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ON THE STRUCTURE OF METABELIAN GALOIS COVERINGS OF COMPLEX ALGEBRAIC VARIETIES

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Abstract. We describe the general structure of metabelian branched covers of smooth complex algebraic varieties. The main question that we consider is the following: given a smooth algebraic variety Y, which "building data" on Y determine a metabelian cover $f : X \to Y$? We prove a structure theorem that answers this question completely. In order to achieve this, we extend the structure theorem for abelian covers of smooth varieties to the case where the total space and the base are normal varieties. For such abelian covers, we also provide a canonical bundle formula. For a metabelian cover $f : X \to Y$, we obtain a formula which relates the canonical bundle of X to that of Y. Finally, we apply our results to the special case of metacyclic covers where we determine their basic invariants as well as their function fields.

1. Introduction. In [CP17], F. Catanese and F. Perroni described the algebro-geometric properties of the dihedral Galois covers of algebraic varieties using the theory of cyclic covers from [C10] to construct building data for the dihedral covers. Earlier results about the dihedral covers of algebraic curves were obtained in [CLP11] by F. Catanese, M. Lönne and F. Perroni. One of the most important properties of the dihedral group, of which the analysis in [CP17] takes advantage, is that D_n is a *metacyclic group.* This means that D_n sits in the short exact sequence of groups

$$0 \to \mathbb{Z}/n\mathbb{Z} \to D_n \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

The above factorization gives a corresponding factorization $\pi : X \xrightarrow{p} Z \xrightarrow{q} Y$, where p, q are cyclic covers and hence the theory in [C10] can be applied. This motivates considering the metacyclic or more generally *metabelian* covers (see Definition 3.1) and trying to generalize the results of [CP17] by techniques of abelian covers. This is the aim of the present paper.

As in [CLP11, CP17] we consider the G-sheaf structure on $\pi_*\mathcal{O}_X$ in order to analyze the structure of the cover. Using the complex representation theory of metabelian and metacyclic groups, we also determine the eigenspaces

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of the action of the group on the direct image sheaf $\pi_*\mathcal{O}_X$. Furthermore, we describe the function fields of metabelian covers. We also explore several examples of metabelian covers along with their properties.

The study of Galois coverings of algebraic varieties, i.e. maps $\pi : X \to Y$ such that π exhibits Y as a quotient X/G by a finite group G has been an interesting and fruitful topic of research with many applications in algebraic geometry. As applications, one can mention the construction of algebraic surfaces with prescribed Chern invariants [Per81] and the use of abelian covers in [C84] to show that certain surfaces constructed as bidouble covers are simply connected, as well as the role of branch covers in the study of Shimura subvarieties in the moduli space of abelian varieties in [MZ18, M21]. For another interesting application, see [P22]. However, in most of these cases the Galois groups of the coverings under consideration were abelian. This is due to a variety of reasons, e.g., simple representation theory of abelian groups or more basic equations of the abelian covers.

According to [CP17], Comessatti [Com30] was the first to study Galois covers with abelian Galois group G and their relations to topology. In [P91] the algebraic structure of abelian covers of smooth algebraic varieties over any algebraically closed field of characteristic coprime to the order of the Galois group is described.

The above mentioned results motivate the study of non-abelian Galois covers using $\mathcal{O}_X[G]$-sheaves to construct building data for the cover. The non-abelian Galois covers are, at least in comparison with the abelian case, relatively unexplored. The studies are mostly dedicated to elementary and small non-abelian groups; see for example [Ea11].

2. The structure of abelian Galois coverings. In this section we discuss the construction of abelian covers of smooth varieties using the building data. Subsequently, we extend these results to the case of abelian covers of normal varieties using the theory of reflexive sheaves.

Let G be a finite abelian group of order m and let $G^* = \mathrm{Hom}(G, \mathbb{C}^*)$ be the group of characters of G. We choose a primitive mth root of unity ξ_m.

Note that if G is a finite abelian group, then G^* is isomorphic to G. To see this, first assume that $G = \mathbb{Z}/N$ is a cyclic group. We fix an isomorphism between $\mathbb{Z}/N$ and the group of Nth roots of unity in $\mathbb{C}^*$ via $1 \mapsto \exp(2\pi i/N)$. Now the group G^* is isomorphic to this latter group via $\chi \mapsto \chi(1)$. In the general case, we can extend this to an isomorphism $\varphi_G : G \xrightarrow{\sim} G^*$ because G is a product of finite cyclic groups. For later applications, we fix this isomorphism between G and G^*.

Let W be a smooth complex algebraic variety and V a normal one and let $f : V \to W$ be an abelian G-Galois covering. By this we mean precisely that there exists a finite abelian group G together with a faithful action of

G on V such that f exhibits W as the quotient of V via G. Such Galois coverings of algebraic varieties have been extensively studied, especially in [C10] and [P91] on which our treatment in this section is based.

Our assumptions imply that there is a decomposition $f_*\mathcal{O}_V = \bigoplus_{\chi\in G^*} L_\chi$, where each L_χ is an invertible sheaf on W on which G acts by the character χ. So in particular, the invariant summand L_1 is isomorphic to $\mathcal{O}_W$. The sheaf $f_*\mathcal{O}_V$ carries a natural structure of a sheaf of $\mathcal{O}_W$-algebras, which is given by the product of regular functions $f_*\mathcal{O}_V \otimes f_*\mathcal{O}_V \to f_*\mathcal{O}_V$ on V. The G-action on V gives $f_*\mathcal{O}_V$ the structure of a G-sheaf by setting $g\cdot h(x) = h(g\cdot x)$ for every local section h of $f_*\mathcal{O}_V$ over an open set U, $x \in f^{-1}(U)$ and $g \in G$. The eigensheaf L_χ consists of regular functions h such that $h(g\cdot x) = \chi(g)h(x)$ for every $g \in G$. The algebra structure on $f_*\mathcal{O}_V$ is then compatible with the G-action. Recall that a G-action on an algebra $\mathcal{A}$ is said to be *compatible* with the algebra structure if $g\cdot(a_1a_2) = (g\cdot a_1)(g\cdot a_2)$ for every $a_1, a_2 \in \mathcal{A}$ and $g \in G$. So the algebra structure on $f_*\mathcal{O}_V$ is determined by $\mathcal{O}_W$-linear multiplication maps $\mu_{\chi\chi'} : L_\chi \otimes L_{\chi'} \to L_{\chi\chi'}$ for all $\chi, \chi' \in G^*$.

Consider the ramification and branch loci R, D of f. Note that R consists precisely of the points in V with non-trivial stabilizer under the action of G. The smoothness of W implies that D is a Cartier divisor and R is $\mathbb{Q}$-Cartier. For every component T of R, the subgroup of G fixing all elements of T pointwise is a cyclic subgroup H_T, called the *inertia group* of T. Also, if D_i is an irreducible component of the branch locus D, then all the components of $f^{-1}(D_i)$ have the same inertia group, since the group G is abelian. So it makes sense to associate to every component D_i a cyclic subgroup H_i of G. The order m_i of H_i is equal to the ramification order of f over D_i and the representation of H_i obtained by taking differentials and restricting to the normal space to D_i is the faithful character χ_i. The generator of H_i which is mapped to $\xi_m^{m/m_i} = \xi_{m_i}$ by χ_i will be denoted by h_i.

