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WEAK TYPE $(1,1)$ *BEHAVIOR FOR THE LITTLEWOOD–PALEY* $\mathfrak{g}$*-FUNCTION*

BY

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Abstract. For $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$, the classical Littlewood–Paley $\mathfrak{g}$-function is defined by

$$\mathfrak{g}(f)(x) = \Big(\int_0^\infty |\nabla u(x,t)|^2 t\,dt\Big)^{1/2},$$

where $u(x,t)$ denotes the Poisson integral of f. The following two weak type $(1,1)$ behaviors for the operator $\mathfrak{g}$ are established:

$$\lambda m(\{x \in \mathbb{R}^n : \mathfrak{g}(f)(x) > \lambda\}) \lesssim n^3 \|f\|_1,$$

$$\lim_{\lambda \to 0_+} \lambda m(\{x \in \mathbb{R}^n : \mathfrak{g}(f)(x) > \lambda\}) = \frac{\sqrt{2}\, c_n \omega_{n-1}}{2n} \Big| \int_{\mathbb{R}^n} f(x)\,dx \Big|,$$

for any $f \in L^1(\mathbb{R}^n)$, where c_n is the constant in the Poisson kernel and ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

1. Introduction. The problem of best constant behavior of operators is an interesting topic in harmonic analysis; see e.g. [1, 2, 3, 5, 4, 7, 8, 11, 13] for the L^p and the weak type $(1,1)$ norms of the Hardy–Littlewood maximal operator and the Hilbert transform. In 1986, E. M. Stein summarized the dimension free properties of centered Hardy–Littlewood maximal function in his survey paper [12]. In addition, he stated similar properties of the Riesz transforms and the classical Littlewood–Paley $\mathfrak{g}$-function.

In this paper, we are interested in the weak type $(1,1)$ norm of the Littlewood–Paley $\mathfrak{g}$-function defined by

$$\mathfrak{g}(f)(x) = \Big(\int_0^\infty |\nabla u(x,t)|^2 t\,dt\Big)^{1/2}, \tag{1.1}$$

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where $u(x,t) = \int_{\mathbb{R}^n} P_t(x-y)f(y)\,dy$ is the Poisson integral of f and

$$P_t(y) = \frac{c_n t}{(|y|^2+t^2)^{(n+1)/2}}, \qquad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}},$$

is the Poisson kernel. Moreover, ∇ denotes the gradient operator on $\mathbb{R}^{n+1}_+$, which means that $|\nabla u(x,t)|^2 = |\partial_t u(x,t)|^2 + \sum_{j=1}^n |\partial_{x_j} u(x,t)|^2$. It is well known that the Littlewood–Paley $\mathfrak{g}$-function is bounded on $L^p(\mathbb{R}^n)$ for all $1<p<\infty$ and of weak type $(1,1)$ (see [10]). In [12], E. M. Stein stated the following result.

THEOREM A. *Let $1<p<\infty$. Then*

$$\|\boldsymbol{R}(f)\|_p \approx \|f\|_p \approx \|\mathfrak{g}(f)\|_p,$$

where $\boldsymbol{R}(f) = (\sum_{j=1}^n |R_j f|^2)^{1/2}$ with R_j $(j=1,\dots,n)$ the Riesz transforms. In particular, the equivalence bounds are independent of the dimension n.

For $p=1$, in 2004, P. Janakiraman [6] showed that the weak type $(1,1)$ bound of R_j (a special singular integral) is at worst $O(\log n)$. Thus, a natural problem is: what is the weak type $(1,1)$ bound of the Littlewood–Paley $\mathfrak{g}$-function? The first goal of this paper is to give an answer to this problem.

THEOREM 1.1. *The weak type $(1,1)$ bound of the $\mathfrak{g}$-function is at worst $O(n^3)$. That is, for any $f\in L^1(\mathbb{R}^n)$ and $\lambda>0$,*

$$\lambda m(\{x\in\mathbb{R}^n : \mathfrak{g}(f)(x)>\lambda\}) \lesssim n^3\|f\|_1.$$

Our second result on the limiting weak type $(1,1)$ behavior of $\mathfrak{g}$ is as follows.

THEOREM 1.2. *Suppose $f\in L^1(\mathbb{R}^n)$. Then we have the following limiting weak type $(1,1)$ behavior:*

$$\lim_{\lambda\to 0_+} \lambda m(\{x\in\mathbb{R}^n : \mathfrak{g}(f)(x)>\lambda\}) = \frac{\sqrt{2}\,c_n\omega_{n-1}}{2n}\Big|\int_{\mathbb{R}^n} f(x)\,dx\Big|, \tag{1.2}$$

where ω_{n-1} is the area measure of $\mathbb{S}^{n-1}$ and c_n is defined as above.

REMARK 1.3. Theorem 1.2 gives an exact constant of limiting weak type $(1,1)$ behavior of $\mathfrak{g}$. Since $\omega_{n-1} = \frac{n\pi^{n/2}}{\Gamma(n/2+1)}$ and $c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$, by using Stirling's formula we have

$$c_n\omega_{n-1} \approx n^{1/2}. \tag{1.3}$$

Thus by Theorems 1.1 and 1.2, $n^{-1/2} \lesssim \|\mathfrak{g}\|_{w(1,1)} \lesssim n^3$.

We give the proofs of Theorems 1.1 and 1.2 in Sections 2 and 3, respectively. Since $\mathfrak{g}$ is a nonlinear operator, the proofs here are more elaborate than that for linear singular integral operators. Here we would like to emphasize that hitherto, to the best knowledge of the author, there has been no serious progress in getting specific information on the weak type constant of the Littlewood–Paley $\mathfrak{g}$-function. The bounds presented in Remark 1.3 may be rough, but they give the first dimension-specific estimations of the weak type constant. We hope this paper will attract more people's attention to the weak type constant for the Littlewood–Paley $\mathfrak{g}$-function. Specially, it is an interesting problem to find the best weak $(1, 1)$ constant for $\mathfrak{g}$ when $n = 1$ since the best constants for the Hilbert transform and the Hardy–Littlewood maximal operator are known in this case (see [3, 7]).

Throughout this paper, the letter C will stand for a positive constant which is independent of the dimension n and not necessarily the same at each occurrence. The notation $A \lesssim B$ means that there exists a constant C independent of the dimension such that $A \le CB$, and $A \lesssim_\varepsilon B$ means the constant depends on of ε but is independent of the dimension. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Proof of Theorem 1.1. Let us begin by recalling a modified Calderón–Zygmund decomposition which plays a key role in the proof of Theorem 1.1.

