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Large values of $n/\varphi(n)$ and $\sigma(n)/n$

by

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Dedicated to the memory of the great mathematician Andrzej Schinzel

1. Introduction. Let n be a positive integer. In this paper, we consider Euler's totient function φ which represents the number of positive integers up to n that are relatively prime to n, and the arithmetic function σ which is defined by

$$\sigma(n) = \sum_{d \mid n} d$$

and denotes the sum of the divisors of n. The two arithmetic functions $n/\varphi(n)$ and $\sigma(n)/n$ are multiplicative with

$$\frac{\sigma(p^a)}{p^a} = \frac{1 + p + \cdots + p^a}{p^a} = \frac{p^{a+1} - 1}{p^a(p-1)} < \frac{p}{p-1} = \frac{p^a}{\varphi(p^a)}. \tag{1.1}$$

Hence $\sigma(n)/n < n/\varphi(n)$ for every $n > 1$. In 1903, Landau [18] (see also [16, Theorem 328]) proved that

$$\limsup_{n\to\infty} \frac{n}{\varphi(n) \log\log n} = e^{\gamma}, \tag{1.2}$$

where $\gamma = 0.577\ldots$ denotes the Euler constant, while in 1913, Gronwall [14, p. 119] (see also [16, Theorem 323]) found the maximal order of σ by showing that

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma}. \tag{1.3}$$

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In the proof of (1.3), Gronwall used the asymptotic formula

$$\prod_{p\le x}\frac{p}{p-1}\sim e^{\gamma}\log x \qquad \text{as } x\to\infty, \tag{1.4}$$

where p runs over primes not exceeding x, which is due to Mertens [19].

In view of (1.2), Rosser and Schoenfeld [30, p. 72] raised the question whether there are infinitely many integers n such that $n/\varphi(n) > e^{\gamma}\log\log n$. In [21, 22], this was answered in the affirmative. In view of (1.3), it is natural to ask the same question for the function $\sigma(n)/n$. In turns out that the answer depends on the truth of the Riemann hypothesis on the non-trivial zeros of the Riemann zeta function. Under the assumption that the Riemann hypothesis is true, Ramanujan [27, p. 143] gave the asymptotic upper bound

$$\frac{\sigma(n)}{n}\le e^{\gamma}\left(\log\log n-\frac{2(\sqrt{2}-1)}{\sqrt{\log n}}+S_1(\log n)+\frac{O(1)}{\sqrt{\log n}\log\log n}\right) \tag{1.5}$$

with $S_1(x)=\sum_\rho x^{\rho-1}/|\rho|^2$, where ρ runs over the non-trivial zeros of the Riemann zeta function. If the Riemann hypothesis is true, one has (see [26, eq. (226)])

$$\begin{aligned}|S_1(x)| &\le \frac{1}{\sqrt{x}}\sum_\rho\frac{1}{\rho(1-\rho)}=\frac{1}{\sqrt{x}}\sum_\rho\left(\frac{1}{\rho}+\frac{1}{1-\rho}\right)\\ &=\frac{2}{\sqrt{x}}\sum_\rho\frac{1}{\rho}=\frac{\tau}{\sqrt{x}},\end{aligned} \tag{1.6}$$

where

$$\tau = 2+\gamma-\log 4\pi = 0.0461914179322420\ldots.$$

So, if the Riemann hypothesis is true, the asymptotic upper bound (1.5) implies that there is a positive integer n_0 such that

$$\frac{\sigma(n)}{n}<e^{\gamma}\log\log n \qquad \text{for } n\ge n_0. \tag{1.7}$$

In 1983, Robin [28, Théorème 1] was able to show that (1.7) is even a sufficient condition for the truth of the Riemann hypothesis. Only one year later, Robin [29, Théorème 1] gave an explicit version of his result. He found that the Riemann hypothesis is true if and only if

$$\frac{\sigma(n)}{n}<e^{\gamma}\log\log n \qquad \text{for } n>5040. \tag{1.8}$$

Hence, there are infinitely many integers n satisfying $\sigma(n)/n > e^{\gamma}\log\log n$ only if the Riemann hypothesis fails.

