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Some geometrical characterizations of L_1-predual spaces

by

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Abstract. Let X be a real Banach space. For a non-empty finite subset F and closed convex subset V of X, we denote by $\mathrm{rad}_X(F)$, $\mathrm{rad}_V(F)$, $\mathrm{cent}_X(F)$ and $d(V, \mathrm{cent}_X(F))$ the Chebyshev radius of F in X, the restricted Chebyshev radius of F in V, the set of Chebyshev centers of F in X and the distance between the sets V and $\mathrm{cent}_X(F)$ respectively. We prove that X is an L_1-predual space if and only if for each four-point subset F of X and non-empty closed convex subset V of X,

$$\mathrm{rad}_V(F) = \mathrm{rad}_X(F) + d(V, \mathrm{cent}_X(F)).$$

Moreover, we explicitly describe the Chebyshev centers of a compact subset of an L_1-predual space. Various new characterizations of ideals in an L_1-predual space are also obtained. In particular, for a compact Hausdorff space S and a subspace $\mathcal{A}$ of $C(S)$ which contains the constant function 1 and separates the points of S, we prove that the state space of $\mathcal{A}$ is a Choquet simplex if and only if $d(\mathcal{A}, \mathrm{cent}_{C(S)}(F)) = 0$ for every four-point subset F of $\mathcal{A}$. We also derive characterizations for a compact convex subset of a locally convex topological vector space to be a Choquet simplex.

1. Introduction. In approximation theory, determining the existence of restricted Chebyshev centers in Banach spaces is one of the classical problems. In this paper, we study this problem in L_1-predual spaces. A Banach space X is said to be an L_1*-predual space* if the dual space of X, denoted by X^*, is isometric to an $L_1(\mu)$ space for some positive measure μ.

The spaces of real-valued continuous functions on a compact Hausdorff space S and those vanishing at infinity on a locally compact Hausdorff space T, denoted by $C(S)$ and $C_0(T)$ respectively, are two important examples of L_1-predual spaces (see [14]). Let $A(K)$ be the space of real-valued affine continuous functions on a compact convex subset K of a locally convex topological vector space (lctvs). We recall that a compact convex subset K

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of a lctvs is a *Choquet simplex* if $A(K)^*$ is a lattice. We refer to [2] for a detailed study of Choquet simplices. It is well-known that $A(K)$ is an L_1-predual space if and only if K is a Choquet simplex (see [12, Theorem 2, p. 185]).

Throughout this paper, we use the following notations and definitions. We consider Banach spaces only over the real field $\mathbb{R}$ and all the subspaces are assumed to be norm closed. Let X be a Banach space. For an element $x \in X$ and $r > 0$, $B(x, r)$ denotes the norm open ball in X centered at x with radius r, and B_X denotes the norm closed unit ball of X. Let $\mathcal{CV}(X)$ denote the class of all non-empty closed convex subsets of X. For a set $V \in \mathcal{CV}(X)$, let $\mathcal{CB}(V)$, $\mathcal{K}(V)$, $\mathcal{F}(V)$ and $\mathcal{F}_4(V)$ denote the classes of all non-empty closed bounded, compact, finite and four-point subsets of V respectively. For bounded subsets A and B of X, we define $d(A, B) = \inf\{\|a - b\| \colon a \in A, b \in B\}$ and $\operatorname{diam}(A) = \sup\{\|a - a'\| \colon a, a' \in A\}$.

Let $V \in \mathcal{CV}(X)$, $B \in \mathcal{CB}(X)$ and $x \in X$. We set $r(x, B) = \sup\{\|x - b\| : b \in B\}$. The number $\operatorname{rad}_V(B) = \inf_{v \in V} r(v, B)$ is the *restricted Chebyshev radius* of B in V. An element $v \in V$ is a *restricted Chebyshev center* of B in V if $\operatorname{rad}_V(B) = r(v, B)$. The set $\{v \in V \colon \operatorname{rad}_V(B) = r(v, B)\}$ is denoted by $\operatorname{cent}_V(B)$. If $V = X$, then $\operatorname{rad}_X(B)$ is the *Chebyshev radius* of B in X and the elements in $\operatorname{cent}_X(B)$ are the *Chebyshev centers* of B in X. Furthermore, for each $\delta > 0$, we define $\operatorname{cent}_V(B, \delta) = \{v \in V \colon r(v, B) \leq \operatorname{rad}_V(B) + \delta\}$.

DEFINITION 1.1 ([16]). Let X be a Banach space, and let $V \in \mathcal{CV}(X)$ and $\mathscr{F} \subseteq \mathcal{CB}(X)$.

(i) We say X *admits centers* for $\mathscr{F}$ if for each $F \in \mathscr{F}$, $\operatorname{cent}_X(F) \neq \emptyset$.
(ii) The set-valued map $\operatorname{cent}_V(\cdot)$ which maps each $F \in \mathscr{F}$ to $\operatorname{cent}_V(F)$ is called the *restricted Chebyshev-center map* on $\mathscr{F}$. In particular, if $V = X$, then $\operatorname{cent}_X(\cdot)$ is called the *Chebyshev-center map* on $\mathscr{F}$.

For $V \in \mathcal{CV}(X)$, the continuity properties of the map $\operatorname{cent}_V(\cdot)$ will be discussed with respect to the Hausdorff metric. Recall that for a Banach space X, the Hausdorff metric d_H is defined in the following manner: for any $B_1, B_2 \in \mathcal{CB}(X)$,

$$d_H(B_1, B_2) = \inf\{a > 0 \colon B_1 \subseteq B_2 + aB(0, 1),\ B_2 \subseteq B_1 + aB(0, 1)\}.$$

It is well-known that an L_1-predual space X admits centers for $\mathcal{K}(X)$, which follows from [3, Corollary 3.4], [15, Theorem 4.5, p. 38 and Theorem 6.1, p. 62]. We use a different approach to prove the existence of Chebyshev centers of a compact subset of an L_1-predual space and describe them explicitly. For a compact Hausdorff space S, Smith and Ward provided the following necessary and sufficient condition in $C(S)$.

THEOREM 1.2 ([18, Theorem 2.2]). *Let S be a compact Hausdorff space, $B \in \mathcal{CB}(C(S))$ and $V \in \mathcal{CV}(C(S))$. Then the set* $\operatorname{cent}_V(B)$ *is non-empty if* *and only if there exist* $x_0 \in \mathrm{cent}_{C(S)}(B)$ *and* $v_0 \in V$ *such that*

$$R\colon = d(V, \mathrm{cent}_{C(S)}(B)) = \|v_0 - x_0\|.$$

Furthermore, $\mathrm{rad}_V(B) = \mathrm{rad}_{C(S)}(B) + R$.

Theorem 1.2 served as a motivation to prove the last equality for the compact subsets of L_1-predual spaces in Theorem 4.9.

There are various characterizations available in the literature for L_1-predual spaces. We state the following important characterization proved by Duan and Lin, which helps us to arrive at the results of Section 4.

Theorem 1.3 ([5, Theorem 2.7]). *Let X be a Banach space. Then X is isometric to an L_1-predual space if and only if for each F in $\mathcal{F}_4(X)$ (or $\mathcal{F}(X)$ or $\mathcal{K}(X)$),* $\mathrm{rad}_X(F) = \frac{1}{2}\,\mathrm{diam}(F)$.

For more characterizations of L_1-predual spaces, we refer to [12, Chapter 7, Section 21]. The authors of [6] characterized the spaces of the type $C(S)$ and $C_0(T)$, where S is a compact Hausdorff space and T is a locally compact Hausdorff space, in terms of the following identities.