LEMMA 2.1 (see [6, pp. 540–541]). *Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. Then the following conclusions hold:*

(i) $\mathbb{R}^n = G \cup E$, $G \cap E = \emptyset$, $E = \bigcup_k Q_k$, *where Q_k are semi-cubes $\prod_{i=1}^n [a_i, b_i]$ whose interiors are disjoint and for every Q_k, there exists $a > 0$ such that $|b_i - a_i| \in \{a, 2a\}$ for all $i = 1, \dots, n$;*

(ii) $\lambda \le \frac{1}{|Q_k|} \int_{Q_k} |f(y)|\, dy \le 2\lambda$ *for each k;*

(iii) $m(E) \le \frac{1}{\lambda} \|f\|_1$;

(iv) $|f(x)| \le \lambda$ *a.e. on G;*

(v) $f = g + b$;

(vi) $\|g\|_2^2 \le 5\lambda \|f\|_1$;

(vii) $b = \sum_k b_k$ *with b_k supported in Q_k, $\int b_k(x)\, dx = 0$ and* $\|b\|_1 = \sum_k \|b_k\|_1 \le 2\|f\|_1$.

Let us now turn to the proof of Theorem 1.1. For $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, we make a modified Calderón–Zygmund decomposition in Lemma 2.1 of f at level λ. By Lemma 2.1(i), we can get semi-cubes $\{Q_j\}_j$ whose interiors are disjoint. An obvious fact is that if Q_j is a semi-cube with center y_0 and side-length a or $2a$ for some $a > 0$, then

$$\frac{a}{2} \le \inf_{y \in \partial Q_j} |y - y_0| \quad \text{and} \quad \sup_{y \in \partial Q_j} |y - y_0| \le a\sqrt{n}.$$

Let $\bar{d}$ be the maximal distance from the corner points of Q_j to the center y_0. Then $\frac{\sqrt{n}}{2} a \le \bar{d} \le a\sqrt{n}$. Denote $Q_j^* = \bigcup_{y \in Q_j} B\big(y, \frac{\bar{d}}{2n^{3/2}}\big)$. Then $Q_j^* \subseteq (1 + 1/n) Q_j$. In fact, if $x \in Q_j^*$, then

$$\mathrm{d}(x, Q_j) \le \bar{d}/(2n^{3/2}) \le \frac{1}{n}\frac{a}{2},$$

where $d(x, Q_j)$ denotes the distance between x and Q_j. Hence we have $x \in (1 + 1/n)Q_j$ since Q_j has side-length a or $2a$. By the fact that $\mathfrak{g}$ is a sublinear operator and Lemma 2.1(v), we have

$$\begin{aligned} &m(\{x \in \mathbb{R}^n : \mathfrak{g}(f)(x) > \lambda\}) \\ &\qquad \le m(\{x \in \mathbb{R}^n : \mathfrak{g}(g)(x) > \lambda/2\}) + m(\{x \in \mathbb{R}^n : \ \mathfrak{g}(b)(x) > \lambda/2\}). \end{aligned}$$

Using Chebyshev's inequality, Lemma 2.1(vi) and Theorem A, we get

$$m(\{x \in \mathbb{R}^n : \mathfrak{g}(g)(x) > \lambda/2\}) \lesssim \lambda^{-1}\|f\|_1. \tag{2.1}$$

Let $H = \bigcup_j Q_j^*$ and $F = H^c$. We have

$$m(\{x \in \mathbb{R}^n : \mathfrak{g}(b)(x) > \lambda/2\}) \le m(H) + m(\{x \in F : \mathfrak{g}(b)(x) > \lambda/2\}).$$

Applying Lemma 2.1(iii) gives

$$m(H) \le \sum_j m(Q_j^*) \le \left(1 + \frac{1}{n}\right)^n \sum_j m(Q_j) \le em(E) \le e\lambda^{-1}\|f\|_1. \tag{2.2}$$

Therefore, it remains to estimate $m(\{x \in F : \mathfrak{g}(b)(x) > \lambda/2\})$. By Lemma 2.1(vii), we have

$$\begin{aligned} m(\{x \in F : \mathfrak{g}(b)(x) > \lambda/2\}) &\le \frac{2}{\lambda}\int_F \mathfrak{g}(b)(x)\,dx \\ &\le \frac{2}{\lambda}\sum_k \int_F \mathfrak{g}(b_k)(x)\,dx. \end{aligned} \tag{2.3}$$

So we only consider a fixed b_k which is supported in Q_k. Without loss of generality, we assume that Q_k is the semi-cube with center y_0 and side-length a or $2a$ for some $a > 0$. Recall that

$$\mathfrak{g}(b_k)(x) = \Big(\int_0^\infty |\nabla P_t * b_k(x)|^2 t\,dt\Big)^{1/2},$$

where $P_t(x) = c_n \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}$ with $c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$ and

$$|\nabla P_t * b_k(x)|^2 = \sum_{1\le i\le n} (\partial_{x_i} P_t * b_k(x))^2 + (\partial_t P_t * b_k(x))^2.$$

Denote $u(x,t) = P_t * b_k(x)$. Since the mean value of b_k vanishes, we can write

$$u(x,t) = \int_{Q_k} b_k(y)\big(P_t(x - y) - P_t(x - y_0)\big)\,dy.$$

By (3.6) we have

$$
\begin{aligned}
&\partial_{x_i} u(x,t)\\
&\quad = \int_{Q_k} b_k(y)(n+1)c_n \frac{(y_i - y_{0,i})t}{(t^2+|x-y|^2)^{\frac{n+3}{2}}}\,dy\\
&\qquad - \int_{Q_k} b_k(y)c_n(n+1)t\left\{\frac{x_i - y_{0,i}}{(t^2+|x-y|^2)^{\frac{n+3}{2}}} - \frac{x_i - y_{0,i}}{(t^2+|x-y_0|^2)^{\frac{n+3}{2}}}\right\}dy\\
&\quad =: A_i(x,t) - B_i(x,t),
\end{aligned}
$$

where y_i, x_i and $y_{0,i}$ are the ith components of y, x and y_0, respectively. By (3.7) we write $\partial_t u(x,t)$ as $D(x,t)+E(x,t)$, where

$$
\begin{aligned}
D(x,t) &= c_n \int_{Q_k} b_k(y)\left[\frac{1}{(t^2+|x-y|^2)^{\frac{n+1}{2}}} - \frac{1}{(t^2+|x-y_0|^2)^{\frac{n+1}{2}}}\right]dy,\\
E(x,t) &= c_n \int_{Q_k} b_k(y)\left[\frac{(n+1)t^2}{(t^2+|x-y_0|^2)^{\frac{n+3}{2}}} - \frac{(n+1)t^2}{(t^2+|x-y|^2)^{\frac{n+3}{2}}}\right]dy.
\end{aligned}
$$

