrix}$$



The representatives of the classes of the non-trivial annihilators of f_r are $\{\overline{4},\overline{7}\}$. We have $A=\{\overline{2},\overline{3},\overline{4},\overline{6},\overline{7}\}$. In the terminology of Theorem 4.5, for $\rho_1=\overline{2}$, $I_1=\sum_{x_{\overline{2}}\notin I_\sigma} I_\sigma = Lx_{\overline{3}}+Lx_{\overline{4}}+Lx_{\overline{5}}+Lx_{\overline{6}}+Lx_{\overline{7}}+Lx_{\overline{8}}$. Similarly for $\rho_2=\overline{3}$, $I_2=Lx_{\overline{4}}+Lx_{\overline{5}}+Lx_{\overline{6}}+Lx_{\overline{7}}+Lx_{\overline{8}}$; for $\rho_3=\overline{4}$, $I_3=Lx_{\overline{6}}+Lx_{\overline{7}}$; for $\rho_4=\overline{6}$, $I_4=Lx_{\overline{2}}+Lx_{\overline{3}}+Lx_{\overline{4}}+Lx_{\overline{7}}+Lx_{\overline{8}}$; and for $\rho_5=\overline{7}$, $I_5=Lx_{\overline{3}}+Lx_{\overline{4}}+Lx_{\overline{8}}$. We know that $f_r=\bigvee_{i=1}^{5} f_{I_i}$. Each f_{I_i}, $i=1,\dots,5$, has $[\rho_i]$ as the unique class of non-trivial annihilators. The corresponding graphs are



5. Cartesian product of elements of $\mathrm{Sl}(G)$**.** In this section we specialize the previous results in the case where $f=f_r$ for some $r\in\mathrm{Sl}(G)$ taking values in some Ω as in the introduction. We denote by $\mathrm{Sl}(G;H)$ the elements of $\mathrm{Sl}(G)$ with $M_r=H$.

Definition 5.1. Let $r\in\mathrm{Sl}(G;H)$ take values in Ω, and $\mathbf{I}=\{I_i\}_{i=1}^{k}$, $k\geq 2$, be a finite sequence of descending ideals of A_{f_r}. Let $r_{\mathbf{I}}:G\to \times^{k+1}\Omega$ be the function defined by

$$r_{\mathbf{I}}(\sigma)=\begin{cases}(\underbrace{r(\sigma),\dots,r(\sigma)}_{k+1\,\text{times}}), & x_\sigma\notin I_1,\\ (\underbrace{r(\sigma),\dots,r(\sigma)}_{k-a+1\,\text{times}},\underbrace{1,\dots,1}_{a\,\text{times}}), & x_\sigma\in I_a\setminus I_{a+1},\ 1\leq a\leq k-1,\\ (r(\sigma),\underbrace{1,\dots,1}_{k\,\text{times}}), & x_\sigma\in I_k.\end{cases}$$

Let $\{\Omega_i, \leq_i\}_{i=1}^k$ be a finite family of multiplicative totally ordered monoids with minimum elements. Then the cartesian product $\times_{i=1}^k \Omega_i = \Omega_1 \times \cdots \times \Omega_k$ is a multiplicative monoid with minimum element $1 = (1_{\Omega_1}, \ldots, 1_{\Omega_k})$ totally ordered by the lexicographic relation $(x_1, \ldots, x_k) \leq (y_1, \ldots, y_k)$ if and only if either $(x_1, \ldots, x_k) = (y_1, \ldots, y_k)$ or there exists $a \in \{1, \ldots, k\}$ such that, for any $i < a$, ($x_i = y_i$ and $x_a < y_a$). If each Ω_i, $i = 1, \ldots, k$, satisfies the relations mentioned in the introduction, then so does Ω.

THEOREM 5.2. *The function in Definition* 5.1 *is an element of* $\mathrm{Sl}(G; H)$.