Let $D_1, \dots, D_r$ be the irreducible components of D. Let $\mathcal{C}$ denote the set of cyclic subgroups of G and for $H \in \mathcal{C}$ denote by S_H the set of generators of the group of characters H^*. Then we may write $D = \sum_{H\in\mathcal{C}}\sum_{\psi\in S_H} D_{H,\psi}$, where $D_{H,\psi}$ is the sum of all components of D that have inertia group H and character ψ. For any $\chi, \chi' \in G^*$, $H \in \mathcal{C}$ and $\psi \in S_H$, we may write $\chi|_H = \psi^{i_\chi}$ and $\chi'|_H = \psi^{i_{\chi'}}$, $i_\chi, i_{\chi'} \in \{0, \dots, |H|-1\}$. Define

$$\epsilon^{H,\psi}_{\chi,\chi'} = \begin{cases} 0, & i_\chi + i_{\chi'} < |H|, \\ 1, & \text{otherwise.} \end{cases}$$

Set $D_{\chi,\chi'} = \sum_{H\in\mathcal{C}}\sum_{\psi\in S_H} \epsilon^{H,\psi}_{\chi,\chi'} D_{H,\psi}$. The line bundles L_χ and divisors D_i are called the building data of the cover. These data are to satisfy the so-called *fundamental relations* and determine the cover $f : V \to W$ up to deck automorphisms. The fundamental relations of the cover are

$$L_\chi + L_{\chi'} \equiv L_{\chi\chi'} + D_{\chi,\chi'}. \tag{2.1}$$

In particular, if $\chi' = \chi^{-1}$, then

$$L_\chi + L_{\chi^{-1}} \equiv D_{\chi,\chi^{-1}}, \tag{2.2}$$

and $D_{\chi,\chi^{-1}}$ is the sum of the components D_i, where $\chi(h_i) \neq 1$. The cover $f : V \to W$ can be constructed from the fundamental relations (2.1) by first defining the variety V inside the vector bundle $\mathcal{L} = \bigoplus_{\chi\neq 1} L_\chi$ by the equations

$$z_\chi z_{\chi'} = \Big(\prod_{H,\psi} s_{H,\psi}^{\epsilon^{H,\psi}_{\chi,\chi'}}\Big) z_{\chi\chi'} \tag{2.3}$$

where z_χ is the fiber coordinate of the bundle L_χ which can also be viewed as the tautological section of the pull-back of the bundle L_χ to $\mathcal{L}$ and $s_{D_{H,\psi}} \in H^0(W, \mathcal{O}_W(D_{H,\psi}))$ is the (pull-back to $\mathcal{L}$ of the) defining equation for $D_{H,\psi}$. This is naturally a W-scheme and is flat over W. Conversely, for every choice of the sections $s_{D_{H,\psi}}$, equations (2.3) define a scheme V flat over W which is smooth if and only if each D_i is smooth, the union $\bigcup D_i$ has at most normal crossing singularities and at any intersection points in $\bigcap D_{H_l,\psi_l}$, the product $\prod_l H_l$ injects into G. We therefore have the following theorem proven in [P91].

THEOREM 2.1. *Let G be a finite abelian group. Let V be a normal algebraic variety and W a smooth one and let $f : V \to W$ be an abelian cover with Galois group G. With the notations as above, the following linear equivalences hold:*

$$L_\chi + L_{\chi'} \equiv L_{\chi\chi'} + D_{\chi,\chi'} \quad \forall \chi, \chi' \in G^*. \tag{2.4}$$

Conversely, a set $\{L_\chi\}_{\chi\in G^*}, \{D_{\chi,\chi'}\}$ of data consisting respectively of invertible sheaves and reduced effective divisors on W satisfying (2.4) determines an abelian cover. If W is furthermore complete, the covering f is determined by the associated building data up to isomorphism.

Next, let V be irreducible normal but W not necessarily smooth. Note that since $W = V/G$, it follows that W is also an irreducible normal variety. Consider the smooth locus W^0 and set $V^0 := f^{-1}(W^0)$. Then the cover $f^0 := f\,|_{V^0} : V^0 \to W^0$ is an abelian cover of the type that we described in Theorem 2.1. In particular, it is determined by the line bundles $(L^0_\chi)_{\chi\in G^*}$ such that $f^0_*\mathcal{O}_{V^0} = \bigoplus_{\chi\in G^*} L^0_\chi$ and reduced effective divisors (D^0_i) on W^0 without common components such that (2.4) holds. Let us denote the natural inclusions by $i : W^0 \to W$ and $\iota : V^0 \to V$. We then have the commutative diagram

$$\begin{array}{ccc} V^0 & \xrightarrow{\iota} & V \\ {\scriptstyle f^0}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ W^0 & \xrightarrow{i} & W \end{array} \tag{2.5}$$

As $\mathcal{O}_V = \iota_*\mathcal{O}_{V^0}$ and $\mathcal{O}_W = i_*\mathcal{O}_{W^0}$, one obtains $f_*\mathcal{O}_V = (f\circ\iota)_*\mathcal{O}_{V^0} = i_*f^0_*\mathcal{O}_{V^0} = \bigoplus_{\chi\in G^*} i_*\mathcal{O}_{W^0}(L^0_\chi)$. This motivates defining $\mathcal{F}_\chi = i_*\mathcal{O}_{W^0}(L^0_\chi)$. Since W is a normal variety and L^0_χ is a line bundle on W^0, it follows that $\mathcal{F}_\chi$ is a reflexive sheaf on W, and any such sheaf is uniquely determined by its restriction to W^0. The multiplication map $\mu : f_*\mathcal{O}_V \otimes_{\mathcal{O}_W} f_*\mathcal{O}_V \to f_*\mathcal{O}_V$ gives rise to $\mu_{\chi\chi'} : \mathcal{F}_\chi \otimes \mathcal{F}_{\chi\prime} \to \mathcal{F}_{\chi\chi'}$ whose restriction to W^0 is (2.4): $\mathcal{O}_{W^0}(L^0_\chi + L^0_{\chi\prime}) = \mathcal{O}_{W^0}(L^0_{\chi\chi'} + D^0_{\chi,\chi'})$. Take $D_i = \overline{D^0_i}$ to be the closure of the divisor D^0_i mentioned above. The multiplication map is fully determined by its restriction to W^0 and the following generalization of Theorem 2.1 holds. We remark that this theorem is proved (somewhat implicitly) for double covers of normal varieties in [CP17, pp. 80, 83] using a theory of double covers developed in [CP17]. Here we establish the result for all abelian covers using instead a result of [H80] which states that a reflexive rank 1 sheaf on a regular scheme is invertible.

THEOREM 2.2. *Let G be an abelian group. Let V be a normal algebraic variety and let $f : V \to W$ be an abelian cover with the Galois group G. With the notations as above, there exist a collection of rank 1 reflexive sheaves $(\mathcal{F}_\chi)_{\chi\in G^*}$ and divisors (D_i) on W such that the following linear equivalences hold:*

$$\mathcal{F}_\chi + \mathcal{F}_{\chi'} \equiv \mathcal{F}_{\chi\chi'} + D_{\chi,\chi'} \qquad \forall \chi,\chi' \in G^*. \tag{2.6}$$

Conversely, the set $\{\mathcal{F}_\chi\}_{\chi\in G^*}, \{D_{\chi,\chi'}\}$ of data as above satisfying (2.6) determines an abelian cover.

Proof. The above explanations prove the first part of the theorem. For the converse, assume that $\{\mathcal{F}_\chi\}_{\chi\in G^*}, \{D_{\chi,\chi'}\}$ are given sets of rank 1 reflexive sheaves and divisors subject to (2.6). Write $V = \mathrm{Spec}(\bigoplus_{\chi\in G^*}\mathcal{F}_\chi)$, where $\bigoplus_{\chi\in G^*}\mathcal{F}_\chi$ is a sheaf of $\mathcal{O}_W$-algebras with the algebra structure given by (2.6). Let $L_\chi = i^*\mathcal{F}_\chi = \mathcal{F}_\chi|_{W^0}$ be the restriction of $\mathcal{F}_\chi$ to W^0. Since $\mathcal{F}_\chi$ is a rank 1 reflexive sheaf, it follows from [H80, Prop. 1.9] that L_χ is an invertible sheaf on W^0. Furthermore, $\mathcal{F}_\chi = i_*L_\chi$ so that L_χ is uniquely determined by $\mathcal{F}_\chi$. By restricting the divisors D_i to divisors D^0_i on W^0, relation (2.6) restricts to relations (2.1): $L_\chi + L_{\chi'} \equiv L_{\chi\chi'} + D^0_{\chi,\chi'}$. Now by Theorem 2.1 these relations determine the abelian cover as claimed. ∎