In the following subsections, we will estimate A_i, B_i, D and E, respectively. First we give some useful inequalities. For $s>0$,

$$
\begin{aligned}
\text{(2.4)}\qquad \left(\int_0^\infty \frac{t^3}{(t^2+s^2)^{n+3}}\,dt\right)^{1/2} &= \frac{1}{s^{n+1}}\left(\int_0^\infty \frac{t^3}{(t^2+1)^{n+3}}\,dt\right)^{1/2}\\
&= \frac{\sqrt{2}}{2s^{n+1}}\left(\int_1^\infty \frac{t-1}{t^{n+3}}\,dt\right)^{1/2}\\
&\lesssim \frac{1}{ns^{n+1}}.
\end{aligned}
$$

Similarly,

$$
\text{(2.5)}\qquad \left(\int_0^\infty \frac{t^3}{(t^2+s^2)^{n+5}}\,dt\right)^{1/2} \lesssim \frac{1}{ns^{n+3}},
$$

$$
\text{(2.6)}\qquad \left(\int_0^\infty \frac{t}{(t^2+s^2)^{n+3}}\,dt\right)^{1/2} \lesssim \frac{1}{n^{1/2}s^{n+2}},
$$

$$
\text{(2.7)}\qquad \left(\int_0^\infty \frac{t}{(t^2+s^2)^{n+1}}\,dt\right)^{1/2} \lesssim \frac{1}{n^{1/2}s^{n}},
$$

$$
\text{(2.8)}\qquad \left(\int_0^\infty \frac{t^5}{(t^2+s^2)^{n+5}}\,dt\right)^{1/2} \lesssim \frac{1}{n^{3/2}s^{n+2}}.
$$

2.1. Estimate of A_i. We split A_i into $A_{i,1}+A_{i,2}$, where

$$A_{i,1}(x,t)=\int_{Q_k\cap\{y\colon |x-y|>n|y-y_0|\}} b_k(y)(n+1)c_n\frac{(y_i-y_{0,i})t}{(t^2+|x-y|^2)^{\frac{n+3}{2}}}\,dy,$$

$$A_{i,2}(x,t)=\int_{Q_k\cap\{y\colon |x-y|\le n|y-y_0|\}} b_k(y)(n+1)c_n\frac{(y_i-y_{0,i})t}{(t^2+|x-y|^2)^{\frac{n+3}{2}}}\,dy.$$

For $A_{i,1}$, note that $|x-y|>n|y-y_0|$ for $x\in F$ and $y\in Q_k$, and we have

$$|x-y|>\frac{n}{n+1}|x-y_0| \quad\text{and}\quad |x-y_0|>\frac{n-1}{n}|x-y|.$$

Since $|y_i-y_{0,i}|\le a$, we have the estimate

$$\begin{aligned}|A_{i,1}(x,t)| &\lesssim c_n n\int_{Q_k}|b_k(y)|\frac{at}{\left(t^2+\left(\frac{n}{n+1}|x-y_0|\right)^2\right)^{\frac{n+3}{2}}}\,dy\\ &=\|b_k\|_1 c_n n\frac{at}{\left(t^2+\left(\frac{n}{n+1}|x-y_0|\right)^2\right)^{\frac{n+3}{2}}}.\end{aligned}$$

Note that this will produce a factor n by taking the sum $\sum_{i=1}^n$. Hence combining with (2.4), we have the estimate

$$\begin{aligned}&\Big(\int_0^\infty\sum_{i=1}^n|A_{i,1}(x,t)|^2t\,dt\Big)^{1/2}\\ &\qquad\le\|b_k\|_1 ac_n n^{3/2}\left(\int_0^\infty\frac{t^3}{\left(t^2+\left(\frac{n}{n+1}|x-y_0|\right)^2\right)^{n+3}}\,dt\right)^{1/2}\\ &\qquad\lesssim\|b_k\|_1 an^{1/2}c_n|x-y_0|^{-n-1}.\end{aligned}$$

Since $x\in F$, we have $|x-y_0|\ge a/2$. Hence

$$\begin{aligned}&\int_F\Big(\int_0^\infty\sum_{i=1}^n|A_{i,1}(x,t)|^2t\,dt\Big)^{1/2}dx \qquad (2.9)\\ &\qquad\lesssim\|b_k\|_1 an^{1/2}c_n\int_{\{x\colon |x-y_0|\ge a/2\}}\frac{1}{|x-y_0|^{n+1}}\,dx\\ &\qquad\lesssim\|b_k\|_1 an^{1/2}c_n\omega_{n-1}\int_{a/2}^\infty\frac{dr}{r^2}\lesssim n\|b_k\|_1,\end{aligned}$$

where in the third inequality we use (1.3).

Now consider the term $A_{i,2}$. Since $x\in F$, $y\in Q_k$ and $|x-y|\le n|y-y_0|$, by the definition of F we have $|x-y|\ge|y-y_0|/(2n^{3/2})$. Using Minkowski's inequality, Fubini's theorem interchanging dy and dx, (2.4), changing to polar coordinates and applying (1.3), we have

$$(2.10)\qquad \int_F \Big(\int_0^\infty \sum_{i=1}^n |A_{i,2}(x,t)|^2 t\,dt\Big)^{1/2} dx$$

$$\lesssim c_n n \int_{Q_k} \int_{F\cap\{x:\,|x-y|\le n|y-y_0|\}} |b_k(y)|\,|y-y_0|\Big(\int_0^\infty \frac{t^3}{(t^2+|x-y|^2)^{n+3}}\,dt\Big)^{1/2} dx\,dy$$

$$\lesssim c_n \int_{Q_k} \Big(\int_{\{x:\,|y-y_0|/(2n^{3/2})\le|x-y|\}} \frac{|y-y_0|\,dx}{|x-y|^{n+1}}\Big)|b_k(y)|\,dy$$

$$\lesssim n^2\|b_k\|_1.$$

2.2. Estimate of B_i. We argue much as before, breaking B_i into two parts:

$$B_i(x,t) = \int_{Q_k} b_k(y)c_n(n+1)t\Big\{\frac{x_i-y_{0,i}}{(t^2+|x-y|^2)^{\frac{n+3}{2}}} - \frac{x_i-y_{0,i}}{(t^2+|x-y_0|^2)^{\frac{n+3}{2}}}\Big\}\,dy$$

$$= \int_{Q_k\cap\{y:\,|x-y|>n|y-y_0|\}} + \int_{Q_k\cap\{y:\,|x-y|\le n|y-y_0|\}} \cdots$$

$$=: B_{i,1}(x,t) + B_{i,2}(x,t).$$

First let us consider $B_{i,1}$. We have $|x-y| > \frac{n}{n+1}|x-y_0|$ and $|x-y_0| > \frac{n-1}{n}|x-y|$ since $|x-y|>n|y-y_0|$. Note $|y-y_0|\le\sqrt{n}\,a$. So by the mean value theorem,

$$|B_{i,1}(x,t)| \lesssim c_n n^2 \int_{Q_k} |b_k(y)|\,|x_i-y_{0,i}|t\frac{|x-y_0|\,|y-y_0|}{\big(t^2+\big(\frac{n}{n+1}|x-y_0|\big)^2\big)^{(n+5)/2}}\,dy$$

$$\lesssim c_n n^{5/2}\|b_k\|_1|x_i-y_{0,i}|\,|x-y_0|at\bigg(t^2+\Big(\frac{n}{n+1}|x-y_0|\Big)^2\bigg)^{-(n+5)/2}.$$

