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Weak limits of fractional Sobolev homeomorphisms are almost injective

by

Armin Schikorra and James M. Scott (Pittsburgh, PA)

Abstract. Let $\Omega \subset \mathbb{R}^n$ be an open set and $f_k \in W^{s,p}(\Omega;\mathbb{R}^n)$ be a sequence of homeomorphisms weakly converging to $f \in W^{s,p}(\Omega;\mathbb{R}^n)$. It is known that if $s = 1$ and $p > n-1$ then f is injective almost everywhere in the domain and the target. In this note we extend such results to the case $s \in (0,1)$ and $sp > n-1$. This in particular applies to C^s-Hölder maps.

1. Introduction and main result. The goal of this note is to prove the following theorem:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be open and let $f : \Omega \to \mathbb{R}^n$ be a weak $W^{s,p}$-limit of Sobolev homeomorphisms $f_j \in W^{s,p}(\Omega;\mathbb{R}^n)$ with $sp > n-1$. Then there is a representative $\widehat{f}$ and a set $\Gamma \subset \mathbb{R}^n$ of Hausdorff dimension at most $\frac{n-1}{s}$ such that $(\widehat{f}\,)^{-1}(y)$ consists of only one point for every $y \in \widehat{f}(\Omega) \setminus \Gamma$.*

For definitions we refer to the next section. An immediate corollary of Theorem 1.1 and the embedding $C^s \hookrightarrow W^{s-\varepsilon,p}_{\mathrm{loc}}$ for any $\varepsilon > 0$ is the following statement for Hölder maps.

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be open and let $f \in C^s(\Omega;\mathbb{R}^n)$ be the pointwise limit of a sequence of equibounded homeomorphisms $f_j \in C^s(\Omega;\mathbb{R}^n)$. If $s > \frac{n}{n-1}$, then there is a set $\Gamma \subset \mathbb{R}^n$ of Hausdorff dimension $\frac{n-1}{s}$ such that $f^{-1}(y)$ consists of only one point for every $y \in f(\Omega) \setminus \Gamma$.*

Observe that for $s \leq \frac{n-1}{n}$ the above statements hold trivially.

This note is inspired by the recent work by Bouchala, Hencl, and Molchanova [4] who proved a corresponding result for $s = 1$.

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THEOREM 1.3 (Bouchala, Hencl, Molchanova). *Let $f : \Omega \to \mathbb{R}^n$ be a weak limit of Sobolev homeomorphisms $f_j \in W^{1,p}(\Omega;\mathbb{R}^n)$ with $p > n-1$. Then there is a representative $\widehat{f}$ and a set $\Gamma \subset \mathbb{R}^n$ of Hausdorff dimension $n-1$ such that $(\widehat{f}\,)^{-1}(y)$ consists of only one point for every $y \in \widehat{f}(\Omega) \setminus \Gamma$.*

While Theorem 1.3 (and in turn our Theorem 1.1) follows by an adaptation of the arguments in the seminal work by Müller and Spector [12], Bouchala, Hencl, and Molchanova [4] also provide an example of the limit case $p = n-1$, where a theorem such as Theorem 1.3 completely fails. Namely they show

THEOREM 1.4 (Bouchala, Hencl, Molchanova). *For $n \geq 3$ there exists $f : [-1,1]^n \to [-1,1]^n$ and a strong limit of Sobolev homeomorphisms $f_k \in W^{1,n-1}([-1,1]^n,\mathbb{R}^n)$ with $f_k(x) = x$ on the boundary $\partial[-1,1]^n$ and such that there exists a set $\Gamma \subset [-1,1]^n$ of positive Lebesgue measure and $f^{-1}(y)$ is a nontrivial continuum for every $y \in \Gamma$.*

As the authors of [4] mention, it may seem surprising that the Hausdorff dimension of the critical set Γ seems to suddenly jump from $n-1$ to n as p changes from $p > n-1$ to $p = n-1$. This question served as one motivation to study the situation for fractional Sobolev spaces. With respect to Theorem 1.1 we see that indeed the consideration of fractional Sobolev space makes the "jump" in dimension of the singular set continuous: As $s \downarrow \frac{n-1}{n}$, the size of the dimension of the singular set Γ varies continuously from $n-1$ to n. It would be interesting to investigate the optimal fractional Sobolev regularity in the limiting examples by Bouchala, Hencl, and Molchanova [4].

Let us stress that Theorem 1.1 follows a very similar argument to the $s = 1$ proof of Theorem 1.3 in [4], which in turn is a streamlined succession of known results and techniques from earlier works; see [3, 12, 13]. Indeed, a crucial fact that is used for $s = 1$ is that on "good slices" ∂B_r the f_k converge in $W^{1,p}(\partial B_r)$, and so by Sobolev–Morrey embedding on these $n-1$-dimensional slices the f_k in fact converge uniformly if $p > n-1$. If $p = n-1$ this uniform convergence may fail.

The same is true if the f_k converge in $W^{s,p}(\partial B_r)$ for good slices ∂B_r and $s \in (0,1)$: if $sp > n-1$ then the convergence is uniform on ∂B_r, and if $sp = n-1$ it may not.

But somewhat surprisingly, a result such as Theorem 1.1 and in particular Corollary 1.2 seems to be unknown to some experts, and the authors thought it important to make it available in the literature.

We try to keep this note as self-contained as possible. In Section 1 we gather the main results on Sobolev spaces that we work with. In Section 3 we discuss the needed notion of degree, and show monotonicity of the degree for limits of homeomorphisms. In Section 4, we collect the corollaries for the topological image from the previous section. In Section 5 we prove our main theorem.

As a last statement of this introduction, let us present an immediate corollary of Theorem 1.1 in the realm of Besov and Triebel–Lizorkin spaces (which we will not pursue further here).

COROLLARY 1.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be open and let $f : \Omega \to \mathbb{R}^n$ be either*

- *a weak $F^{s,p}_q$-limit of Sobolev homeomorphisms $f_j \in F^{s,p}_q(\Omega;\mathbb{R}^n)$ with $sp > n-1$ and $q \in [1,\infty]$, or*
- *a weak $B^{s,p}_q$-limit of Sobolev homeomorphisms $f_j \in F^{s,p}_q(\Omega;\mathbb{R}^n)$ with $sp > n-1$ and $q \in [1,p]$.*

Then there is a representative $\widehat{f}$ and a set $\Gamma \subset \mathbb{R}^n$ of Hausdorff dimension at most $\frac{n-1}{s}$ such that $(\widehat{f}\,)^{-1}(y)$ consists of only one point for every $y \in \widehat{f}(\Omega) \setminus \Gamma$.

These statements follow from the embedding $F^{s,p}_q \hookrightarrow F^{t,p}_p = W^{t,p}$ for any $0 < t < s$ and $q \in [1,\infty]$, and $B^{s,p}_q \hookrightarrow B^{s,p}_p = F^{s,p}_p = W^{s,p}$ for any $q \leq p$. See [14].

2. Preliminaries on Sobolev spaces, capacities etc. First we establish notation. For $s \in (0,1)$ and $p \in (1,\infty)$ we define the fractional Sobolev space $W^{s,p}(\Omega;\mathbb{R}^n)$ to be the class of functions $u : \Omega \to \mathbb{R}^n$ for which the Gagliardo seminorm

$$[u]^p_{W^{s,p}(\Omega)} := \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x \tag{2.1}$$

is finite; we equip this space with the norm $\|u\|^p_{W^{s,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)}$.

We denote the n-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^n$ by $\mathcal{L}^n(A)$, and for $\beta > 0$ we denote the β-dimensional Hausdorff measure by $\mathscr{H}^\beta(A)$. We write $A \precsim B$ whenever there exists a constant C such that $A \leq CB$.

For a function f continuous on a compact set A, we define the oscillation

$$\operatorname{osc}_A f := \sup_{x,y\in A} (f(x) - f(y)) = \operatorname{diam} f(A).$$

We denote uniform convergence by $\rightrightarrows$.

