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HOVEY TRIPLES ARISING FROM TWO COTORSION PAIRS OF EXTRIANGULATED CATEGORIES

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Abstract. Assume that $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is an extriangulated category satisfying Condition (WIC). Let $(\mathcal{Q},\widetilde{\mathcal{R}})$ and $(\widetilde{\mathcal{Q}},\mathcal{R})$ be two hereditary cotorsion pairs with $\widetilde{\mathcal{R}}\subseteq\mathcal{R}$, $\widetilde{\mathcal{Q}}\subseteq\mathcal{Q}$ and $\widetilde{\mathcal{Q}}\cap\mathcal{R}=\mathcal{Q}\cap\widetilde{R}$. Then there exists a unique thick class $\mathcal{W}$ for which $(\mathcal{Q},\mathcal{W},\mathcal{R})$ is a Hovey triple. This result generalizes the work by Gillespie in an exact case. Moreover, it highlights new phenomena when applied to triangulated categories.

1. Introduction. Let $\mathcal{C}$ be an abelian category. Hovey [5] showed that there exists a one-to-one correspondence between abelian model structures on $\mathcal{C}$ and triples $(\mathcal{Q},\mathcal{W},\mathcal{R})$ of subclasses of $\mathcal{C}$ such that $\mathcal{W}$ is thick and $(\mathcal{Q},\mathcal{W}\cap\mathcal{R})$ and $(\mathcal{Q}\cap\mathcal{W},\mathcal{R})$ are two complete cotorsion pairs. In this case, $\mathcal{Q}$ is precisely the class of cofibrant objects, $\mathcal{R}$ is the class of fibrant objects, and $\mathcal{W}$ is the class of trivial objects in the abelian model structure on $\mathcal{C}$. Given such a triple $(\mathcal{Q},\mathcal{W},\mathcal{R})$, called a Hovey triple, denote the associated cotorsion pairs by $(\mathcal{Q},\widetilde{\mathcal{R}})=(\mathcal{Q},\mathcal{W}\cap\mathcal{R})$ and $(\widetilde{\mathcal{Q}},\mathcal{R})=(\mathcal{Q}\cap\mathcal{W},\mathcal{R})$. Then (1) $\widetilde{\mathcal{R}}\subseteq\mathcal{R},\widetilde{Q}\subseteq\mathcal{Q}$ and (2) $\widetilde{Q}\cap\mathcal{R}=\mathcal{Q}\cap\widetilde{\mathcal{R}}$. This holds in the general setting when $\mathcal{C}$ is a weakly idempotent complete exact category. Gillespie [3] proved that Hovey's correspondence carries over to this setting.

Gillespie [4] proved that there exists a converse when the cotorsion pairs above are hereditary. That is, if $(\mathcal{Q},\widetilde{\mathcal{R}})$ and $(\widetilde{\mathcal{Q}},\mathcal{R})$ are complete hereditary cotorsion pairs satisfying conditions (1) and (2) above, then there is a unique thick class $\mathcal{W}$ for which $(\mathcal{Q},\mathcal{W},\mathcal{R})$ is a Hovey triple.

Extriangulated categories were recently introduced by Nakaoka and Palu [14] by extracting those properties of Ext^1 on exact categories (which is itself a generalization of the concept of a module category and an abelian category) and on triangulated categories that seem relevant from the point of view of cotorsion pairs. In particular, exact categories and triangulated categories are extriangulated categories. There are a lot of examples of extri-

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angulated categories which are neither exact nor triangulated [14, 16, 9, 17]. Hence, many results which hold on exact categories and triangulated categories can be unified in the same framework. Recently, Herschend, Liu and Nakaoka [7] introduced the notion of n-exangulated categories. It should be noted that the case $n = 1$ corresponds to extriangulated categories. As a typical result, n-exact and $(n+2)$-angulated categories are n-exangulated [7, Propositions 4.5 and 4.34]. There are however n-exangulated categories which are neither n-exact nor $(n+2)$-angulated [8].

Motivated by Hovey's correspondence, we introduce the notion of extriangulated model structures, which are a generalization of abelian model structures.

DEFINITION 1.1. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, and $\mathcal{Q}, \mathcal{W}, \mathcal{R}$ three classes of objects in $\mathcal{C}$.

(1) $\mathcal{W}$ is called *thick* if it is closed under direct summands and satisfies the 2-out-of-3 property: for any $\mathbb{E}$-triangle $A \to B \to C \dashrightarrow$ in $\mathcal{C}$ with two terms in $\mathcal{W}$, the third term belongs to $\mathcal{W}$ as well.
(2) A triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is an *extriangulated model structure* if $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are two cotorsion pairs and $\mathcal{W}$ is thick. In this case, $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is also called a *Hovey* triple.

One of the main results of the paper is the following theorem (see also Theorem 3.9) that extends the main result of Gillespie [4] from exact categories to the extriangulated category level.

THEOREM 1.2. *Assume that* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *is an extriangulated category satisfying Condition* (WIC) *defined in Definition* 3.1. *Let* $(\mathcal{Q}, \widetilde{\mathcal{R}})$ *and* $(\widetilde{\mathcal{Q}}, \mathcal{R})$ *be hereditary cotorsion pairs such that* $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$, $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$ *and* $\widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{\mathcal{R}}$. *Then the following statements hold:*

(a) *There exists a unique thick class* $\mathcal{W}$ *such that* $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ *is a Hovey triple.*
(b) *The thick class* $\mathcal{W}$ *is described as follows:*

$$\begin{aligned}\mathcal{W} &= \{X \in \mathcal{C} \mid \exists \text{ an } \mathbb{E}\text{-triangle } X \to R \to Q \dashrightarrow \text{ with } R \in \widetilde{\mathcal{R}},\ Q \in \widetilde{\mathcal{Q}}\} \\ &= \{X \in \mathcal{C} \mid \exists \text{ an } \mathbb{E}\text{-triangle } R' \to Q' \to X \dashrightarrow \text{ with } R' \in \widetilde{\mathcal{R}},\ Q' \in \widetilde{\mathcal{Q}}\}.\end{aligned}$$

The theorem is proved in Section 3, where its generality is illustrated by Example 3.12. In Section 2, we review some basic concepts and results concerning extriangulated categories.

Throughout this paper we freely use the standard terminology and notation applied in [14, 11, 10, 18, 15, 12].