Proof. As $x_h \notin I_1$ for $h \in H$, we have $r_{\mathbf{I}}(h) = 1$. In particular, $r_{\mathbf{I}}(1) = 1$. For ease of calculations we set $I_0 = J_{f_r}$ and $I_{k+1} = \{0\}$. Then $r_{\mathbf{I}}$ takes the form

$$r_{\mathbf{I}}(\sigma) = \begin{cases} (\underbrace{1, \ldots, 1}_{k+1 \text{ times}}), & \sigma \in H, \\ (\underbrace{r(\sigma), \ldots, r(\sigma)}_{k-a+1 \text{ times}}, \underbrace{1, \ldots, 1}_{a \text{ times}}), & x_\sigma \in I_a \setminus I_{a+1},\ 0 \leq a \leq k. \end{cases}$$

First we show that $r_{\mathbf{I}}(h\sigma) = r_{\mathbf{I}}(\sigma h) = r_{\mathbf{I}}(\sigma)$ for $\sigma \in G$ and $h \in H$. If $\sigma \in H$, then $h\sigma, \sigma h \in H$ and so $r_{\mathbf{I}}(h\sigma) = r_{\mathbf{I}}(\sigma h) = r_{\mathbf{I}}(\sigma) = 1$. If $x_\sigma \in I_a \setminus I_{a+1}$, $0 \leq a \leq k$, then also $x_{h\sigma}, x_{\sigma h} \in I_a \setminus I_{a+1}$ and hence

$$r_{\mathbf{I}}(h\sigma) = (\underbrace{r(h\sigma), \ldots, r(h\sigma)}_{k-a+1 \text{ times}}, \underbrace{1, \ldots, 1}_{a \text{ times}}) = r_{\mathbf{I}}(\sigma);$$

similarly $r_{\mathbf{I}}(\sigma h) = r_{\mathbf{I}}(\sigma)$.

Now we show that $r_{\mathbf{I}}(\sigma\tau) \leq r_{\mathbf{I}}(\sigma) r_{\mathbf{I}}(\tau)$ for $\sigma, \tau \in G$. If $\sigma \in H$, then $r_{\mathbf{I}}(\sigma\tau) = r_{\mathbf{I}}(\tau) = r_{\mathbf{I}}(\sigma) r_{\mathbf{I}}(\tau)$, and similarly if $\tau \in H$. If $\sigma\tau \in H$, then $r_{\mathbf{I}}(\sigma\tau) = 1 \leq r_{\mathbf{I}}(\sigma) r_{\mathbf{I}}(\tau)$. Next let $\sigma, \tau, \sigma\tau \in G^*$. For $\rho \in G^*$ we set $s(\rho) = \{a \in \mathbb{N} : x_\rho \in I_a \setminus I_{a+1},\ 0 \leq a \leq k\}$. We distinguish two cases. First, if $r(\sigma\tau) < r(\sigma) r(\tau)$ for $\sigma, \tau \in G^*$, then

$$r_{\mathbf{I}}(\sigma\tau) \leq (r(\sigma\tau), \ldots, r(\sigma\tau)) < (r(\sigma) r(\tau), 1, \ldots, 1) \leq r_{\mathbf{I}}(\sigma) r_{\mathbf{I}}(\tau).$$

Next suppose that $r(\sigma\tau) = r(\sigma) r(\tau)$ for $\sigma, \tau \in G^*$. Then $f_r(\sigma, \tau) = 1$. Since $x_\sigma \in I_{s(\sigma)}$, it follows that $x_\sigma x_\tau = x_{\sigma\tau} \in I_{s(\sigma)}$, and so $s(\sigma) \leq s(\sigma\tau)$ and similarly $s(\tau) \leq s(\sigma\tau)$. We thus have

$$r_{\mathbf{I}}(\sigma\tau) = (\underbrace{r(\sigma\tau), \ldots, r(\sigma\tau)}_{k-s(\sigma\tau)+1}, \underbrace{1, \ldots, 1}_{s(\sigma\tau)}) = (\underbrace{r(\sigma)r(\tau), \ldots, r(\sigma)r(\tau)}_{k-s(\sigma\tau)+1}, \underbrace{1, \ldots, 1}_{s(\sigma\tau)}).$$