Theorem 1.4 ([6]). *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X is isometric to either a $C(S)$ space or a $C_0(T)$ space, for some compact Hausdorff space S or locally compact Hausdorff space T.*
(ii) *For all $V \in \mathcal{CV}(X)$ and $B \in \mathcal{CB}(X)$,*
$$\mathrm{rad}_V(B) = \mathrm{rad}_X(B) + d(V, \mathrm{cent}_X(B)).$$
(iii) *For all $V \in \mathcal{CV}(X)$ and $B \in \mathcal{CB}(X)$,*
$$\mathrm{rad}_V(B) = \mathrm{rad}_X(B) + \lim_{\delta\to 0^+} d(V, \mathrm{cent}_X(B, \delta)).$$

The characterization above validates a corresponding investigation in L_1-predual spaces. In fact, we prove that if conditions (ii) and (iii) of Theorem 1.4 are weakened by replacing $\mathcal{CB}(X)$ with $\mathcal{F}_4(X)$, then these conditions characterize an L_1-predual space; see Theorem 4.10.

Let us recall the following notions.

Definition 1.5. Let X be a Banach space.

(i) ([8]) A subspace Y of X is an *ideal* in X if the annihilator of Y in X^* is the kernel of a linear projection of norm 1 in X^*.
(ii) ([9]) A subspace J of X is an *M-ideal* in X if there exists a linear projection $P\colon X^* \to X^*$ such that for each $x^* \in X^*$, $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$ and the range of P is the annihilator of J in X^*.
(iii) ([9]) A subspace J of X is an *M-summand* in X if there exists a linear projection $P\colon X \to X$ such that the range of P is J and for each $x \in X$, $\|x\| = \max\{\|Px\|, \|x - Px\|\}$.

Clearly, M-summands are M-ideals (see the discussion in [9, p. 2]) and M-ideals are ideals. Rao [17, Proposition 1] proved that a subspace Y of an L_1-predual space is an ideal in X if and only if Y itself is an L_1-predual. Let S be a compact Hausdorff space, J an M-ideal in $C(S)$, and $B \in \mathcal{CB}(C(S))$. Then from [19, Theorem 1], it follows that $\mathrm{cent}_J(B) \neq \emptyset$. We also recall the description of $\mathrm{cent}_{C(S)}(B)$ provided in [20, Theorem I.2.2] as the set of those functions in $C(S)$ which are interposed between two special functions defined in terms of the elements of B. Motivated by the facts above, we investigate the possibility to obtain a similar description for $\mathrm{cent}_J(B)$.

We now recall the concept of state spaces.

Definition 1.6 ([2]). Let S be a compact Hausdorff space and $\mathcal{A}$ be a subspace of $C(S)$ that contains the constant function 1 and separates the points of S. The *state space* of a subspace $\mathcal{A}$, denoted by $\mathcal{S}_{\mathcal{A}}$, is defined as

$$\mathcal{S}_{\mathcal{A}} = \{L \in \mathcal{A}^* \colon L(1) = 1 = \|L\|\}.$$

It is easily seen that $\mathcal{S}_{\mathcal{A}}$ is a weak*-compact convex subset of $\mathcal{A}^*$. The result in [7, Theorem 1.4] provides plenty of examples of subspaces $\mathcal{A}$ of $C(S)$ such that $\mathcal{S}_{\mathcal{A}}$ is a Choquet simplex.

We now provide a brief account of each section.

In Section 2, we discuss a few preliminaries and observations which lay the groundwork for the results in the subsequent sections.

In Section 3, for a compact Hausdorff space S, an M-summand J in $C(S)$ and $B \in \mathcal{CB}(J)$, we prove that $\mathrm{cent}_J(B) = \mathrm{cent}_{C(S)}(B) \cap J$; see Theorem 3.5. We illustrate with examples that, in general, the equality above may not hold in the case of M-ideals that are not M-summands.

In Section 4, for an L_1-predual space X and $F \in \mathcal{K}(X)$, we describe the elements in $\mathrm{cent}_X(F)$ via a well-known isometric identification of X to a subspace of $A(B_{X^*})$, where B_{X^*} is endowed with the weak* topology (see Lemma 2.4). This description leads to the following consequences of interest. We prove that for an L_1-predual space X, and any $F \in \mathcal{K}(X)$ and $V \in \mathcal{CV}(X)$, the set $\mathrm{cent}_V(F)$ is non-empty if and only if the infimum defining $d(V, \mathrm{cent}_X(F))$ is attained. Furthermore, we characterize an L_1-predual space X as a Banach space which satisfies the identity $\mathrm{rad}_V(F) = \mathrm{rad}_X(F) + d(V, \mathrm{cent}_X(F))$ for all $V \in \mathcal{CV}(X)$ and F in $\mathcal{F}_4(X)$ (or $\mathcal{F}(X)$ or $\mathcal{K}(X)$).

In Section 4, we also establish that for an L_1-predual X, the map $\mathrm{cent}_X(\cdot)$ is 2-Lipschitz continuous on $\mathcal{K}(X)$ in the Hausdorff metric. Finally, we prove that a subspace Y of an L_1-predual space X is an ideal in X if and only if for each $F \in \mathcal{F}_4(Y)$, $d(Y, \mathrm{cent}_X(F)) = 0$. For an ideal Y in an L_1-predual X, we also prove that $\mathrm{cent}_Y(F) = \mathrm{cent}_X(F) \cap Y$ for each $F \in \mathcal{K}(Y)$.

2. Preliminaries. Let S be a compact Hausdorff space and $\mathcal{A}$ be a subspace of $C(S)$. Let $B \in \mathcal{CB}(C(S))$. We define the following functions in the same way as in [18]. For each $t \in S$,

$$\begin{aligned} m_B(t) &= \inf\{b(t) \colon b \in B\}, \\ n_B(t) &= \liminf_{s \to t} m_B(s), \\ M_B(t) &= \sup\{b(t) \colon b \in B\}, \\ N_B(t) &= \limsup_{s \to t} M_B(s). \end{aligned} \tag{2.1}$$

Let us also define

$$r_B = \tfrac{1}{2} \sup\{N_B(t) - n_B(t) \colon t \in S\}. \tag{2.2}$$

Remark 2.1. We list a few properties of the objects defined in (2.1) and (2.2), which are easy to verify.

(i) If $B \in \mathcal{CB}(C(S))$, then the functions N_B, m_B are upper semicontinuous and n_B, M_B are lower semicontinuous on S.
(ii) If $F \in \mathcal{K}(C(S))$, then the functions M_F and m_F are continuous on S, and consequently $N_F = M_F$ and $n_F = m_F$.
(iii) For each $B \in \mathcal{CB}(C(S))$, $r_B \leq \mathrm{rad}_{\mathcal{A}}(B)$; a proof can be found in [20, Lemma I.2.1].

Lemma 2.2. *Let S be a compact Hausdorff space. If $B \in \mathcal{CB}(C(S))$, then $\frac{1}{2}\,\mathrm{diam}(B) \leq r_B$. Moreover, if $F \in \mathcal{K}(C(S))$, then $\frac{1}{2}\,\mathrm{diam}(F) = r_F$.*

Proof. Let $B \in \mathcal{CB}(C(S))$ and $L = \sup\{N_B(t) - n_B(t) \colon t \in S\} = 2r_B$. By the upper semicontinuity of m_B and $-M_B$, for all $t \in S$ and $b \in B$ we have $b(t) \leq M_B(t) \leq N_B(t)$ and $n_B(t) \leq m_B(t) \leq b(t)$. Therefore, for all $b_1, b_2 \in B$ and $t \in S$,

$$\pm(b_1(t) - b_2(t)) \leq N_B(t) - n_B(t) \leq L.$$

It follows that $\mathrm{diam}(B) \leq L$.