Since $x\in F$, we have $|x-y_0|\ge a/2$. Hence

$$(2.11)\qquad \int_F \Big(\int_0^\infty \sum_{i=1}^n |B_{i,1}(x,t)|^2 t\,dt\Big)^{1/2} dx$$

$$\lesssim c_n n^{5/2}\|b_k\|_1 a\int_F\Big(\int_0^\infty \frac{t^3}{\big(t^2+\big(\frac{n}{n+1}|x-y_0|\big)^2\big)^{n+5}}\,dt\Big)^{1/2}|x-y_0|^2\,dx$$

$$\lesssim c_n n^{3/2}\|b_k\|_1 a \int_{\{x:\,|x-y_0|>a/2\}} \frac{dx}{|x-y_0|^{n+1}} \lesssim n^2\|b_k\|_1,$$

where we use (2.5) in the second inequality and (1.3) in the third.

Now we consider $B_{i,2}$. Since $x \in F$, $y \in Q_k$ and $|x-y| \le n|y-y_0|$, we have $|x-y| \ge |y-y_0|/(2n^{3/2})$ by the definition of F. In particular, $|x-y_0| \ge |y-y_0|/(2n^{3/2})$. It is easy to see that

$$|x-y_0| \le (n+1)|y-y_0|.$$

Applying Minkowski's inequality, (2.4), Fubini's theorem and (1.3), we have

$$\begin{aligned}
(2.12)\qquad &\int_F \Big(\int_0^\infty \sum_{i=1}^n |B_{i,2}(x,t)|^2 t\,dt\Big)^{1/2} dx \\
&\lesssim c_n n \int_F \int_{Q_k\cap\{y:\,|x-y|\le n|y-y_0|\}} |x-y_0|\,|b_k(y)| \\
&\quad\times \bigg[\Big(\int_0^\infty \frac{t^3}{(t^2+|x-y|^2)^{n+3}}\,dt\Big)^{1/2} + \Big(\int_0^\infty \frac{t^3}{(t^2+|x-y_0|^2)^{n+3}}\,dt\Big)^{1/2}\bigg] dy\,dx \\
&\lesssim c_n \int_{Q_k} |b_k(y)| \bigg[\int_{\{x:\,|y-y_0|/(2n^{3/2})\le|x-y|\}} \frac{|x-y_0|}{|x-y|^{n+1}}\,dx \\
&\quad + \int_{\{x:\,|y-y_0|/(2n^{3/2})\le|x-y_0|\le(n+1)|y-y_0|\}} |x-y_0|^{-n}\,dx\bigg] dy \\
&\lesssim n^3\|b_k\|_1.
\end{aligned}$$

2.3. Estimate of D. We split D into two parts:

$$\begin{aligned}
D(x,t) &= \int_{Q_k} b_k(y) c_n \bigg[\frac{1}{(t^2+|x-y|^2)^{\frac{n+1}{2}}} - \frac{1}{(t^2+|x-y_0|^2)^{\frac{n+1}{2}}}\bigg] dy \\
&= \int_{Q_k\cap\{y:\,|x-y|>n|y-y_0|\}} + \int_{Q_k\cap\{y:\,|x-y|\le n|y-y_0|\}} \cdots \\
&=: D_1(x,t) + D_2(x,t).
\end{aligned}$$

We consider D_1 first. Since $x \in F$, $y \in Q_k$ and $|x-y| > n|y-y_0|$, we have

$$|x-y| > \frac{n}{n+1}|x-y_0|, \quad |x-y_0| \ge \frac{a}{2} \quad \text{and} \quad |y-y_0| \le n^{1/2}a.$$

Using the mean value theorem, we get

$$\begin{aligned}
(2.13)\qquad &|D_1(x,t)| \\
&\lesssim c_n \int_{Q_k\cap\{y:\,|x-y|>n|y-y_0|\}} |b_k(y)| \frac{(n+1)|x-y_0|\,|y-y_0|}{\big(t^2+\big(|x-y_0|\frac{n}{n+1}\big)^2\big)^{n+3/2}}\,dy \\
&\lesssim c_n n^{3/2} a \|b_k\|_1 \frac{|x-y_0|}{\big(t^2+\big(|x-y_0|\frac{n}{n+1}\big)^2\big)^{n+3/2}}.
\end{aligned}$$

Thus, using (2.13), (2.6), changing to polar coordinates and (1.3),

$$\int_F \Big(\int_0^\infty |D_1(x,t)|^2 t\,dt\Big)^{1/2} dx \tag{2.14}$$
$$\lesssim c_n n^{3/2} a \|b_k\|_1 \int_F \Big(\int_0^\infty \frac{|x-y_0|^2 t}{\big(t^2+\big(\frac{n}{n+1}|x-y_0|\big)^2\big)^{n+3}}\,dt\Big)^{1/2} dx$$
$$\lesssim c_n n a \|b_k\|_1 \int_{\{x:\, a/2\le |x-y_0|\}} \frac{dx}{|x-y_0|^{n+1}}$$
$$\lesssim n^{3/2}\|b_k\|_1.$$

Consider the term D_2. Since $x\in F$, $y\in Q_k$ and $|x-y|\le n|y-y_0|$, as in estimating $B_{i,2}$ we have $|y-y_0|/(2n^{3/2})\le |x-y|\le n|y-y_0|$ and $\frac{|y-y_0|}{2n^{3/2}}\le |x-y_0|\le (n+1)|y-y_0|$.