2. Preliminaries. Let us briefly recall some definitions and basic properties of extriangulated categories from [14].

Let $\mathcal{C}$ be an additive category. Suppose that $\mathcal{C}$ is equipped with an additive bifunctor

$$\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab,$$

where Ab is the category of abelian groups. For any objects $A, C \in \mathcal{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Thus formally, an $\mathbb{E}$-extension is a triplet (A, δ, C). Let (A, δ, C) be an $\mathbb{E}$-extension. Since $\mathbb{E}$ is a bifunctor, for any $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C', C)$ we have $\mathbb{E}$-extensions

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \quad \text{and} \quad \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).$$

We simply denote them by $a_*\delta$ and $c^*\delta$. For any $A, C \in \mathcal{C}$, the zero element $0 \in \mathbb{E}(C, A)$ is called the *split* $\mathbb{E}$-*extension*.

Definition 2.1 ([14, Definition 2.3]). Let (A, δ, C) and (A', δ', C') be any $\mathbb{E}$-extensions. A *morphism*

$$(a, c)\colon (A, \delta, C) \to (A', \delta', C')$$

of $\mathbb{E}$-extensions is a pair of morphisms $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C, C')$ in $\mathcal{C}$ satisfying $a_*\delta = c^*\delta'$. We denote it simply as $(a, c)\colon \delta \to \delta'$.

Definition 2.2 ([14, Definition 2.6]). Let $\delta = (A, \delta, C)$, $\delta' = (A', \delta', C')$ be $\mathbb{E}$-extensions. Let

$$C \xrightarrow{\iota_C} C \oplus C' \xleftarrow{\iota_{C'}} C' \quad \text{and} \quad A \xleftarrow{p_A} A \oplus A' \xrightarrow{p_{A'}} A'$$

be the coproduct and product in $\mathcal{B}$, respectively. Since $\mathbb{E}$ is biadditive, we have a natural isomorphism

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A').$$

Let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ correspond to $(\delta, 0, 0, \delta')$ through the above isomorphism. This is the unique element which satisfies

$$\mathbb{E}(\iota_C, p_A)(\delta \oplus \delta') = \delta, \qquad \mathbb{E}(\iota_C, p_{A'})(\delta \oplus \delta') = 0,$$
$$\mathbb{E}(\iota_{C'}, p_A)(\delta \oplus \delta') = 0, \qquad \mathbb{E}(\iota_{C'}, p_{A'})(\delta \oplus \delta') = \delta'.$$

Let $A, C \in \mathcal{C}$ be any objects. Sequences of morphisms in $\mathcal{C}$

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B \xrightarrow{y'} C$$

are said to be *equivalent* if there exists an isomorphism $b \in \mathcal{C}(B, B')$ which makes the following diagram commutative:

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \cong \downarrow b & & \| \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

For any $A, C \in \mathcal{C}$, we denote $0 = [A \xrightarrow{\binom{1}{0}} A \oplus C \xrightarrow{(0,1)} C]$.

For any two equivalence classes, we denote

$$[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].$$

DEFINITION 2.3 ([14, Definition 2.9]). Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if it satisfies the following condition:

- Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be $\mathbb{E}$-extensions with
$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$
Then, for any morphism $(a, c)\colon \delta \to \delta'$, there exists $b \in \mathcal{C}(B, B')$ which makes the following diagram commutative:
$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

In the above situation, we say that the triplet (a, b, c) *realizes* (a, c).

DEFINITION 2.4 ([14, Definition 2.10]). A realization $\mathfrak{s}$ of $\mathbb{E}$ is called *additive* if it satisfies the following conditions:

(1) For any $A, C \in \mathcal{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C, A)$ satisfies $\mathfrak{s}(0) = 0$.
(2) For any $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$,
$$\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta').$$

DEFINITION 2.5 ([14, Definition 2.12]). A triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is called an *externally triangulated category* (or *extriangulated category* for short) if it satisfies the following conditions:

(ET1) $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ is a biadditive functor.
(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.
(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any $\mathbb{E}$-extensions, realized as
$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$
For any commutative square
$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$
in $\mathcal{C}$, there exists a morphism $(a, c)\colon \delta \to \delta'$ satisfying $cy = y'b$.
(ET3)$^{\mathrm{op}}$ Dual of (ET3).

(ET4) Let (A, δ, D) and (B, δ', F) be $\mathbb{E}$-extensions realized by

$$A \xrightarrow{f} B \xrightarrow{f'} D \quad \text{and} \quad B \xrightarrow{g} C \xrightarrow{g'} F$$

respectively. Then there exists an object $E \in \mathcal{C}$, a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\ & & F & = & F \end{array}$$

in $\mathcal{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities:

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $f'_*\delta'$,
(ii) $d^*\delta'' = \delta$,
(iii) $f_*\delta'' = e^*\delta'$.

$(\text{ET4})^{\text{op}}$ Dual of (ET4).

We will use the following terminology from [14].

Definition 2.6. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

(1) A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. In this case, x is called an *inflation* and y is called a *deflation*.
(2) If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C, A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an $\mathbb{E}$-*triangle*, and write it in the following way:

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

We usually do not write the "δ" if it is not used in the argument.
(3) Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be $\mathbb{E}$-triangles. If a triplet (a, b, c) realizes $(a, c)\colon \delta \to \delta'$, then we write it as

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} & \end{array}$$

and call (a, b, c) a *morphism of* $\mathbb{E}$-*triangles*.

Assume that $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is an extriangulated category. By Yoneda's lemma, any $\mathbb{E}$-extension $\delta\in\mathbb{E}(C,A)$ induces natural transformations

$$\delta_\sharp\colon \mathcal{C}(-,C)\Rightarrow\mathbb{E}(-,A)\quad\text{and}\quad\delta^\sharp\colon\mathcal{C}(A,-)\Rightarrow\mathbb{E}(C,-).$$

For any $X\in\mathcal{C}$, these $(\delta_\sharp)_X$ and $\delta^\sharp_X$ are given as follows:

(1) $(\delta_\sharp)_X\colon\mathcal{C}(X,C)\to\mathbb{E}(X,A),\ f\mapsto f^*\delta$.
(2) $\delta^\sharp_X\colon\mathcal{C}(A,X)\to\mathbb{E}(C,X),\ g\mapsto g_*\delta$.

When there is no danger of confusion, we will sometimes use, instead of

$$\mathrm{Hom}_{\mathcal{C}}(X,A)\xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X,\ f)}\mathrm{Hom}_{\mathcal{C}}(X,B),$$

the simplified form

$$\mathcal{C}(X,A)\xrightarrow{\mathcal{C}(X,\ f)}\mathcal{C}(X,B).$$

LEMMA 2.7. *Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category, and*

$$A\overset{x}{\to}B\overset{y}{\to}C\overset{\delta}{\dashrightarrow}$$

an $\mathbb{E}$-triangle. Then we have the following long exact sequences:

$$\mathcal{C}(-,A)\xrightarrow{\mathcal{C}(-,x)}\mathcal{C}(-,B)\xrightarrow{\mathcal{C}(-,y)}\mathcal{C}(-,C)\xrightarrow{\delta_\sharp}\mathbb{E}(-,A)\xrightarrow{\mathbb{E}(-,x)}\mathbb{E}(-,B),$$

$$\mathcal{C}(C,-)\xrightarrow{\mathcal{C}(y,-)}\mathcal{C}(B,-)\xrightarrow{\mathcal{C}(x,-)}\mathcal{C}(A,-)\xrightarrow{\delta^\sharp}\mathbb{E}(C,-)\xrightarrow{\mathbb{E}(y,-)}\mathbb{E}(B,-).$$

Proof. This follows from Proposition 3.3. ∎

We recall the following result, which (together with its dual) will be used later.