2.1. A canonical bundle formula. In this subsection, we prove a canonical bundle formula for abelian covers $f : V \to W$ with V and W normal varieties. The case where V, W are smooth and the cover is cyclic is standard and is treated for example in [BHPV04, Lemma 17.1]. We first assume that W is smooth and extend this result to the case of abelian covers. In this case, the branch locus D is a divisor whose irreducible components will be denoted by D_i. Note the decomposition $f_*\mathcal{O}_V = \bigoplus_{\chi\in G^*} L_\chi$. We have already remarked that the scheme V is given inside the (total space of the) vector bundle $\mathcal{L} = \bigoplus_{\chi\in G^*\setminus\{1\}} L_\chi$ by equations (2.3) in terms of the tautological section z_χ of the pull-back of L_χ to $\mathcal{L}$ and the defining equation $s_i \in H^0(W, \mathcal{O}_W(D_i))$ for D_i. One can embed W in $\mathcal{L}$ by the zero section of $p : \mathcal{L} \to W$, where p denotes the bundle projection (we will use the same notation for $\mathcal{L}$ and its total space). Let the branch divisor D be reduced. As a closed subscheme, W is given inside $\mathcal{L}$ by the equations $z_\chi = 0$. Let R be the ramification locus of f. Recall that this means that R is the set of points of V with non-trivial stabilizer. We remark again that since W is smooth and V is normal, R is a $\mathbb{Q}$-Cartier divisor. Suppose $H_j = \langle h_j\rangle$, $j = 1, \ldots, s$, are the (non-trivial) inertia groups of f and let R_j be the divisorial part of the reduced ramification divisor R_{red} consisting of all points that have H_j as their stabilizer. Then $e_j = |H_j|$ is the ramification index of R_j. Consider the group of characters G^* and suppose χ_j is the character corresponding to h_j under the isomorphism $\varphi_G : G \xrightarrow{\sim} G^*$ mentioned on page 3. Let $L_j = L_{\chi_j}$ be the line bundle associated to χ_j for $j = 1, \ldots, s$. It is clear that $R_{\mathrm{red}} = R_1 + \cdots + R_s$. Note that (2.3) together with the smoothness of W implies that V has at most singularities over singular points of the branch divisor D.

Theorem 2.3. *Let G be a finite abelian group. Let V be a normal algebraic variety and W a smooth one and let $f : V \to W$ be an abelian cover of degree n with Galois group G such that the linear equivalences* (2.4) *hold. Then:*

(1) $\mathcal{O}_V(R_{\mathrm{red}}) = f^*(\bigotimes_{j=1}^s L_j)$.
(2) $f^*D = \sum e_j R_j$.
(3) $K_V = f^*(K_W \otimes \bigotimes_{j=1}^s L_j^{e_j-1})$.

Proof. Consider the vector bundle $\mathcal{L} = \bigoplus_{\chi\in G^*\setminus\{1\}} L_\chi$ with projection $p : \mathcal{L} \to W$ as above and note that $f = p|_V$. Now consider the subbundle $L_j \subset \mathcal{L}$ as above for every $j = 1, \ldots, s$ and the projection $p_j = p|_{L_j} : L_j \to W$. One can embed W in $\mathcal{L}$ and in L_j via the zero section. Then W is given by the equation $z_j = z_{\chi_j} = 0$ inside L_j and by the equations $z_\chi = 0$ for all $\chi \in G^*$ inside $\mathcal{L}$. In particular, W is the divisor of z_j in L_j and hence $\mathcal{O}_{L_j}(W) = p_j^* L_j$. Inside $\mathcal{L}$, consider the intersection $V \cap W$ which, as a subset of V, coincides with the reduced divisor R_{red} on V. For the jth component R_j we have $\mathcal{O}_V(R_j) = \mathcal{O}_{L_j}(W)|_V = f^*(L_j)$. All of the assertions now follow from this together with the equality $R_{\mathrm{red}} = R_1 + \cdots + R_s$, the definition of the ramification index and the canonical bundle formula. ∎

Now we consider the case where W is normal but not necessarily smooth and $f : V \to W$ is an abelian cover with V normal. Let us recall that since W is normal, the branch locus D of f is precisely the set of points of W over which f is not a local isomorphism. Grauert–Remmert's extension of Riemann's existence theorem [GR58] asserts that there is a bijection between the set of isomorphism classes of finite coverings $f : V \to W$ between normal varieties V and W and isomorphism classes of maximal connected topological coverings $V^* \to U$, where $U \subseteq W$ is a Zariski open set in W. In view of this, the branch locus D is the complement of the maximal open subset U over which f is a topological (unramified) covering. Recall that if W is smooth, then D is an algebraic subset of pure codimension 1 [Zar58]. As an example where W is singular, suppose W is a surface with an A_1 singularity. This singularity has the local model $x^2 + y^2 + z^2 = 0$. Such a surface can be obtained as follows: Let $G = \mathbb{Z}_2$ act on $\mathbb{A}^2$ by $u, v \mapsto -u, -v$, where u and v are local coordinates. The quotient $\mathbb{A}^2 \to \mathbb{A}^2/G$ is a double cover and the only singularity of $\mathbb{A}^2/G$ is an A_1 singularity. One can identify $\mathbb{A}^2/G$ with $W : (xz = y^2) \subset \mathbb{A}^3$. Here W is a cone with vertex $P = (0, 0)$. To resolve the singularity, one can blow up W at the singular point P. The cone W is a union of generating lines l through P and the resolution $\varphi : Z \to W$ is the disjoint union of these lines and hence can be identified with a cylinder. The exceptional curve Γ is a rational (-2)-curve. Note that one can resolve the singularity of W by first blowing up $\mathbb{A}^2$ and at P and then dividing out by the action of G. However, in general there may be no relation between resolving the singularities of a quotient variety V/G, the blow-ups of V and the action of G. Moreover, $K_Z = \varphi^* K_X$. In other words, K_Z is trivial on a neighborhood of the exceptional locus.

We use the notations and assumptions of Theorem 2.2. Let $\iota : V^0 \hookrightarrow V$ and $i : W^0 \hookrightarrow W$ be as in (2.5). By Theorem 2.2 there exist reflexive rank 1 shaves $\mathcal{F}_\chi$, $\chi \in G^*$, such that $V = \operatorname{Spec}(\bigoplus_{\chi \in G^*} \mathcal{F}_\chi)$, where $\bigoplus_{\chi \in G^*} \mathcal{F}_\chi$ is regarded as a sheaf of $\mathcal{O}_W$-algebras with the algebra structure given by relations (2.6). Let $L^0_\chi = i^* \mathcal{F}_\chi = \mathcal{F}_\chi|_{W^0}$ be the restriction of $\mathcal{F}_\chi$ on W^0. We have seen that L^0_χ is an invertible sheaf on W^0 and $\mathcal{F}_\chi = i_* L_\chi$ so that L^0_χ is uniquely determined by $\mathcal{F}_\chi$. Let $R^0 = \sum e_j R^0_j$ be the ramification divisor of the cover $f^0 : V^0 \to W^0$, with notations of Theorem 2.3. Let $R^0_{\mathrm{red}} = R^0_1 + \cdots + R^0_s$ be the irreducible decomposition of the reduced divisor. The closure R of R^0 in V is also a Weil divisor on V. We set $\mathcal{L}' = \bigotimes_{j=1}^s (L^0_j)^{e_j - 1}$ and correspondingly $\mathcal{F}' = i_* \mathcal{L}' = \bigotimes_{j=1}^s \mathcal{F}_j^{e_j - 1}$. With these notations we have

Proposition 2.4. *Let V and W be normal varieties and $f : V \to W$ be an abelian cover with Galois group G. Then $K_V = f^*(K_W \otimes \mathcal{F}')$.*

Proof. Let $V' = \mathrm{Reg}(V) \cap f^{-1}(\mathrm{Reg}(W)) = V^0 \cap f^{-1}(W^0)$ and $W' = f(V')$. As both V and W are normal and f is finite by assumption, it follows that the complement of V' and hence that of W' is of codimension ≥ 2. As in the proof of Theorem 2.2, the sheaf $\mathcal{F}'$ defined above is reflexive and on W' restricts to $\mathcal{L}'$. Furthermore, $K_V = \iota_* K_{V'}$ and $K_W = i_* K_{W'}$, where $\iota : V' \hookrightarrow V$ (resp. $i : W' \hookrightarrow W$) is the natural inclusion as in (2.5). The claim now follows from this together with Theorem 2.3(3). ■