Define the *precise representative* of a measurable function f by

$$f^*(x) := \begin{cases} \lim_{r\to 0^+} ⨍_{B_r(x)} f(y)\,\mathrm{d}y & \text{when the limit exists,}\\ 0 & \text{otherwise.}\end{cases} \tag{2.2}$$

Many properties of the precise representative for functions in the *Bessel potential* spaces are accessible in the literature. The corresponding statements can then be obtained for fractional Sobolev functions via embedding theorems for the Triebel–Lizorkin spaces $F^s_{p,q}$; see [14]. For completeness, we give a summary of the statements we will need.

We denote the Bessel potential spaces $H^{s,p}$ by

(2.3)
$$H^{s,p}(\mathbb{R}^n;\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R}^n : \|\mathcal{F}^{-1}((1+|\xi|^2)^{s/2}(\mathcal{F}f)(\xi))\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse respectively. The following is a corollary of a classical embedding theorem for the spaces $F^s_{p,q}$ [14, Section 2.2.3], [15, Theorem 2.14, Remark 2.4], which we will use multiple times throughout the work for various parameters of integrability, differentiability, and dimension. We are additionally using the identifications $F^s_{p,2} = H^{s,p}$ and $F^s_{p,p} = W^{s,p}$.

THEOREM 2.1. *Let $N \geq 1$. Let $p \in (1,\infty)$ and $s \in (0,1)$, and suppose that $t \in (0,1)$ and $p_t \in (1,\infty)$ satisfy*

$$s - \frac{N}{p} < t < s, \qquad p_t := \frac{Np}{N-(s-t)p}.$$

Then

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow H^{t,p_t}(\mathbb{R}^N), \quad or \quad [f]_{H^{t,p_t}(\mathbb{R}^N)} \precsim [f]_{W^{s,p}(\mathbb{R}^N)}. \tag{2.4}$$

Note that if we write the definition of p_t as

$$sp - N = \frac{p}{p_t}(tp_t - N), \tag{2.5}$$

then it becomes clear that if $sp > N$ then $tp_t > N$ for any $t \in (0,s)$.

With this embedding we can prove some useful properties of the precise representative:

PROPOSITION 2.2. *Suppose $f \in W^{s,p}(\mathbb{R}^n;\mathbb{R}^n)$ with $sp \in [1,n)$. Let $p^* = \frac{np}{n-sp}$. Define*

$$A_f := \{x \in \mathbb{R}^n : x \text{ is not a Lebesgue point of } f\}. \tag{2.6}$$

Then the following hold:

(i) $\dim_{\mathscr{H}}(A_f) \leq n - sp$.
(ii) *For any $x \in \mathbb{R}^n \setminus A_f$,*

$$\lim_{r\to 0^+} \fint_{B_r(x)} |f(y) - f^*(x)|^q \, \mathrm{d}y = 0 \tag{2.7}$$

for every $q \in [1,p^*)$.

(iii) *If φ_ε is the family of standard mollifiers then*

$$\varphi_\varepsilon * f(x) \to f^*(x)$$

for each $x \in \Omega \setminus A_f$.

Proof. Let $\varepsilon > 0$ be arbitrary; we will show that

$$\mathscr{H}^{n-sp+\varepsilon}(A_f) = 0, \tag{2.8}$$

which will imply (i). We use Theorem 2.1 with $N = n$; choose $t \in (0, s)$ so that

$$n - tp_t = n - sp + \varepsilon;$$

this is possible since by definition $n - tp_t > n - sp$ for $sp \in [1, n)$ and for any $t \in (0, s)$. Then $f \in H^{t,p_t}(\mathbb{R}^n; \mathbb{R}^n)$ and so [1, Proposition 6.1.2, Theorem 5.1.13] implies $\mathscr{H}^{\beta}(A_f) = 0$ for all $\beta \geq n - tp_t = n - sp + \varepsilon$, hence (2.8) is established.

To see (ii), use Theorem 2.1 with $N = n$ again; note that any $q \in (p, p_*)$ can be written as $q = p_t$ for some $t \in (0, s)$. Then $f \in H^{t,p_t}(\mathbb{R}^n; \mathbb{R}^n)$ for every $t \in (0, s)$, and so [1, Theorem 6.2.1] applies, which is precisely (ii). We obtain (2.7) for the range $q \in [1, p]$ using Hölder's inequality.

For a proof of (iii) see [7, Theorem 4.1(iv)]. ∎

Lemma 2.3. *Suppose $f \in W^{s,p}(\mathbb{R}^n; \mathbb{R}^n)$ with $sp \in [1, n)$, and suppose $f^*(x) \in E$ for every $x \in \mathbb{R}^n \setminus M$, where $\mathcal{L}^n(M) = 0$ and $E \subset \mathbb{R}^n$ is a closed set. Then $f^*(x) \in E$ for every $x \in \mathbb{R}^n \setminus A_f$.*

Proof. Suppose to the contrary that $f^*(x) \in \mathbb{R}^n \setminus E$ for some $x \in \mathbb{R}^n \setminus A_f$. Then there exists $\varepsilon > 0$ such that $B(f^*(x), \varepsilon) \subset \mathbb{R}^n \setminus E$. By the assumption that $f^*(y) \in E$ for $y \in \mathbb{R}^n \setminus M$ we have

$$\fint_{B(x,r)} |f(y) - f^*(x)|^p \, \mathrm{d}y = \fint_{B(x,r)\setminus M} |f(y) - f^*(x)|^p \, \mathrm{d}y \geq \varepsilon^p$$

uniformly as $r \to 0$, which is a contradiction since f^* satisfies (2.7) for every $x \in \mathbb{R}^n \setminus A_f$. ∎

We will need information on the Hausdorff dimension of images of spheres embedded in $\mathbb{R}^n$. The following is a special case of such a result in [10] for Bessel potential functions, which will then apply to functions in $W^{s,p}$ via Theorem 2.1:

Proposition 2.4 ([10, Theorem 1.1]). *Let $N, K \in \mathbb{N}$, $t \in (0, 1)$ and $q \in (1, \infty)$ with $tq > N$ and $\alpha \in (0, N]$. Define $\beta := \frac{\alpha q}{tq - N + \alpha}$. Suppose $g \in H^{t,q}(\mathbb{R}^N; \mathbb{R}^K)$ is a continuous representative and $A \subset \mathbb{R}^N$ is a set with $\dim_{\mathscr{H}}(A) \leq \alpha$. Then $\dim_{\mathscr{H}}(g(A)) \leq \beta$.*

We then have as a corollary

Theorem 2.5. *Let $n \geq 2$, $s \in (0, 1)$, and $p > 1$ with $n - 1 < sp < n$. Let $r > 0$, $a \in \mathbb{R}^n$ with $\partial B \equiv \partial B(a, r)$ and $g \in W^{s,p}(\partial B; \mathbb{R}^n)$ be a continuous representative. Then $\dim_{\mathscr{H}}(g(\partial B)) \leq \frac{n-1}{s}$.*

Proof. It suffices to show that

$$\mathscr{H}^{\frac{n-1}{s}+\varepsilon}(g(\partial B)) = 0$$

for $\varepsilon > 0$ arbitrarily small. Cover ∂B by sets S_i diffeomorphic to $\mathbb{R}^{n-1}$ (2^n hemispheres will do), and let $\psi_i : \mathbb{R}^{n-1} \to S_i$ be the corresponding diffeomorphisms. So $\partial B \subset \bigcup_{i=1}^M S_i$, and the functions

$$g_i := g \circ \psi_i$$

belong to $W^{s,p}(\mathbb{R}^{n-1};\mathbb{R}^n)$, and hence belong to $H^{t,p_t}(\mathbb{R}^{n-1};\mathbb{R}^n)$ by Theorem 2.1 for any $t \in \big(s - \frac{n-1}{p}, s\big)$ and for $p_t = \frac{(n-1)p}{(n-1)-(s-t)p}$.