Applying Minkowski's inequality, (2.7), Fubini's theorem and (1.3), we have

$$\int_F \Big(\int_0^\infty |D_2(x,t)|^2 t\,dt\Big)^{1/2} dx \tag{2.15}$$
$$\lesssim c_n \int_F \int_{Q_k\cap\{y:\, |x-y|\le n|y-y_0|\}} |b_k(y)|$$
$$\times \Big[\Big(\int_0^\infty \frac{t}{(t^2+|x-y|^2)^{n+1}}\,dt\Big)^{1/2} + \Big(\int_0^\infty \frac{t}{(t^2+|x-y_0|^2)^{n+1}}\,dt\Big)^{1/2}\Big]\,dy\,dx$$
$$\lesssim c_n n^{-1/2} \int_{Q_k} |b_k(y)| \Big[\int_{\{x:\, \frac{|y-y_0|}{2n^{3/2}}\le |x-y|\le n|y-y_0|\}} \frac{1}{|x-y|^n}\,dx$$
$$+ \int_{\{x:\, \frac{|y-y_0|}{2n^{3/2}}\le |x-y_0|\le (n+1)|y-y_0|\}} \frac{1}{|x-y_0|^n}\,dx\Big]\,dy$$
$$\lesssim \log n \|b_k\|_1.$$

2.4. Estimate of E. We split E into two parts:

$$E(x,t) = \int_{Q_k} b_k(y) c_n \Big[\frac{(n+1)t^2}{(t^2+|x-y_0|^2)^{\frac{n+3}{2}}} - \frac{(n+1)t^2}{(t^2+|x-y|^2)^{\frac{n+3}{2}}}\Big]\,dy$$
$$= \int_{Q_k\cap\{y:\, |x-y|>n|y-y_0|\}} + \int_{Q_k\cap\{y:\, |x-y|\le n|y-y_0|\}} \cdots$$
$$=: E_1(x,t)+E_2(x,t).$$

For the term E_1, we have $|x-y| > \frac{n}{n+1}|x-y_0|$, $|x-y_0| \geq \frac{a}{2}$ and $|y-y_0| \leq n^{1/2}a$. So, by the mean value theorem,

$$\begin{aligned}
&|E_1(x,t)| \\
&\quad \leq c_n \int\limits_{Q_k\cap\{y:\,|x-y|>n|y-y_0|\}} |b_k(y)|(n+1)(n+3)t^2 \frac{|x-y_0|\,|y-y_0|}{\big(t^2+\big(\frac{n}{n+1}|x-y_0|\big)^2\big)^{\frac{n+5}{2}}}\,dy \\
&\quad \lesssim c_n n^{5/2} a\|b_k\|_1 \frac{|x-y_0|t^2}{\big(t^2+\big(\frac{n}{n+1}|x-y_0|\big)^2\big)^{\frac{n+5}{2}}}.
\end{aligned}$$

Therefore by (2.8), we get

$$\begin{aligned}
(2.16)\qquad &\int\limits_F \Big(\int\limits_0^\infty |E_1(x,t)|^2 t\,dt\Big)^{1/2} dx \\
&\lesssim c_n n^{5/2} a\|b_k\|_1 \int\limits_F \Big(\int\limits_0^\infty \frac{|x-y_0|^2 t^5}{\big(t^2+\big(\frac{n}{n+1}|x-y_0|\big)^2\big)^{n+5}}\,dt\Big)^{1/2} dx \\
&\lesssim a c_n n\|b_k\|_1 \int\limits_{\{x:\,|x-y_0|\geq a/2\}} \frac{dx}{|x-y_0|^{n+1}} \\
&\lesssim n^{3/2}\|b_k\|_1.
\end{aligned}$$

Now we consider E_2. In the same way as in estimating $B_{i,2}$ or D_2 before, we have

$$\begin{aligned}
(2.17)\qquad &\int\limits_F \Big(\int\limits_0^\infty |E_2(x,t)|^2 t\,dt\Big)^{1/2} dx \\
&\lesssim c_n n \int\limits_F \int\limits_{Q_k\cap\{y:\,|x-y|\leq n|y-y_0|\}} |b_k(y)| \\
&\quad\times \Bigg[\Big(\int\limits_0^\infty \frac{t^5}{(t^2+|x-y|^2)^{n+3}}\,dt\Big)^{1/2} + \Big(\int\limits_0^\infty \frac{t^5}{(t^2+|x-y_0|^2)^{n+3}}\,dt\Big)^{1/2}\Bigg] dy\,dx \\
&\lesssim c_n n^{-1/2} \int\limits_{Q_k} |b_k(y)| \Bigg[\int\limits_{\{x:\,\frac{|y-y_0|}{2n^{3/2}}\leq|x-y|\leq n|y-y_0|\}} \frac{1}{|x-y|^n}\,dx \\
&\quad + \int\limits_{\{x:\,\frac{|y-y_0|}{2n^{3/2}}\leq|x-y_0|\leq(n+1)|y-y_0|\}} \frac{1}{|x-y_0|^n}\,dx\Bigg] dy \\
&\lesssim (\log n)\|b_k\|_1.
\end{aligned}$$

Applying the inequality $(|a|+|b|)^{1/2} \leq |a|^{1/2}+|b|^{1/2}$, and then (2.9)–(2.12) and (2.14)–(2.17), we have

$$
\begin{aligned}
\int_F \mathfrak{g}(b_k)(x)\,dx &= \int_F \Big(\int_0^\infty \Big[\sum_{i=1}^n (\partial_{x_i} u(x,t))^2 + (\partial_t u(x,t))^2 \Big] t\,dt \Big)^{1/2} dx \\
&\lesssim \int_F \Big(\int_0^\infty \sum_{i=1}^n (A_{i,1}(x,t))^2 t\,dt \Big)^{1/2} dx + \int_F \Big(\int_0^\infty \sum_{i=1}^n (A_{i,2}(x,t))^2 t\,dt \Big)^{1/2} dx \\
&\quad + \int_F \Big(\int_0^\infty \sum_{i=1}^n (B_{i,1}(x,t))^2 t\,dt \Big)^{1/2} dx + \int_F \Big(\int_0^\infty \sum_{i=1}^n (B_{i,2}(x,t))^2 t\,dt \Big)^{1/2} dx \\
&\quad + \int_F \Big(\int_0^\infty (D_1(x,t))^2 t\,dt \Big)^{1/2} dx + \int_F \Big(\int_0^\infty (D_2(x,t))^2 t\,dt \Big)^{1/2} dx \\
&\quad + \int_F \Big(\int_0^\infty (E_1(x,t))^2 t\,dt \Big)^{1/2} dx + \int_F \Big(\int_0^\infty (E_2(x,t))^2 t\,dt \Big)^{1/2} dx \\
&\lesssim n^3 \|b_k\|_1.
\end{aligned}
$$

Combining (2.1)–(2.3) with the above estimate, we get the bound of Theorem 1.1.

Remark 2.2. It seems that by using the method presented here, the upper bound n^3 in Theorem 1.1 cannot be improved.