LEMMA 2.8 ([13, Proposition 1.20]). *Assume that $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is an extriangulated category. Let $A\overset{x}{\to}B\overset{y}{\to}C\overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle, and $f\colon A\to D$ be any morphism in $\mathcal{C}$. Then there exists a morphism g which gives a morphism of $\mathbb{E}$-triangles*

$$\begin{array}{ccccccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \| & & \\
D & \xrightarrow{d} & F & \xrightarrow{e} & C & \overset{f_*\delta}{\dashrightarrow} &
\end{array}$$

and moreover the sequence $A\xrightarrow{\binom{-x}{f}}B\oplus D\xrightarrow{(g,\ d)}F\overset{e^*\delta}{\dashrightarrow}$ becomes an $\mathbb{E}$-triangle.

Lemma 2.9. *Assume that* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *is an extriangulated category. Let*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \overset{\delta}{\dashrightarrow} & \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \| & & \\ D & \xrightarrow{u} & F & \xrightarrow{v} & C & \overset{\eta}{\dashrightarrow} & \end{array}$$

be a morphism of $\mathbb{E}$*-triangles. Then there exists an isomorphism* $w\colon F \to F$ *which makes*

$$A \xrightarrow{\binom{-f}{a}} B \oplus D \xrightarrow{(b,\ u)} F \overset{\theta}{\dashrightarrow}$$

an $\mathbb{E}$*-triangle, where* $\theta = (w^{-1})^* v^* \delta$.

Proof. By Lemma 2.8, we have the commutative diagram of $\mathbb{E}$-triangles

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \overset{\delta}{\dashrightarrow} & \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b'} & & \| & & \\ D & \xrightarrow{x} & F' & \xrightarrow{y} & C & \overset{a_*\delta}{\dashrightarrow} & \end{array}$$

and moreover

$$A \xrightarrow{\binom{-f}{a}} B \oplus D \xrightarrow{(b',\ x)} F' \overset{y^*\delta}{\dashrightarrow}$$

is an $\mathbb{E}$-triangle. By the definition of realization and [14, Corollary 3.6], there exists an isomorphism $c\colon F' \to F$ which gives a morphism of $\mathbb{E}$-triangles.

$$\begin{array}{ccccccc} D & \xrightarrow{x} & F' & \xrightarrow{y} & C & \overset{a_*\delta}{\dashrightarrow} & \\ \| & & \downarrow{\scriptstyle c} & & \| & & \\ D & \xrightarrow{u} & F & \xrightarrow{v} & C & \overset{\eta}{\dashrightarrow} & \end{array}$$

Consider the isomorphism $c^{-1}\colon F \to F'$; by [14, Proposition 3.7], the sequence

$$A \xrightarrow{\binom{-f}{a}} B \oplus D \xrightarrow{(cb',\ u)} F \overset{v^*\delta}{\dashrightarrow}$$

is an $\mathbb{E}$-triangle. Noting that $(b, u)\binom{-f}{a} = 0$, there is a morphism $w\colon F \to F$ such that $(b, u) = w(cb', u)$ and so $b = wcb'$ and $u = wu$. Since $(1 - w)u = 0$, there exists a morphism $d\colon C \to F$ such that $1 - w = dv$. It follows that

$$(1 + dv)(1 - dv)u = (1 + dv)wu = (1 + dv)u = u.$$

Observe that $v(b - cb') = vb - vcb' = g - yb' = 0$. Hence there exists a morphism $m\colon B \to D$ such that $b - cb' = um$ and so $dvb = dv(um + cb') = dvcb'$. Thus we have $(1+dv)b = b+dvb = wcb'+dvcb' = (w+dv)cb' = cb'$, which implies

$$(1 + dv)(1 - dv)cb' = (1 + dv)wcb' = (1 + dv)b = cb'.$$

Hence we have a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\binom{-f}{a}} & B\oplus D & \xrightarrow{(cb',\ u)} & F & \overset{v^*\delta}{\dashrightarrow} & \\ \| & & \| & & \downarrow{\scriptstyle (1+dv)(1-dv)} & & \\ A & \xrightarrow{\binom{-f}{a}} & B\oplus D & \xrightarrow{(cb',\ u)} & F & \overset{v^*\delta}{\dashrightarrow} & \end{array}$$

of $\mathbb{E}$-triangles. By [14, Corollary 3.6], the morphism $(1 + dv)(1 - dv)$ is an isomorphism. As $(1 + dv)(1 - dv) = (1 - dv)(1 + dv)$, the morphism $1 - dv = w$ is an isomorphism. Consider the isomorphism $w^{-1}\colon F \to F$. By [14, Proposition 3.7], the sequence

$$A \xrightarrow{\binom{-f}{a}} B\oplus D \xrightarrow{(b,\ u)} F \overset{\theta}{\dashrightarrow}$$

is an $\mathbb{E}$-triangle, where $\theta = (w^{-1})^* v^*\delta$. ∎

3. Hovey triples arising from two cotorsion pairs. One of our aims in this section is to prove Theorem 3.9 that is a slightly modified version of Theorem 1.2. We begin by recalling from [14, Condition 5.8] the definition of Condition (WIC) used in Theorem 1.2. We also collect a couple of preparatory results used in the proof of Theorem 3.9.

DEFINITION 3.1. An extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies *Condition* (WIC) if the following two conditions hold:

(a) If $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ is a pair of composable morphisms in $\mathcal{C}$ and gf is an inflation then so is f.
(b) If $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ is a pair of composable morphisms in $\mathcal{C}$ and gf is a deflation then so is g.

REMARK 3.2. (i) If the category $\mathcal{C}$ is exact then $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies Condition (WIC) if and only if $\mathcal{C}$ is weakly idempotent complete [1, Proposition 7.6].

(ii) If $\mathcal{C}$ is a triangulated category then Condition (WIC) is automatically satisfied.

LEMMA 3.3. *Assume that* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *is an extriangulated category satisfying Condition* (WIC). *Let*

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\ \| & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} & \end{array}$$

be a morphism of $\mathbb{E}$*-triangles. If* c *is a deflation, then so is* b.