Now, let $F \in \mathcal{K}(C(S))$. By Remark 2.1(ii), $n_F = m_F$ and $N_F = M_F$. Now, for each $t \in S$, the evaluation functional δ_t, defined as $\delta_t(f) = f(t)$ for each $f \in C(S)$, is norm continuous on $C(S)$. Therefore, due to the compactness of F, for each $t \in S$ there exist $z_1, z_2 \in F$ such that

$$M_F(t) - m_F(t) = \max_{z' \in F} \delta_t(z') - \min_{z' \in F} \delta_t(z') = z_1(t) - z_2(t) \leq \mathrm{diam}(F).$$

It follows that $2r_F = \sup\{M_F(t) - m_F(t) \colon t \in S\} \leq \mathrm{diam}(F)$. ∎

For a subspace $\mathcal{A}$ of $C(S)$ and $B \in \mathcal{CB}(C(S))$, we now provide a description of $\mathrm{cent}_{\mathcal{A}}(B)$.

Proposition 2.3. *Let S be a compact Hausdorff space and $\mathcal{A}$ be a subspace of $C(S)$. If $B \in \mathcal{CB}(C(S))$, then*

$$\mathrm{cent}_{\mathcal{A}}(B) = \{x \in \mathcal{A} \colon N_B - \mathrm{rad}_{\mathcal{A}}(B) \leq x \leq n_B + \mathrm{rad}_{\mathcal{A}}(B)\}.$$

Proof. Let $B \in \mathcal{CB}(\mathcal{A})$. Without loss of generality, we assume that $\mathrm{cent}_{\mathcal{A}}(B) \neq \emptyset$. Suppose $x \in \mathrm{cent}_{\mathcal{A}}(B)$. Then $\mathrm{rad}_{\mathcal{A}}(B) = r(x, B)$. It follows that for all $t \in S$ and $b \in B$,

$$b(t) - \mathrm{rad}_{\mathcal{A}}(B) \leq x(t) \leq b(t) + \mathrm{rad}_{\mathcal{A}}(B).$$

Therefore, from the definitions in (2.1), for each $t \in S$,

$$N_B(t) - \mathrm{rad}_{\mathcal{A}}(B) \leq x(t) \leq n_B(t) + \mathrm{rad}_{\mathcal{A}}(B). \tag{2.3}$$

Now, if $x \in \mathcal{A}$ is such that (2.3) holds, then using Remark 2.1(i), it is easy to deduce that $\mathrm{rad}_{\mathcal{A}}(B) = r(x, B)$. ∎

We refer to [1, Section 4] for examples of subspaces $\mathcal{A}$ of $C(S)$ such that for each $B \in \mathcal{CB}(C(S))$, $\mathrm{cent}_{\mathcal{A}}(B) \neq \emptyset$.

Let X be a Banach space. By Banach–Alaoglu's theorem, B_{X^*} endowed with the weak* topology is a weak*-compact Hausdorff space. There exists a natural affine homeomorphism from B_{X^*} onto B_{X^*} given by $\sigma(x^*) = -x^*$ for $x^* \in B_{X^*}$. Define

$$A_\sigma(B_{X^*}) = \{a \in A(B_{X^*}) \colon a = -a \circ \sigma\}.$$

Clearly, $A_\sigma(B_{X^*})$ is a closed subspace of $A(B_{X^*})$. The following result identifies X with $A_\sigma(B_{X^*})$.

LEMMA 2.4 ([12, Lemma 8, p. 213]). *Let X be a Banach space. Then X is isometric and linearly isomorphic to $A_\sigma(B_{X^*})$ under the mapping $x \mapsto \overline{x}$, where $\overline{x}(x^*) = x^*(x)$ for all $x^* \in B_{X^*}$ and $x \in X$.*

REMARK 2.5. Let X be a Banach space and $F \in \mathcal{K}(X)$. By Lemma 2.4, F can be viewed as a compact subset of $A_\sigma(B_{X^*}) \subseteq C(B_{X^*})$. Hence, for each $x^* \in B_{X^*}$, the norm continuity of x^* on X and the compactness of F imply that

$$\begin{aligned} M_F(x^*) &= \max\{x^*(x) \colon x \in F\}, \\ m_F(x^*) &= \min\{x^*(x) \colon x \in F\}, \\ r_F &= \tfrac{1}{2}\max\{M_F(x^*) - m_F(x^*) \colon x^* \in B_{X^*}\}. \end{aligned} \tag{2.4}$$

3. Chebyshev centers in M-ideals in $C(S)$. Let S be a compact Hausdorff space. We use the following notation for M-ideals in $C(S)$. Consider a subset D of S and define

$$J_D = \{h \in C(S) \colon h(t) = 0 \text{ for each } t \in D\}.$$

From [9, Example 1.4(a), p. 3], J is an M-ideal in $C(S)$ if and only if there exists a closed subset D of S such that $J = J_D$, and J is an M-summand in $C(S)$ if and only if there exists a clopen subset D of S such that $J = J_D$.

We now make the following easy observation.

Proposition 3.1. *Let S be a compact Hausdorff space. If D is a closed subset of S, then the subspace J_D of $C(S)$ admits centers for $\mathcal{K}(J_D)$ and for each $F \in \mathcal{K}(J_D)$, we have* $\operatorname{rad}_{J_D}(F) = r_F$ *and*

$$\operatorname{cent}_{J_D}(F) = \{h \in J_D \colon M_F - r_F \leq h \leq m_F + r_F\}. \tag{3.1}$$

Proof. Let $F \in \mathcal{K}(J_D)$. From Remark 2.1(ii), M_F and m_F are continuous functions on S. Clearly, for each $t \in D$, $M_F(t) = 0 = m_F(t)$. Therefore,

$$\frac{M_F + m_F}{2} \in \{h \in J_D \colon M_F - r_F \leq h \leq m_F + r_F\}.$$

It follows that $\operatorname{rad}_{J_D}(F) \leq r_F$. Thus from Remark 2.1(iii) and Proposition 2.3, we see that $\operatorname{rad}_{J_D}(F) = r_F$ and

$$\operatorname{cent}_{J_D}(F) = \{h \in J_D \colon M_F - r_F \leq h \leq m_F + r_F\}. \blacksquare$$

Let T be a locally compact Hausdorff space. It is known that $C_0(T)$ admits centers for $\mathcal{CB}(C_0(T))$; see [6]. Let T_∞ denote the one-point compactification of T and let t_∞ denote the *point at infinity*. Then the subspace $J_{\{t_\infty\}} = \{x \in C(T_\infty) \colon x(t_\infty) = 0\}$ of $C(T_\infty)$ is isometrically isomorphic to $C_0(T)$. Hence the following result follows immediately from Proposition 3.1.

Corollary 3.2. *Let T be a locally compact Hausdorff space. Then for each $F \in \mathcal{K}(C_0(T))$, we have* $\operatorname{cent}_{C_0(T)}(F) \neq \emptyset$, $\operatorname{rad}_{C_0(T)}(F) = r_F$ *and*

$$\operatorname{cent}_{C_0(T)}(F) = \{x \in C_0(T) \colon M_F - r_F \leq x \leq m_F + r_F\},$$

where for each $t \in T$, $M_F(t) = \sup\{f(t) \colon f \in F\}$, $m_F(t) = \inf\{f(t) \colon f \in F\}$ and $r_F = \frac{1}{2}\sup\{M(t) - m(t) \colon t \in T\}$.

We now recall the well-known insertion theorem by Katětov.