3. Proof of Theorem 1.2. Before the proof, we make some observations.

Define

$$
f_s(y) = \frac{1}{s^n} f\Big(\frac{y}{s}\Big) \quad \text{for } s > 0.
$$

Then $\mathfrak{g}(f_s)(x)$ is equal to

$$
\begin{aligned}
&\Big(\int_0^\infty \Big[\sum_{i=1}^n \Big(\int_{\mathbb{R}^n} (\partial_{x_i} P_t)(x-y) f_s(y)\,dy \Big)^2 + \Big(\int_{\mathbb{R}^n} (\partial_t P_t)(x-y) f_s(y)\,dy \Big)^2 \Big] t\,dt \Big)^{1/2} \\
&= \Big(\int_0^\infty \Big[\sum_{i=1}^n \Big(\int_{\mathbb{R}^n} (\partial_{x_i} P_t)(x-sy) f(y)\,dy \Big)^2 + \Big(\int_{\mathbb{R}^n} (\partial_t P_t)(x-sy) f(y)\,dy \Big)^2 \Big] t\,dt \Big)^{1/2} \\
&= \frac{1}{s^n} \Big(\int_0^\infty \Big[\sum_{i=1}^n \Big(\int_{\mathbb{R}^n} (\partial_{x_i} P_t)\Big(\frac{x}{s} - y\Big) f(y)\,dy \Big)^2 \\
&\quad + \Big(\int_{\mathbb{R}^n} (\partial_t P_t)\Big(\frac{x}{s} - y\Big) f(y)\,dy \Big)^2 \Big] t\,dt \Big)^{1/2} \\
&= \frac{1}{s^n} \mathfrak{g}(f)\Big(\frac{x}{s}\Big).
\end{aligned}
$$

Therefore

$$m(\{x \in \mathbb{R}^n : \mathfrak{g}(f_s)(x) > 1\}) = m\left(\left\{x \in \mathbb{R}^n : \frac{1}{s^n}\mathfrak{g}(f)\left(\frac{x}{s}\right) > 1\right\}\right)$$
$$= s^n m(\{x \in \mathbb{R}^n : \mathfrak{g}(f)(x) > s^n\}).$$

Now by setting $\lambda = s^n$, we get Theorem 1.2 if we can prove the equality

$$\lim_{s\to 0_+} m(\{x \in \mathbb{R}^n : \mathfrak{g}(f_s)(x) > 1\}) = \frac{\sqrt{2}\, c_n\omega_{n-1}}{2n}\Big|\int_{\mathbb{R}^n} f(x)\, dx\Big| \tag{3.1}$$

for any $f \in L^1(\mathbb{R}^n)$.

In the rest of this section, we will prove (3.1). Without loss of generality, we may assume $\|f\|_1 = 1$. Let δ be small enough, $0 < \delta \ll 1$. Assume that $0 < \epsilon < \delta/2$. Since $\|f\|_1 = 1$, there exists an a_ϵ with $0 < a_\epsilon < \infty$ such that $\int_{B(0,a_\epsilon)} |f(x)|\, dx = 1 - \epsilon$. Set $f^0 = f\chi_{B(0,a_\epsilon)}$ and $f^\infty = f - f^0$, where χ_E is the characteristic function of E. For $s > 0$, let

$$f_s^0(\cdot) = \frac{1}{s^n} f^0\left(\frac{\cdot}{s}\right) \quad \text{and} \quad f_s^\infty(\cdot) = \frac{1}{s^n} f^\infty\left(\frac{\cdot}{s}\right).$$

Then $f_s = f_s^0 + f_s^\infty$, $\|f_s^0\|_1 = 1 - \epsilon$ and $\|f_s^\infty\| = \epsilon$.

For any $\lambda > 0$, define

$$D^s = \{x \in \mathbb{R}^n : \mathfrak{g}(f_s)(x) > 1\},$$
$$E_\lambda^s = \{x \in \mathbb{R}^n : \mathfrak{g}(f_s^\infty)(x) > \lambda\},$$
$$F_\lambda^s = \{x \in \mathbb{R}^n : \mathfrak{g}(f_s^0)(x) > \lambda\}.$$

Note that the Littlewood–Paley $\mathfrak{g}$-function is a sublinear operator, hence

$$F_{1+\delta}^s \subset E_\delta^s \cup D^s \quad \text{and} \quad D^s \subset E_\delta^s \cup F_{1-\delta}^s.$$

Therefore

$$m(F_{1+\delta}^s) - m(E_\delta^s) \le m(D^s) \le m(E_\delta^s) + m(F_{1-\delta}^s). \tag{3.2}$$

Since the Littlewood–Paley $\mathfrak{g}$-function is of weak type $(1,1)$, we have

$$m(E_\delta^s) = m(\{x \in \mathbb{R}^n : \mathfrak{g}(f_s^\infty)(x) > \delta\}) \le \frac{C_n}{\delta}\|f_s^\infty\|_1 = \frac{C_n\epsilon}{\delta}, \tag{3.3}$$

where C_n is a constant depending on n (here there is no need to care about the relationship between C_n and n; see below).

By the choice of δ and ϵ, $m(F_{1-\delta}^s)$ and $m(F_{1+\delta}^s)$ should approximate $m(D^s)$ as $s \to 0$. This will be exploited in what follows.

Choose $\eta > 2sa_\epsilon$. It is easy to see that

$$m(F_{1-\delta}^s) - v_n\eta^n \le m(F_{1-\delta}^s \cap B(0,\eta)^c) \le m(F_{1-\delta}^s),$$

where v_n is the measure of the unit ball in $\mathbb{R}^n$. Therefore $m(F_{1-\delta}^s \cap B(0,\eta)^c))$ approximates $m(F_{1-\delta}^s)$ as $\eta \to 0$. Similarly, $m(F_{1+\delta}^s \cap B(0,\eta)^c)$ approximates

$m(F^s_{1+\delta})$ as $\eta \to 0$. To get the estimate of $m(D^t)$, we should give a lower estimate of $m(F^s_{1+\delta})$ and an upper estimate of $m(F^s_{1-\delta)})$. Write

$$\begin{aligned}
\mathfrak{g}(f^0_s)(x) &= \Big(\int_0^\infty \Big[\sum_{i=1}^n \Big(\int_{\mathbb{R}^n} (\partial_{x_i} P_t)(x-y) f^0_s(y)\,dy\Big)^2 \\
&\quad + \Big(\int_{\mathbb{R}^n} (\partial_t P_t)(x-y) f^0_s(y)\,dy\Big)^2\Big] t\,dt\Big)^{1/2} \\
&\le \Big\{\int_0^\infty \Big[\sum_{i=1}^n \Big(\int_{\mathbb{R}^n} \big((\partial_{x_i} P_t)(x-y) - (\partial_{x_i} P_t)(x)\big) f^0_s(y)\,dy\Big)^2 \\
&\quad + \Big(\int_{\mathbb{R}^n} \big((\partial_t P_t)(x-y) - (\partial_t P_t)(x)\big) f^0_s(y)\,dy\Big)^2\Big] t\,dt\Big\}^{1/2} \\
&\quad + \Big(\int_0^\infty \Big[\sum_{i=1}^n (\partial_{x_i} P_t(x))^2 + (\partial_t P_t(x))^2\Big] t\,dt\Big)^{1/2} \Big|\int_{\mathbb{R}^n} f^0_s(y)\,dy\Big| \\
&=: H(f^0_s)(x) + J(x)\Big|\int_{\mathbb{R}^n} f^0_s(y)\,dy\Big|.
\end{aligned} \tag{3.4}$$