Also, for $\sigma, \tau \in G$,

$$r_{\mathbf{I}}(\sigma) r_{\mathbf{I}}(\tau) = (\underbrace{r(\sigma), \ldots, r(\sigma)}_{k-s(\sigma)+1}, \underbrace{1, \ldots, 1}_{s(\sigma)})(\underbrace{r(\tau), \ldots, r(\tau)}_{k-s(\tau)+1}, \underbrace{1, \ldots, 1}_{s(\tau)}).$$

Again we distinguish two cases, for $\sigma, \tau \in G^*$:

(a) $s(\sigma\tau) > \max\{s(\sigma), s(\tau)\}$. Then $k-s(\sigma\tau)+1 < k-s(\sigma)+1$ and $k-s(\sigma\tau)+1 < k-s(\tau)+1$. Since $1 < r(\sigma)r(\tau)$, it follows that $r_{\mathbf{I}}(\sigma\tau) < r_{\mathbf{I}}(\sigma)r_{\mathbf{I}}(\tau)$.
(b) $s(\sigma\tau) = \max\{s(\sigma), s(\tau)\}$. If $s(\sigma) = s(\tau)$, then $k-s(\sigma\tau)+1 = k-s(\sigma)+1$ and so $r_{\mathbf{I}}(\sigma\tau)=r_{\mathbf{I}}(\sigma)r_{\mathbf{I}}(\tau)$. If $s(\sigma)>s(\tau)$, then $k-s(\sigma\tau)+1 = k-s(\sigma)+1$ and $s(\sigma)-s(\tau) > 0$. Since $1 < r(\tau)$, it follows that $r_{\mathbf{I}}(\sigma\tau) < r_{\mathbf{I}}(\sigma)r_{\mathbf{I}}(\tau)$. Similarly if $s(\sigma) < s(\tau)$.

Finally, if $\sigma \in M_{r_{\mathbf{I}}}$, then $r_{\mathbf{I}}(\sigma) = 1$. By definition, the only possibility is $r(\sigma) = 1$ and so $\sigma \in H$. ∎

Corollary 5.3. *Let $\mathbf{I} = \{I_i\}_{i=1}^k$, $k \ge 2$, be a finite descending sequence of ideals of A_{f_r}. Then $(f_r)_{\mathbf{I}} \le f_{r_{\mathbf{I}}} \le f_r$.*

Proof. We set $(f_r)_{\mathbf{I}}=f'$. We set $H=H(f')=H(f_r)=M_r=M_{r_{\mathbf{I}}}=H(f_{r_{\mathbf{I}}})$. Let $\sigma,\tau \in G^*$ be such that $f'(\sigma,\tau) = 1$. Then $f_r(\sigma,\tau) = 1$ and $x_\sigma, x_\tau, x_{\sigma\tau} \in I_a \setminus I_{a+1}$ for some $1 \le a \le k-1$. Thus $r(\sigma\tau) = r(\sigma)r(\tau)$

$$r_{\mathbf{I}}(\sigma\tau) = (\underbrace{r(\sigma),\dots,r(\sigma)}_{k-a+1 \text{ times}},\underbrace{1,\dots,1}_{a \text{ times}})(\underbrace{r(\tau),\dots,r(\tau)}_{k-a+1 \text{ times}},\underbrace{1,\dots,1}_{a \text{ times}}) = r_{\mathbf{I}}(\sigma)r_{\mathbf{I}}(\tau).$$