Theorem 3.3 ([10, 11]). *Let S be a compact Hausdorff space. If $g, -f$ are upper semicontinuous functions on S such that $g \leq f$ on S, then there exists $h \in C(S)$ such that $g \leq h \leq f$ on S.*

We need the following auxiliary result to prove the main result of this section. It is a variant of Theorem 3.3.

Lemma 3.4. *Let D be a clopen subset of a compact Hausdorff space S. If $g, -f$ are upper semicontinuous functions on S such that $g \leq f$ on S and for each $t \in D$, $g(t) \leq \alpha \leq f(t)$, then there exists $h \in C(S)$ such that $g \leq h \leq f$ on S and for each $t \in D$, $h(t) = \alpha$.*

Proof. Without loss of generality, we assume that D is a non-empty proper subset of S. By our assumption, both D and $S \setminus D$ are clopen in S. Thus by Theorem 3.3, there exists $k \in C(S \setminus D)$ such that $g \leq k \leq f$ on $S \setminus D$. Define $h : S \to \mathbb{R}$ as $h = k$ on $S \setminus D$ and $h(t) = \alpha$ for each $t \in D$. It is easy to see that $h \in C(S)$ and $g \leq h \leq f$ on S. $\blacksquare$

Theorem 3.5. *Let S be a compact Hausdorff space and J be an M-summand in $C(S)$. Then J admits centers for $\mathcal{CB}(J)$. Moreover, for each $B \in \mathcal{CB}(J)$, we have* $\mathrm{rad}_J(B) = r_B$ *and*

$$\mathrm{cent}_J(B) = \{h \in J\colon N_B - r_B \leq h \leq n_B + r_B\}. \tag{3.2}$$

Proof. Let J be an M-summand in $C(S)$. Thus there exists a clopen subset D of S such that $J = J_D$. Let $B \in \mathcal{CB}(J_D)$. Clearly, for each $t \in D$, $M_B(t) = 0 = m_B(t)$. Using the assumption that D is clopen in S, by the definitions of the functions N_B and $-n_B$ and Remark 2.1(i), it follows that for each $t \in D$, $N_B(t) = 0 = n_B(t)$. Hence for each $t \in D$,

$$N_B(t) - r_B \leq 0 \leq n_B(t) + r_B.$$

By the definition of r_B, Remark 2.1(i) and Lemma 3.4, there exists $h \in J_D$ such that $N_B - r_B \leq h \leq n_B + r_B$. This shows that the set on the right-hand side in (3.2) is non-empty. It follows that $\mathrm{rad}_{J_D}(B) \leq r_B$. Therefore, by Remark 2.1(iii) and Proposition 2.3, $\mathrm{rad}_{J_D}(B) = r_B$ and

$$\mathrm{cent}_{J_D}(B) = \{h \in J_D\colon N_B - r_B \leq h \leq n_B + r_B\}. \ ■$$

Consider the notations and hypotheses used in Theorem 3.5. For a $B \in \mathcal{CB}(C(S))$, even though the set $\mathrm{cent}_{J_D}(B)$ is non-empty, it need not have a description as in (3.2). One can construct many examples to illustrate this fact. We provide one such example below.

Example 3.6. Consider the subspace $J = \{h \in C(\{0,1\})\colon h(0) = 0\}$ of $C(\{0,1\})$. Define the functions $f, g\colon \{0,1\} \to \mathbb{R}$ as $f(0) = 2$, $f(1) = 0$, $g(0) = 3$ and $g(1) = 1$. Let $B = \{f, g\} \subseteq C(\{0,1\}) - J$. It is easy to see that if $h \in C(\{0,1\})$ is such that $M_B - r_B \leq h \leq m_B + r_B$, then $h(0) = \frac{5}{2}$ and $h(1) = \frac{1}{2}$ and hence $h \notin J$.

If the set D is not clopen as given in Theorem 3.5, then $\mathrm{cent}_{J_D}(B)$ need not have the description as in (3.2) for each $B \in \mathcal{CB}(J_D)$. The following example supports this fact.

Example 3.7. Let $0 < a < b < 1$. Consider the space $C([0,1])$ and $D = [a,b]$. Let

$$B = \{f \in J_D\colon 0 \leq f(t) \leq 1 \text{ for each } t \in [0,1]\} \in \mathcal{CB}(J_D) - \mathcal{K}(J_D).$$

It is easy to see that for each $t \in [0,1] - (a,b)$ we have $N_B(t) = 1$ and $n_B(t) = 0$, and for each $t \in (a,b)$, $N_B(t) = 0 = n_B(t)$. Hence $r_B = \frac{1}{2}$. If $h \in C([0,1])$ with $N_B - r_B \leq h \leq n_B + r_B$, then for each $t \in [0,1] - (a,b)$, $h(t) = \frac{1}{2}$ and thus $h \notin J_D$.

4. Restricted Chebyshev centers in L_1-predual spaces. We begin this section by proving the existence and providing a description of a Chebyshev center of a compact subset of an L_1-predual space using the isometric identification in Lemma 2.4. To this end, we recall a separation theorem in an L_1-predual space, proved by Ka-Sing Lau.

THEOREM 4.1 ([13, Theorem 2.3]). *Let X be an L_1-predual space. If f is a real-valued weak*-lower semicontinuous concave function on B_{X^*} such that for each $x^* \in B_{X^*}$, $f(x^*) + f(-x^*) \geq 0$, then there exists $a \in A_\sigma(B_{X^*})$ such that $a \leq f$.*

THEOREM 4.2. *Let X be an L_1-predual space. If $F \in \mathcal{K}(A_\sigma(B_{X^*}))$, then* $\mathrm{cent}_{A_\sigma(B_{X^*})}(F) \neq \emptyset$, $\mathrm{rad}_{A_\sigma(B_{X^*})}(F) = r_F$ *and*

$$\mathrm{cent}_{A_\sigma(B_{X^*})}(F) = \{a \in A_\sigma(B_{X^*}) \colon M_F - r_F \leq a \leq m_F + r_F\}. \tag{4.1}$$

Proof. Let $F \in \mathcal{K}(A_\sigma(B_{X^*}))$. Define $f = m_F + r_F$ on B_{X^*}. It follows from Remark 2.5 that f is a weak*-continuous concave function on B_{X^*}. Let $x^* \in B_{X^*}$. Then,

$$\begin{aligned} f(x^*) + f(-x^*) &= m_F(x^*) + r_F + m_F(-x^*) + r_F \\ &= (m_F(x^*) + r_F) - (M_F(x^*) - r_F) \geq 0. \end{aligned}$$

Therefore, by Theorem 4.1, there exists $a \in A_\sigma(B_{X^*})$ such that $a \leq m_F + r_F$. Now, for each $x^* \in B_{X^*}$, $a(-x^*) \leq m_F(-x^*) + r_F$, and hence $M_F(x^*) - r_F \leq a(x^*)$. Therefore, $M_F - r_F \leq a \leq m_F + r_F$ on B_{X^*}. It follows that $\mathrm{rad}_{A_\sigma(B_{X^*})}(F) = r_F$, and from Proposition 2.3 we can deduce that $\mathrm{cent}_{A_\sigma(B_{X^*})}(F) \neq \emptyset$ and

$$\mathrm{cent}_{A_\sigma(B_{X^*})}(F) = \{a \in A_\sigma(B_{X^*}) \colon M_F - r_F \leq a \leq m_F + r_F\}. \blacksquare$$

REMARK 4.3. Let X be an L_1-predual space and $F \in \mathcal{K}(X)$. Using Lemma 2.4, we can view F as a compact subset of $A_\sigma(B_{X^*})$. Therefore, applying Theorem 4.2, it is easy to see that $\mathrm{cent}_X(F) \neq \emptyset$ and $\mathrm{rad}_X(F) = r_F$. Moreover, $\mathrm{cent}_X(F) = \{x \in X : M_F - r_F \leq \overline{x} \leq m_F + r_F \text{ on } B_{X^*}\}$.