Applying Proposition 2.4 to each g_i with $q = p_t$ and $N = \alpha = n-1$ gives

$$\mathscr{H}^{\gamma}(g_i(\mathbb{R}^{n-1})) = 0 \quad \text{ for every } \gamma > \tfrac{n-1}{t},\ i = \{1,\dots,M\}.$$

Choose $t < s$ close enough to s so that $\frac{n-1}{t} < \frac{n-1}{s} + \varepsilon$. Then

$$\mathscr{H}^{\frac{n-1}{s}+\varepsilon}(g(\partial B)) \le \sum_{i=1}^M \mathscr{H}^{\frac{n-1}{s}+\varepsilon}(g(S_i)) = \sum_{i=1}^M \mathscr{H}^{\frac{n-1}{s}+\varepsilon}(g_i(\mathbb{R}^{n-1})) = 0,$$

as desired. ■

Throughout this note we additionally require control of fractional Sobolev functions on spheres in $\mathbb{R}^n$. In the case $s = 1$, this control is obtained straightforwardly; for example, using Fubini's theorem for a smooth function f on $\overline{B(a,r)}$, we obtain

$$\int_0^r \int_{\partial B(a,\rho)} |\widetilde{\nabla} f(\rho\omega)|^p \,\mathrm{d}\mathscr{H}^{n-1}(w)\,\mathrm{d}\rho \le \int_{B(a,r)} |\nabla f(x)|^p \,\mathrm{d}x,$$

where $\widetilde{\nabla} f$ denotes the tangential derivative of $f|_{\partial B(a,\rho)}$. The following Besov-type inequality serves as a fractional analogue:

Lemma 2.6. *Let $B(a,r) \subset \mathbb{R}^n$, with $p \in [1,\infty)$ and $s \in (0,1)$. Then there exists a constant $C = C(n,s,p)$ such that for every $f \in W^{s,p}(B(a,r);\mathbb{R}^n)$,*

$$\int_{r/2}^r \int_{\partial B(a,\rho)} \int_{\partial B(a,\rho)} \frac{|f(x)-f(y)|^p}{|x-y|^{n-1+sp}} \,\mathrm{d}\mathscr{H}^{n-1}(y)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,\mathrm{d}\rho \le C[f]^p_{W^{s,p}(B(a,r))}. \tag{2.9}$$

These types of estimates are well-known to experts (see for example [5]), but for the sake of completeness we have included a proof in the Appendix. The following corollary to the lemma reveals finer properties of Sobolev functions:

Corollary 2.7. *Let $1 < sp < n$, let $x_0 \in \Omega \subset \mathbb{R}^n$, and suppose $f \in W^{s,p}(\Omega;\mathbb{R}^n)$. Then there exists a set $N_{x_0} \subset (0, \operatorname{dist}(x_0,\partial\Omega))$ with $\mathscr{L}^1(N_{x_0}) = 0$ such that for every $r \in (0,\operatorname{dist}(x_0,\partial\Omega)) \setminus N_{x_0}$ the function*

$f^*|_{\partial B(x_0,r)}$ *belongs to* $W^{s,p}(\partial B(x_0,r);\mathbb{R}^n)$, *where* f^* *is the precise representative defined in* (2.2). *If in addition* $sp > n-1$ *then* $f^*|_{\partial B(x_0,r)}$ *is continuous. In general the singular set depends on* x_0.

Proof. For $\varepsilon > 0$ let φ_ε be the standard mollifier, and let $f^\varepsilon := \varphi_\varepsilon * f^*$. Then f^ε converges to f^* in $W^{s,p}(B(x_0,r))$ for any $r \in (0, \operatorname{dist}(x_0,\partial\Omega))$, and by Lemma 2.6,

$$\int_{r/2}^{r} [f^\varepsilon - f^*]^p_{W^{s,p}(\partial B(x_0,\rho))}\,\mathrm{d}\rho \le C[f^\varepsilon - f^*]^p_{W^{s,p}(B(x_0,r))} \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus for $\mathscr{L}^1$-almost every $r \in (0, \operatorname{dist}(x_0,\partial\Omega))$ the smooth functions $f^\varepsilon|_{\partial B(x_0,r)}$ converge to a function $g_r \in W^{s,p}(\partial B(x_0,r))$. On the other hand, Proposition 2.2 applies to f since we can find a Sobolev extension domain K satisfying $B(x_0,r) \subset K \subset \Omega$. Thus since $sp > 1$, we deduce from Proposition 2.2(iii) that for every $r \in (0, \operatorname{dist}(x_0,\partial\Omega))$,

$$f^\varepsilon(x) \to f^*(x) \quad \text{on } B(x_0,r) \setminus A_f, \qquad \text{where } \mathscr{H}^{n-1}(A_f) = 0.$$

Hence for $\mathscr{L}^1$-almost every $r \in (0, \operatorname{dist}(x_0,\partial\Omega))$ the functions $f^\varepsilon|_{\partial B(x_0,r)}(x)$ converge to $f^*(x)$ for $\mathscr{H}^{n-1}$-almost every $x \in \partial B(x_0,r)$. So for $\mathscr{L}^1$-almost every $r \in (0, \operatorname{dist}(x_0,\partial\Omega))$ the function $f^*|_{\partial B(x_0,r)}$ agrees with g_r up to a set of $\mathscr{H}^{n-1}$-measure zero, hence $f^*|_{\partial B(x_0,r)}$ belongs to $W^{s,p}(\partial B(x_0,r))$.

Now if $sp > n-1$, then $f^\varepsilon \to g_r$ locally uniformly on $\partial B(x_0,r)$ by the Sobolev compact embedding theorem (see for example [14, Theorem 2, p. 82], [17, Lemma 41.4]), and additionally $\mathscr{H}^1(A_f) = 0$. Therefore for $\mathscr{L}^1$-almost every $r \in (0, \operatorname{dist}(x_0,\partial\Omega))$ the sequence $f^\varepsilon(x)$ converges to $f^*(x)$ *for every* $x \in \partial B(x_0,r)$, and so $f^*(x)$ agrees with the continuous function $g_r(x)$ *for every* $x \in \partial B(x_0,r)$. ∎

The following is an adaptation of [11, Proposition 3.1], which in turn is an extension of an argument in [16].

Proposition 2.8. *Let* $\Omega \subset \mathbb{R}^n$, $s \in (0,1)$ *and* $p \in (1,\infty)$ *with* $n-1 < sp < n$. *Assume that* $f \in W^{s,p}(\Omega;\mathbb{R}^n)$ *satisfies the following: for any* $x_0 \in \Omega$ *there exists a set* N_{x_0} *satisfying* $\mathscr{L}^1(N_{x_0}) = 0$ *such that for all radii* $r, \rho \in (0, \operatorname{dist}(x_0,\partial\Omega)) \setminus N_{x_0}$ *with* $r < \rho$, *for some* $\Lambda \ge 1$ *independent of* r, ρ *and* x_0,

$$\operatorname{osc}_{\partial B(x_0,r)} f^* \le \Lambda \operatorname{osc}_{\partial B(x_0,\rho)} f^*,$$

where f^* *is the continuous representative of* f *defined in* (2.2). *Then there exists a singular set* $\Sigma \subset \Omega$ *with* $\mathscr{H}^{(n-sp)_+}(\Sigma) = 0$ *such that* $f^* : \Omega\setminus\Sigma \to \mathbb{R}^n$ *is continuous.*

Proof. Without loss of generality assume $sp < n$. The case $n = sp$ can be found in [11, Proposition 4.1], and the case $n < sp$ is obvious by Sobolev–Morrey embedding; see [6].

By Corollary 2.7 for any y_0 and $R > 0$ with $B(y_0, R) \subset \Omega$ and $\mathscr{L}^1$-almost any ρ such that $r < \rho < R$, the function $f^*|_{\partial B(y_0,\rho)}$ belongs to $W^{s,p}(B(y_0, \rho))$. As in [11, Proposition 3.1], by Sobolev–Morrey embedding,

$$
\begin{aligned}
(\operatorname{osc}_{\partial B(y_0,r)} f^*)^p &\leq \Lambda(\operatorname{osc}_{\partial B(y_0,\rho)} f^*)^p \\
&\leq C\rho^{sp-(n-1)} \int\limits_{\partial B(y_0,\rho)} \int\limits_{\partial B(y_0,\rho)} \frac{|f^*(x) - f^*(y)|^p}{|x-y|^{(n-1)+sp}} \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Multiplying by $\rho^{-sp+(n-1)}$ and integrating in ρ we obtain, using Lemma 2.6,

$$c(s,p)(R^{n-sp} - r^{n-sp})(\operatorname{osc}_{\partial B(y_0,r)} f^*)^p \leq [f^*]^p_{W^{s,p}(B(y_0,R))}.$$

In particular we have

$$(\operatorname{osc}_{\partial B(y_0,r)} f^*)^p \leq R^{sp-n}[f^*]^p_{W^{s,p}(B(y_0,R))} \tag{2.10}$$

for any $y_0 \in \Omega$, $R \in (0, \operatorname{dist}(y_0, \partial\Omega))$ and for every $r \in (0, R/2) \setminus N_{y_0}$.