Similarly, it can be showed that

$$\mathfrak{g}(f^0_s)(x) \ge J(x)\Big|\int_{\mathbb{R}^n} f^0_s(y)\,dy\Big| - H(f^0_s)(x). \tag{3.5}$$

Recall that

$$\partial_{x_i} P_t(x) = -c_n \frac{x_i(n+1)t}{(t^2+|x|^2)^{\frac{n+3}{2}}}, \tag{3.6}$$

$$\partial_t P_t(x) = c_n \Big[\frac{1}{(t^2+|x|^2)^{\frac{n+1}{2}}} - \frac{(n+1)t^2}{(t^2+|x|^2)^{\frac{n+3}{2}}}\Big]. \tag{3.7}$$

Let us first manipulate $J(x)$. Note that

$$\begin{aligned}
\int_0^\infty \sum_{i=1}^n (\partial_{x_i} P_t(x))^2 t\,dt &= c_n^2 (n+1)^2 |x|^2 \int_0^\infty \frac{t^3}{(t^2+|x|^2)^{n+3}}\,dt \\
&= \frac{c_n^2}{|x|^{2n}} (n+1)^2 \int_0^\infty \frac{t^3}{(1+t^2)^{n+3}}\,dt = \frac{c_n^2}{|x|^{2n}} \frac{n+1}{2(n+2)}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty (\partial_t P_t(x))^2 t\,dt &= \frac{c_n^2}{|x|^{2n}} \int_0^\infty \Big[\frac{t}{(1+t^2)^{n+1}} + \frac{(n+1)^2 t^5}{(1+t^2)^{n+3}} - \frac{2(n+1)t^3}{(1+t^2)^{n+2}}\Big]\,dt \\
&= \frac{c_n^2}{2|x|^{2n}} \Big(\frac{1}{n} + \frac{2(n+1)}{n(n+2)} - \frac{2}{n}\Big) = \frac{c_n^2}{|x|^{2n}} \frac{1}{2(n+2)}.
\end{aligned}$$

Hence we have

$$J(x)=\Big(\int_0^\infty\sum_{i=1}^n(\partial_{x_i}P_t(x))^2t\,dt+\int_0^\infty(\partial_tP_t(x))^2t\,dt\Big)^{1/2}=\frac{\sqrt{2}}{2}\frac{c_n}{|x|^n}. \tag{3.8}$$

On the other hand, write

$$\begin{aligned}(\partial_{x_i}P_t)(x-y)-(\partial_{x_i}P_t)(x)\\ &=c_n\frac{y_i(n+1)t}{(t^2+|x-y|^2)^{\frac{n+3}{2}}}-c_n\Big(\frac{x_i(n+1)t}{(t^2+|x-y|^2)^{\frac{n+3}{2}}}-\frac{x_i(n+1)t}{(t^2+|x|^2)^{\frac{n+3}{2}}}\Big)\\ &=:K_{i,1}(x,y,t)-K_{i,2}(x,y,t)\end{aligned}$$

and

$$\begin{aligned}(\partial_tP_t)(x-y)-(\partial_tP_t)(x)&=c_n\Big(\frac{1}{(t^2+|x-y|^2)^{\frac{n+1}{2}}}-\frac{1}{(t^2+|x|^2)^{\frac{n+1}{2}}}\Big)\\ &\quad-c_n\Big(\frac{(n+1)t^2}{(t^2+|x-y|^2)^{\frac{n+3}{2}}}-\frac{(n+1)t^2}{(t^2+|x|^2)^{\frac{n+3}{2}}}\Big)\\ &=:K_3(x,y,t)-K_4(x,y,t).\end{aligned}$$

Denote

$$G_s=\{x\in B(0,\eta)^c:H(f_s^0)(x)\geq\delta\}.$$

Then, by Chebyshev's inequality and Fubini's theorem, we have

$$\begin{aligned}m(G_s)&\leq\frac{1}{\delta}\int_{B(0,\eta)^c}\Big(\int_0^\infty\sum_{i=1}^n\Big(\int_{\mathbb{R}^n}K_{i,1}(x,y,t)f_s^0(y)\,dy\Big)^2t\,dt\Big)^{1/2}dx\\ &\quad+\frac{1}{\delta}\int_{B(0,\eta)^c}\Big(\int_0^\infty\sum_{i=1}^n\Big(\int_{\mathbb{R}^n}K_{i,2}(x,y,t)f_s^0(y)\,dy\Big)^2t\,dt\Big)^{1/2}dx\\ &\quad+\frac{1}{\delta}\int_{B(0,\eta)^c}\Big(\int_0^\infty\Big(\int_{\mathbb{R}^n}K_3(x,y,t)f_s^0(y)\,dy\Big)^2t\,dt\Big)^{1/2}dx\\ &\quad+\frac{1}{\delta}\int_{B(0,\eta)^c}\Big(\int_0^\infty\Big(\int_{\mathbb{R}^n}K_4(x,y,t)f_s^0(y)\,dy\Big)^2t\,dt\Big)^{1/2}dx\\ &=:I+II+III+IV.\end{aligned}$$

Consider the term I first. Since $|y|<sa_\epsilon$, $|x|>\eta$ and $\eta>2sa_\epsilon$, we get

$$\tfrac{1}{2}|x|<|x-y|<\tfrac{3}{2}|x|. \tag{3.9}$$

Combining (3.9) and Minkowski's inequality with variable t, we have

$$I \leq \frac{C_n}{\delta} \int\limits_{B(0,\eta)^c} \Bigg(\int\limits_0^\infty \Bigg(\int\limits_{B(0,sa_\varepsilon)} \frac{|y|t}{(t^2+(|x|/2)^2)^{\frac{n+3}{2}}} |f_s^0(y)|\,dy \Bigg)^2 t\,dt \Bigg)^{1/2} dx \tag{3.10}$$
$$\leq \frac{C_n}{\delta} \int\limits_{B(o,\eta)^c} \int\limits_{B(0,sa_\varepsilon)} \Bigg(\int\limits_0^\infty \frac{t^3}{(t^2+(|x|/2)^2)^{n+3}}\,dt \Bigg)^{1/2} |y|\,|f_s^0(y)|\,dy\,dx$$
$$\leq C_n \frac{sa_\epsilon}{\delta} \int\limits_{B(0,sa_\varepsilon)} \int\limits_{B(0,\eta)^c} \frac{|f_s^0(y)|}{|x|^{n+1}}\,dx\,dy$$
$$\leq C_n \frac{sa_\epsilon}{\delta\eta},$$

where in the third inequality we use $|y| \leq sa_\epsilon$, Fubini's theorem and (2.4).