Proof. Since c is a deflation, there exists an $\mathbb{E}$-triangle

$$D \xrightarrow{d} C \xrightarrow{c} C' \overset{\eta}{\dashrightarrow}$$

in $\mathcal{C}$. By [14, Proposition 3.15], we obtain a commutative diagram of $\mathbb{E}$-triangles

$$\begin{array}{ccccccc}
 & & D & = & D & & \\
 & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle d} & & \\
A & \xrightarrow{g} & M & \xrightarrow{h} & C & \dashrightarrow & \\
\| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle c} & & \\
A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} & \\
 & & \downarrow & & \downarrow{\scriptstyle \eta} & & \\
\end{array}$$

It follows that $ch = y'f$ and so $(c, y')\binom{-h}{f} = 0$. By the dual of Lemma 2.9, the sequence

$$B \xrightarrow{\binom{-y}{b}} C \oplus B' \xrightarrow{(c,\ y')} C' \overset{\theta}{\dashrightarrow}$$

is an $\mathbb{E}$-triangle. So there exists a morphism $k\colon M \to B$ such that $\binom{-y}{b}k = \binom{-h}{f}$. In particular, $f = bk$. By Condition (WIC) and since f is a deflation, b becomes a deflation. ∎

LEMMA 3.4. *Assume that* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *is an extriangulated category satisfying Condition* (WIC). *Let*

$$\begin{array}{ccccccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & \\
A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow}
\end{array}$$

be a morphism of $\mathbb{E}$*-triangles. If* a *and* c *are deflations, then there exists a deflation* $b'\colon B \to B'$ *which gives the following morphism of* $\mathbb{E}$*-triangles:*

$$\begin{array}{ccccccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle c} & \\
A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow}
\end{array}$$

Proof. Since a is a deflation, there exists an $\mathbb{E}$-triangle

$$D \xrightarrow{d} A \xrightarrow{a} A' \overset{\eta}{\dashrightarrow}$$

in $\mathcal{C}$. By (ET4), we have a commutative diagram

$$\begin{array}{ccccccc}
D & \xrightarrow{d} & A & \xrightarrow{a} & A' & \overset{\eta}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle u} & & \\
D & \xrightarrow{s} & B & \xrightarrow{t} & M & \dashrightarrow & \\
 & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle v} & & \\
 & & C & = & C & & \\
 & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle a_*\delta} & &
\end{array}$$

of $\mathbb{E}$-triangles. By the dual of Lemma 2.8, we have a commutative diagram

$$\begin{array}{ccccccc}
A' & \xrightarrow{p} & L & \xrightarrow{q} & C & \overset{c^*\delta'}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle c} & & \\
A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} &
\end{array}$$

of $\mathbb{E}$-triangles. By Lemma 3.3, g is a deflation since c is. By the definition of realization and [14, Corollary 3.6], there exists an isomorphism $h\colon M \to L$ which gives a morphism of $\mathbb{E}$-triangles

$$\begin{array}{ccccccc}
A' & \xrightarrow{u} & M & \xrightarrow{v} & C & \overset{a_*\delta}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle h} & & \| & & \\
A' & \xrightarrow{p} & L & \xrightarrow{q} & C & \overset{c^*\delta'}{\dashrightarrow} &
\end{array}$$

Hence we have a commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle ght} & & \downarrow{\scriptstyle c} & & \\
A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} &
\end{array}$$

of $\mathbb{E}$-triangles. Since t, h and g are deflations, so is the composite

$$b' := ght.$$

This completes the proof. ■

We recall from [9, Lemma 4.14] the following result.

LEMMA 3.5. *Assume that* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *is an extriangulated category satisfying Condition* (WIC). *Let*

$$\begin{array}{ccccccc}
K & & K' & & K'' & & \\
{\scriptstyle k}\downarrow & & {\scriptstyle k'}\downarrow & & {\scriptstyle k''}\downarrow & & \\
A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\
{\scriptstyle a}\downarrow & \circlearrowleft & {\scriptstyle b}\downarrow & \circlearrowleft & {\scriptstyle c'}\downarrow & & \\
A' & \xrightarrow[x']{} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} & \\
{\scriptstyle \kappa}\downarrow & & {\scriptstyle \kappa'}\downarrow & & {\scriptstyle \kappa''}\downarrow & &
\end{array}$$

be a diagram of $\mathbb{E}$*-triangles. If* y *is a retraction, then there exists an* $\mathbb{E}$*-triangle*

$$K \xrightarrow{m} K' \xrightarrow{n'} K'' \overset{\delta''}{\dashrightarrow}$$

which makes the following diagram commutative:

$$\begin{array}{ccccccc}
K & \xrightarrow{m} & K' & \xrightarrow{n'} & K'' & \overset{\delta''}{\dashrightarrow} & \\
{\scriptstyle k}\downarrow & \circlearrowleft & {\scriptstyle k'}\downarrow & \circlearrowleft & {\scriptstyle k''}\downarrow & & \\
A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\
{\scriptstyle a}\downarrow & \circlearrowleft & {\scriptstyle b}\downarrow & \circlearrowleft & {\scriptstyle c'}\downarrow & & \\
A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} & \\
{\scriptstyle \kappa}\downarrow & & {\scriptstyle \kappa'}\downarrow & & {\scriptstyle \kappa''}\downarrow & &
\end{array}$$

in which $(k, k', k''), (a, b, c'), (m, x, x')$ *and* (n', y, y') *are morphisms of* $\mathbb{E}$*-triangles.*

Definition 3.6. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{W}$ a class of objects in $\mathcal{C}$.

(1) $\mathcal{W}$ is *closed under cocones of deflations* if for any $\mathbb{E}$-triangle $A \overset{x}{\to} B \overset{y}{\to} C \overset{\delta}{\dashrightarrow}$ which satisfies $B, C \in \mathcal{W}$, we have $A \in \mathcal{W}$.
(2) $\mathcal{W}$ is *closed under cones of inflations* if for any $\mathbb{E}$-triangle $A \overset{x}{\to} B \overset{y}{\to} C \overset{\delta}{\dashrightarrow}$ which satisfies $A, B \in \mathcal{W}$, we have $C \in \mathcal{W}$.
(3) $\mathcal{W}$ is *closed under extensions* if for any $\mathbb{E}$-triangle $A \overset{x}{\to} B \overset{y}{\to} C \overset{\delta}{\dashrightarrow}$ which satisfies $A, C \in \mathcal{W}$, we have $B \in \mathcal{W}$.