Let

$$X := \Big\{x \in \Omega : \limsup_{R\to 0^+} R^{sp-n}[f^*]^p_{W^{s,p}(B(x,R))} > 0\Big\}. \tag{2.11}$$

By Frostman's Lemma (see [18, Corollary 3.2.3]) we have $\mathscr{H}^{(n-sp)_+}(X) = 0$.

Now we may assume without loss of generality that for any $x, y \in \Omega$ there exist $y_0 \in \mathbb{Q}^n \cap \Omega$ and $r \in (0, \operatorname{dist}(y_0, \partial\Omega))$ such that $x, y \in \partial B(y_0, r)$. Set

$$Y := \bigcup_{y_0 \in \mathbb{Q}^n} \bigcup_{r \in N_{y_0}} \partial B(y_0, r).$$

As a countable union of $\mathcal{L}^n$-measure zero sets, Y is an $\mathcal{L}^n$-measure zero set.

Define

$$\Sigma := A_f \cup X \tag{2.12}$$

where A_f is the set of non-Lebesgue points of f as defined in Proposition 2.2. We have

$$\mathscr{H}^{(n-sp)_+}(\Sigma) = 0.$$

Let $x_0 \in \Omega \setminus \Sigma$ and fix $\varepsilon > 0$. Since $x_0 \notin X$, there must be some $R = R(x_0, \varepsilon) > 0$ such that

$$R^{sp-n}[f^*]^p_{W^{s,p}(B(x_0,R))} < \varepsilon.$$

Consequently, for any $y_0 \in B(x_0, R/2)$,

$$(R/2)^{sp-n}[f^*]^p_{W^{s,p}(B(y_0,R/2))} < C_{s,p,n}\varepsilon.$$

Thus, from (2.10),

$$\sup_{r \in (0,R/2)\setminus N_{y_0}} \operatorname{osc}_{\partial B(y_0,r)} f^* < C_{s,p,n}\varepsilon \quad \forall y_0 \in B(x_0, R/2). \tag{2.13}$$

Now let $x, y \in B(x_0, R/8) \setminus Y$. Then there exist $y_0 \in \mathbb{Q}^n \cap B(x_0, R/2)$ and $r \in (0, R/2) \setminus N_{y_0}$ such that $x, y \in \partial B(y_0, r)$. From (2.13) we then have

$$|f^*(x) - f^*(y)| \leq \operatorname{osc}_{\partial B(y_0,r)} f^* < C_{s,p,n}\varepsilon \quad \forall x, y \in B(x_0, R/8) \setminus Y,$$

that is,

$$\sup_{x,y \in B(x_0,R/8)\setminus Y} |f^*(x) - f^*(y)| \leq C_{s,p,n}\varepsilon.$$

Hence f^* is continuous in $\Omega \setminus (\Sigma \cup Y)$. However, $\Sigma \cup Y$ is too large. To remedy this, we use the definition of f^* in $\Omega \setminus A_f$. For all $x, y \in B(x_0, R/8) \setminus \Sigma \subset B(x_0, R/8) \setminus A_f$ there exist $r_x, r_y \in (0, R/8)$ such that

$$\begin{aligned}
|f^*(x) - f^*(y)| &\leq \varepsilon + \Big| \fint_{B(X,r_x)} f(z)\,\mathrm{d}z - \fint_{B(y,r_y)} f(w)\,\mathrm{d}w \Big| \\
&\leq \varepsilon + \fint_{B(x,r_x)\setminus Y} \fint_{B(y,r_y)\setminus Y} |f^*(z) - f^*(w)|\,\mathrm{d}w\,\mathrm{d}z \\
&\leq \varepsilon + \sup_{z,w \in B(x_0,R/8)\setminus Y} |f^*(z) - f^*(w)| \\
&\leq (C_{s,p,n} + 1)\varepsilon. \ \blacksquare
\end{aligned}$$

3. Degree and monotonicity estimates. Let $B = B(x_0, r) \subset \mathbb{R}^n$ and let $f : \partial B \to \mathbb{R}^n$ be continuous. For $y \notin f(\partial B)$ define the *degree*

$$\deg(f, \partial B, y) := \deg_{\mathbb{S}^{n-1}}(\psi) \quad \text{where} \quad \psi := \frac{f\big(\frac{x-x_0}{r}\big) - y}{\big|f\big(\frac{x-x_0}{r}\big) - y\big|} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$$

and $\deg_{\mathbb{S}^{n-1}}$ computes the homotopy class of ψ in $\pi_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z}$.

The main topological ingredient is the following lemma (which is well-known). Items (i) and (iii) are essentially a rewritten version of [4, Lemma 5.1], and (ii) is a consequence of (i) motivated by [16, 8, 11].

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Assume that $B_1 := B(x_1, r_1)$ and $B_2 := B(x_2, r_2)$ are two open balls $\subset\subset \Omega$ and $f, f_k : \partial B_1 \cup \partial B_2 \to \mathbb{R}^n$ are continuous maps, $k \in \mathbb{N}$, such that f_k uniformly converges to f on $\partial B_1 \cup \partial B_2$. If each f_k can be extended to a homeomorphism $F_k : \Omega \to \mathbb{R}^n$ then the following hold:*

(i) *If $B_1 \subset B_2$ then*

$$\begin{aligned}
&f(\partial B_1) \cup \{y \in \mathbb{R}^n \setminus f(\partial B_1) : \deg(f, \partial B_1, y) \neq 0\} \\
&\qquad \subset f(\partial B_2) \cup \{y \in \mathbb{R}^n \setminus f(\partial B_2) : \deg(f, \partial B_2, y) \neq 0\}.
\end{aligned}$$

(ii) *If $B_1 \subset B_2$ then we have monotonicity of oscillation:*

$$\operatorname{osc}_{\partial B_1} f \leq 8 \operatorname{osc}_{\partial B_2} f,$$

and

$$\operatorname{diam}\{y \in \mathbb{R}^n \setminus f(\partial B_1) : \deg(f, \partial B_1, y) \neq 0\} \leq 8 \operatorname{osc}_{\partial B_2} f.$$

(iii) *If* $B_1 \cap B_2 = \emptyset$ *then the sets*

$$\{y \in \mathbb{R}^n \setminus f(\partial B_1) : \deg(f, \partial B_1, y) \neq 0\},$$
$$\{y \in \mathbb{R}^n \setminus f(\partial B_2) : \deg(f, \partial B_2, y) \neq 0\}$$

have empty intersection.

Proof. To prove (i), assume $B_1 \subset B_2$ and let

$$y \in f(\partial B_1) \cup \{y \in \mathbb{R}^n \setminus f(\partial B_1) : \deg(f, \partial B_1, y) \neq 0\}.$$

If $y \in f(\partial B_2)$ there is nothing to show, so we may assume that $y \notin f(\partial B_2)$. By uniform convergence, $y \notin f_k(\partial B_2)$ for all large k.

We use a contradiction argument; assume that $\deg(f, \partial B_2, y) = 0$. By the uniform convergence and since $y \notin f_k(\partial B_2)$, we have $\deg(f_k, \partial B_2, y) = 0$ for large k.