So $f_{r_{\mathbf{I}}}(\sigma,\tau) = 1$, which proves the first part of the inequality. For the second part, if $f_r(\sigma,\tau) = 0$, then $r(\sigma\tau) < r(\sigma)r(\tau)$ and as in the proof of Theorem 5.2, we deduce that $r_{\mathbf{I}}(\sigma\tau) < r_{\mathbf{I}}(\sigma)r_{\mathbf{I}}(\tau)$ and so $f_{r_{\mathbf{I}}}(\sigma,\tau) = 0$. Therefore $f_{r_{\mathbf{I}}} \le f_r$. ∎

Proposition 5.4. *Let $\mathbf{I} = \{I_i\}_{i=1}^k$, $k \ge 2$, be a finite descending sequence of ideals of A_{f_r} such that $I_1 = J_f$ and $I_k = 0$. Then $(f_r)_{\mathbf{I}} = f_{r_{\mathbf{I}}}$.*

Proof. We set $f_1 = (f_r)_{\mathbf{I}}$ and $f_2 = f_{r_{\mathbf{I}}}$. We must show that $f_2 \le f_1$. Let $\sigma,\tau \in G$ be such that $f_2(\sigma,\tau) = 1$. Let $\sigma,\tau \in G^*$. We have $r_{\mathbf{I}}(\sigma\tau) = r_{\mathbf{I}}(\sigma)r_{\mathbf{I}}(\tau)$. From the proof of Theorem 5.2 (case (b)), we deduce that $r(\sigma\tau) = r(\sigma)r(\tau)$ and $x_\sigma,x_\tau,x_{\sigma\tau} \in J_f \setminus I_1$ or $x_\sigma,x_\tau,x_{\sigma\tau} \in I_a \setminus I_{a+1}$ for some $1 \le a \le k-1$ or $x_\sigma,x_\tau,x_{\sigma\tau} \in I_k$. Since $I_1 = J_f$ and $I_k = 0$, the only possibility is $x_\sigma,x_\tau,x_{\sigma\tau} \in I_a \setminus I_{a+1}$ for some $1 \le a \le k-1$. So $f_r(\sigma,\tau) = 1$ and, by Definition 2.2, $f_1(\sigma,\tau) = 1$ as required. ∎

Theorem 5.5. *Let $r \in \mathrm{Sl}(G)$ in any Ω and let $\mathbf{I} = \{I_i\}_{i=1}^k$ be a finite descending sequence of ideals of A_f. There exists $r' \in \mathrm{Sl}(G)$ in $\times^l\Omega$, $l \in \mathbb{N}^*$, such that $(f_r)_{\mathbf{I}} = f_{r'}$.*

Proof. Since J, I_k are nilpotent, let a,b be the smallest positive integers such that $J^{2^a} \subseteq I_1$ and $I_k^{2^b} = 0$, $a \ge 2$ and $b \ge 1$. From Proposition 2.7 we notice that $J^2 \subseteq J^2 + I_1$ and so $(f_r)_{\{J,J^2+I_1\}} = f_0$. Also $(J^{2^{i-1}} + I_1)^2 \subseteq (J^{2^{i-1}})^2 + I_1 = J^{2^i} + I_1$ for $i \in \{2,\dots,a\}$, and so $(f_r)_{\{J^{2^{i-1}}+I_1,J^{2^i}+I_1\}} = f_0$. Finally, $(I_k^{2^{i-1}})^2 \subseteq I_k^{2^i}$ for $i \in \{1,\dots,b\}$, and so $(f_r)_{\{I_k^{2^{i-1}},I_k^{2^i}\}} = f_0$. Then from Corollary 2.10 we have

$$\begin{aligned}(f_r)_{\{J,J^2+I_1,\dots,J^{2^{a-1}}+I_1,I_1,\dots,I_k,I_k^2,\dots,I_k^{2^{b-1}},I_k^{2^b}\}} \\ &= (f_r)_{\{J,J^2+I_1\}} \vee \dots \vee (f_r)_{\{J^{2^{a-1}}+I_1,I_1\}} \\ &\quad \vee (f_r)_{\{I_1,\dots,I_k\}} \vee (f_r)_{\{I_k,I_k^2\}} \vee \dots \vee (f_r)_{\{I_k^{2^{b-1}},0\}} \\ &= (f_r)_{\mathbf{I}}.\end{aligned}$$