We now give a few applications of Theorem 4.2. For a Banach space X, if $V \in \mathcal{CV}(X)$, then the following result tells us that the map $B \mapsto \mathrm{rad}_V(B)$ for $B \in \mathcal{CB}(X)$ is continuous on $\mathcal{CB}(X)$ in the Hausdorff metric. A proof can be found in [4, Theorem 2.5].

LEMMA 4.4. *Let X be a Banach space. If $V \in \mathcal{CV}(X)$ and $B_1, B_2 \in \mathcal{CB}(X)$, then $|\mathrm{rad}_V(B_1) - \mathrm{rad}_V(B_2)| \leq d_H(B_1, B_2)$.*

For an L_1-predual space X, we prove a continuity property of the map $\mathrm{cent}_X(\cdot)$ on $\mathcal{K}(X)$ in the Hausdorff metric.

THEOREM 4.5. *Let X be an L_1-predual space. Then the map $\mathrm{cent}_X(\cdot)$ is 2-Lipschitz continuous on $\mathcal{K}(X)$ in the Hausdorff metric.*

Proof. From Lemma 2.4, it suffices to prove that $\mathrm{cent}_{A_\sigma(B_{X^*})}(\cdot)$ is 2-Lipschitz continuous on $\mathcal{K}(A_\sigma(B_{X^*}))$ in the Hausdorff metric.

Let $\varepsilon > 0$ and $F_1, F_2 \in \mathcal{K}(A_\sigma(B_{X^*}))$ be such that $d_H(F_1, F_2) < \varepsilon/2$. Then there exists $0 < \delta < \varepsilon/2$ such that

$$F_1 \subseteq F_2 + \delta B(0,1) \quad \text{and} \quad F_2 \subseteq F_1 + \delta B(0,1). \tag{4.2}$$

Thus by Theorem 4.2, for each $i = 1, 2$,

$$\operatorname{cent}_{A_\sigma(B_{X^*})}(F_i) = \{a \in A_\sigma(B_{X^*})\colon M_{F_i} - r_{F_i} \le a \le m_{F_i} + r_{F_i}\}, \tag{4.3}$$

and $r_{F_i} = \operatorname{rad}_{A_\sigma(B_{X^*})}(F_i)$. By Lemma 4.4, $|r_{F_1} - r_{F_2}| \le d_H(F_1, F_2)$. Hence we can choose δ such that $r_{F_1} - \delta < r_{F_2} < r_{F_1} + \delta$ and (4.2) hold true.

Let $a \in \operatorname{cent}_{A_\sigma(B_{X^*})}(F_1)$ and $z_2 \in F_2$. It follows from (4.2) that there exist $z_1 \in F_1$ and $z_0 \in B(0,1)$ such that $z_2 = z_1 + \delta z_0$. Thus using (4.3), for each $x^* \in B_{X^*}$,

$$\begin{aligned} z_2(x^*) - r_{F_2} &= z_1(x^*) + \delta z_0(x^*) - r_{F_2} \\ &< z_1(x^*) + \delta - r_{F_1} + \delta \le a(x^*) + 2\delta, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} z_2(x^*) + r_{F_2} &= z_1(x^*) + \delta z_0(x^*) + r_{F_2} \\ &> z_1(x^*) - \delta + r_{F_1} - \delta \ge a(x^*) - 2\delta. \end{aligned} \tag{4.5}$$

It follows from (4.4), (4.5) and the definitions of M_{F_2} and m_{F_2} that $M_{F_2} - r_{F_2} \le a + 2\delta$ and $a - 2\delta \le m_{F_2} + r_{F_2}$ on B_{X^*}. Define $g = \max\{M_{F_2} - r_{F_2}, a - 2\delta\}$ and $f = \min\{a + 2\delta, m_{F_2} + r_{F_2}\}$ on B_{X^*}. Clearly, $g \le f$ and $-g, f$ are weak*-continuous concave functions on B_{X^*}. For each $x^* \in B_{X^*}$, by using the fact that $M_{F_2}(-x^*) = -m_{F_2}(x^*)$, it is easy to see that

$$f(x^*) + f(-x^*) = f(x^*) - g(x^*) \ge 0.$$

Therefore, by Theorem 4.1, there exists $a' \in A_\sigma(B_{X^*})$ such that $a' \le f$. Now, for each $x^* \in B_{X^*}$, since $f(x^*) = -g(-x^*)$, it follows that $g(-x^*) \le -a'(x^*) = a'(-x^*)$. Thus $g \le a' \le f$ on B_{X^*}. It follows that $a' \in \operatorname{cent}_{A_\sigma(B_{X^*})}(F_2)$ with $\|a' - a\| \le 2\delta < \varepsilon$. This proves that $\operatorname{cent}_{A_\sigma(B_{X^*})}(F_1) \subseteq \operatorname{cent}_{A_\sigma(B_{X^*})}(F_2) + \varepsilon B(0,1)$. Similarly, $\operatorname{cent}_{A_\sigma(B_{X^*})}(F_2) \subseteq \operatorname{cent}_{A_\sigma(B_{X^*})}(F_1) + \varepsilon B(0,1)$. Hence $d_H(\operatorname{cent}_{A_\sigma(B_{X^*})}(F_1), \operatorname{cent}_{A_\sigma(B_{X^*})}(F_2)) \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $d_H(\operatorname{cent}_{A_\sigma(B_{X^*})}(F_1), \operatorname{cent}_{A_\sigma(B_{X^*})}(F_2)) \le 2 d_H(F_1, F_2)$. ∎

REMARK 4.6. In Theorem 4.5, the Lipschitz constant 2 is best possible. For example, consider $\mathbb{R}^2$ equipped with the supremum norm. Let $F = \{(-1,0), (1,0)\}$ and $G = \{(0,1)\}$. Then $\operatorname{cent}_{\mathbb{R}^2}(F) = \{(0,\lambda)\colon -1 \le \lambda \le 1\}$ and $\operatorname{cent}_{\mathbb{R}^2}(G) = \{(0,1)\}$. Moreover,

$$d_H(F, G) = 1 \quad \text{and} \quad d_H(\operatorname{cent}_{\mathbb{R}^2}(F), \operatorname{cent}_{\mathbb{R}^2}(G)) = 2.$$

Next, we provide a necessary and sufficient condition for the existence of restricted Chebyshev centers in L_1-predual spaces. Before proceeding, we need the following definition of a convex symmetric lower semicontinuous set-valued function.

Definition 4.7 ([14]). Let C be a convex subset of a lctvs and E be another lctvs. Let Φ be a set-valued map from C to the family of non-empty convex subsets of E. The map Φ is said to be

(i) *convex* if for all $0 \leq \alpha \leq 1$ and $c_1, c_2 \in C$,

$$\alpha\Phi(c_1) + (1-\alpha)\Phi(c_2) \subseteq \Phi(\alpha c_1 + (1-\alpha)c_2);$$

(ii) *lower semicontinuous* if for each open subset U of E, the set $\{c \in C : \Phi(c) \cap U \neq \emptyset\}$ is a relatively open subset of C; and

(iii) *symmetric* if $\Phi(-c) = -\Phi(c)$ whenever $c, -c \in C$.

We also need a selection theorem by Lazar and Lindenstrauss.