Definition 3.7 ([14, Definition 4.1]). Assume that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category. Let $\mathcal{Q}$ and $\mathcal{R}$ be two classes of objects in $\mathcal{C}$. We call $(\mathcal{Q}, \mathcal{R})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathbb{E}(\mathcal{Q}, \mathcal{R}) = 0$.

(b) For any $C \in \mathcal{C}$, there are $\mathbb{E}$-triangles

$$R_C \to Q_C \to C \dashrightarrow \quad \text{and} \quad C \to R^C \to Q^C \dashrightarrow$$

satisfying $Q_C, Q^C \in \mathcal{Q}$ and $R_C, R^C \in \mathcal{R}$.

A cotorsion pair $(\mathcal{Q}, \mathcal{R})$ is called *hereditary* if $\mathcal{Q}$ is closed under cocones of deflations and $\mathcal{R}$ is closed under cones of inflations.

REMARK 3.8. If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an exact category, then the above cotorsion pair is just a complete cotorsion pair in the sense of [6] (see also [12]).

Now we state and prove our main result.

THEOREM 3.9. *Assume that* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *is an extriangulated category satisfying Condition* (WIC). *Let* $(\mathcal{Q}, \widetilde{\mathcal{R}})$ *and* $(\widetilde{\mathcal{Q}}, \mathcal{R})$ *be hereditary cotorsion pairs such that* $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$, $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$ *and* $\widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{\mathcal{R}}$. *Then the following statements hold:*

(a) *The following two classes coincide:*

$$\mathcal{W}_1 = \{X \in \mathcal{C} \mid \exists \text{ an } \mathbb{E}\text{-triangle } X \to R \to Q \dashrightarrow \text{ with } R \in \widetilde{\mathcal{R}},\ Q \in \widetilde{\mathcal{Q}}\},$$
$$\mathcal{W}_2 = \{X \in \mathcal{C} \mid \exists \text{ an } \mathbb{E}\text{-triangle } R' \to Q' \to X \dashrightarrow \text{ with } R' \in \widetilde{\mathcal{R}},\ Q' \in \widetilde{\mathcal{Q}}\}.$$

(b) *The class* $\mathcal{W} := \mathcal{W}_1 = \mathcal{W}_2$ *is thick.*
(c) *The class* $\mathcal{W} := \mathcal{W}_1 = \mathcal{W}_2$ *is a unique thick class such that* $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ *is a Hovey triple.*

Proof. (1) We show $\mathcal{W}_1 = \mathcal{W}_2$.

For any $X \in \mathcal{W}_1$, there exists an $\mathbb{E}$-triangle $X \to R \to Q \dashrightarrow$ where $R \in \widetilde{\mathcal{R}}$ and $Q \in \widetilde{\mathcal{Q}}$. Since $(\mathcal{Q}, \widetilde{\mathcal{R}})$ is cotorsion pair, there exists an $\mathbb{E}$-triangle $R' \to Q' \to R \dashrightarrow$ where $R' \in \widetilde{\mathcal{R}}$ and $Q' \in \mathcal{Q}$. By (ET4$^{\mathrm{op}}$), we have the following commutative diagram of $\mathbb{E}$-triangles:

$$\begin{array}{ccccc}
R' & \longrightarrow & M & \longrightarrow & X \dashrightarrow \\
\| & & \downarrow & & \downarrow \\
R' & \longrightarrow & Q' & \longrightarrow & R \dashrightarrow \\
 & & \downarrow & & \downarrow \\
 & & Q & = & Q \\
 & & \downarrow & & \downarrow \\
 & & & &
\end{array}$$

Since $\widetilde{\mathcal{R}}$ is closed under extensions, we have $Q' \in \widetilde{\mathcal{R}}$. It follows that $Q' \in \mathcal{Q} \cap \widetilde{R} = \widetilde{\mathcal{Q}} \cap \mathcal{R}$ and so $Q' \in \widetilde{\mathcal{Q}}$. Now because $Q', Q \in \widetilde{\mathcal{Q}}$ and $\widetilde{Q}$ is closed under cocones of deflations, we conclude that $M \in \widetilde{\mathcal{Q}}$. Thus we find an $\mathbb{E}$-triangle

$R' \to M \to X \dashrightarrow$ where $R' \in \widetilde{\mathcal{R}}$ and $M \in \widetilde{\mathcal{Q}}$. Hence $\mathcal{W}_1 \subseteq \mathcal{W}_2$. A similar argument will show that $\mathcal{W}_2 \subseteq \mathcal{W}_1$.

(2) We show that $\mathcal{W} = \mathcal{W}_1 = \mathcal{W}_2$ is thick.

- We first prove that $\mathcal{W}$ is closed under direct summands.

Suppose $W \in \mathcal{W}$ and $X \xrightarrow{i} W \xrightarrow{p} X$ satisfies $pi = 1_X$. We want to show that $X \in \mathcal{W}$.

Since $W \in \mathcal{W}$, there exists an $\mathbb{E}$-triangle

$$W \xrightarrow{x} \widetilde{R} \xrightarrow{y} \widetilde{Q} \dashrightarrow$$

where $\widetilde{R} \in \widetilde{\mathcal{R}}$ and $\widetilde{Q} \in \widetilde{\mathcal{Q}}$. Since $(\widetilde{\mathcal{Q}}, \mathcal{R})$ is a cotorsion pair, there exists an $\mathbb{E}$-triangle

$$X \xrightarrow{k} R \xrightarrow{h} \widetilde{Q'} \dashrightarrow$$

where $R \in \mathcal{R}$ and $\widetilde{Q'} \in \widetilde{\mathcal{Q}}$.

Applying the functor $\mathrm{Hom}_{\mathcal{C}}(-, \widetilde{R})$ to the $\mathbb{E}$-triangle $X \xrightarrow{k} R \xrightarrow{h} \widetilde{Q'} \dashrightarrow$, by Lemma 2.7 we have an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(R, \widetilde{R}) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(k, \widetilde{R})} \mathrm{Hom}_{\mathcal{C}}(X, \widetilde{R}) \to \mathbb{E}(\widetilde{Q'}, \widetilde{R}) = 0.$$

Thus there exists a morphism $j\colon R \to \widetilde{R}$ such that $xi = jk$.