From Proposition 5.4 it follows that the theorem is true for

$$r' = r_{\{J,J^2+I_1,\dots,J^{2^{a-1}}+I_1,I_1,\dots,I_k,I_k^2,\dots,I_k^{2^{b-1}},0\}}.$$

There are $a+k+b-2$ ideals, so $l = a+k+b-1$. If $a=0$ ($J=I_1$) or $a=1$ ($J^2 \subseteq I_1$) or $b=0$ ($I_k=0$), the proof is identical by omitting the irrelevant terms. ■

Example 5.6. We return to Example 4.6. We notice that $I_3^2 = I_5^2 = 0$, $I_1^2 = I_2^2 = I_4^2 = Lx_{\overline{4}} \neq 0$ and $I_1^4 = I_2^4 = I_4^4 = 0$. We set $r_i = r_{\{J,I_i,0\}}$ for $i = 3,5$ and $r_i = r_{\{J,I_i,I_i^2,0\}}$ for $i = 1,2,4$. Then $f_{I_i} = (f_r)_{\{J,I_i\}} = (f_r)_{\{J,I_i,0\}} = f_{r_i}$ for $i = 3,5$, and $f_{I_i} = (f_r)_{\{J,I_i,I_i^2,0\}} = f_{r_i}$ for $i = 1,2,4$. More specifically, for $i = 1$, $x_{\overline{1}}, x_{\overline{2}} \in J \setminus I_1$, $x_{\overline{3}}, x_{\overline{5}}, x_{\overline{6}}, x_{\overline{7}}, x_{\overline{8}} \in I_1 \setminus I_1^2$, $x_{\overline{4}} \in I_1^2$; for $i = 2$, $x_{\overline{1}}, x_{\overline{2}}, x_{\overline{3}} \in J \setminus I_2$, $x_{\overline{5}}, x_{\overline{6}}, x_{\overline{7}}, x_{\overline{8}} \in I_2 \setminus I_2^2$, $x_{\overline{4}} \in I_2^2$; for $i = 3$, $x_{\overline{1}}, x_{\overline{2}}, x_{\overline{3}}, x_{\overline{4}}, x_{\overline{5}}, x_{\overline{8}} \in J \setminus I_3$, $x_{\overline{6}}, x_{\overline{7}} \in I_3$; for $i = 4$, $x_{\overline{1}}, x_{\overline{5}}, x_{\overline{6}} \in J \setminus I_4$, $x_{\overline{2}}, x_{\overline{3}}, x_{\overline{7}}, x_{\overline{8}} \in I_4 \setminus I_4^2$, $x_{\overline{4}} \in I_4^2$; and for $i = 5$, $x_{\overline{1}}, x_{\overline{2}}, x_{\overline{5}}, x_{\overline{6}}, x_{\overline{7}} \in J \setminus I_5$, $x_{\overline{3}}, x_{\overline{4}}, x_{\overline{8}} \in I_5$. So we have the following table:

σ	$r(\sigma)$	$r_1(\sigma)$	$r_2(\sigma)$	$r_3(\sigma)$	$r_4(\sigma)$	$r_5(\sigma)$
$\overline{0}$	0	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0,0)$	$(0,0,0,0)$
$\overline{1}$	1	$(1,1,1,1,0)$	$(1,1,1,1,0)$	$(1,1,1,0)$	$(1,1,1,1,0)$	$(1,1,1,0)$
$\overline{2}$	2	$(2,2,2,2,0)$	$(2,2,2,2,0)$	$(2,2,2,0)$	$(2,2,2,0,0)$	$(2,2,2,0)$
$\overline{3}$	3	$(3,3,3,0,0)$	$(3,3,3,3,0)$	$(3,3,3,0)$	$(3,3,3,0,0)$	$(3,3,0,0)$
$\overline{4}$	4	$(4,4,0,0,0)$	$(4,4,0,0,0)$	$(4,4,4,0)$	$(4,4,0,0,0)$	$(4,4,0,0)$
$\overline{5}$	1	$(1,1,1,0,0)$	$(1,1,1,0,0)$	$(1,1,1,0)$	$(1,1,1,1,0)$	$(1,1,1,0)$
$\overline{6}$	2	$(2,2,2,0,0)$	$(2,2,2,0,0)$	$(2,2,0,0)$	$(2,2,2,2,0)$	$(2,2,2,0)$
$\overline{7}$	3	$(3,3,3,0,0)$	$(3,3,3,0,0)$	$(3,3,0,0)$	$(3,3,3,0,0)$	$(3,3,3,0)$
$\overline{8}$	3	$(3,3,3,0,0)$	$(3,3,3,0,0)$	$(3,3,3,0)$	$(3,3,3,0,0)$	$(3,3,0,0)$

6. The ideal I_g. Once we decompose f into idempotent 2-cocycles having a unique class of non-trivial annihilators, we can proceed further based on the different generators of those non-trivial annihilators. Let B_f^* be the set of generators of the elements of G^* with respect to f which are non-trivial annihilators of f (for more details on the set B_f see [LT17, Section 6]). The elements of B_f^* are maximal inside Γ_f with respect to inclusion of ordered sets. For $g \in \Gamma_f$, we denote $I_g = \sum_{\sigma \in g} I_\sigma$.

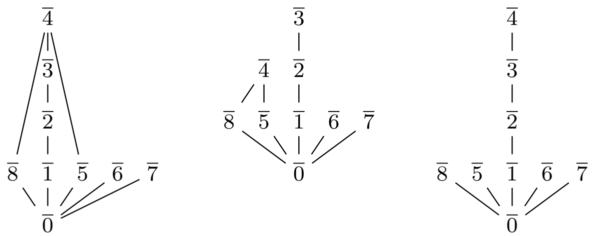
PROPOSITION 6.1. *Let* $f \in E^2(G, L; H)$. *Then* $f = \bigvee_{\gamma \in B_f^*} f_{\{J, I_\gamma, 0\}}$.

Proof. We set $f_\gamma = f_{\{J,I_\gamma,0\}}$, $B^* = B_f^*$ and $H = H(f) = H(f_\gamma)$. Since $I_\gamma \lhd A_f$, we know that $f_\gamma \le f$ for every $\gamma \in B^*$ and so $\bigvee_{\gamma \in B^*} f_\gamma \le f$. For the opposite direction, let $\sigma, \tau \in G$ be such that $f(\sigma, \tau) = 1$. If $f = f_0$, then the equality is immediate. So suppose that $f \ne f_0$ (i.e. there exists a generator with at least two elements). If $\sigma \in H$, then $f_\gamma(\sigma, \tau) = 1$ for every $\gamma \in B^*$, and similarly if $\tau \in H$. If $\sigma\tau \in H$, then $\sigma, \tau \in H$ and so again $f_\gamma(\sigma, \tau) = 1$. So let $\sigma, \tau, \sigma\tau \in G^*$. Moreover, let $g_\sigma, g_\tau \in \Gamma_f$, where $g_\sigma = (\sigma_1, \ldots, \sigma_a)$, $g_\tau = (\tau_1, \ldots, \tau_b)$, $a, b \ge 1$. Since $f(\sigma, \tau) = 1$, we have $g_\sigma g_\tau = g_{\sigma\tau} \in \Gamma_f$ [LT17, Remark 6.2]. We extend $g_{\sigma\tau}$ to an element of B^*, say $\gamma = g_1 g_{\sigma\tau} g_2$ (or $\gamma = g_1 g_{\sigma\tau}$ or $\gamma = g_{\sigma\tau} g_2$ or $\gamma = g_{\sigma\tau}$ if already $g_{\sigma\tau} \in B^*$). We note that $x_\sigma \in I_{\sigma_1} \subseteq I_\gamma$ and that $x_\tau \in I_{\tau_1} \subseteq I_\gamma$. Then $x_{\sigma\tau} = x_\sigma x_\tau \in I_\gamma$, and so by definition $f_\gamma(\sigma, \tau) = 1$, which proves that $f \le \bigvee_{\gamma \in B^*} f_\gamma$. ∎