The equivalent criterion (1.8) for the Riemann hypothesis is called *Robin's criterion* and the inequality (1.8) is called *Robin's inequality.* Robin's inequality (1.8) has been slightly improved in [24, Corollary 1.2]. Even Robin's inequality remains open in general: so far it has been shown to hold unconditionally in many cases (see, for instance, [11, 15, 5]). In particular, Robin's inequality has been proven for several m-free integers (cf. [11, 32, 9, 20, 3]). Here a positive integer n is called *m-free* if n is not divisible by the mth power of any prime number.

Let $f(n)$ be a positive arithmetical function, i.e. a function defined on the positive integers with positive values. A positive integer n is said to be *an f-champion* if $1 \le m < n$ implies $f(m) < f(n)$. The champions for $\sigma(n)/n$ are said to be *superabundant* (SA for short), i.e. the number n is SA if

$$m < n \quad \text{implies} \quad \frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}. \tag{1.9}$$

The SA numbers have been introduced and studied by Alaoglu and Erdős [1, Sect. 4]. They were also studied by Ramanujan [27, Sect. 59], who called them *generalized highly composite*. It is possible to adapt the algorithm described in [24, Sect. 3.4] to compute a table of SA numbers [34]. Let p_k denote the kth prime and

$$M_{p_k} = p_1 \dots p_k$$

the kth *primorial*, i.e. the product of the first k primes. If $n < M_{p_k}$ then the standard factorization of n can be written as $n = q_1^{\alpha_1} \dots q_j^{\alpha_j}$ with $q_1 < \dots < q_j$, $j < k$ and $q_i \ge p_i$ for $1 \le i \le j$. Therefore,

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{j} \frac{q_i}{q_i - 1} \le \prod_{i=1}^{j} \frac{p_i}{p_i - 1} < \prod_{i=1}^{k} \frac{p_i}{p_i - 1} = \frac{M_{p_k}}{\varphi(M_{p_k})} \tag{1.10}$$

and we can see that if $f(n) = n/\varphi(n)$ then the f-champions are the numbers M_{p_k} for $k \ge 1$.

The aim of this paper is to study and compare the large values taken by the two functions $n/\varphi(n)$ and $\sigma(n)/n$. Let X be a positive real number. It is convenient to introduce

$$\Sigma(X) = \max_{n \le X} \frac{\sigma(n)}{n} \quad \text{and} \quad \Phi(X) = \max_{n \le X} \frac{n}{\varphi(n)}, \tag{1.11}$$

so that $\Phi(X)/\Sigma(X) = 1$ for $1 \le X < 2$ and $\Phi(X)/\Sigma(X) = 4/3$ for $2 \le X < 4$. We prove

Theorem 1.1. *For every real $X \ge 4$, we have*

$$1 + \frac{2\sqrt{2}}{\sqrt{\log X}\log\log X} - \frac{4.143}{\sqrt{\log X}\,(\log\log X)^2} \le \frac{\Phi(X)}{\Sigma(X)} \le 1 + \frac{2\sqrt{2}}{\sqrt{\log X}\log\log X} + \frac{3.17}{\sqrt{\log X}\,(\log\log X)^2}. \tag{1.12}$$

However, the ratios $\sigma(n)/n$ and $n/\varphi(n)$ cannot be too large. Rosser and Schoenfeld [30, Theorem 15] showed that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \leq e^{\gamma} \log\log n + \frac{2.50636\ldots}{\log\log n} \tag{1.13}$$

unconditionally for every integer $n \geq 3$, with equality for $n = 223\,092\,870 = \prod_{2\leq p\leq 23} p$. The advantage of inequality (1.13) over Robin's inequality (1.8) is that (1.13) holds for every positive integer n with $\log\log n$ positive. Robin [29, Théorème 2] (see also [8, eq. (7.83), p. 212]) used a lower bound for Chebyshev's θ-function $\theta(x) = \sum_{p\leq x} \log p$, where p runs over primes not exceeding x, to improve (1.13) by showing that

$$\frac{n}{\varphi(n)} < e^{\gamma} \log\log n + \frac{0.6}{\log\log n}$$

holds unconditionally for every integer $n \geq \prod_{2\leq p<20000} p$ and that