Applying the functor $\mathrm{Hom}_{\mathcal{C}}(-, R)$ to the $\mathbb{E}$-triangle $W \xrightarrow{x} \widetilde{R} \xrightarrow{y} \widetilde{Q} \dashrightarrow$, by Lemma 2.7 we have an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(\widetilde{R}, R) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(x, R)} \mathrm{Hom}_{\mathcal{C}}(W, R) \to \mathbb{E}(\widetilde{Q}, R) = 0.$$

Thus there exists a morphism $q\colon \widetilde{R} \to \mathcal{R}$ such that $kp = qx$. By (ET3), we have a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{k} & R & \xrightarrow{h} & \widetilde{Q'} & \dashrightarrow & \\
\big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} & & \big\downarrow & & \\
W & \xrightarrow{x} & \widetilde{R} & \xrightarrow{y} & \widetilde{Q} & \dashrightarrow & \\
\big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} & & \big\downarrow & & \\
X & \xrightarrow{k} & R & \xrightarrow{h} & \widetilde{Q'} & \dashrightarrow &
\end{array}$$

of $\mathbb{E}$-triangles. It follows that $(1_R - qj)k = k - qjk = k - kpi = 0$. So there exists a morphism $t\colon \widetilde{Q'} \to R$ such that $th = 1_R - qj$. That is, $1_R - qj$ factors through $\widetilde{Q'}$. We claim that $1_R - qj$ factors through some object in $\widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{R}$. In fact, since $(\widetilde{\mathcal{Q}}, \mathcal{R})$ is a cotorsion pair, there exists an $\mathbb{E}$-triangle $\widetilde{Q'} \xrightarrow{u} R' \xrightarrow{v} \widetilde{Q''} \dashrightarrow$ where $R' \in \mathcal{R}$ and $\widetilde{Q''} \in \widetilde{\mathcal{Q}}$. Since $\widetilde{\mathcal{Q}}$ is closed under extensions, $R' \in \widetilde{\mathcal{Q}}$ implies $R' \in \widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{R}$. Hence $R' \in \widetilde{\mathcal{R}}$.

Applying $\operatorname{Hom}_{\mathcal{C}}(-,R)$ to the $\mathbb{E}$-triangle $\widetilde{Q'} \xrightarrow{u} R' \xrightarrow{v} \widetilde{Q''} \dashrightarrow$, by Lemma 2.7 we have an exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(R',R) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(u,R)} \operatorname{Hom}_{\mathcal{C}}(\widetilde{Q'},R) \to \mathbb{E}(\widetilde{Q''},R) = 0.$$

Thus there exists a morphism $\beta\colon R' \to R$ such that $t = \beta u$ and so

$$1_R - qj = th = \beta(uh).$$

It follows that $(q,\beta)\binom{j}{uh} = 1_R$. Namely, the composition

$$R \xrightarrow{\binom{j}{uh}} \widetilde{R} \oplus R' \xrightarrow{(q,\beta)} R$$

is the identity 1_R. This means that R is a direct summand of $\widetilde{R} \oplus R'$. Since $\widetilde{\mathcal{R}}$ is closed under direct sums and direct summands, we obtain $R \in \widetilde{\mathcal{R}}$. This shows that $X \in \mathcal{W}$, as required.

- We show that $\mathcal{W}$ is closed under extensions.

Noting that $\widetilde{\mathcal{Q}} \subseteq \mathcal{W}$ and $\widetilde{\mathcal{R}} \subseteq \mathcal{W}$, we have the following claim.

Claim I. *Let $R \xrightarrow{m} Y \xrightarrow{n} W \dashrightarrow$ be an $\mathbb{E}$-triangle with $R \in \widetilde{\mathcal{R}}$ and $W \in \mathcal{W}$. Then there exists a commutative diagram*

$$\begin{array}{ccccccc}
\widetilde{R} & \longrightarrow & \widetilde{R''} & \longrightarrow & \widetilde{R'} & \dashrightarrow & \\
\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow & & \\
\widetilde{Q} & \longrightarrow & \widetilde{Q''} & \longrightarrow & \widetilde{Q'} & \dashrightarrow & \\
\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow & & \\
R & \longrightarrow & Y & \longrightarrow & W & \dashrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\end{array}$$

of $\mathbb{E}$-triangles, where $\widetilde{Q}, \widetilde{Q'}, \widetilde{Q''} \in \widetilde{\mathcal{Q}}$ and $\widetilde{R}, \widetilde{R'}, \widetilde{R''} \in \widetilde{\mathcal{R}}$.

Indeed, since $R, W \in \mathcal{W}$, there are $\mathbb{E}$-triangles

$$\widetilde{R} \xrightarrow{b} \widetilde{Q} \xrightarrow{a} R \dashrightarrow \quad \text{and} \quad \widetilde{R'} \xrightarrow{d} \widetilde{Q'} \xrightarrow{c} W \dashrightarrow$$

where $\widetilde{R}, \widetilde{R'} \in \widetilde{\mathcal{R}}$ and $\widetilde{Q}, \widetilde{Q'} \in \widetilde{\mathcal{Q}}$. Since $R \in \widetilde{\mathcal{R}} \subseteq \mathcal{R}$, we have $\mathbb{E}(\widetilde{Q'}, R) = 0$.

Applying the functor $\operatorname{Hom}_{\mathcal{C}}(\widetilde{Q'},-)$ to the $\mathbb{E}$-triangle $R \xrightarrow{m} Y \xrightarrow{n} W \dashrightarrow$, by Lemma 2.7 we have an exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(\widetilde{Q'},Y) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(\widetilde{Q'},n)} \operatorname{Hom}_{\mathcal{C}}(\widetilde{Q'},W) \to \mathbb{E}(\widetilde{Q'},R) = 0.$$

Thus there exists a morphism $w\colon \widetilde{Q'} \to Y$ such that $c = nw$. Hence we have a commutative diagram

$$\begin{array}{ccccccc}
\widetilde{Q} & \xrightarrow{\binom{1}{0}} & \widetilde{Q}\oplus\widetilde{Q'} & \xrightarrow{(0,1)} & \widetilde{Q'} & \dashrightarrow & \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle (ma,\ w)} & & \downarrow{\scriptstyle c} & & \\
R & \xrightarrow{m} & Y & \xrightarrow{n} & W & \dashrightarrow &
\end{array}$$

of $\mathbb{E}$-triangles. Since a, c are deflations, by Lemma 3.4 there exists a deflation $b'\colon \widetilde{Q}\oplus\widetilde{Q'} \to Y$ which gives the following morphism of $\mathbb{E}$-triangles:

$$\begin{array}{ccccccc}
\widetilde{Q} & \xrightarrow{\binom{1}{0}} & \widetilde{Q}\oplus\widetilde{Q'} & \xrightarrow{(0,1)} & \widetilde{Q'} & \dashrightarrow & \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle c} & & \\
R & \xrightarrow{m} & Y & \xrightarrow{n} & W & \dashrightarrow &
\end{array}$$

Since b' is a deflation, there exists an $\mathbb{E}$-triangle

$$\widetilde{R''} \to \widetilde{Q}\oplus\widetilde{Q'} \to Y \dashrightarrow .$$

By Lemma 3.5, we have a commutative diagram

$$\begin{array}{ccccccc}
\widetilde{R} & \longrightarrow & \widetilde{R''} & \longrightarrow & \widetilde{R'} & \dashrightarrow & \\
\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow & & \\
\widetilde{Q} & \xrightarrow{\binom{1}{0}} & \widetilde{Q}\oplus\widetilde{Q'} & \xrightarrow{(0,1)} & \widetilde{Q'} & \dashrightarrow & \\
\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow & & \\
R & \longrightarrow & Y & \longrightarrow & W & \dashrightarrow & \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & &
\end{array}$$

of $\mathbb{E}$-triangles. Since $\widetilde{\mathcal{Q}}$ is closed under direct sums and extensions, we have $\widetilde{Q}\oplus\widetilde{Q'} \in \widetilde{\mathcal{Q}}$. Since $\widetilde{\mathcal{R}}$ is closed under extensions, we have $\widetilde{R''} \in \widetilde{\mathcal{R}}$. So we have proved our claim.