EXAMPLE 6.2. Consider the idempotent 2-cocycle f_{I_3} of Example 4.6 with $\overline{4}$ as unique non-trivial annihilator. In Example 5.6 we had $f_{I_3} = f_{r_3}$. We now observe that $f_{r_3} = f_{r'}$ for $r' = \{0, 9, 18, 27, 36, 9, 17, 24, 27\}$. The generators of the elements of G^* with respect to $f_{r'}$ are

$$\Gamma_{f_{r'}} = \{\{(\overline{1})\}, \{(\overline{1}, \overline{1})\}, \{(\overline{1}, \overline{1}, \overline{1})\}, \{(\overline{5}, \overline{8}), (\overline{8}, \overline{5}), (\overline{1}, \overline{1}, \overline{1}, \overline{1})\}, \{(\overline{5})\}, \{(\overline{6})\}, \{(\overline{7})\}, \{(\overline{8})\}\}.$$

For the three generators of $x_{\overline{4}}$, $\gamma_1 = (\overline{5}, \overline{8})$, $\gamma_2 = (\overline{8}, \overline{5})$ and $\gamma_3 = (\overline{1}, \overline{1}, \overline{1}, \overline{1})$, we have $P_1 = I_{\gamma_{\overline{1}}} = I_{\gamma_{\overline{2}}} = I_{\overline{5}} + I_{\overline{8}} = Lx_{\overline{4}} + Lx_{\overline{5}} + Lx_{\overline{8}}$ and $P_2 = I_{\gamma_{\overline{3}}} = Lx_{\overline{1}} + Lx_{\overline{2}} + Lx_{\overline{3}} + Lx_{\overline{4}}$. Since $(f_{r'})_{\{J,P_i,0\}} = f_{r'_{\{J,P_i,0\}}}$, $i = 1, 2$ (Proposition 5.4), from Proposition 6.1 we have $f_{r'} = f_{r'_{\{J,P_1,0\}}} \vee f_{r'_{\{J,P_2,0\}}}$ with the respective graphs



and generators

$$\Gamma_{f_{r'_{\{J,P_1,0\}}}} = \{\{(\overline{1})\}, \{(\overline{1}, \overline{1})\}, \{(\overline{1}, \overline{1}, \overline{1})\}, \{(\overline{5}, \overline{8}), (\overline{8}, \overline{5})\}, \{(\overline{5})\}, \{(\overline{6})\}, \{(\overline{7})\}, \{(\overline{8})\}\},$$

$$\Gamma_{f_{r'_{\{J,P_2,0\}}}} = \{\{(\overline{1})\}, \{(\overline{1}, \overline{1})\}, \{(\overline{1}, \overline{1}, \overline{1})\}, \{(\overline{1}, \overline{1}, \overline{1}, \overline{1})\}, \{(\overline{5})\}, \{(\overline{6})\}, \{(\overline{7})\}, \{(\overline{8})\}\}.$$ ∎