Let $F_k : \Omega \to \mathbb{R}^n$ be a homeomorphism such that $f_k = F_k|_{\partial B_2}$. Then $\deg(f_k, \partial B_2, y) = 0$ implies that $y \notin F_k(\overline{B_2})$. Since $\overline{B_1} \subset \overline{B_2}$ this implies that $y \notin F_k(\overline{B_1})$ and thus

$$\deg(f_k, \partial B_1, y) = 0 \quad \text{for large } k.$$

This leads to a contradiction as $k \to \infty$ unless $y \in f(\partial B_1)$. However, since $F_k : \overline{B_2} \to \mathbb{R}^n$ is a homeomorphism, it is an open map so if $y \in (\partial B_1) \setminus F_k(\overline{B_2})$ there must be $q_k \in \partial B_2$ such that

$$\operatorname{dist}(y, F_k(\overline{B_2})) = |y - f_k(q_k)|.$$

Since $y \notin f(\partial B_2)$, we conclude via uniform convergence that

$$\liminf_{k \to \infty} \operatorname{dist}(y, F_k(\overline{B_2})) > 0$$

and thus

$$\operatorname{dist}(y, f(\partial B_1)) = \liminf_{k \to \infty} \operatorname{dist}(y, f_k(\partial B_1)) \geq \liminf_{k \to \infty} \operatorname{dist}(y, F_k(\overline{B_2})) > 0,$$

and consequently $y \notin f(\partial B_1)$.

To prove (ii), we first observe that

$$f(\partial B_1) \subset f(\partial B_2) \cup \{y \in \mathbb{R}^n \setminus f(\partial B_2) : \deg(f, \partial B_2, y) \neq 0\}.$$

Let $D := \operatorname{diam} f(\partial B_2)$ and pick any $x_0 \in \partial B_2$. Then $f(\partial B_2) \subset B(f(x_0), 3D)$. Moreover, let $\pi : \mathbb{R}^n \to B(f(x_0), 4D)$ be Lipschitz such that $\pi|_{B(f(x_0),3D)} = \mathrm{id}$. Since the degree depends only on the boundary values, for any $y \notin f(\partial B_2)$ we have

$$\deg(f, \partial B_2, y) = \deg(\pi \circ f, \partial B_2, y).$$

Since a necessary condition for the degree to be nonzero at a point y is that y belongs to the image, we conclude that

$$\{y \in \mathbb{R}^n \setminus f(\partial B_2) : \deg(f, \partial B_2, y) \neq 0\} \subset B(f(x_0), 4D).$$

In conclusion, we have shown

$$f(\partial B_1) \subset B(f(x_0), 4D)$$

and thus

$$\operatorname{diam} f(\partial B_1) \leq 8D = 8 \operatorname{diam} f(\partial B_2).$$

For (iii), assume that $y \in \mathbb{R}^n \setminus (f(\partial B_1) \cup f(\partial B_2))$ and

$$\deg(f, \partial B_1, y) \neq 0, \quad \deg(f, \partial B_2, y) \neq 0.$$

By uniform convergence, $y \in \mathbb{R}^n \setminus (f_k(\partial B_1) \cup f_k(\partial B_2))$ for eventually all $k \in \mathbb{N}$, and

$$\deg(f_k, \partial B_1, y) \neq 0, \quad \deg(f_k, \partial B_2, y) \neq 0.$$

This means that $y \in F_k(B_1) \cap F_k(B_2)$, which is a contradiction to F_k being a homeomorphism. ■

4. Corollaries for limits of homeomorphisms. We need the following result, which is a fractional analogue of [12, Lemma 2.9]:

Lemma 4.1. *Let $n \geq 2$, and let $p \in (1, \infty)$ and $s \in (0, 1)$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain, and let*

$$f_k \rightharpoonup f \quad \text{in } W^{s,p}(\Omega; \mathbb{R}^n). \tag{4.1}$$

Let $x_0 \in \Omega$, and define $r_{x_0} := \operatorname{dist}(x_0, \partial\Omega)$. Then there is a set $N_{x_0} \subset \mathbb{R}$ with $\mathscr{L}^1(N_{x_0}) = 0$ such that for any $r \in (0, r_{x_0}) \setminus N_{x_0}$ there exists a subsequence f_k such that

$$f_k^* \rightharpoonup f^* \quad \text{in } W^{s,p}(\partial B(x_0, r); \mathbb{R}^n). \tag{4.2}$$

If $sp > n - 1$ then

$$f_k^* \rightrightarrows f^* \quad \text{on } \partial B(x_0, r). \tag{4.3}$$

In general the subsequence depends on r.

Proof. First, by compact embedding there is a subsequence $f_k \to f$ in $L^p(B(x_0, r_{x_0}); \mathbb{R}^n)$ and so Fubini's theorem implies

(4.4)
$$f_k^* \to f^* \text{ in } L^p(\partial B(x_0, r); \mathbb{R}^n) \text{ for every } r \in (0, r_{x_0}) \setminus N_1 \text{ with } \mathscr{L}^1(N_1) = 0.$$

Next, define

$$\Phi_k(r) := \int_{\partial B(x_0,r)} \int_{\partial B(x_0,r)} \frac{|f_k^*(x) - f_k^*(y)|^p}{|x - y|^{n-1+sp}} \, d\mathscr{H}^{n-1}(y) \, d\mathscr{H}^{n-1}(x), \tag{4.5}$$

with

$$\Phi(r) := \liminf_{k\to\infty} \Phi_k(r). \tag{4.6}$$

Then by Fatou's Lemma and by Lemma 2.6,

$$\int_{r/2}^{r} \Phi(r)\,\mathrm{d}r \le \liminf_{k\to\infty} \int_{r/2}^{r} \Phi_k(r)\,\mathrm{d}r \le \liminf_{j\to\infty} [f_k^*]^p_{W^{s,p}(B(x_0,r))} < \infty$$

for every $r \in (0, r_{x_0})$. Define $N_2 := \{r \in (0, r_{x_0}) : \Phi(r) = \infty\}$, and define $N_{x_0} := N_1 \cup N_2$; note $\mathscr{L}^1(N_{x_0}) = 0$. Then let $r \in (0, r_{x_0}) \setminus N_{x_0}$, and choose a subsequence (not relabeled) satisfying

$$\Phi(r) = \lim_{k\to\infty} \Phi_k(r).$$

Then $f_k^* \to f^*$ strongly in $L^p(\partial B(x_0, r); \mathbb{R}^n)$ and $\lim_{k\to\infty}[f_k^*]_{W^{s,p}(\partial B(x_0,r))} < \infty$, and so (4.2) follows.

In the event that $sp > n-1$ the uniform convergence follows from the compact Sobolev embedding theorem. ∎

The following is a corollary of the Sobolev compact embedding theorem, Lemma 3.1 and Proposition 2.8:

Corollary 4.2. *Let $f_k \in W^{s,p}(\Omega; \mathbb{R}^n)$ be a sequence of homeomorphisms weakly converging in $W^{s,p}(\Omega; \mathbb{R}^n)$ to f. If $sp > n-1$, there exists a set $\Sigma \subset \Omega$ with $\mathscr{H}^{n-sp}(\Sigma) = 0$ such that*

(i) *$f^* : \Omega \setminus \Sigma \to \mathbb{R}^n$ is continuous,*
(ii) *the set $\{f^*(x)\}$ coincides with the topological image $(f^*)^T(x)$ for every $x \in \Omega \setminus \Sigma$, where $(f^*)^T(x)$ is defined as*

$$(f^*)^T(x) := \bigcap_{r\in(0,r_x)\setminus N_x} f^*(\partial B(x,r)) \cup \{y \in \mathbb{R}^n \setminus f^*(\partial B(x,r)) : \deg(f^*, B(x,r), y) \ne 0\},$$

and r_x and N_x have been defined in Lemma 4.1.

Proof. By Lemma 4.1 and Corollary 2.7, the assumptions of Lemma 3.1 are satisfied for all $x_1, x_2 \in \Omega$ and for almost all $r_1 \in (0, r_{x_1}) \setminus N_{x_1}$ and $r_2 \in (0, r_{x_2}) \setminus N_{x_2}$. It follows that the assumptions of Proposition 2.8 are satisfied, and so f^* is continuous on a $\mathscr{H}^{n-sp}$-null set Σ; see (2.6), (2.11) and (2.12) for the definitions. Thus (i) is proved.