Theorem 4.8 ([14, Theorem 2.2]). *Let X be an L_1-predual space. If $\Phi\colon B_{X^*} \to \mathcal{CV}(\mathbb{R})$ is a convex symmetric weak*-lower semicontinuous set-valued map on B_{X^*}, then Φ admits a selection from $A_\sigma(B_{X^*})$, that is, there exists $a \in A_\sigma(B_{X^*})$ such that for each $x^* \in B_{X^*}$, $a(x^*) \in \Phi(x^*)$.*

We now prove the following necessary and sufficient condition in L_1-predual spaces.

Theorem 4.9. *Let X be an L_1-predual space. If $V \in \mathcal{CV}(X)$ and $F \in \mathcal{K}(X)$, then*

$$\operatorname{rad}_V(F) = \operatorname{rad}_X(F) + d(V, \operatorname{cent}_X(F)). \tag{4.6}$$

Furthermore, a necessary and sufficient condition for $\operatorname{cent}_V(F) \neq \emptyset$ is that there exist $v_0 \in V$ and $x_0 \in \operatorname{cent}_X(F)$ such that $d(V, \operatorname{cent}_X(F)) = \|v_0 - x_0\|$.

Proof. Using Lemma 2.4, it suffices to prove the following claims. Let $F \in \mathcal{K}(A_\sigma(B_{X^*}))$ and $V \in \mathcal{CV}(A_\sigma(B_{X^*}))$.

Claim 1. *We have*

$$\operatorname{rad}_V(F) = \operatorname{rad}_{A_\sigma(B_{X^*})}(F) + d(V, \operatorname{cent}_{A_\sigma(B_{X^*})}(F)). \tag{4.7}$$

Claim 2. $\operatorname{cent}_V(F) \neq \emptyset$ *if and only if* $d(V, \operatorname{cent}_{A_\sigma(B_{X^*})}(F)) = \|v_0 - a_0\|$ *for some $v_0 \in V$ and $a_0 \in \operatorname{cent}_{A_\sigma(B_{X^*})}(F)$.*

From Theorem 4.2, $\operatorname{cent}_{A_\sigma(B_{X^*})}(F) \neq \emptyset$ and $\operatorname{rad}_{A_\sigma(B_{X^*})}(F) = r_F$. Let $R = d(V, \operatorname{cent}_{A_\sigma(B_{X^*})}(F))$. In order to prove (4.7), it suffices to show that $\operatorname{rad}_V(F) \geq r_F + R$ because the reverse inequality is a simple consequence of the triangle inequality. We apply the techniques used in [18, proof of Theorem 2.2].

Let $S_n = \operatorname{rad}_V(F) - r_F + 1/n$ for $n = 1, 2, \ldots$. There exists $v_n \in V$ such that

$$r(v_n, F) < \operatorname{rad}_V(F) + 1/n.$$

Hence for all $z \in F$ and $x^* \in B_{X^*}$,

$$z(x^*) - \operatorname{rad}_V(F) - 1/n < v_n(x^*) < z(x^*) + \operatorname{rad}_V(F) + 1/n.$$

Therefore, from the inequalities above and (2.4), for each $x^* \in B_{X^*}$,

$$M_F(x^*) - \mathrm{rad}_V(F) - 1/n \le v_n(x^*) \le m_F(x^*) + \mathrm{rad}_V(F) + 1/n. \tag{4.8}$$

This implies

$$M_F(x^*) - r_F - S_n \le v_n(x^*) \le m_F(x^*) + r_F + S_n. \tag{4.9}$$

For each $x^* \in B_{X^*}$, we define

$$\Phi_n(x^*) = [v_n(x^*) - S_n,\ v_n(x^*) + S_n] \cap [M_F(x^*) - r_F, m_F(x^*) + r_F]. \tag{4.10}$$

Clearly, $\Phi_n(x^*)$ is closed, convex and bounded for each $x^* \in B_{X^*}$. Moreover, (4.9) guarantees that $\Phi_n(x^*)$ is non-empty for each $x^* \in B_{X^*}$. The set-valued map Φ_n can be proved to be weak*-lower semicontinuous on B_{X^*} using the same argument as in [18, proof of Theorem 2.2].

Now, let $0 \le \alpha \le 1$ and $x_1^*, x_2^* \in B_{X^*}$. Since $v_n \in A_\sigma(B_{X^*})$ and the functions M_F and $-m_F$ are convex on B_{X^*}, it is easy to verify that

$$\alpha\Phi(x_1^*) + (1-\alpha)\Phi(x_2^*) \subseteq \Phi(\alpha x_1^* + (1-\alpha)x_2^*).$$

Moreover, if $x^* \in B_{X^*}$, then from the facts that $v_n \in A_\sigma(B_{X^*})$ and $M_F(-x^*) = -m_F(x^*)$, it follows that $-\Phi_n(x^*) = \Phi_n(-x^*)$. Therefore, Φ_n is convex, symmetric and weak*-lower semicontinuous on B_{X^*}, and hence by Theorem 4.8 there exists $a_n \in A_\sigma(B_{X^*})$ such that $a_n(x^*) \in \Phi_n(x^*)$ for each $x^* \in B_{X^*}$. Therefore, for each $x^* \in B_{X^*}$,

$$v_n(x^*) - S_n \le a_n(x^*) \le v_n(x^*) + S_n,$$
$$M_F(x^*) - r_F \le a_n(x^*) \le m_F(x^*) + r_F.$$

It follows that $a_n \in \mathrm{cent}_{A_\sigma(B_{X^*})}(F)$ with $\|v_n - a_n\| \le S_n$. Hence $R \le S_n$ for each $n = 1, 2, \ldots$. This proves (4.7).

We now prove Claim 2. Suppose $R = \|v_0 - a_0\|$ for some $v_0 \in V$ and $a_0 \in \mathrm{cent}_{A_\sigma(B_{X^*})}(F)$. This implies

$$r(v_0, F) \le \sup_{z\in F}\{\|v_0 - a_0\| + \|a_0 - z\|\} \le R + r_F = \mathrm{rad}_V(F).$$

Therefore, $\mathrm{rad}_V(F) = r(v_0, F)$ and hence $v_0 \in \mathrm{cent}_V(F)$.

Conversely, if $v_0 \in \mathrm{cent}_V(F)$, then an argument similar to the one above proves that the set-valued map defined as $\Phi(x^*) = [v_0(x^*) - R, v_0(x^*) + R] \cap [M_F(x^*) - r_F, m_F(x^*) + r_F]$, for $x^* \in B_{X^*}$, is convex, symmetric and weak*-lower semicontinuous on B_{X^*}. Hence, by Theorem 4.8, there exists a selection $a_0 \in \mathrm{cent}_{A_\sigma(B_{X^*})}(F)$ such that $R = \|v_0 - a_0\|$. ■

We now provide a geometrical characterization of L_1-predual spaces in terms of the identity in (4.6).

THEOREM 4.10. *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X is an L_1-predual space.*
(ii) *For all $V \in \mathcal{CV}(X)$ and $F \in \mathcal{F}_4(X)$,*

$$\mathrm{rad}_V(F) = \mathrm{rad}_X(F) + d(V, \mathrm{cent}_X(F)).$$

(iii) *For all $V \in \mathcal{CV}(X)$ and $F \in \mathcal{F}_4(X)$,*

$$\mathrm{rad}_V(F) = \mathrm{rad}_X(F) + \lim_{\delta\to 0^+} d(V, \mathrm{cent}_X(F, \delta)).$$

Proof. (i)⇒(ii) follows from Theorem 4.9, and (ii)⇒(iii) follows from the following inequalities, holding for each $F \in \mathcal{K}(X)$:

$$\mathrm{rad}_V(F) \le \mathrm{rad}_X(F) + \lim_{\delta\to 0^+} d(V, \mathrm{cent}_X(F, \delta)) \le \mathrm{rad}_X(F) + d(V, \mathrm{cent}_X(F)).$$

In order to prove (iii)⇒(i), by Theorem 1.3 it suffices to show that for each $F \in \mathcal{F}_4(X)$, $\mathrm{rad}_X(F) = \frac{1}{2}\,\mathrm{diam}(F)$. The idea is similar to that in [6, proof of Theorem 3.4].