By arguing with the term I with the estimate (2.4) replaced by (2.5), (2.6) and (2.8) respectively, we can get $II, III, IV \leq C_n \frac{sa_\epsilon}{\delta\eta}$. Combining the above estimates, we have

$$m(G_s) \leq C_n \frac{sa_\epsilon}{\delta\eta}. \tag{3.11}$$

Now consider $x \in B(0,\eta)^c \cap G_s^c$, so that $H(f_s^0)(x) \leq \delta$. By (3.4) and (3.5), we have

$$J(x)\Big|\int\limits_{\mathbb{R}^n} f_s^0(y)\,dy\Big| - \delta \leq \mathfrak{g}(f_s^0)(x) \leq \delta + J(x)\Big|\int\limits_{\mathbb{R}^n} f_s^0(y)\,dy\Big|. \tag{3.12}$$

Therefore

$$\{x \in B(0,\eta)^c \cap G_s^c : \mathfrak{g}(f_s^0)(x) > 1-\delta\}$$
$$\subset \Big\{x \in B(0,\eta)^c \cap G_s^c : J(x)\Big|\int\limits_{\mathbb{R}^n} f_s^0(y)\,dy\Big| > 1-2\delta\Big\}$$

and

$$\{x \in B(0,\eta)^c \cap G_s^c : \mathfrak{g}(f_s^0)(x) > 1+\delta\}$$
$$\supset \Big\{x \in B(0,\eta)^c \cap G_s^c : J(x)\Big|\int\limits_{\mathbb{R}^n} f_s^0(y)\,dy\Big| > 1+2\delta\Big\}.$$

Note that

$$m(\{x \in \mathbb{R}^n : 1/|x|^n > \lambda\}) = \frac{\omega_{n-1}}{n\lambda}. \tag{3.13}$$

By the definition of f^0 and f^∞, we have

$$\Big|\int\limits_{\mathbb{R}^n} f(y)\,dy\Big| - \epsilon \leq \Big|\int\limits_{\mathbb{R}^n} f_s^0(y)\,dy\Big| \leq \Big|\int\limits_{\mathbb{R}^n} f(y)\,dy\Big| + \epsilon. \tag{3.14}$$

Then by (3.8), (3.13) and (3.14),

$$
\begin{aligned}
(3.15)\qquad m(\{x\in B(0,\eta)^c\cap G_s^c : \mathfrak{g}(f_s^0)(x)>1-\delta\}) \\
&\le m\Big(\Big\{x\in\mathbb{R}^n : J(x)\Big|\int_{\mathbb{R}^n} f_s^0(y)\,dy\Big|>1-2\delta\Big\}\Big)\\
&\le \frac{\sqrt{2}\,c_n\omega_{n-1}}{2n(1-2\delta)}\Big(\Big|\int_{\mathbb{R}^n} f(y)\,dy\Big|+\epsilon\Big).
\end{aligned}
$$

Similarly, by (3.8), (3.13), (3.14) and (3.11),

$$
\begin{aligned}
(3.16)\qquad & m(\{x\in B(0,\eta)^c\cap G_s^c : \mathfrak{g}(f_s^0)(x)>1+\delta\})\\
&\ge m\Big(\Big\{x\in\mathbb{R}^n : J(x)\Big|\int_{\mathbb{R}^n} f_s^0(y)\,dy\Big|>1+2\delta\Big\}\Big)-v_n\eta^n-m(G_s)\\
&\ge \frac{\sqrt{2}\,c_n\omega_{n-1}}{2n(1+2\delta)}\Big(\Big|\int_{\mathbb{R}^n} f(y)\,dy\Big|-\epsilon\Big)-v_n\eta^n-C\frac{sa_\epsilon}{\eta\delta}.
\end{aligned}
$$

Combining (3.3), (3.2), (3.11), (3.15) and (3.16), we have

$$
\begin{aligned}
(3.17)\qquad & m(D^s)\le m(F^s_{1-\delta})+C_n\frac{\epsilon}{\delta}\\
&\le m(\{x\in B(x,\eta)^c\cap G_s^c : \mathfrak{g}(f_s^0)(x)>1-\delta\})+v_n\eta^n+m(G_s)+C_n\frac{\epsilon}{\delta}\\
&\le \frac{\sqrt{2}\,c_n\omega_{n-1}}{2n(1-2\delta)}\Big(\Big|\int_{\mathbb{R}^n} f(y)\,dy\Big|+\epsilon\Big)+v_n\eta^n+C_n\frac{sa_\epsilon}{\eta\delta}+C_n\frac{\epsilon}{\delta}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.18)\qquad & m(D^s)\ge m(F^s_{1+\delta})-C_n\frac{\epsilon}{\delta}\\
&\ge m(\{x\in B(x,\eta)^c\cap G_s^c : \mathfrak{g}(f_s^0)(x)>1+\delta\})-C_n\frac{\epsilon}{\delta}\\
&\ge \frac{\sqrt{2}\,c_n\omega_{n-1}}{2n(1+2\delta)}\Big(\Big|\int_{\mathbb{R}^n} f(y)\,dy\Big|-\epsilon\Big)-v_n\eta^n-C_n\frac{sa_\epsilon}{\eta\delta}-C_n\frac{\epsilon}{\delta}.
\end{aligned}
$$

Let $s\to 0$. Then by (3.17),

$$
\begin{aligned}
&\limsup_{s\to 0_+} m(\{x\in\mathbb{R}^n : \mathfrak{g}(f_s)(x)>1\})\\
&\qquad\le \frac{\sqrt{2}\,c_n\omega_{n-1}}{2n(1-2\delta)}\Big(\Big|\int_{\mathbb{R}^n} f(y)\,dy\Big|+\epsilon\Big)+v_n\eta^n+C_n\frac{\epsilon}{\delta}.
\end{aligned}
$$

Note $\epsilon\le\delta/2$. Let first $\epsilon\to 0$, then $\delta\to 0$, and finally $\eta\to 0$. Then we get

$$
\limsup_{s\to 0_+} m(\{x\in\mathbb{R}^n : \mathfrak{g}(f_s)(x)>1\})\le \frac{\sqrt{2}\,c_n\omega_{n-1}}{2n}\Big|\int_{\mathbb{R}^n} f(y)\,dy\Big|.
$$

Arguing similarly with (3.18), we have

$$\liminf_{s\to 0_+} m(\{x\in\mathbb{R}^n : \mathfrak{g}(f_s)(x)>1\}) \geq \frac{\sqrt{2}\, c_n\omega_{n-1}}{2n}\Big|\int_{\mathbb{R}^n} f(y)\,dy\Big|.$$

Combining these two estimates we have

$$\lim_{s\to 0_+} m(\{x\in\mathbb{R}^n : \mathfrak{g}(f_s)(x)>1\}) = \frac{\sqrt{2}\, c_n\omega_{n-1}}{2n}\Big|\int_{\mathbb{R}^n} f(y)\,dy\Big|.$$

So we have proved (3.1). This finishes the proof of Theorem 1.2.

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