Suppose that we are given an idempotent 2-cocycle $f \in E^2(G, L)$ and we want to find some $s \in Sl(G)$ such that $f = f_s$. For this we must find a function $s' : N_1(f) \to \Omega \setminus \{1\}$ and apply [LT17, Proposition 6.10]. For f of Example 4.6, the elements $\{s'(\overline{1}), s'(\overline{5}), s'(\overline{8})\}$ must satisfy the equations $s'(\overline{5})s'(\overline{8}) = s'(\overline{8})s'(\overline{5}) = s'(\overline{1})^4$ and $s'(\overline{1})s'(\overline{1})s'(\overline{5}) = s'(\overline{1})s'(\overline{5})s(\overline{1}) = s'(\overline{5})s'(\overline{1})s'(\overline{1})$ (choosing a commutative Ω simplifies the situation but may not lead to the desired s, if it exists, as shown in [LT17, Example 8.5]). But if f has a unique class of non-trivial annihilators, then we are left with a single equation. For f_{I_3} of the same example, we would only have the first of the two equations mentioned above. Simpler yet, for the idempotent 2-cocycle $f_{\{J,P_1,0\}}$ of Example 6.2 we have the equation $s'(\overline{5})s'(\overline{8}) = s'(\overline{8})s(\overline{5})$ and the trivial equation $s'(\overline{1})^3 =$ constant, and for $f_{\{J,P_2,0\}}$ the trivial equation $s'(\overline{1})^4 =$ constant.

Remark 6.3. Suppose that f has a unique class of non-trivial annihilators, say $[\rho]$. If I_γ is a constant for every generator γ of ρ, then the equality in Proposition 6.1 is trivial, since every term $f_{\{J,I_\gamma,0\}}$ of the second part equals f, as in the following example.

For $g_1, g_2 \in \Gamma_f$, the relation $g_1 \leq g_2$ if and only if g_1 is an ordered part of g_2 is a partial ordering with least element the empty word (). We call the Hasse diagram with regard to this ordering the *graph of generators* of f.

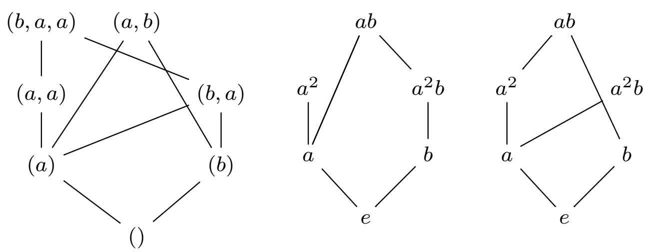
Example 6.4. Let

$$D_3 = \{a, b : a^3 = b^2 = e,\ bab = a^{-1}\} = \{e, a, a^2, b, ab, a^2b\}$$

be the dihedral group of order 6. Let $f \in E^2(G, L)$ be defined by the table

	e	a	a^2	b	ab	a^2b
e	1	1	1	1	1	1
a	1	1	0	1	0	0
a^2	1	0	0	0	0	0
b	1	1	1	0	0	0
ab	1	0	0	0	0	0
a^2b	1	1	0	0	0	0

with generators $\Gamma_a = \{(a)\}$, $\Gamma_b = \{(b)\}$, $\Gamma_{a^2b} = \{(b, a)\}$, $\Gamma_{a^2} = \{(a, a)\}$, $\Gamma_{ab} = \{(a, b), (b, a, a)\}$ and graphs (of generators, left, right)



We notice that the only non-trivial annihilator of f is ab with generators $\gamma_1 = (a, b)$ and $\gamma_2 = (b, a, a)$. Since they contain exactly the same letters (a and b with different multiplicities) we have $I_{\gamma_1} = I_{\gamma_2} = I_a + I_b$. No $r \in \mathrm{Sl}(G)$ is now such that $f = f_r$. To prove that no such r exists, one must prove that there does not exist a monoid Ω with the properties mentioned in the introduction such that $r(b)r(a)r(a) = r(a)r(b)$.

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