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n + \frac{0.648\ldots}{\log\log n}$$

unconditionally for every $n \geq 3$, with equality for $n = 12$. In [2, Theorem 1.1], the same method with improved effective estimates for Chebyshev's θ-function is used to see that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < e^{\gamma} \log\log n + \frac{0.1209}{(\log\log n)^2} \qquad \text{for } n \geq 10^{10^{10}}. \tag{1.14}$$

Robin [28] proved that if Robin's inequality (1.8) is satisfied for two consecutive colossally abundant numbers N and N' (see Sect. 3), then it is also satisfied for all integers n between N and N'. By computing all CA numbers up to $10^{10^{10}}$, Briggs [6] proved that

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n \qquad \text{for } 5040 < n \leq 10^{10^{10}}. \tag{1.15}$$

Combined with (1.14), this shows that

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n + \frac{0.1209}{(\log\log n)^2} \qquad \text{for } n > 5040.$$

For $k = 999\,999\,476\,056$ and $p_k = 29\,996\,208\,012\,611$, we define

$$M^{(0)} = M_{p_k} = \exp(29\,996\,203\,625\,537.226167\ldots) = 10^{10^{13.114850604\ldots}}. \tag{1.16}$$

Morrill and Platt [20] improved Briggs' result (1.15) by showing that

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n \qquad \text{for } 5040 < n \leq M^{(0)}. \tag{1.17}$$

In [3], it has been shown that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < e^{\gamma} \log\log n + \frac{\alpha_0}{(\log\log n)^2} \qquad \text{for } n \geq M^{(0)}, \tag{1.18}$$

where

$$\alpha_0 = 0.0094243 \times e^\gamma = 0.0167853\ldots. \tag{1.19}$$

In the proof of (1.18), improved effective estimates for the product in (1.4) were utilized. Now, let $i = 564\,397\,542$. Then

$$M_{p_{i-1}} = \exp(\theta(p_{i-1})) = \exp(12\,530\,479\,255.595893\ldots),$$
$$M_{p_i} = \exp(\theta(p_i)) = \exp(12\,530\,479\,278.847331\ldots).$$

Further, let

$$X^{(0)} = \exp(12\,530\,479\,255.595931). \tag{1.20}$$

Note that $X^{(0)}$ satisfies $M_{p_{i-1}} < X^{(0)} < M_{p_i}$ and is much smaller than $M^{(0)}$. In the following theorem, we use Theorem 1.1 to see that the right-hand inequality in (1.18) also holds with the same constant 0.0094243 for every integer $n \geq X^{(0)}$.

THEOREM 1.2. *Let α_0 and $X^{(0)}$ be defined as in* (1.19) *and* (1.20), *respectively. Then, for every integer $n \geq X^{(0)}$, we have*

$$\frac{n}{\varphi(n)} \leq e^\gamma \log\log n + \frac{\alpha_0}{(\log\log n)^2}, \tag{1.21}$$

but this does not hold for $n = M_{p_{i-1}}$.

If we combine (1.17) with (1.18), we can see that

$$\frac{\sigma(n)}{n} < e^\gamma \log\log n + \frac{\alpha_0}{(\log\log n)^2} \quad \text{for } n > 5040. \tag{1.22}$$

Under the assumption that the Riemann hypothesis is true, an effective form of the asymptotic upper bound (1.5) was shown in [24], namely

$$\frac{\sigma(n)}{n} \leq e^\gamma \biggl(\log\log n - \frac{2(\sqrt{2}-1)}{\sqrt{\log n}} + S_1(\log n) + \frac{3.789}{\sqrt{\log n}\log\log n} + \frac{0.026\log\log n}{\log^{2/3} n} \biggr) \tag{1.23}$$

for every integer $n \geq 3$. It turns out that the validity of this inequality for every $n \geq 3$ even provides an equivalent criterion for the Riemann hypothesis. Under the assumption that the Riemann hypothesis is true, inequality (1.23) combined with (1.6) yields

$$\frac{\sigma(n)}{n} \leq e^\gamma \biggl(\log\log n + \frac{-2(\sqrt{2}-1)+\tau}{\sqrt{\log n}} + \frac{3.789}{\sqrt{\log n}\log\log n} + \frac{0.026\log\log n}{\log^{2/3} n} \biggr) \tag{1.24}$$

for every integer $n \geq 3$. Finally, we can utilize Theorems 1.1 and 1.2 to give the following unconditional result concerning (1.24), which simultaneously provides an improvement of (1.22).