2.2. Field extensions

2.2.1. *Kummer theory.* Let K be a field that contains n distinct nth roots of unity with $n > 1$. A *Kummer extension* of K is a (finite) Galois field extension L/K whose Galois group G is abelian. Kummer theory asserts that any such Galois extension is formed by adjoining nth roots of elements of K^*. In other words, there are $a_1, \ldots, a_t \in K^*$ such that $L = K(a_1^{1/n}, \ldots, a_t^{1/n})$. Moreover, if K^* is the multiplicative group of K, then Kummer extensions of exponent n are in bijection with subgroups of the group $K^*/(K^*)^n$. The correspondence can be described explicitly as follows. Given a subgroup

$$\Delta \subseteq K^*/(K^*)^n$$

the corresponding extension is given by

$$L = K(\Delta^{1/n}),$$

where $\Delta^{1/n} = \{a^{1/n} \mid a \in K^*,\ a\cdot (K^*)^n \in \Delta\}$. We also have

$$\mathrm{Gal}(L/K) \cong \Delta.$$

Now let $f : V \to W$ be a G-Galois covering of algebraic varieties. It corresponds to a Galois extension $\mathbb{C}(W) \subseteq \mathbb{C}(V)$ such that $\mathrm{Gal}(\mathbb{C}(V)/\mathbb{C}(W)) \cong G$. If $G = \langle \sigma_1, \ldots, \sigma_s \rangle$ is furthermore an abelian group, then $\mathbb{C}(V)/\mathbb{C}(W)$ is a Kummer extension. Let us use the same notation for elements of G, elements of $\mathbb{C}(V)/\mathbb{C}(W)$ and the corresponding automorphisms of V. Setting $\mathrm{ord}(\sigma_i) = m_i$, by the above description one has

$$\mathbb{C}(V) = \mathbb{C}(W)(\sqrt[m_1]{f_1}, \ldots, \sqrt[m_s]{f_s}) = \mathbb{C}(W)(v_1, \ldots, v_s)$$

with $f_i \in \mathbb{C}(W)$ and $\sigma_j \cdot v_i = \xi_{m_i}^{\delta_{ij}} \cdot v_i$, where δ_{ij} is the Kronecker delta and $\xi_{m_i} = \xi_m^{m/m_i}$ is a primitive m_ith root of unity (ξ_m being a primitive mth root of unity).

3. The structure of metabelian Galois coverings

DEFINITION 3.1. A *metabelian* (resp. *metacyclic*) group G is a group that has an abelian (resp. cyclic) normal subgroup A such that G/A is also abelian (resp. cyclic). In other words, it is an extension of an abelian (resp. cyclic) group by an abelian (resp. cyclic) group.

The above definition is equivalent to saying that metabelian groups are precisely the solvable groups of derived length at most 2.

Suppose

$$0 \to A \to G \to N \to 0 \tag{3.1}$$

is the extension mentioned in Definition 3.1 with A, N abelian. It is straightforward to see that the definition of a metabelian group implies that G is metabelian if and only if G has the following presentation:

$$\begin{aligned}\langle \sigma_1, \dots, \sigma_s, \tau_1, \dots, \tau_l \mid \sigma_i\sigma_j = \sigma_j\sigma_i,\, \overline{\tau}_i\overline{\tau}_j = \overline{\tau}_j\overline{\tau}_i,\, \sigma_i^{m_i} = 1,\\ \sigma_i\tau_j = \tau_j\sigma_1^{r_{1ij}}\cdots\sigma_s^{r_{sij}}, \tau_j^{a_j} = \sigma_1^{k_{1j}}\cdots\sigma_s^{k_{sj}}\rangle.\end{aligned} \tag{3.2}$$

Here $A = \langle \sigma_1, \dots, \sigma_s\rangle$ and $N = \langle \overline{\tau}_1, \dots, \overline{\tau}_l\rangle$, and $\overline{\tau}_j$ denotes the image of τ_j in $N = G/A$.

Now let G be a finite metabelian group as above, and X a normal algebraic variety over $\mathbb{C}$ with $G \subset \mathrm{Aut}(X)$ such that $Y := X/G$ is a smooth complex algebraic variety. We are interested in the quotient map $\pi : X \to Y$. The factorization (3.1) gives rise to a factorization $\pi : X \xrightarrow{p} Z \xrightarrow{q} Y$ where p, q are the corresponding intermediate abelian covers, i.e., $p : X \to Z = X/A$ is an abelian Galois covering with Galois group A and $q : Z \to Y = Z/N$ is an abelian covering with Galois group N. Therefore to study the Galois covering $\pi : X \to Y$, it is helpful to study these intermediate abelian coverings. We will use the theory of abelian Galois coverings that we explained in Section 2 to study $\pi : X \to Y$ by looking at these intermediate abelian coverings.

Explicitly, $Z = X/A$ is a normal variety, but in general not smooth. By Theorem 2.2, the cover $p : X \to Z$ is determined by the existence of rank 1 reflexive sheaves $(\mathcal{F}_\chi)_{\chi\in A^*}$, and reduced effective divisors (D_i) on Z without common components such that (2.6) holds. The multiplication map $\mu_{\chi\chi'} : \mathcal{F}_\chi \otimes \mathcal{F}_{\chi'} \to \mathcal{F}_{\chi\chi'}$ is fully determined by its restriction to Z^0.

Let $p^0 : X^0 \to Z^0$ be the restriction of p to the smooth locus and $X^0 = p^{-1}(Z^0)$. This is an abelian cover with branch divisor $D^0 = \sum D_k^0$. Let H_k be the inertia group of order m_k associated to D_k^0 and let h_k be the generator of H_k chosen on page 4. For an irreducible character χ_i of A, let a_i^k be the smallest positive integer such that $\chi_i(h_k) = \xi_m^{ma_i^k/m_k}$. Set

$$\epsilon_{ij}^k = \begin{cases} 1, & a_i^k + a_j^k \geq m_k \\ 0, & a_i^k + a_j^k < m_k. \end{cases}$$

Before stating our structure theorem, let us introduce another notation: Suppose χ is an irreducible character of the abelian group $A = \langle \sigma_1, \dots, \sigma_s\rangle$. Since A is a normal subgroup of G, we have $\tau_j^{-1}\sigma_u\tau_j \in A$ for every $u = 1, \dots, s$ with τ_j as in (3.2). We define a new character $\chi_j^{(1)}$ of A by $\chi_j^{(1)}(\sigma_u) =$

$\chi(\tau_j^{-1}\sigma_u\tau_j)$ for every $u = 1, \ldots, s$. Since χ is irreducible, so is $\chi_j^{(1)}$. In the same manner, one can define a character $\chi_j^{(\gamma)}$ of A for each $\gamma \in \mathbb{N}$ by setting $\chi_j^{(\gamma)}(\sigma_u) = \chi(\tau_j^{-\gamma}\sigma_u\tau_j^{\gamma})$ for every $u = 1, \ldots, s$. By presentation (3.2), it is clear that $\chi_j^{(a_j)} = \chi$.

Now, we are ready to state our first main theorem which describes the structure of metabelian Galois covers.

Theorem 3.2 (Structure theorem for metabelian covers). *A metabelian Galois cover $\pi : X \to Y$ is determined by the following data:*

(1) *Line bundles $(L_\eta)_{\eta \in N^*}$ and reduced effective divisors $B_1, \ldots, B_l$ on Y such that $L_\eta + L_{\eta'} \equiv \sum \epsilon^i_{\eta\eta'} B_i$.*
(2) *Reduced effective Weil divisors $D_1, \ldots, D_n$ on $Z = \mathrm{Spec}(\bigoplus L_\eta^{-1})$ without common components such that $\overline{\tau_j}(D_{\chi_i}) = D_{\chi_{ij}^{(1)}}$, where $\chi_{ij}^{(1)}$ is the character of A associated to χ_i defined above.*
(3) *Rank 1 reflexive sheaves $\mathcal{F}_{\chi_1}, \ldots, \mathcal{F}_{\chi_n}$ on Z such that the linear equivalence (2.6) holds and for every $\gamma \in \mathbb{N}$, $\overline{\tau_j}^{\gamma}(\mathcal{F}_{\chi_i}) = \mathcal{F}_{\chi_{ij}^{(\gamma)}}$, where $\chi_{ij}^{(\gamma)}$ is defined above. Furthermore, $\overline{\tau_j}^{a_j}$ acts on the local sections of $\mathcal{F}_{\chi_i}$ as the identity map.*