Let $F = \{x_1, x_2, x_3, x_4\} \subseteq X$. Without loss of generality, let $\mathrm{diam}(F) = \|x_1 - x_2\|$. Define $R = \frac{1}{2}\,\mathrm{diam}(F)$. Now, let $F' = \{x_1, x_2\}$ and $V = \{x_3\}$. Then $\mathrm{rad}_X(F') = R$. By our assumption,

$$\mathrm{rad}_{\{x_3\}}(F') = \mathrm{rad}_X(F') + \lim_{\delta\to 0^+} d(\{x_3\}, \mathrm{cent}_X(F', \delta)).$$

Therefore, $2R \ge R + \lim_{\delta\to 0^+} d(\{x_3\}, \mathrm{cent}_X(F', \delta))$ and hence, for each $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that $r(x_\varepsilon, \{x_1, x_2, x_3\}) \le R + \varepsilon$. It follows that $\mathrm{rad}_X(\{x_1, x_2, x_3\}) = R$. We next consider $F' = \{x_1, x_2, x_3\}$ and $V = \{x_4\}$ and follow the arguments above to obtain $\mathrm{rad}_X(F) = R$. ∎

The following result follows directly from Theorems 4.9 and 4.10.

Corollary 4.11. *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X is an L_1-predual space.*
(ii) *For all $V \in \mathcal{CV}(X)$ and $F \in \mathcal{F}(X)$,*

$$\mathrm{rad}_V(F) = \mathrm{rad}_X(F) + d(V, \mathrm{cent}_X(F)).$$

(iii) *For all $V \in \mathcal{CV}(X)$ and $F \in \mathcal{F}(X)$,*

$$\mathrm{rad}_V(F) = \mathrm{rad}_X(F) + \lim_{\delta\to 0^+} d(V, \mathrm{cent}_X(F, \delta)).$$

(iv) *For all $V \in \mathcal{CV}(X)$ and $F \in \mathcal{K}(X)$,*

$$\mathrm{rad}_V(F) = \mathrm{rad}_X(F) + d(V, \mathrm{cent}_X(F)).$$

(v) *For all $V \in \mathcal{CV}(X)$ and $F \in \mathcal{K}(X)$,*

$$\mathrm{rad}_V(F) = \mathrm{rad}_X(F) + \lim_{\delta\to 0^+} d(V, \mathrm{cent}_X(F, \delta)).$$

We conclude this paper with one more application of Theorem 4.2: we derive some characterizations for the ideals in an L_1-predual.

THEOREM 4.12. *Let Y be a subspace of an L_1-predual space X. Then the following statements are equivalent:*

(i) *Y is an ideal in X.*
(ii) *For each $F \in \mathcal{F}_4(Y)$,*

$$\mathrm{cent}_Y(F) \neq \emptyset \quad \textit{and} \quad \mathrm{cent}_Y(F) = \mathrm{cent}_X(F) \cap Y.$$

(iii) *For each $F \in \mathcal{F}_4(Y)$, $d(Y, \mathrm{cent}_X(F)) = 0$.*

Proof. We first prove (i)⇒(ii). By [17, Proposition 1], Y is an L_1-predual space. Thus by Lemma 2.4, Y and X are isometrically isomorphic to $A_\sigma(B_{Y^*})$ and $A_\sigma(B_{X^*})$ respectively.

Let $F = \{y_1, y_2, y_3, y_4\} \subseteq Y$. Clearly, $\mathrm{cent}_X(F) \cap Y \subseteq \mathrm{cent}_Y(F)$. Under the mapping given in Lemma 2.4, let $\overline{F} = \{\overline{y_1}, \overline{y_2}, \overline{y_3}, \overline{y_4}\} \in \mathcal{F}(A_\sigma(B_{Y^*}))$. For each $y^* \in B_{Y^*}$, we define

$$\begin{aligned} &\tilde{m}_{\overline{F}}(y^*) = \min\{y^*(y_i) : 1 \leq i \leq 4\}, \\ &\tilde{M}_{\overline{F}}(y^*) = \max\{y^*(y_i) : 1 \leq i \leq 4\} \\ &\qquad \tilde{r}_{\overline{F}} = \tfrac{1}{2} \max\{M_{\overline{F}}(y^*) - m_{\overline{F}}(y^*) : y^* \in B_{Y^*}\}. \end{aligned} \tag{4.11}$$

Then by Theorems 1.3 and 4.2,

$$\mathrm{cent}_{A_\sigma(B_{Y^*})}(\overline{F}) = \{a \in A_\sigma(B_{Y^*}) : \tilde{M}_{\overline{F}} - \tilde{r}_{\overline{F}} \leq a \leq \tilde{m}_{\overline{F}} + \tilde{r}_{\overline{F}}\} \neq \emptyset$$

and $\mathrm{rad}_{A_\sigma(B_{Y^*})}(\overline{F}) = \tilde{r}_{\overline{F}} = \frac{1}{2}\,\mathrm{diam}(\overline{F})$.

Since $\overline{F} \in \mathcal{F}(A_\sigma(B_{X^*}))$, we define, for each $x^* \in B_{X^*}$,

$$\begin{aligned} &m_{\overline{F}}(x^*) = \min\{x^*(y_i) : 1 \leq i \leq 4\}, \\ &M_{\overline{F}}(x^*) = \max\{x^*(y_i) : 1 \leq i \leq 4\} \\ &\qquad r_{\overline{F}} = \tfrac{1}{2} \max\{M_{\overline{F}}(x^*) - m_{\overline{F}}(x^*) : x^* \in B_{X^*}\} \end{aligned} \tag{4.12}$$

Then again by Theorems 1.3 and 4.2,

$$\mathrm{cent}_{A_\sigma(B_{X^*})}(\overline{F}) = \{a \in A_\sigma(B_{X^*}) : M_{\overline{F}} - r_{\overline{F}} \leq a \leq m_{\overline{F}} + r_{\overline{F}}\} \neq \emptyset$$

and $\mathrm{rad}_{A_\sigma(B_{X^*})}(\overline{F}) = r_{\overline{F}} = \frac{1}{2}\,\mathrm{diam}(\overline{F})$.

Let $y_0 \in Y$ be such that $\overline{y_0} \in \mathrm{cent}_{A_\sigma(B_{Y^*})}(\overline{F})$. Then for each $x^* \in B_{X^*}$,

$$\begin{aligned} M_{\overline{F}}(x^*) - r_{\overline{F}} = \tilde{M}_{\overline{F}}(x^*|_Y) - \tilde{r}_{\overline{F}} \leq x^*|_Y(y_0) &= x^*(y_0) \\ &\leq \tilde{m}_{\overline{F}}(x^*|_Y) + \tilde{r}_{\overline{F}} \\ &= m_{\overline{F}}(x^*) + r_{\overline{F}}. \end{aligned}$$

Therefore, $\overline{y_0} \in \mathrm{cent}_{A_\sigma(B_{X^*})}(\overline{F})$. It follows from Remark 4.3 that $\mathrm{cent}_Y(F) \subseteq \mathrm{cent}_X(F) \cap Y$.

(ii)⇒(iii) is easy to observe.