THEOREM 1.3. *Let* α_0 *be as in* (1.19)*. For every* $n \geq \exp(26\,318\,064\,420)$, *we have*

$$\frac{\sigma(n)}{n} < e^\gamma \log\log n + \frac{\alpha_0}{(\log\log n)^2} - \frac{2\sqrt{2}\,e^\gamma}{\sqrt{\log n}} + \frac{4.143e^\gamma}{\sqrt{\log n}\log\log n}. \tag{1.25}$$

1.1. Notation

- $\theta(x) = \sum_{p \leq x} \log p$ is the Chebyshev θ-function.
- $\pi(x) = \sum_{p \leq x} 1$ is the prime counting function.
- p_j is the jth prime with $p_1 = 2$, $p_2 = 3, \ldots$. For p prime and $n \in \mathbb{N}$, $v_p(n)$ denotes the largest exponent such that $p^{v_p(n)}$ divides n.
- $M_{p_k} = p_1 \ldots p_k$ is the kth primorial. If p is the kth prime then $M_p = M_{p_k}$.
- CA numbers are defined in Sect. 3, SA numbers in (1.9) and HR numbers in Sect. 3.1.

We use the following constants:

- $\alpha_0 = 0.0094243 \times e^\gamma = 0.0167853\ldots$ (see (1.19)).
- $\xi^{(0)} = 10^9 + 7$ is the smallest prime exceeding 10^9, $\log \xi^{(0)} = 20.723265\ldots$ (see (3.15) and (3.17)).
- $M^{(0)}$ is defined as in (1.16).
- $N^{(0)}$ is defined in (3.16) and $(\xi_k^{(0)})_{2 \leq k \leq 33}$ in (3.20).

All the computations have been carried out in Maple [34].

2. Useful results.

We shall use the following results: for $u, v > 0$ and $w \in \mathbb{R}$,

$$t \mapsto \frac{(\log t - w)^u}{t^v} \text{ is decreasing for } t > \exp(w + u/v), \tag{2.1}$$

$$-\log(1-w) \leq w\left(1 + \frac{w}{2(1-w_0)}\right) \leq \frac{w}{1-w_0} \quad \text{for } 0 \leq w \leq w_0 < 1, \tag{2.2}$$

$$1 + u \leq \exp(u) \quad \text{for } u \in \mathbb{R}, \tag{2.3}$$

$$\exp(u) \leq 1 + u + \frac{u^2}{2(1-u_0)} \quad \text{for } 0 \leq u \leq u_0 < 1. \tag{2.4}$$

REMARK 2.1. Note that inequality (2.4) is sharp if u_0 is close to zero, but it is useless if u_0 is close to 1. This inequality does appear in the proof of Theorem 1.1 with $u_0 = 0.000005$.

LEMMA 2.2. *Let $\theta(x)$ be the Chebyshev function. For every $x \geq 0$, one has*

$$\theta(x) < (1 + 1.93378 \times 10^{-8})x, \tag{2.5}$$

and for every $x \geq 41113$, one has

$$|\theta(x) - x| < \frac{0.0806x}{\log x}. \tag{2.6}$$

Further, for every $x \geq 10^9$, one has

$$|\theta(x) - x| \leq \frac{0.42065x}{\log^3 x}. \tag{2.7}$$

Proof. Inequality (2.5) is a result of Broadbent et al. [7, Corollary 2.1]. From [7, Table 15], we know that inequality (2.6) holds for every $x \geq 10^5$. A direct computer check provides that (2.6) also holds for every x with $41113 \leq x < 10^5$. According to [7, Table 15], we know that inequality (2.7) is fulfilled for every $x \geq 10^9$. ■

REMARK 2.3. It should be noted that, compared to the estimates given in Lemma 2.2, there are asymptotically stronger, and still explicit, estimates for Chebyshev's θ-function that improve (2.5)–(2.7) for very large x. For instance, Fiori, Kadiri, and Swidinsky [13, Corollary 14] found that

$$|\theta(x) - x| \leq 121.0961\left(\frac{\log x}{R}\right)^{3/2} \exp\left(-2\sqrt{\frac{\log x}{R}}\right)$$

for every $x \geq 2$, where $R = 5.5666305$.