Proof. (1) yields a flat abelian cover $q : Z \to Y$, where $Z = \mathrm{Spec}(\bigoplus L_\eta^{-1})$. Next, define $X := \mathrm{Spec}(\bigoplus_{\chi \in A^*} \mathcal{F}_\chi)$. The $\mathcal{O}_Z$-algebra structure is given by the morphisms $\mathcal{F}_\chi \otimes \mathcal{F}_{\chi'} \to \mathcal{F}_{\chi\chi'}$ which are uniquely determined by their restriction to Z^0 as Z is normal. Let g_k be the local equation for D_k, i.e., a function on Z such that $D_k = \{g_k = 0\}$. As $\mathcal{F}_{\chi_i}$ is locally free on Z^0 [H80, Prop. 1.9], choose local generators e_{χ_i} and set $e_{\chi_{ij}^{(\gamma)}} = \overline{\tau_j}^{\gamma *}(e_{\chi_i})$. Let ϵ^k_{ij} be as defined on the previous page. The algebra structure on the restriction of $\bigoplus_{\chi \in N^*} \mathcal{F}_\chi$ to this open subset is given by

$$e_{\chi_i} e_{\chi_j} = e_{\chi_i\chi_j} \prod g_k^{\epsilon^k_{ij}}.$$

If we choose different generators $\tilde{e}_{\chi_i}$ satisfying the same equations $\tilde{e}_{\chi_{ij}^{(\gamma)}} = \overline{\tau_j}^{\gamma *}(\tilde{e}_{\chi_i})$, then we obtain an algebra canonically isomorphic to the previous one. Due to the relation $\overline{\tau_j}(D_{\chi_i}) = D_{\chi_{ij}^{(1)}}$, one concludes that the morphism $\overline{\tau_j}^*$ defines a morphism of $\mathcal{O}_Z$-algebras on $\bigoplus_{\chi \in N^*} \mathcal{F}_\chi$. Finally, the last claim of (3) ensures that the $\overline{\tau_j}^{a_j *}$ satisfy the relations of (3.2). ∎

With the notations of Theorem 3.2, we define $U_\chi := q_*(\mathcal{F}_\chi)$ for every $\chi \in A^*$. These sheaves will be very useful in what follows.

Proposition 3.3. *$\tau_j : X \to X$ induces an isomorphism $\tau_j^* : U_{\chi_i} \to U_{\chi_{ij}^{(1)}}$ such that $\overline{\tau}_j^{*a_j} : U_{\chi_i} \to U_{\chi_i}$ is the identity.*

Proof. Let $V \subset Y$ be an open set. Then $U_{\chi_i}(V) = \mathcal{F}_{\chi_i}(q^{-1}(V))$ is the set of regular functions $f \in \mathcal{O}_X(\pi^{-1}(V))$ such that $\sigma_u^* f = \chi_i(\sigma_u) f$ for every $u = 1, \ldots, s$. It follows from the presentation (3.2) that $f \mapsto \tau_j^* f$ induces a morphism $\tau^* : U_i \to U_{\chi_{ij}^{(1)}}$ of $\mathcal{O}_Y$-modules. Since $\overline{\tau}_j^{a_j} = 1$, we have $(\overline{\tau}_j^*)^{a_j} = 1$ and $\overline{\tau}_j$ is an isomorphism. ■

Using Proposition 3.3, we can analyze the local behavior of the intermediate branch divisor D_p (of the map $p : X \to Z$) and the sheaves U_χ as follows: Since $\overline{\tau}_j$ induces a bijection $A^* \to A^*$ by $\chi_i \mapsto \chi_{ij}^{(1)}$ it follows that it also induces $\overline{\tau}_j^* D_{p^0} = D_{p^0}$, where $p^0 : X^0 \to Z^0$ is the restriction of p to the smooth locus. So $D_{p^0} = \sum_\alpha \left(D_{p^0}\right)_\alpha$, as a Weil divisor on Z^0, is invariant under the action of $\overline{\tau}_j : Z^0 \to Z^0$ induced by τ_j for every j and hence there exists an effective Cartier divisor $\mathit{\Delta}_{p^0}^j$ such that $(\tau_j^0)^* \mathit{\Delta}_{p^0}^j = D_{p^0}$, where $\tau_j^0 = \tau_j|_{Z^0}$. Let $\mathit{\Delta}_{p^0}^{j,\alpha} = (\tau_j^0)^* \left(D_{p^0}\right)_\alpha$. We define

$$\mathit{\Delta}_p^j = \overline{\mathit{\Delta}_{p^0}^j}, \qquad \mathit{\Delta}_p^{j,\alpha} = \overline{\mathit{\Delta}_{p^0}^{j,\alpha}}.$$

Let $\mu_{\chi,\chi'} : U_\chi \otimes U_{\chi'} \to U_{\chi\chi'}$ be the multiplication map. Notice that $U_1 = q_* \mathcal{O}_Z$, so in particular

$$\mu_{\chi,\chi^{-1}} : U_\chi \otimes U_{\chi^{-1}} \to U_1 = \mathcal{O}_Y \oplus L_1 \oplus \cdots \oplus L_{t-1},$$

where the L_i are line bundles related to the abelian cover q as in Theorem 3.2. Let us consider the multiplication map $\mu_{ij} := \mu_{\chi_i, (\chi_{ij}^{(1)})^{-1}}$. By using the isomorphism in Proposition 3.3, we obtain a map $\mu_{ij} : U_{\chi_i} \otimes U_{\chi_i^{-1}} \to U_1 = q_* \mathcal{O}_Z = \mathcal{O}_Y \oplus L_1 \oplus \cdots \oplus L_{t-1}$. Write

$$\mu_{ij}^\alpha = \mathrm{pr}^\alpha \circ \mu_{ij} : U_{\chi_i} \otimes U_{\chi_i^{-1}} \to L_\alpha \quad \text{for } 0 \le \alpha \le t-1.$$

We may therefore consider $\mu_{ij}^\alpha \in \mathrm{H}^0(Y, U_{\chi_i} \otimes U_{\chi_i^{-1}} \otimes L_\alpha^{-1})$. For each $y \in Y$, let μ_{ij}^α be the stalk of μ_{ij}^α at y. Note that since Z is normal, the singular locus is of codim ≥ 2, so the divisor of zeros of μ_{ij}^α is determined by its restriction to Z^0 and Y^0. So we may (and do) assume that Z is smooth. If $u_{\chi_i}, u_{\chi_i^{-1}}$ are basis elements of U_{χ_i} and $U_{\chi_i^{-1}}$ respectively, then $u_{\chi_i} \otimes u_{\chi_i^{-1}}$ is a basis for $U_{\chi_i} \otimes U_{\chi_i^{-1}}$. Note that u_{χ_i} and $u_{\chi_i^{-1}}$ are considered as regular functions on a neighborhood of $\pi^{-1}(y)$ such that $\sigma_r^*(u_{\chi_i}) = \chi_i(\sigma_r) u_{\chi_i}$ (analogous for $u_{\chi_i^{-1}}$). Choose local analytic coordinates such that Z is given locally by the equation $z^t = y_1$. Suppose e_{χ_i} is a basis of $\mathcal{F}_{\chi_i}$ and $e_{\chi_i^{-1}}$ is a basis of $\mathcal{F}_{\chi_i^{-1}}$. Let

$$u_{\chi_i} = 1 \cdot e_{\chi_i}, \qquad u_{\chi_i^{-1}} = z \cdot e_{\chi_i^{-1}}.$$

Notice that $\tau_j^*(e_{\chi_i^{-1}}) = e_{(\chi_{ij}^{(1)})^{-1}}$ and $\tau_j^*(z) = \xi_{a_j} \cdot z$. Substituting from the above equalities, we get

$$\mu_{ij}^{\alpha}(u_{\chi_i} \otimes u_{\chi_i^{-1}}) = e_{\chi_i} \cdot \tau_j^*(e_{(\chi_{ij}^{(1)})^{-1}}) = \xi_t \cdot (e_{\chi_i} e_{\chi_i^{-1}})_{\alpha} = z b_{p,\alpha},$$

where $b_{p,\alpha}$ is a local equation for $(D_{p^0})_\alpha$ and the last equality is due to (2.6).