Now we show that $\mathcal{W}$ is closed under extensions. Let

$$W \to Y \to W' \dashrightarrow$$

be an $\mathbb{E}$-triangle with $W, W' \in \mathcal{W}$. We need to show that $Y \in \mathcal{W}$ too.

Since $W \in \mathcal{W}$, there exists an $\mathbb{E}$-triangle $W \to R \to Q \dashrightarrow$ where $R \in \widetilde{\mathcal{R}}$ and $Q \in \widetilde{\mathcal{Q}}$. By [14, Proposition 3.15], we have a commutative diagram

$$\begin{array}{ccccc}
W & \longrightarrow & Y & \longrightarrow & W' \dashrightarrow \\
\downarrow & & \downarrow & & \| \\
R & \longrightarrow & M & \longrightarrow & W' \dashrightarrow \\
\downarrow & & \downarrow & & \\
Q & = & Q & & \\
\downarrow & & \downarrow & &
\end{array}$$

of $\mathbb{E}$-triangles. By Claim I, there exists a commutative diagram

$$\begin{array}{ccccc}
\widetilde{R} & \longrightarrow & \widetilde{R''} & \longrightarrow & \widetilde{R'} \dashrightarrow \\
\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\
\widetilde{Q} & \longrightarrow & \widetilde{Q''} & \longrightarrow & \widetilde{Q'} \dashrightarrow \\
\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\
R & \longrightarrow & M & \longrightarrow & W' \dashrightarrow \\
\downarrow & & \downarrow & & \downarrow
\end{array}$$

of $\mathbb{E}$-triangles, where $\widetilde{Q}, \widetilde{Q'}, \widetilde{Q''} \in \widetilde{\mathcal{Q}}$ and $\widetilde{R}, \widetilde{R'}, \widetilde{R''} \in \widetilde{\mathcal{R}}$. By (ET4$^{\text{op}}$), we have a commutative diagram

$$\begin{array}{ccccc}
\widetilde{R''} & \longrightarrow & L & \longrightarrow & Y \dashrightarrow \\
\| & & \downarrow & & \downarrow \\
\widetilde{R''} & \longrightarrow & \widetilde{Q''} & \longrightarrow & M \dashrightarrow \\
 & & \downarrow & & \downarrow \\
 & & Q & = & Q \\
 & & \downarrow & & \downarrow
\end{array}$$

of $\mathbb{E}$-triangles. Since $\widetilde{\mathcal{Q}}$ is closed under cocones of deflations, we have $L \in \widetilde{\mathcal{Q}}$. Therefore $Y \in \mathcal{W}$. This shows that $\mathcal{W}$ is closed under extensions.

- Now we show that $\mathcal{W}$ satisfies the 2-out-of-3 property.

Let $W \to W' \to Z \dashrightarrow$ be an $\mathbb{E}$-triangle with $W, W' \in \mathcal{W}$. We need to show that $Z \in \mathcal{W}$ too. A dual argument shows that if $X \to W' \to W \dashrightarrow$ is an $\mathbb{E}$-triangle with $W, W' \in \mathcal{W}$, then $X \in \mathcal{W}$.

Since $W \in \mathcal{W}$, there exists an $\mathbb{E}$-triangle $W \to \widetilde{R} \to \widetilde{Q} \dashrightarrow$ where $\widetilde{R} \in \widetilde{\mathcal{R}}$ and $\widetilde{Q} \in \widetilde{\mathcal{Q}}$. By [14, Proposition 3.15], we have a commutative diagram

$$\begin{array}{ccccc}
W & \longrightarrow & W' & \longrightarrow & Z \dashrightarrow \\
\downarrow & & \downarrow & & \| \\
\widetilde{R} & \longrightarrow & M & \longrightarrow & Z \dashrightarrow \\
\downarrow & & \downarrow & & \\
\widetilde{Q} & = & \widetilde{Q} & & \\
\vdots & & \vdots & & \\
\downarrow & & \downarrow & &
\end{array}$$

of $\mathbb{E}$-triangles. Since $\mathcal{W}$ is closed under extensions, we have $M \in \mathcal{W}$. So there exists an $\mathbb{E}$-triangle $M \to \widetilde{R'} \to \widetilde{Q'} \dashrightarrow$ where $\widetilde{R'} \in \widetilde{\mathcal{R}}$ and $\widetilde{Q'} \in \widetilde{\mathcal{Q}}$. By (ET4), we have a commutative diagram

$$\begin{array}{ccccc}
\widetilde{R} & \longrightarrow & M & \longrightarrow & Z \dashrightarrow \\
\| & & \downarrow & & \downarrow \\
\widetilde{R} & \longrightarrow & \widetilde{R'} & \longrightarrow & L \dashrightarrow \\
 & & \downarrow & & \downarrow \\
 & & \widetilde{Q'} & = & \widetilde{Q'} \\
 & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow
\end{array}$$

of $\mathbb{E}$-triangles. Since $\widetilde{\mathcal{R}}$ is closed under cones of inflations, we have $L \in \widetilde{\mathcal{R}}$. Hence $Z \in \mathcal{W}$.

This completes the proof that $\mathcal{W}$ is thick.

(3) Now we show that $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is a Hovey triple. It suffices to show $\mathcal{W} \cap \mathcal{R} = \widetilde{R}$ and $\mathcal{Q} \cap \mathcal{W} = \widetilde{Q}$.

We only prove that $\mathcal{W} \cap \mathcal{R} = \widetilde{R}$; the other proof is similar.