To prove (ii) it suffices to show that

(a) $f^*(x) \in (f^*)^T(x)$ for every $x \in \Omega \setminus \Sigma$,
(b) the diameter of the set $(f^*)^T(x)$ is zero for every $x \in \Omega \setminus \Sigma$.

To see (a) we start by proving the following stronger statement:

(a′) For all $x_0 \in \Omega$ and $r \in (0, r_{x_0}) \setminus N_{x_0}$,

$$f^*(x) \in f^*(\partial B(x_0, r)) \cup \{y \in \mathbb{R}^n \setminus f^*(\partial B(x_0, r)) : \deg(f^*, B(x_0, r), y) \neq 0\} \quad \text{for every } x \in B(x_0, r) \setminus \Sigma.$$

Then (a) follows easily from (a′) by choosing $x_0 \in \Omega \setminus \Sigma$. By definition of Σ and by Lemma 2.3 it in turn suffices to show that

(a″) There exists a set M with $\mathscr{L}^n(M) = 0$ such that for all $x_0 \in \Omega$ and $r \in (0, r_{x_0}) \setminus N_{x_0}$,

$$f^*(x) \in f^*(\partial B(x_0, r)) \cup \{y \in \mathbb{R}^n \setminus f^*(\partial B(x_0, r)) : \deg(f^*, B(x_0, r), y) \neq 0\} \quad \text{for every } x \in B(x_0, r) \setminus M.$$

Let $\delta > 0$ be arbitrary. Then by the Sobolev compact embedding theorem and by Egorov's theorem there exists a subsequence (not relabeled) f_k converging uniformly to f^* on $B(x_0, r) \setminus M_\delta$ with $\mathscr{L}^n(M_\delta) < \delta$. Now let $x \in \Omega \setminus M_\delta$. It suffices to show that if $f^*(x) \notin f^*(\partial B(x_0, r))$ then $\deg(f^*, B(x_0, r), f^*(x)) \neq 0$. Since $f_k \rightrightarrows f^*$ on $\partial B(x_0, r)$, $f^*(x) \notin f_k(\partial B(x, r))$ for all k sufficiently large. So there exists $\varepsilon > 0$ such that $B(f^*(x), \varepsilon)$ does not intersect $f^*(\partial B(x_0, r))$ or $f_k(\partial B(x_0, r))$ for k sufficiently large. Then since the f_k are homeomorphisms, it must be that $\deg(f_k, \partial B(x_0, r), p)$ is a nonzero constant for all k sufficiently large and for all $p \in B(f^*(x), \varepsilon)$. In addition, $f_k \rightrightarrows f^*$ on $B(x_0, r) \setminus M_\delta$ so $f_k(x) \in B(f^*(x), \varepsilon)$ for k sufficiently large, uniformly in x. Thus the continuity of the degree yields

$$\deg(f^*, B(x_0, r), f^*(x)) = \lim_{k \to \infty} \deg(f_k, B(x_0, r), f_k(x)).$$

Since $\deg(f_k, B(x_0, r), f_k(x))$ is a nonzero constant for all k sufficiently large, we have proved that

- for every $x_0 \in \Omega$ and $r \in (0, r_{x_0}) \setminus N_{x_0}$,

$$f^*(x) \in f^*(\partial B(x_0, r)) \cup \{y \in \mathbb{R}^n \setminus f^*(\partial B(x_0, r)) : \deg(f^*, B(x_0, r), y) \neq 0\} \quad \text{for every } x \in B(x_0, r) \setminus M_\delta \text{ with } \mathscr{L}^n(M_\delta) < \delta.$$

Since $\delta > 0$ is arbitrary, (a″) is proved.

To see (b), let $x_0 \in \Omega \setminus \Sigma$, and let $\varepsilon > 0$. Then by definition of the set X there exists $R = R(x_0, \varepsilon) \in (0, r_{x_0})$ such that

$$R^{sp-n}[f^*]_{W^{s,p}(B(x_0,R))} < \varepsilon.$$

So by Lemma 3.1(ii) and (2.10),

$$\begin{aligned}\operatorname{diam}(f^*)^T(x_0) &\le \operatorname{diam}\big(f^*(\partial B(x_0,r)) \cup \{y : \deg(f^*, B(x_0,r), y) \neq 0\}\big) \\ &< C\varepsilon\end{aligned}$$

for every $r \in (0, R/4) \setminus N_{x_0}$. Therefore by definition $\operatorname{diam}(f^*)^T(x_0) < \varepsilon$. The proof is complete. ∎

REMARK 4.3. We can define a representative $\widehat{f}$ of f as

$$\widehat{f}(x) := \begin{cases} f^*(x) & \text{when } x \in \Omega \setminus \Sigma, \\ \text{any element of } f^T(x) & \text{otherwise.} \end{cases} \tag{4.7}$$

Then $\widehat{f}$ agrees with f^* everywhere outside Σ, and $\widehat{f}$ has the added property that $\widehat{f}(x) \in (\widehat{f}\,)^T(x)$ *for every* $x \in \Omega$.

5. Proof of Theorem 1.1. We proceed identically to [4]. Assume that $f = \widehat{f}$. We argue by contradiction; suppose that there is a $\delta > 0$ such that the set

$$\Gamma := \{y \in \mathbb{R}^n : \operatorname{diam} f^{-1}(\{y\}) > 0\} \tag{5.1}$$

satisfies $\mathscr{H}^{\frac{n-1}{s}+\delta}(\Gamma) > 0$. Then there exists $K \in \mathbb{N}$ such that the set

$$\Gamma_K := \{y \in \mathbb{R}^n : \operatorname{diam} f^{-1}(\{y\}) > 1/K\} \tag{5.2}$$

satisfies $\mathscr{H}^{\frac{n-1}{s}+\delta}(\Gamma_K) > 0$, since $F = \bigcup_{k\in\mathbb{N}} \Gamma_k$. For each x there exists $r < \frac{1}{2K}$ such that $f|_{\partial B(x,r)} \in W^{s,p}(\partial B(x,r);\mathbb{R}^n) \cap C^0(\partial B(x,r);\mathbb{R}^n)$ by Lemma 4.1. Then choosing a covering of Ω with such a collection $\mathcal{B} := (B(x_i,r_i))_{i=1}^\infty$, by Theorem 2.5 we have $\dim_{\mathscr{H}}(f(\partial B(x_i,r_i))) < \frac{n-1}{s}$, so $\mathscr{H}^{\frac{n-1}{s}+\delta}(f(\partial B(x_i,r_i))) = 0$. Therefore, the set

$$E := \bigcup_{i=1}^{\infty} f(\partial B(x_i,r_i)) \tag{5.3}$$

satisfies $\mathscr{H}^{\frac{n-1}{s}+\delta}(E) = 0$. We will show that $\Gamma_K \subset E$, which contradicts the statement $\mathscr{H}^{\frac{n-1}{s}+\delta}(\Gamma_K) > 0$.

Assume $y \in \Gamma_K \setminus E$. Then there must exist z_1 and z_2 in Ω with $f(z_1) = f(z_2) = y$ and with $\operatorname{dist}(z_1,z_2) > 1/K$. Fix an element $B(x_i,r_i)$ from the collection $\mathcal{B}$ with $z_1 \in B(x_i,r_i)$ and $z_2 \notin B(x_i,r_i)$. Combining Lemma 3.1(i) with the fact that

$$f(x) \in f^T(x) \subset f(\partial B(x,r)) \cup \{q \in \mathbb{R}^n \setminus f(\partial B(x,r)) : \deg(f, \partial B(x,r), q) \neq 0\}$$

for all $x \in \Omega$ and $r \in (0, \operatorname{dist}(x,\partial\Omega)) \setminus N_x$, we get

$$y = f(z_1) \in f(\partial B(x_i,r_i)) \cup \{q \in \mathbb{R}^n \setminus f(\partial B(x_i,r_i)) : \deg(f, B(x_i,r_i), q) \neq 0\}.$$

However, $y \notin E$ so $y \notin f(\partial B(x_i, r_i))$, and thus

$$y = f(z_1) \in \{q \in \mathbb{R}^n \setminus f(\partial B(x_i, r_i)) : \deg(f, B(x_i, r_i), q) \neq 0\}.$$

At the same time, a similar argument using Lemma 3.1(iii) gives

$$y = f(z_2) \in f^T(z_2) \subset \mathbb{R}^n \setminus \{q \in \mathbb{R}^n \setminus f(\partial B(x_i, r_i)) : \deg(f, B(x_i, r_i), q) \neq 0\},$$

which is a contradiction. ∎

Appendix. Proof of Lemma 2.6. It suffices to prove (2.9) for $a = 0$ and $r = 1$. In the case of general a and r we can apply (2.9) for $a = 0$, $r = 1$ to the function

$$g(x) := f(a + rx) \in W^{s,p}(B(0,1))$$

and obtain (2.9) for general a and r by a change of variables.