3.1. Invariants of metabelian covers. Consider the metabelian cover $\pi : X \xrightarrow{p} Z \xrightarrow{q} Y$ with X, Y and Z smooth and p, q abelian covers of degree m and t respectively. Using Proposition 2.4, one can compute the invariants of the cover. Indeed, let $\mathcal{L}'_q$ be the reflexive sheaf associated to the abelian cover q by Proposition 2.4 and let $\mathcal{F}_i = \mathcal{F}_{\chi_i^{-1}}$ be the reflexive sheaves in the structure Theorem 3.2. We have

(3.3)
$$\begin{aligned}\pi_*\omega_X = q_*(p_*\omega_X) = q_*(\omega_Z \otimes \bigoplus \mathcal{F}_i)) &= q_*(q^*(\omega_Y \otimes \mathcal{F}'_q) \otimes \bigoplus \mathcal{F}_i) \\ &= q_*((\bigoplus \mathcal{F}_i) \otimes q^*(\omega_Y \otimes \mathcal{F}'_q)) = q_*(\bigoplus \mathcal{F}_i) \otimes \omega_Y \otimes \mathcal{F}'_q \\ &= (\bigoplus U_i) \otimes \omega_Y \otimes \mathcal{F}'_q = \bigoplus(\omega_Y \otimes \mathcal{F}'_q \otimes U_i)).\end{aligned}$$

In the above, $\mathcal{F}'_q = \bigotimes_{j=1}^{s} \mathcal{F}_{\chi_j}^{e_j - 1}$ (e_j is the ramification index as in Theorem 2.3), $U_i = q_*(\mathcal{F}_i)$ and we have used [P91, Prop. 4.1] and the projection formula together with the identification $\omega_Z = q^*(\omega_Y \otimes \mathcal{F}'_q)$ by Theorem 2.3.

The results for metacyclic covers. As mentioned in Definition 3.1, a finite metacyclic group G is an extension of a finite cyclic group by a finite cyclic group, namely the following special case of (3.1):

$$0 \to \langle \sigma \rangle \to G \to \langle \overline{\tau} \rangle \to 0. \tag{3.4}$$

Here $\overline{\tau}$ is the image of some $\tau \in G$ in $G/\langle \sigma \rangle$. As a special case of metabelian groups, metacyclic groups are precisely the groups with the following presentation:

$$G = G_{m,k,t,r} := \langle \sigma, \tau \mid \sigma^m = 1,\ \sigma^k = \tau^t,\ \sigma\tau = \tau\sigma^r \rangle. \tag{3.5}$$

The numbers m, k, t and r are subject to the following conditions:

$$r^t \equiv 1 \pmod{m}, \qquad kr \equiv k \pmod{m}, \qquad \gcd(r, m) = 1. \tag{3.6}$$

Note that it follows from (3.5) that $|G| = mt$. Let us remark that in (3.5), we always assume that $\mathrm{ord}(\sigma) = m$ and $\mathrm{ord}(\overline{\tau}) = t$. This is equivalent to saying that t is the smallest positive integer such that $\tau^t \in \langle \sigma \rangle$. This assumption together with (3.5) and (3.6) implies that $\mathrm{ord}(\tau) = \frac{mt}{\gcd(k,m)}$. In particular, if $\gcd(k, m) = 1$, then G is cyclic. Furthermore, G is abelian if and only if $r \equiv 1 \pmod{m}$. Since we are mainly interested in non-abelian covers, we may (and do) henceforth assume that

$$\gcd(k, m) > 1 \quad \text{and} \quad m \nmid r - 1. \tag{3.7}$$

Moreover, we emphasize that the quadruple (m,k,t,r) is *not* an invariant of the group, i.e., it may happen that $G_{m,k,t,r} \cong G_{m',k',t',r'}$, but $\{m,k,t,r\} \neq \{m',k',t',r'\}$. For instance $G_{36,6,0,19} \cong G_{12,18,0,7}$. If $t = \mathrm{ord}(\tau)$, then we take $k = m$ and the group $G_{m,k,t,r}$ is called *split*. In this case, $G = AN$, where $A = \langle\sigma\rangle$ and $N = \langle\overline{\tau}\rangle$.

Before exploring the metacyclic covers in detail, let us mention that the structure theorem for metabelian covers, Theorem 3.2, takes the following special form when the Galois group is metacyclic.

Theorem 3.4 (Structure theorem for metacyclic covers). *Let Y be a smooth algebraic variety. The following data determine a $G_{m,k,t,r}$-cover $\pi : X \to Y$.*

(1) *A line bundle $\mathcal{L}$ and an effective reduced divisor B_q such that $\mathcal{L}^{\otimes t} = \mathcal{O}_Y(-B_q)$*
(2) *Reduced effective Weil divisors $D_1, \dots, D_m$ on $Z := \mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^{t-1})$ such that $\overline{\tau}(D_i) = D_{\overline{ri}}$. Here $\overline{\tau}$ is an automorphism of order t of $q : Z \to Y$.*
(3) *Rank 1 reflexive sheaves $\mathcal{F}_1, \dots, \mathcal{F}_m$ on Z flat over $\mathcal{O}_Y$ such that $\overline{\tau}(\mathcal{F}_i) = \mathcal{F}_{\overline{ri}}$ and such that $\overline{\tau}^t$ acts on the local sections of $\mathcal{F}_i$ as the identity.*

Proof. This is a special case of Theorem 3.2. However, we remark that in this case, $X := \mathrm{Spec}(\mathcal{O}_Z \oplus \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_{m-1})$ and the $\mathcal{O}_Z$-algebra structure is given by the morphisms $\mathcal{F}_i \otimes \mathcal{F}_j \to \mathcal{F}_{\overline{i+j}}$ (Here $\overline{i+j} \in \{0, 1, \dots, m-1\}$ and $\overline{i+j} = i + j \pmod{m}$) which is uniquely determined by its restriction to Z^0 as Z is normal. As $\mathcal{F}_i$ is locally free on Z^0, there are local generators e_i and we set $e_{\overline{ri}} = \overline{\tau}^*(e_i)$. The algebra structure on the restriction of $\mathcal{O}_Z \oplus \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_{m-1}$ on this local open subset is given by

$$e_i e_j = e_{\overline{i+j}} \prod g_s^{\epsilon_{ij}^s},$$

where g_s is the local equation for D_s and ϵ_{ij}^s was defined before Theorem 3.2. If we choose different generators $\tilde{e}_i$ satisfying the same equations $\tilde{e}_{\overline{ri}} = \overline{\tau}^*(\tilde{e}_i)$, then the algebra structure is isomorphic to the above algebra. Due to relation $\overline{\tau}(D_i) = D_{\overline{ri}}$ one concludes that the morphism $\overline{\tau}^*$ defines a morphism of $\mathcal{O}_Z$-algebras on $\mathcal{O}_Z \oplus \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_{m-1}$. The last assertion follows from (3.5) since $\overline{\tau}^{*t} = \sigma^{k^*}$. ∎

3.2. Complex representations of metacyclic groups. In this section we consider representations of the metacyclic group $G_{m,k,t,r}$. In particular the relation (3.6) holds true. Complex irreducible representations of finite metacyclic groups have been determined in [B79]. Let us explain the results of [B79], albeit with slightly different notation. Let $U_m = \{\zeta \in \mathbb{C} \mid \zeta^m = 1\}$ and consider the map $\alpha_r : U_m \to U_m$, $\zeta \mapsto \zeta^r$. So the group $\langle\alpha_r\rangle$ acts on U_m. For any $\zeta \in U_m$, let $t(\zeta)$ be the size of the orbit of ζ under the action of $\langle\alpha_r\rangle$.

Then $t(\zeta) \mid t$. Let $\{\zeta_1, \dots, \zeta_s\}$ be a set of representatives of the distinct orbits of U_m. Complex irreducible representations are classified as follows. Consider the matrices $T_{\zeta,\theta} \in \mathrm{GL}(t(\zeta), \mathbb{C})$ given by

$$T_{\zeta,\theta}(\sigma) = \begin{bmatrix} \zeta & & & 0 \\ & \zeta^r & & \\ & & \ddots & \\ 0 & & & \zeta^{r^{t(\zeta)-1}} \end{bmatrix}, \quad T_{\zeta,\theta}(\tau) = \begin{bmatrix} 0 & 0 & & \dots & \theta \\ 1 & 0 & & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{3.8}$$

The (inequivalent) complex irreducible representations of $G_{m,k,t,r}$ are precisely $T_{\zeta_i,\theta} \in \mathrm{GL}(t(\zeta_i), \mathbb{C})$ where $1 \le i \le s$ and θ runs over the solutions of $\theta^{t/t(\zeta_i)} = \zeta_i^k$. Therefore the number of (inequivalent) complex irreducible representations of $G_{m,k,t,r}$ is equal to $\nu = t \sum_{i=1}^{s} 1/t(\zeta_i)$. Let $\zeta_i = \xi_m^{l_i}$, where ξ_m is a primitive mth root of unity as before. We have

THEOREM 3.5. *Let Y be a smooth variety and $\pi : X \to Y$ a flat $G_{m,k,t,r}$-cover with X normal and let $p : X \to Z$ and $q : Z \to Y$ be the intermediate coverings of degree m and t respectively. Then*

$$\pi_* \mathcal{O}_X = \bigoplus_{i=1}^{\nu} (\pi_* \mathcal{O}_X)_i, \tag{3.9}$$

where the summands are as follows: If ζ_i is a representative of an orbit with $t(\zeta_i) = t$, then

$$(\pi_* \mathcal{O}_X)_i = U_{l_i} \oplus U_{rl_i} \oplus \cdots \oplus U_{r^{(t(\zeta_i)-1)} l_i}, \tag{3.10}$$

is precisely the eigensheaf corresponding to the irreducible representation $\rho_i = T_{\zeta_i, \zeta_i^k}$ in (3.8), where $U_j = q_*(\mathcal{F}_{\bar{j}})$. If $t(\zeta_i) < t$, then $(\pi_* \mathcal{O}_X)_i$ is the sum of eigensheaves comming from metacyclic intermediate covers $X/H \to Y$.