Finally, we prove (iii)⇒(i). Let $F \in \mathcal{F}_4(Y)$. By our assumption, $d(Y, \mathrm{cent}_X(F)) = 0$. Using the identity in Theorem 4.10 and by Theorem 1.3,

$\mathrm{rad}_Y(F) = \mathrm{rad}_X(F) = \frac{1}{2}\,\mathrm{diam}(F)$. Therefore, (i) follows from Theorem 1.3 and [17, Proposition 1]. ■

Using an argument similar to that in Theorem 4.12 and by Theorem 1.3, we can deduce the following result.

COROLLARY 4.13. *Let Y be a subspace of an L_1-predual space X. Then the following statements are equivalent:*

(i) *Y is an ideal in X.*
(ii) *For each $F \in \mathcal{F}(Y)$,*

$$\mathrm{cent}_Y(F) \neq \emptyset \quad \text{and} \quad \mathrm{cent}_Y(F) = \mathrm{cent}_X(F) \cap Y.$$

(iii) *For each $F \in \mathcal{F}(Y)$, $d(Y, \mathrm{cent}_X(F)) = 0$.*
(iv) *For each $F \in \mathcal{K}(Y)$,*

$$\mathrm{cent}_Y(F) \neq \emptyset \quad \text{and} \quad \mathrm{cent}_Y(F) = \mathrm{cent}_X(F) \cap Y.$$

(v) *For each $F \in \mathcal{K}(Y)$, $d(Y, \mathrm{cent}_X(F)) = 0$.*

The following result is a special case of Theorem 4.12.

PROPOSITION 4.14. *Let S be a compact Hausdorff space and $\mathcal{A}$ be a subspace of $C(S)$ that contains the constant function 1 and separates the points of S. Then the following statements are equivalent:*

(i) *$\mathcal{S}_{\mathcal{A}}$ is a Choquet simplex.*
(ii) *For each $F \in \mathcal{F}_4(\mathcal{A})$,*

$$\mathrm{cent}_{\mathcal{A}}(F) \neq \emptyset \quad \text{and} \quad \mathrm{cent}_{\mathcal{A}}(F) = \mathrm{cent}_{C(S)}(F) \cap \mathcal{A}.$$

(iii) *For each $F \in \mathcal{F}_4(\mathcal{A})$, $d(\mathcal{A}, \mathrm{cent}_{C(S)}(F)) = 0$.*
(iv) *For each $F \in \mathcal{F}(\mathcal{A})$,*

$$\mathrm{cent}_{\mathcal{A}}(F) \neq \emptyset \quad \text{and} \quad \mathrm{cent}_{\mathcal{A}}(F) = \mathrm{cent}_{C(S)}(F) \cap \mathcal{A}.$$

(v) *For each $F \in \mathcal{F}(\mathcal{A})$, $d(\mathcal{A}, \mathrm{cent}_{C(S)}(F)) = 0$.*
(vi) *For each $F \in \mathcal{K}(\mathcal{A})$,*

$$\mathrm{cent}_{\mathcal{A}}(F) \neq \emptyset \quad \text{and} \quad \mathrm{cent}_{\mathcal{A}}(F) = \mathrm{cent}_{C(S)}(F) \cap \mathcal{A}.$$

(vii) *For each $F \in \mathcal{K}(\mathcal{A})$, $d(\mathcal{A}, \mathrm{cent}_{C(S)}(F)) = 0$.*

Proof. By [2, Theorem 4.9, p. 14], $\mathcal{A}$ is order isometric to the space $A(\mathcal{S}_{\mathcal{A}})$. Therefore, $\mathcal{S}_{\mathcal{A}}$ is a Choquet simplex if and only if $A(\mathcal{S}_{\mathcal{A}})$ is an L_1-predual space if and only if $\mathcal{A}$ is an L_1-predual space if and only if $\mathcal{A}$ is an ideal in $C(S)$. Now by applying Theorem 4.12 and Corollary 4.13, we get the desired equivalence. ■

If K is a compact convex subset of a lctvs, then it follows from [2, Theorem 4.7, p. 14] that K is affinely homeomorphic to $\mathcal{S}_{A(K)}$. Hence the following result is a direct consequence of Proposition 4.14.

Corollary 4.15. *Let K be a compact convex subset of a lctvs. Then the following statements are equivalent:*

(i) *K is a Choquet simplex.*

(ii) *For each F in $\mathcal{F}_4(A(K))$ (or $\mathcal{F}(A(K))$ or $\mathcal{K}(A(K))$),*

$$\mathrm{cent}_{A(K)}(F) \neq \emptyset \quad \textit{and} \quad \mathrm{cent}_{A(K)}(F) = \mathrm{cent}_{C(K)}(F) \cap A(K).$$

(iii) *For each F in $\mathcal{F}_4(A(K))$ (or $\mathcal{F}(A(K))$ or $\mathcal{K}(A(K))$),*

$$d(A(K), \mathrm{cent}_{C(K)}(F)) = 0.$$

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References

[1] D. Amir, J. Mach, and K. Saatkamp, *Existence of Chebyshev centers, best n-nets and best compact approximants*, Trans. Amer. Math. Soc. 271 (1982), 513–524.

[2] L. Asimow and A. J. Ellis, *Convexity Theory and its Applications in Functional Analysis*, London Math. Soc. Monogr. 16, Academic Press, London, 1980.

[3] P. Bandyopadhyay and T. S. S. R. K. Rao, *Central subspaces of Banach spaces*, J. Approx. Theory 103 (2000), 206–222.

[4] S. Daptari and T. Paul, *Some geometric properties of relative Chebyshev centres in Banach spaces*, in: Recent Trends in Operator Theory and Applications, Contemp. Math. 737, Amer. Math. Soc., 2019, 77–87.

[5] Y. Duan and B.-L. Lin, *Characterizations of L^1-predual spaces by centerable subsets*, Comment. Math. Univ. Carolin. 48 (2007), 239–243.

[6] R. Espínola, A. Wiśnicki, and J. Wośko, *A geometrical characterization of the $C(K)$ and $C_0(K)$ spaces*, J. Approx. Theory 105 (2000), 87–101.

[7] A. Gleit, *On the existence of simplex spaces*, Israel J. Math. 9 (1971), 199–209.

[8] G. Godefroy, N. J. Kalton, and P. D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. 104 (1993), 13–59.

[9] P. Harmand, D. Werner, and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.

[10] M. Katětov, *On real-valued functions in topological spaces*, Fund. Math. 38 (1951), 85–91.

[11] M. Katětov, *Correction to "On real-valued functions in topological spaces" (Fund. Math. 38 (1951), 85–91)*, Fund. Math. 40 (1953), 203–205.

[12] H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Grundlehren Math. Wiss. 208, Springer, New York, 1974.

[13] K. S. Lau, *The dual ball of a Lindenstrauss space*, Math. Scand. 33 (1973), 323–337.

[14] A. J. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1 spaces and their representing matrices*, Acta Math. 126 (1971), 165–193.

[15] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. 48 (1964), 112 pp.

[16] D. V. Pai and P. T. Nowroji, *On restricted centers of sets*, J. Approx. Theory 66 (1991), 170–189.

[17] T. S. S. R. K. Rao, *On ideals in Banach spaces*, Rocky Mountain J. Math. 31 (2001), 595–609.

[18] P. W. Smith and J. D. Ward, *Restricted centers in $C(\Omega)$*, Proc. Amer. Math. Soc. 48 (1975), 165–172.
[19] P. W. Smith and J. D. Ward, *Restricted centers in subalgebras of $C(X)$*, J. Approx. Theory 15 (1975), 54–59.
[20] J. D. Ward, *Existence and uniqueness of Chebyshev centers in certain Banach spaces*, Ph.D. thesis, Purdue Univ., 1973.

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