REMARK 2.4. Under the assumption that the Riemann hypothesis is true, von Koch [33] deduced the stronger asymptotic formula $\theta(x) = x + O(\sqrt{x}\log^2 x)$ as $x \to \infty$. An explicit version of that result was given by Schoenfeld [31, Theorem 10]. Under the assumption that the Riemann hypothesis is true, he found that

$$|\theta(x) - x| < \frac{\sqrt{x}}{8\pi}\log^2 x \qquad \text{for } x \geq 599. \tag{2.8}$$

Büthe [10, Theorem 2] showed that inequality (2.8) holds unconditionally for every x such that $599 \leq x \leq 1.4 \times 10^{25}$. Büthe's result was improved by Platt and Trudgian [25, Corollary 1]. They proved that (2.8) holds unconditionally for every x satisfying $599 \leq x \leq 2.169 \times 10^{25}$. Recently, Johnston [17, Corollary 3.3] extended this last range to $599 \leq x \leq 1.101 \times 10^{26}$.

LEMMA 2.5. *For $x \geq x_0 = 10^9$, one has*

$$\pi(x(1 + 0.069/\log^2 x)) - \pi(x) \geq 2.083\frac{\sqrt{x}}{\log x}. \tag{2.9}$$

Proof. Applying Lemma 2.3 of [12] (that is a consequence of (2.7)) with $K = 2$, $\alpha = 0.42065$, $x_0 = 10^9$, and $a = 0.069$, we get

$$\pi(x(1 + 0.069/\log^2 x)) - \pi(x) \geq b\frac{x}{\log^3 x} \qquad \text{for } x \geq 10^9$$

where

$$b = \left(1 - \frac{a}{\log^{K+1} x_0}\right)\left(a - \frac{2\alpha}{\log x_0} - \frac{\alpha a}{\log^{K+1} x_0}\right) \geq 0.0283.$$

Finally, for $x \geq x_0$,

$$0.0283\frac{x}{\log^3 x} \geq 0.0283\frac{\sqrt{x_0}}{\log^2 x_0}\frac{\sqrt{x}}{\log x} = 2.083\ldots\frac{\sqrt{x}}{\log x},$$

which completes the proof. ∎

LEMMA 2.6. *Let a be a positive real number and let*

$$h = h(a,t) = ae^{t/2}t^2 - 2t\sqrt{2}. \tag{2.10}$$

(i) *For $t > 2\sqrt{2} - 4 = -1.17\ldots$, the mapping $t \mapsto h(a,t)$ is convex and $h' = \partial h/\partial t$ is increasing on a and on t.*
(ii) *If t_0 is a number > 2.54557 such that $h(a,t_0) > -5$, then $h(a,t)$ is increasing in t for $t > t_0$.*

Proof. One has

$$h' = \frac{\partial h}{\partial t} = \frac{a}{2}e^{t/2}t^2 + 2ae^{t/2}t - 2\sqrt{2}$$

and

$$h'' = \frac{\partial^2 h}{\partial t^2} = ae^{t/2}\left(\frac{t^2}{4} + 2t + 2\right).$$

The trinomial $t^2/4 + 2t + 2$ is positive for $t > 2\sqrt{2} - 4$, which proves (i).