Proof. If $t(\zeta_i) = t$, then it is clear from (3.8) that the group $G_{m,k,t,r}$ acts on the sheaf $(\pi_* \mathcal{O}_X)_i$ in (3.10) via the irreducible representation ρ_i. On the other hand, such an eigensheaf is a free $\mathcal{O}_Y[G_{m,k,t,r}]$-module of rank $(\dim \rho_i)^2$. Since $t(\zeta_i) = t$, $(\pi_* \mathcal{O}_X)_i$ is of rank $t^2 = t(\zeta_i)^2 = (\dim \rho_i)^2$. Consequently, $(\pi_* \mathcal{O}_X)_i$ is the eigensheaf associated to ρ_i. If $t(\zeta_i) < t$, then the sections of the line bundles associated to irreducible representations in (3.8), are invariant under a non-trivial subgroup H, hence they descend to regular functions on X/H. As any subgroup of a metacyclic group is itself metacyclic, the cover $X/H \to Y$ mentioned above is again a metacyclic cover with Galois group H. ∎

EXAMPLE 3.6. (Catanese–Perroni [CP17, p. 80]) Let $\pi : X \to Y$ be a D_n-cover. If n is odd, there is only one orbit $\{1\}$ in U_n such that $t(1) = 1$ and there are $(n-1)/2$ orbits with $t(\zeta_i) = 2$. The sections arising from the bundle associated to the orbit $\{1\}$ are invariant under the cyclic subgroup $H = \langle\sigma\rangle$, so they descend to sections on Z, hence $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L} \bigoplus_{i=1}^{(n-1)/2}(\pi_*\mathcal{O}_X)_i$, where $\mathcal{O}_Y \oplus \mathcal{L} = q_*\mathcal{O}_Z$. If n is even, there are two orbits $\{1\}, \{-1\}$ such that $t(1) = t(-1) = 1$ and there are $(n-2)/2$ orbits with $t(\zeta_i) = 2$. The sections arising from the bundle associated to the orbits $\{1\}, \{-1\}$ are invariant under the subgroup $H = \langle\sigma^2, \tau\rangle$, so they descend to sections of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover $X/H \to Y$. Consequently, $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L} \oplus \mathcal{M} \oplus \mathcal{N} \bigoplus_{i=1}^{(n-2)/2}(\pi_*\mathcal{O}_X)_i$, where the summands $\mathcal{L}, \mathcal{M}$ and $\mathcal{N}$ correspond to the intermediate abelian cover X/H.

If t is a prime number, Theorem 3.5 gives more information about $G_{m,k,t,r}$-covers.

Corollary 3.7. *Let Y be a smooth complex algebraic variety and $\pi : X \to Y$ a flat $G_{m,k,t,r}$-cover with X normal and suppose that $p : X \to Z$ and $q : Z \to Y$ are the intermediate coverings of degrees m and t respectively, where t is a prime number. Then*

$$\pi_*\mathcal{O}_X = \bigoplus_{i=1}^{\nu}(\pi_*\mathcal{O}_X)_i. \tag{3.11}$$

The eigensheaves $(\pi_*\mathcal{O}_X)_i$ are described as follows: Let $b = \gcd(r-1, m)$ and $h = (m-b)/t$ (this is an integer). If $i \geq h$, then

$$(\pi_*\mathcal{O}_X)_i = U_{l_i} \oplus U_{rl_i} \oplus \cdots \oplus U_{r^{(t(\zeta_i)-1)}l_i} \tag{3.12}$$

is the eigensheaf associated to the irreducible representation T_{ζ_i,ζ_i^k} of (3.8), where $U_j = q_*(\mathcal{F}_{\overline{j}})$.

Proof. By the description of the irreducible representations in (3.8), an $\langle\alpha_r\rangle$-orbit with representative $\zeta_i = \xi_m^{l_i}$ satisfies $t(\zeta_i) = 1$ if and only if $m \mid l_i(r-1)$. Therefore there are $b = \gcd(r-1, m)$ orbits with $t(\zeta_i) = 1$. As t is a prime number and $t(\zeta_i) \mid t$, all the other orbits have size $t(\zeta_i) = t$ and there are $h = (m-b)/t$ of these. Note that (3.7) implies that if G is non-abelian then $h \neq 0$. Theorem 3.5 then yields the remaining assertions. ∎

Proposition 3.8. *The automorphism $\tau : X \to X$ induces an isomorphism $\tau^* : U_i \to U_{\overline{ri}}$ and $(\overline{\tau}^*)^t : U_i \to U_i$ is the identity.*

Proof. Let $V \subset Y$ be an open set. Then $U_i(V) = \mathcal{F}_i(q^{-1}(V))$ is the set of regular functions $f \in \mathcal{O}_X(\pi^{-1}(V))$ such that $\sigma^* f = \exp(\frac{2\pi\sqrt{-1}i}{m})f$. The relation $\sigma\tau = \tau\sigma^r$ implies that $f \mapsto \tau^* f$ induces a morphism $\tau^* : U_i \to U_{\overline{ri}}$ of $\mathcal{O}_Y$-modules. The last claim follows since $r^t \equiv 1 \pmod{m}$ and $\tau^t = \sigma^k$. ∎

3.3. Function fields of metacyclic covers. Consider a metacyclic Galois covering $f : X \to Y$ with Galois group $G_{m,k,t,r}$ defined as above with the intermediate cyclic coverings $p : X \to Z$ and $q : X \to Z$ of degrees m and t respectively. Recall from Section 2.2.1 that these morphisms give rise to field extensions $\mathbb{C}(Y) \subset \mathbb{C}(Z) \subset \mathbb{C}(X)$. Let $A = \langle\sigma\rangle$ and $N = \langle\overline{\tau}\rangle$ with $\operatorname{ord}(\sigma) = m$ and $\operatorname{ord}(\overline{\tau}) = t$. Then

$$\mathbb{C}(X) = \mathbb{C}(Z)(\sqrt[m]{g}) = \mathbb{C}(Z)(x), \quad g \in \mathbb{C}(Z), \quad \sigma\cdot x = \xi_m\cdot x, \tag{3.13}$$

$$\mathbb{C}(Z) = \mathbb{C}(Y)(\sqrt[t]{f}) = \mathbb{C}(Y)(z), \quad f \in \mathbb{C}(Y), \quad \overline{\tau}.z = \xi_t\cdot z, \tag{3.14}$$

where ξ_m (resp. ξ_t) denotes a primitve mth (resp. tth) root of unity. In particular, $\mathbb{C}(X) = \mathbb{C}(Y)(v, w)$. In this subsection we describe the above function fields and field extensions following [CP17], [P91].

Proposition 3.9. *Let $\pi : X \to Y$ be a metacyclic Galois covering with Galois group $G_{m,k,t,r}$ and intermediate cyclic coverings $p : X \to Z$ and $q : Z \to Y$. Then:*

(1) *There exists $\alpha \in \mathbb{C}(Z)$ such that $\tau(x) = \alpha x^r$.*
(2) *Let $c = \frac{r^t-1}{m}$. Then*

$$g^c = \frac{\xi_m^k}{\prod_{i=1}^t \overline{\tau}^{t-i}(\alpha)\alpha^{r^{t-1}}}.$$

Proof. (1) The element $x^{-r}\tau(x)$ is invariant under σ:

$$\sigma(x^{-r}\tau(x)) = \sigma(x^{-r})\sigma\tau(x) = \sigma(x^{-r})\tau\sigma^r(x) = \xi_m^{-r}x^{-r}\tau(\xi_m^r x) = x^{-r}\tau(x),$$

i.e., $x^{-r}\tau(x) \in \mathbb{C}(X)^{\langle\sigma\rangle} = \mathbb{C}(Z)$.