It is clear that $\widetilde{\mathcal{R}} \subseteq \mathcal{W} \cap \mathcal{R}$. Conversely, for any $X \in \mathcal{W} \cap \mathcal{R}$, there exists an $\mathbb{E}$-triangle $X \to \widetilde{R} \to \widetilde{Q} \dashrightarrow$ where $\widetilde{R} \in \widetilde{\mathcal{R}}$ and $\widetilde{Q} \in \widetilde{\mathcal{Q}}$. Since $X \in \mathcal{R}$, we have $\mathbb{E}(\widetilde{Q}, X) = 0$. We conclude that the above $\mathbb{E}$-triangle splits. Thus X is a direct summand of $\widetilde{R}$. It follows that $X \in \widetilde{\mathcal{R}}$. This completes the proof that $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is a Hovey triple.

(4) Now we show that $\mathcal{W}$ is unique.

Suppose that $(\mathcal{Q}, \mathcal{V}, \mathcal{R})$ is an arbitrary Hovey triple. Then $\mathcal{V} \cap \mathcal{R} = \widetilde{\mathcal{R}} = \mathcal{W} \cap \mathcal{R}$, $\mathcal{Q} \cap \mathcal{V} = \widetilde{Q} = \mathcal{Q} \cap \mathcal{W}$, and by applying the definitions of $\mathcal{W}$ and $\mathcal{V}$ we obtain $\mathcal{W} = \mathcal{V}$. This completes the proof of the theorem. ∎

When our main result is applied to an exact category, we obtain

COROLLARY 3.10 ([4, Theorem 1.2]). *Assume that $\mathcal{C}$ is a weakly idempotent complete exact category. Let $(\mathcal{Q}, \widetilde{\mathcal{R}})$ and $(\widetilde{\mathcal{Q}}, \mathcal{R})$ be hereditary cotorsion pairs with $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$, $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$ and $\widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{R}$. Then there exists a unique thick class $\mathcal{W}$ for which $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is a Hovey triple. Moreover, $\mathcal{W}$ can be described in the following two ways:*

$$\begin{aligned}\mathcal{W} &= \{X \in \mathcal{C} \mid \exists \text{ an exact sequence } X \rightarrowtail R \twoheadrightarrow Q \text{ with } R \in \widetilde{\mathcal{R}},\ Q \in \widetilde{\mathcal{Q}}\} \\ &= \{X \in \mathcal{C} \mid \exists \text{ an exact sequence } R' \rightarrowtail Q' \twoheadrightarrow X \text{ with } R' \in \widetilde{\mathcal{R}},\ Q' \in \widetilde{\mathcal{Q}}\}.\end{aligned}$$

Applying our main result to a triangulated category, we get

COROLLARY 3.11. *Assume that $\mathcal{C}$ is a triangulated category. Let $(\mathcal{Q}, \widetilde{\mathcal{R}})$ and $(\widetilde{\mathcal{Q}}, \mathcal{R})$ be hereditary cotorsion pairs with $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$, $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$ and $\widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{R}$. Then there exists a unique thick class $\mathcal{W}$ for which $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is a Hovey triple. Moreover, $\mathcal{W}$ can be described in the following two ways:*

$$\begin{aligned}\mathcal{W} &= \{X \in \mathcal{C} \mid \exists \text{ a triangle } X \to R \to Q \to X[1] \text{ with } R \in \widetilde{\mathcal{R}},\ Q \in \widetilde{\mathcal{Q}}\} \\ &= \{X \in \mathcal{C} \mid \exists \text{ a triangle } R' \to Q' \to X \to R'[1] \text{ with } R' \in \widetilde{\mathcal{R}},\ Q' \in \widetilde{\mathcal{Q}}\}.\end{aligned}$$

We end this article with an example.

EXAMPLE 3.12. Assume that R is a unitary ring. Denote by $\operatorname{Mod} R$ the category of left R-modules. Following [2], given a cotorsion pair $(\mathscr{X}, \mathscr{Y})$ in $\operatorname{Mod} R$, define the following three classes in $\operatorname{Mod} R$:

(1) $dw\mathscr{X}$ is the class of all chain complexes of R-modules with $X_n \in \mathscr{X}$ for any $n \in \mathbb{Z}$.
(2) $\widetilde{\mathscr{X}}$ is the class of all exact chain complexes of R-modules such that $Z_n(X) \in \mathscr{X}$ for any $n \in \mathbb{Z}$.
(3) $dg\mathscr{X}$ is the class of all chain complexes of R-modules such that $X_n \in \mathscr{X}$ for any $n \in \mathbb{Z}$ and every chain map $f\colon X \to Y$ is null homotopic whenever $Y \in \widetilde{\mathscr{Y}}$.

Analogously we define the classes $dw\mathscr{Y}, \widetilde{\mathscr{Y}}$ and $dg\mathscr{Y}$.

It is well known that $(\operatorname{Mod} R, \mathcal{I})$ is a hereditary complete cotorsion pair in $\operatorname{Mod} R$.

By [2, Lemma 4.2], we know that $(\mathcal{E}, dg\mathcal{I})$ and $({}^{\perp}dw\mathcal{I}, dw\mathcal{I})$ are hereditary cotorsion pairs in $\mathbf{K}(R)$, where ${}^{\perp}dw\mathcal{I}$ is the class of all complexes X such that $\operatorname{Ext}^1(X, Y) = 0$ for all $Y \in dw\mathcal{I}$. It is clear that $dg\mathcal{I} \subseteq dw\mathcal{I}$, ${}^{\perp}dw\mathcal{I} \subseteq \mathcal{E}$ and

$$\widetilde{\mathcal{I}} = \mathcal{E} \cap dg\mathcal{I} = {}^{\perp}dw\mathcal{I} \cap dw\mathcal{I}.$$

Thus Corollary 3.11 yields a model structure on $\mathbf{K}(R)$ starting from the single cotorsion pair $(\operatorname{Mod} R, \mathcal{I})$ in $\operatorname{Mod} R$.

By [2, Lemma 4.5], we know that $(dw\mathcal{P}, dw\mathcal{P}^{\perp})$ and $(dg\mathcal{P}, \mathcal{E})$ are hereditary cotorsion pairs in $\mathbf{K}(R)$, where $dw\mathcal{P}^{\perp}$ is the class of all complexes Y such that $\mathrm{Ext}^1(X, Y) = 0$ for all $X \in dw\mathcal{P}$. It is clear that $dg\mathcal{P} \subseteq dw\mathcal{P}$, $dw\mathcal{P}^{\perp} \subseteq \mathcal{E}$, and

$$\widetilde{\mathcal{P}} = dw\mathcal{P} \cap dw\mathcal{P}^{\perp} = dg\mathcal{P} \cap \mathcal{E}.$$

Thus Corollary 3.11 yields a model structure on $\mathbf{K}(R)$ starting from the single cotorsion pair $(\mathcal{P}, \mathrm{Mod}\, R)$ in $\mathrm{Mod}\, R$.

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