If $h(a,t_0) > -5$ then $ae^{t_0/2}t_0^2 > 2\sqrt{2}\,t_0 - 5$, which implies

$$\begin{aligned} h'(a,t_0) &= \frac{a}{2}e^{t_0/2}t_0^2\left(1 + \frac{4}{t_0}\right) - 2\sqrt{2} > \left(\sqrt{2}\,t_0 - \frac{5}{2}\right)\left(1 + \frac{4}{t_0}\right) - 2\sqrt{2} \\ &= \frac{1}{t_0}\left(\sqrt{2}\,t_0^2 - \left(\frac{5}{2} - 2\sqrt{2}\right)t_0 - 10\right) \end{aligned}$$

and the above trinomial is positive for $t_0 > 2.545565\ldots$. Since $h(a,t)$ is convex in t, the derivative $h'(a,t)$ is increasing in t and therefore, as it is positive for $t = t_0$, it is also positive for $t > t_0$, which proves (ii). ∎

3. Colossally abundant (CA) numbers. A positive integer N is said to be *colossally abundant* (or a *CA number*) if there exists a real number $\varepsilon > 0$ such that

$$\frac{\sigma(n)}{n^{1+\varepsilon}} \leq \frac{\sigma(N)}{N^{1+\varepsilon}}$$

for every positive integer n. The number ε is called a *parameter* of the CA number N. The colossally abundant numbers were introduced in 1944 by Alaoglu and Erdős [1, Sect. 3] (see also [27, Sect. 59], [8, Sect. 6.3], and [24, Sect. 4]). Below, some properties of CA numbers are recalled.

If t is a real number with $t > 1$ and k is a positive integer, one defines

$$F(t,k) = \frac{\log(1+1/(t^k+t^{k-1}+\cdots+t))}{\log t} = \frac{\log(1+(t-1)/(t^{k+1}-t))}{\log t}. \tag{3.1}$$

Note that the second formula allows us to calculate $F(t,u)$ for u real positive and that $F(t,u)$ is decreasing in t for u fixed and in u for t fixed.

We consider the set

$$\mathcal{E} = \{F(p,k) : p \text{ prime}, k \text{ integer } \geq 1\}. \tag{3.2}$$

It is convenient to arrange the elements of $\mathcal{E} \cup \{\infty\}$ defined in (3.2) in the decreasing sequence

$$\varepsilon_1 = \infty > \varepsilon_2 = F(2,1) = \frac{\log(3/2)}{\log 2} = 0.58\ldots > \varepsilon_3 = \frac{\log(4/3)}{\log 3} = 0.26\ldots > \cdots > \varepsilon_i = F(q_i,k_i) > \cdots. \tag{3.3}$$

In the set $\mathcal{E}$ defined by (3.2), there could exist elements admitting two representations (cf. [24, Sect. 4.1])

$$\varepsilon_i = F(q_i,k_i) = F(q_i',k_i') \tag{3.4}$$

with $k_i > k_i' \geq 1$ and $q_i < q_i'$. An element $\varepsilon_i \in \mathcal{E}$ satisfying (3.4) is said to be *extraordinary*, but none is known. If ε_i is not extraordinary, it is said to be *ordinary* and it satisfies

$$\varepsilon_i = F(q_i,k_i) \tag{3.5}$$

in only one way. To $\varepsilon_i \in \mathcal{E}$, we attach the number $\xi = \xi_1$ defined by $F(\xi,1) = \varepsilon_i$ and, for $k \geq 1$, the numbers ξ_k defined by

$$F(\xi_k,k) = \frac{\log(1+1/(\xi_k+\xi_k^2+\cdots+\xi_k^k))}{\log \xi_k} = F(\xi,1) = \varepsilon_i, \tag{3.6}$$

the integer

$$K = \text{ the largest } k \text{ such that } \xi_k \geq 2 \tag{3.7}$$

and the CA number

$$N_{\varepsilon_i} = \prod_{k=1}^{K} \prod_{\xi_{k+1} < p \le \xi_k} p^k = \prod_{k=1}^{K} \prod_{p \le \xi_k} p,$$
$$\frac{\sigma(N_{\varepsilon_i})}{N_{\varepsilon_i}} = \prod_{k=1}^{K} \prod_{\xi_{k+1} < p \le \xi_k} \frac{1 - 1/p^{k+1}}{1 - 1/p}. \tag{3.8}$$

REMARK 3.1. Note that $\xi = \xi_1 > \cdots > \xi_K$. Also, ξ and ξ_k (for k fixed) and K do not decrease when ε_i decreases, i.e., when N_{ε_i} increases.