(2) Using the formula in (1) with $\alpha \in \mathbb{C}(Z)$, one calculates

$$\xi_m^k x = \sigma^k(x) = \tau^t(x) = \prod_{i=1}^t \tau^{t-i}(\alpha)\alpha^{r^{t-1}}x^{r^t} = \prod_{i=1}^t \overline{\tau}^{t-i}(\alpha)\alpha^{r^{t-1}}g^c x,$$

which settles (2). ∎

Lemma 3.10.

(1) *If $h \in \mathbb{C}(Z)$, then $F := h\tau(h)\tau^2(h)\cdots\tau^{t-1}(h) \in \mathbb{C}(Y)$.*
(2) *Let $P := x\tau(x)\tau^2(x)\cdots\tau^{t-1}(x)$. Then there exists $u \in \mathbb{N}$ such that $u \mid \gcd(r-1, m)$ and $P^u \in \mathbb{C}(Y)$.*

Proof. (1) Since $h \in \mathbb{C}(Z)$, we have

$$F = h\tau(h)\tau^2(h)\cdots\tau^{t-1}(h) = h\overline{\tau}(h)\overline{\tau}^2(h)\cdots\overline{\tau}^{t-1}(h) \in \mathbb{C}(Z).$$

It therefore suffices to show that F is invariant under the action of τ. Indeed,

$$\tau(F) = \tau(h)\tau^2(h)\cdots\tau^{t-1}(h)\tau^t(h). \tag{3.15}$$

But $\tau^t = \sigma^k$ and so $\tau^t(h) = \sigma^k(h) = h$, because $h \in \mathbb{C}(Z)$ by assumption and hence it is invariant under σ.

(2) By Proposition 3.9, $\alpha = x^{-r}\tau(x) \in \mathbb{C}(Z)$. So using (3.15), we have

$$\alpha\tau(\alpha)\tau^2(\alpha)\cdots\tau^{t-1}(\alpha) = \xi_m^k(x\tau(x)\tau^2(x)\cdots\tau^{t-1}(x))^{-(r-1)} \in \mathbb{C}(Y).$$

Note that $\tau^t(x) = \sigma^k(x) = \xi_m^k x$. For the last assertion, notice that $x^m = g \in \mathbb{C}(Z)$ and it follows by substituting $h = x^m$ in (1) that

$$h\tau(h)\tau^2(h)\cdots\tau^{t-1}(h) = (x\tau(x)\tau^2(x)\cdots\tau^{t-1}(x))^m \in \mathbb{C}(Y).$$

Now take u to be the smallest positive integer such that $(x\tau(x)\tau^2(x)\cdots\tau^{t-1}(x))^u \in \mathbb{C}(Y)$. By the minimality of u, it follows that $u \mid \gcd(r-1, m)$. ■

In the special case $t = 2$, the above results can help to determine the structure of metacyclic extensions.

Proposition 3.11. *Let $\mathbb{C}(Y) \subset \mathbb{C}(X)$ be a $G_{m,k,2,r}$-extension. There exist $a \in \mathbb{C}(Y)$ and $x, P \in \mathbb{C}(X)$ with*

$$x^{2m} - 2ax^m + P^m = 0,$$

such that $P^u \in \mathbb{C}(Y)$ for some $u \mid \gcd(r-1, m)$ (in particular, $P^m \in \mathbb{C}(Y)$) and $\mathbb{C}(X) = \mathbb{C}(Y)(x)$. The $G_{m,k,2,r}$-action is given by $\sigma(x) = \xi_m x$ where ξ_m is a primitive mth root of unity in $\mathbb{C}$ and $\tau(x) = P/x$.

Conversely, given m, k, r satisfying (3.6) (with $t = 2$) and $a \in \mathbb{C}(Y)$, if $P = \alpha x^{r+1}$ is such that $\alpha \in \mathbb{C}(Y)[x^m]$ and $x^{2m} - 2ax^m + P^m$ is irreducible (in $\mathbb{C}(Y)[x]$) then $\frac{\mathbb{C}(Y)[x]}{(x^{2m}-2ax^m+P^m)}$ is a $G_{m,k,2,r}$-Galois extension of $\mathbb{C}(Y)$ with the $G_{m,k,2,r}$-action given by $\sigma(x) = \xi_m x$ and $\tau(x) = P/x = \alpha x^r$.

Proof. Let $P := x\tau(x) \in \mathbb{C}(X)$. The existence of $u \in \mathbb{N}$ with the claimed properties follows from Lemma 3.10(2). Since $t = 2$, we may write $x^m = g = a + z$ with $a \in \mathbb{C}(Y)$ and we have

$$P^m = (x\tau(x))^m = x^m\overline{\tau}(x^m) = (a+z)(a-z) = a^2 - z^2 = a^2 - f.$$

So $f = a^2 - P^m$. On the other hand, $x^m = a + z$ and the above calculation gives

$$x^{2m} - 2ax^m + P^m = 0.$$

Of course, by Proposition 3.9, we know that $P = \alpha x^{r+1}$ with $\alpha \in \mathbb{C}(Z)$.

For the converse statement, one checks that by the choice of P, there is an action of $G_{m,k,2,r}$ on $\mathbb{C}(X) = \frac{\mathbb{C}(Y)[x]}{(x^{2m}-2ax^m+P^m)}$. This action is Galois as the latter contains all of the conjugates of x under $G_{m,k,2,r}$. ■

A special case: dicyclic covers. The dicyclic group is in our compact notation the metacyclic group $G_{2n,n,2,-1}$. They are therefore precisely the extensions of $\mathbb{Z}_2$ by cyclic groups. In this case, as in the proof of Proposition 3.9(1), we have $x\tau(x) \in \mathbb{C}(X)^{\langle\sigma\rangle} = \mathbb{C}(Z)$ and hence $x\tau(x) = a + bz$ with $a, b \in \mathbb{C}(Y)$. Futhermore, $\overline{\tau}(x\tau(x)) = -x\tau(x)$, which forces $a = 0$, and we may assume $b = 1$. That is, $x\tau(x) = z$ or $\tau(x) = z/x$. In particular, $\tau(x)^2 = f/x^2$ with $z^2 = f \in \mathbb{C}(Y)$. Now, $x^{2n} = g \in \mathbb{C}(Z)$ and hence $x^{2n} = g = c + dz$ with $c, d \in \mathbb{C}(Y)$. Consequently,

$$f^n = z^{2n} = x^{2n}\tau(x^{2n}) = x^{2n}\overline{\tau}(x^{2n}) = x^{2n}(c - dz) = (c+dz)(c-dz) = c^2 - d^2 f.$$

On the other hand, since $x^{2n} = g = c + dz$, the above computations imply

$$x^{4n} - 2cx^{2n} + c^2 - d^2 f = x^{4n} - 2cx^{2n} + f^n = 0$$

with $c, d, f \in \mathbb{C}(Y)$. We summarize this discussion in the following

COROLLARY 3.12. *Let $\mathbb{C}(Y) \subset \mathbb{C}(X)$ be a $G_{2n,n,2,-1}$-extension. Then there exist $c, f \in \mathbb{C}(Y)$ and $x \in \mathbb{C}(X)$ such that $\mathbb{C}(X) = \mathbb{C}(Y)(x)$ and the following equation is satisfied:*

$$x^{4n} - 2cx^{2n} + c^2 - d^2 f = x^{4n} - 2cx^{2n} + f^n = 0.$$

The $G_{2n,n,2,-1}$-action on $\mathbb{C}(X)$ is given by $\sigma(x) = \xi_{2n} x$ and $\tau(x)^2 = f/x^2$, where ξ_{2n} is a primitive $2n$th root of unity in $\mathbb{C}$. Conversely, given $c, f \in \mathbb{C}(Y)$ and $x \in \mathbb{C}(X)$ such that $x^{4n} - 2cx^{2n} + f^n$ is irreducible in $\mathbb{C}(Y)[x]$, then $\frac{\mathbb{C}(Y)[x]}{(x^{4n}-2cx^{2n}+f^n)}$ is a $G_{2n,n,2,-1}$-extension of $\mathbb{C}(Y)$ with the group action described above.

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