Since the function $f(x) - ⨍_{B(0,1)} f(y)\, \mathrm{d}y$ also belongs to $W^{s,p}(B(0,1))$, we can assume without loss of generality that

$$⨍_{B(0,1)} f(y)\, \mathrm{d}y = 0.$$

Thus by the Poincaré inequality it suffices to show that there exists a constant $C = C(n, s, p) > 0$ such that

$$\text{(A.1)} \quad \int_{1/2}^{1} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \rho^{n-1-sp} \frac{|f(\rho x) - f(\rho y)|^p}{|x-y|^{n-1+sp}} \, \mathrm{d}\mathscr{H}^{n-1}(y)\, \mathrm{d}\mathscr{H}^{n-1}(x)\, \mathrm{d}\rho \le C \|f\|^p_{W^{s,p}(B(0,1))};$$

note that we used polar coordinates to rewrite the integral.

We prove (A.1) by splitting the domain of the left-hand side integral and estimating each piece. Each domain of integration is locally homeomorphic to a Euclidean ball in $\mathbb{R}^{n-1}$, which allows us to apply translation arguments in the spirit of [2, Lemma 7.44]. Any local diffeomorphism between $\mathbb{S}^{n-1}$ and $\mathbb{R}^{n-1}$ will do, but we make this argument explicit by using the stereographic projection.

Step 1. For each $\mu \in [0,1)$ define the spherical cap $H_\mu := \{x \in \mathbb{S}^{n-1} : x_n < \mu\}$. We will show that for every $\mu \in [0,1)$ there exists a constant $C = C(n, s, p)$ such that

$$\text{(A.2)} \quad \int_{1/2}^{1} \int_{H_\mu} \int_{H_\mu} \rho^{n-1-sp} \frac{|f(\rho x) - f(\rho y)|^p}{|x-y|^{n-1+sp}} \, \mathrm{d}\mathscr{H}^{n-1}(y)\, \mathrm{d}\mathscr{H}^{n-1}(x)\, \mathrm{d}\rho \le C \left(\frac{1+\mu}{1-\mu}\right)^{1+sp} \|f\|^p_{W^{s,p}(B(0,1))}.$$

Throughout the proof, for any $\lambda > 0$ we write $B_{n-1}(0,\lambda)$ for the ball in $\mathbb{R}^{n-1}$ centered at 0 of radius λ. We next establish notation for the stereographic projection $\psi : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1} \setminus \{(0,\dots,0,1)\}$ to prove (A.2). Details on the stereographic projection can be found in several places, for instance in [9, Appendix D.6]. We use the definition

$$\psi(x_1,\dots,x_{n-1}) := \left(\frac{2x_1}{1+|x|^2},\dots,\frac{2x_{n-1}}{1+|x|^2},\frac{|x|^2-1}{1+|x|^2}\right)$$

so that we have the correspondence of domains

$$\psi(B_{n-1}(0,\lambda)) = H_\mu, \quad \text{where} \quad \lambda = \sqrt{\frac{1+\mu}{1-\mu}}.$$

The formula for the Jacobian $J_\psi(x) := \left(\frac{2}{1+|x|^2}\right)^{n-1}$ will be used throughout in order to ensure that quantities such as $J_\psi(x-y)$ and $|J_\psi(x) - J_\psi(y)|$ remain bounded above and below uniformly for x and y in $B_{n-1}(0,\lambda)$, with bounds depending only on n and λ.

To prove (A.2) we need to show that for every $\lambda \in [1,\infty)$ and for every ball $B_{n-1}(0,\lambda) \subset \mathbb{R}^{n-1}$,

$$\text{(A.3)} \quad \int_{1/2}^{1} \int_{B_{n-1}(0,\lambda)} \int_{B_{n-1}(0,\lambda)} \rho^{n-1-sp} \frac{|f(\rho\psi(x)) - f(\rho\psi(y))|^p}{|\psi(x)-\psi(y)|^{n-1+sp}} \times J_\psi(x) J_\psi(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\rho \precsim_{n,s,p} \lambda^{2+2sp} \|f\|^p_{W^{s,p}(B(0,1))}.$$

We proceed using a technique found in [2, Lemma 7.44]. Let $\sigma \in [1/2,1]$, and integrate the inequality

$$|f(\rho\psi(x)) - f(\rho\psi(y))|^p \precsim_p \left|f(\rho\psi(x)) - f\left(\sigma\psi\left(\tfrac{x+y}{2}\right)\right)\right|^p + \left|f\left(\sigma\psi\left(\tfrac{x+y}{2}\right)\right) - f(\rho\psi(y))\right|^p$$

with respect to σ over $B\left(r, \frac{|\psi(x)-\psi(y)|}{2}\right) \cap \left[\frac{1}{2},1\right] \subset \mathbb{R}$ to get

$$\begin{aligned}
&|f(\rho\psi(x)) - f(\rho\psi(y))|^p \\
&\precsim_p \frac{1}{|\psi(x)-\psi(y)|} \int_{\{|\sigma-\rho|\le|\psi(x)-\psi(y)|/2\}\cap[1/2,1]} \left|f(\rho\psi(x)) - f\left(\sigma\psi\left(\tfrac{x+y}{2}\right)\right)\right|^p \mathrm{d}\sigma \\
&\quad + \frac{1}{|\psi(x)-\psi(y)|} \int_{\{|\sigma-\rho|\le|\psi(x)-\psi(y)|/2\}\cap[1/2,1]} \left|f\left(\sigma\psi\left(\tfrac{x+y}{2}\right)\right) - f(\rho\psi(y))\right|^p \mathrm{d}\sigma.
\end{aligned}$$

For $\eta \in \mathbb{R}^{n-1}$ set

$$\Upsilon(\eta) := \int_{\{|\sigma-\rho|\le|\psi(x)-\psi(y)|/2\}\cap[1/2,1]} \left|f(\rho\psi(\eta)) - f\left(\sigma\psi\left(\tfrac{x+y}{2}\right)\right)\right|^p \mathrm{d}\sigma;$$

therefore

(A.4)
$$\begin{aligned}
&\int_{1/2}^{1} \rho^{n-1-sp} \int_{B_{n-1}(0,\lambda)} \int_{B_{n-1}(0,\lambda)} \frac{|f(\rho\psi(x)) - f(\rho\psi(y))|^p}{|\psi(x)-\psi(y)|^{n-1+sp}} J_\psi(x) J_\psi(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\rho \\
&\quad \precsim_p \int_{1/2}^{1} \rho^{n-1-sp} \int_{B_{n-1}(0,\lambda)} \int_{B_{n-1}(0,\lambda)} \frac{\Upsilon(x)+\Upsilon(y)}{|\psi(x)-\psi(y)|^{n+sp}} J_\psi(x) J_\psi(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\rho \\
&\quad =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