If ε_i is ordinary and satisfies (3.5), then $v_{q_i}(N_{\varepsilon_i}) = k_i$, $q_i = \xi_{k_i}(N_{\varepsilon_i})$, and

$$N_{\varepsilon_{i-1}} = \frac{N_{\varepsilon_i}}{q_i}, \qquad \frac{\sigma(N_{\varepsilon_{i-1}})}{N_{\varepsilon_{i-1}}} = \frac{1 - 1/\xi_{k_i}^{k_i}}{1 - 1/\xi_{k_i}^{k_i+1}} \frac{\sigma(N_{\varepsilon_i})}{N_{\varepsilon_i}}. \tag{3.9}$$

If ε_i is extraordinary and satisfies (3.4), we have $v_{q_i}(N_{\varepsilon_i}) = k_i$, $q_i = \xi_{k_i}(N_{\varepsilon_i})$, $v_{q_i'}(N_{\varepsilon_i}) = k_i'$, $q_i' = \xi_{k_i'}(N_{\varepsilon_i})$, and

$$N_{\varepsilon_{i-1}} = \frac{N_{\varepsilon_i}}{q_i q_i'}, \qquad \frac{\sigma(N_{\varepsilon_{i-1}})}{N_{\varepsilon_{i-1}}} = \frac{1 - 1/\xi_{k_i}^{k_i}}{1 - 1/\xi_{k_i}^{k_i+1}} \frac{1 - 1/\xi_{k_i'}^{k_i'}}{1 - 1/\xi_{k_i'}^{k_i'+1}} \frac{\sigma(N_{\varepsilon_i})}{N_{\varepsilon_i}}. \tag{3.10}$$

In both cases, we get

$$N_{\varepsilon_i}/\xi^2 < N_{\varepsilon_{i-1}}. \tag{3.11}$$

Moreover, for $k \ge 2$, one has (see [24, Proposition 4.7])

$$\xi^{1/k} \le \xi_k \le (k\xi)^{1/k}. \tag{3.12}$$

For $k = 2$, more precise estimates hold:

$$\xi_2 \ge \sqrt{2\xi}\left(1 - \frac{\log 2}{2\log\xi}\right) > \sqrt{2\xi}\left(1 - \frac{0.347}{\log\xi}\right) \quad \text{for } \xi \ge 1530 \tag{3.13}$$

and

$$\xi_2 \le \sqrt{2\xi}\left(1 - \frac{0.323}{\log\xi}\right) \quad \text{for } \xi \ge 10^9. \tag{3.14}$$

The following data (from [24, Sect. 4.4]) will be used in Lemma 3.2 below:

$$\xi = \xi^{(0)} = \xi_1^{(0)} = 10^9 + 7 \tag{3.15}$$

(the smallest prime exceeding 10^9) and $\varepsilon_i = F(\xi, 1)$, the CA number N_{ε_i} is

$$N^{(0)} = 2^{33} 3^{21} 5^{14} 7^{11} 11^9 13^8 \times \prod_{p=17}^{23} p^7 \prod_{p=29}^{41} p^6 \prod_{p=43}^{83} p^5 \prod_{p=89}^{241} p^4 \prod_{p=251}^{1409} p^3 \prod_{p=1423}^{44021} p^2 \prod_{p=44027}^{1000000007} p, \tag{3.16}$$

(3.17) $\log N^{(0)} = 1000014552.11\ldots, \quad \log\log N^{(0)} = 20.7232\ldots,$

(3.18) $\sigma(N^{(0)})/N^{(0)} = 36.909618566\ldots,$

(3.19) $K = K(N^{(0)}) = 33,$

(3.20)
$$\begin{aligned}&\xi_2=\xi_2^{(0)}=44023.5\ldots,\ \ \xi_3=\xi_3^{(0)}=1418.3\ldots,\ \ \xi_4=\xi_4^{(0)}=247.3\ldots,\\&\xi_5=\xi_5^{(0)}=85.6\ldots,\ \ \ldots,\ \ \xi_{33}=\xi_{33}^{(0)}=2.033\ldots,\ \ \xi_{34}=\xi_{34}^{(0)}=1.991\ldots,\end{aligned}$$

and for $N_{\varepsilon_i} \geq N^{(0)}$ one has (see [24, Lemma 4.10])