By change of variables and by the formula for J_ψ as well as the formula

$$|\psi(x)-\psi(y)| = \frac{2|x-y|}{(1+|x|^2)^{1/2}(1+|y|^2)^{1/2}},$$

which is valid for all $x, y \in \mathbb{R}^{n-1}$, we have

$$\begin{aligned}
\mathrm{I} &= \int_{1/2}^{1} \int_{B_{n-1}(0,\lambda)} \int_{B_{n-1}(0,\lambda)} \int_{\{|\sigma-\rho|\le \frac{|\psi(x)-\psi(y)|}{2}\}\cap[1/2,1]} \rho^{n-1-sp} \\
&\qquad\qquad \times \frac{\left|f(\rho\psi(x)) - f\left(\sigma\psi\left(\frac{x+y}{2}\right)\right)\right|^p}{|\psi(x)-\psi(y)|^{n+sp}} J_\psi(x) J_\psi(y)\,\mathrm{d}\sigma\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\rho \\
&= \int_{1/2}^{1}\int_{1/2}^{1} \int_{B_{n-1}(0,\lambda)} \int_{\{|2z-x|\le\lambda\}\cap\{|\psi(x)-\psi(2z-x)|\ge 2|\sigma-\rho|\}} \rho^{n-1-sp} \\
&\qquad\qquad \times \frac{|f(\rho\psi(x)) - f(\sigma\psi(z))|^p}{|\psi(x)-\psi(2z-x)|^{n+sp}} J_\psi(x) J_\psi(2z-x)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}\rho \\
&= \frac{1}{2^{(n+sp)/2}} \int_{1/2}^{1}\int_{1/2}^{1} \int_{B_{n-1}(0,\lambda)} \int_{\{|2z-x|\le\lambda\}\cap\{|\psi(x)-\psi(z)|\ge G(x,z)|\sigma-\rho|\}} \rho^{n-1-sp} \\
&\qquad\qquad \times \frac{|f(\rho\psi(x)) - f(\sigma\psi(z))|^p}{|\psi(x)-\psi(z)|^{n+sp}} J_\psi(x) J_\psi(z) G(x,z)^{2+sp-n}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}\rho,
\end{aligned}$$

where

$$G(x,z) := \left(\frac{1+|2z-x|^2}{1+|x|^2}\right)^{1/2}.$$

Now, since $|x| \le \lambda$ and $|2z-x| \le \lambda$, we have the uniform bound

$$\frac{1}{\sqrt{1+\lambda^2}} \le G(x,z) \le \sqrt{1+\lambda^2},$$

and so I can be majorized by

(A.5)

$$C(n,s,p)\lambda^{2+sp-n}\int_{1/2}^{1}\int_{1/2}^{1}\int_{B_{n-1}(0,\lambda)}\int_{\{|2z-x|\le\lambda\}\cap\{|\psi(x)-\psi(z)|\ge\frac{|\sigma-\rho|}{\sqrt{1+\lambda^2}}\}}\rho^{n-1-sp}$$

$$\times\frac{|f(\rho\psi(x))-f(\sigma\psi(z))|^p}{|\psi(x)-\psi(z)|^{n+sp}}J_\psi(x)J_\psi(z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}\rho.$$

Finally, on the domain of integration in (A.5) the estimate

$$\begin{aligned}|\rho\psi(x)-\sigma\psi(z)|&\le\rho|\psi(x)-\psi(z)|+|\psi(z)|\,|\rho-\sigma|\\&\le|\psi(x)-\psi(z)|+\sqrt{1+\lambda^2}|\psi(x)-\psi(z)|\le 2\lambda|\psi(x)-\psi(z)|\end{aligned}$$

holds, and since ρ and σ are bounded away from zero we have

$$\begin{aligned}\mathrm{I}&\le C\lambda^{2+2sp}\int_{1/2}^{1}\int_{1/2}^{1}\int_{B_{n-1}(0,\lambda)}\int_{\{|2z-x|\le\lambda\}\cap\{|\psi(x)-\psi(z)|\ge\frac{|\sigma-\rho|}{\sqrt{1+\lambda^2}}\}}\rho^{n-1}\sigma^{n-1}\\&\qquad\times\frac{|f(\rho\psi(x))-f(\sigma\psi(z))|^p}{|\rho\psi(x)-\sigma\psi(z)|^{n+sp}}J_\psi(x)J_\psi(z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}\rho\\&\le C\lambda^{2+2sp}\int_0^1\int_0^1\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\rho^{n-1}\sigma^{n-1}\frac{|f(\rho\psi(x))-f(\sigma\psi(z))|^p}{|\rho\psi(x)-\sigma\psi(z)|^{n+sp}}\\&\qquad\times J_\psi(x)J_\psi(z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}\rho\\&=C(n,s,p)\lambda^{2+2sp}\int_{B(0,1)}\int_{B(0,1)}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

A similar estimate holds for the quantity II in (A.4). Therefore (A.3), and thus (A.2), is proved.

Step 2. We conclude the proof. Split the integral on the left-hand side of (A.1) via a change of variables and symmetry as

$$\begin{aligned}&\int_{1/2}^{1}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\rho^{n-1-sp}\frac{|f(\rho x)-f(\rho y)|^p}{|x-y|^{n-1+sp}}\,\mathrm{d}\mathscr{H}^{n-1}(y)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,\mathrm{d}\rho\\&\quad=\int_{1/2}^{1}\int_{H_0}\int_{H_0}\ldots+\int_{1/2}^{1}\int_{\mathbb{S}^{n-1}\backslash H_0}\int_{\mathbb{S}^{n-1}\backslash H_0}\ldots+2\int_{1/2}^{1}\int_{\mathbb{S}^{n-1}\backslash H_0}\int_{H_0}\ldots\\&\quad=:\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}\tag{A.6}$$

Clearly, by (A.2) with $\mu=0$, we have

$$\mathrm{I}\precsim_{n,s,p}[f]^p_{W^{s,p}(B(0,1))}.\tag{A.7}$$

Now, let $Q:\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ be the matrix $\operatorname{diag}(1,\dots,1,-1)$. Setting $h(x)=f(Qx)$ for any $x\in B_1(0)$, a change of variables gives

$$\mathrm{II}=\int_{1/2}^{1}\int_{H_0}\int_{H_0}\rho^{n-1-sp}\frac{|h(\rho x)-h(\rho y)|^p}{|x-y|^{n-1+sp}}\,\mathrm{d}\mathscr{H}^{n-1}(y)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,\mathrm{d}\rho.$$

Thus by (A.2) with $\mu=0$ and by another change of variables,

$$\begin{aligned}\mathrm{II}&\le C\int_{B(0,1)}\int_{B(0,1)}\frac{|h(x)-h(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\\&=C\int_{B(0,1)}\int_{B(0,1)}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}\tag{A.8}$$

For the last integral, we have

$$\mathrm{III}=2\int_{1/2}^{1}\int_{H_{1/2}\setminus H_0}\int_{H_0}\dots+2\int_{1/2}^{1}\int_{\mathbb{S}^{n-1}\setminus H_{1/2}}\int_{H_0}\dots=:\mathrm{III}_1+\mathrm{III}_2.$$

Using $H_0\subset H_{1/2}$ along with (A.2) for $\mu=1/2$, we find that

$$\begin{aligned}\mathrm{III}_1&\le\int_{1/2}^{1}\int_{H_{1/2}}\int_{H_{1/2}}\rho^{n-1-sp}\frac{|f(\rho x)-f(\rho y)|^p}{|x-y|^{n-1+sp}}\,\mathrm{d}\mathscr{H}^{n-1}(y)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,\mathrm{d}\rho\\&\le[f]^p_{W^{s,p}(B(0,1))}.\end{aligned}\tag{A.9}$$

Since $\operatorname{dist}(\overline{\mathbb{S}^{n-1}\setminus H_{1/2}},\overline{H_0})=C(n)>0$, we see that $|x-y|\ge C(n)>0$ for all $x\in\mathbb{S}^{n-1}\setminus H_{1/2}$ and for all $y\in H_0$, and so the integral III_2 can be estimated by

$$\mathrm{III}_2\lesssim_{n,s,p}\int_{1/2}^{1}\int_{\mathbb{S}^{n-1}}|f(\rho x)|^p\,\mathrm{d}\mathscr{H}^{n-1}(x)\,\mathrm{d}\rho\lesssim_{n,s,p}\|f\|_{L^p(B(0,1))}.\tag{A.10}$$

Combining (A.6) with estimates (A.7)–(A.10) gives (A.1). ∎

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Armin Schikorra, James M. Scott
Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260, USA
E-mail: armin@pitt.edu
james.scott@pitt.edu