(3.21)
$$T_3 = \sum_{k=3}^{K} \xi_k \leq 1.9769\xi^{1/3}.$$

LEMMA 3.2. *Let X be a real number satisfying $X \geq N^{(0)}$ (see (3.16)). For simplicity, write L for $\log X$, λ for $\log\log X$, N' for $N_{\varepsilon_{i-1}}$, and N for N_{ε_i} (see (3.8)). One defines ε_i and ε_{i-1} belonging to $\mathcal{E}$ (see (3.2) and (3.3)) such that*

(3.22)
$$N' = N_{\varepsilon_{i-1}} \leq X < N_{\varepsilon_i} = N.$$

To ε_i, one associates ξ, ξ_k and K (see (3.6) and (3.7)). Then

(3.23)
$$\begin{aligned}0.999952\xi < \xi - \frac{0.42065\xi}{\log^3 \xi}\\ \leq L = \log X < \xi + \frac{0.8302\xi}{\log^3\xi} < 1.000094\xi,\end{aligned}$$

(3.24)
$$0.9999L < L/1.000094 < \xi < L/0.999952 < 1.000049L,$$

(3.25)
$$0.999995\lambda \leq \log\xi \leq 1.0000024\lambda,$$

(3.26)
$$|L-\xi| \leq \frac{0.8302\xi}{\log^3\xi} \leq \frac{0.8303\,L}{\lambda^3},$$

(3.27)
$$\left|\frac{1}{\sqrt{L}\,\lambda} - \frac{1}{\sqrt{\xi}\log\xi}\right| \leq \frac{0.456}{\sqrt{L}\,\lambda^4} \leq \frac{0.0011}{\sqrt{L}\,\lambda^2},$$

(3.28)
$$\left|\frac{1}{\sqrt{L}\,\lambda^2} - \frac{1}{\sqrt{\xi}\log^2\xi}\right| \leq \frac{0.496}{\sqrt{L}\,\lambda^5} \leq \frac{0.000056}{\sqrt{L}\,\lambda^2}.$$

Proof. Since ξ is non-decreasing in N (see Remark 3.1) and $N > X \geq N^{(0)}$ is assumed, we get $\xi \geq \xi^{(0)} > 10^9$ (see (3.15)). Using (3.21), (3.14), and (2.1), it follows that

(3.29)
$$\begin{aligned}T_2 = \sum_{k=2}^{K}\xi_k = \xi_2 + T_3 \leq \sqrt{2\xi}\left(1 - \frac{0.323}{\log\xi}\right) + 1.9769\xi^{1/3}\\ = \sqrt{2\xi}\left(1 + \frac{1}{\log\xi}\left(\frac{1.9769\log\xi}{\sqrt{2}\,\xi^{1/6}} - 0.323\right)\right)\end{aligned}$$

$$\leq \sqrt{2\xi}\left(1 + \frac{1}{\log \xi}\left(\frac{1.9769 \log 10^9}{\sqrt{2}\, 10^{9/6}} - 0.323\right)\right) \leq \sqrt{2\xi} + 0.5931 \frac{\sqrt{2\xi}}{\log \xi}$$
$$\leq \sqrt{2\xi} + \frac{0.8388\sqrt{\xi}}{\log \xi} \leq \sqrt{\xi}\left(\sqrt{2} + \frac{0.8388}{\log 10^9}\right) \leq 1.4547\sqrt{\xi}.$$

From (3.22), (3.8), (2.5), (2.7), (3.29), and (2.1), we may write

$$(3.30) \qquad L < \log N = \sum_{k=1}^{K} \theta(\xi_k) \leq \theta(\xi) + (1 + 1.93378 \times 10^{-8}) T_2$$
$$\leq \xi + \frac{0.42065\xi}{\log^3 \xi} + 1.455\sqrt{\xi} = \xi\left(1 + \frac{1}{\log^3 \xi}\left(0.42065 + \frac{1.455 \log^3 \xi}{\sqrt{\xi}}\right)\right)$$
$$\leq \xi\left(1 + \frac{1}{\log^3 \xi}\left(0.42065 + \frac{1.455 \log^3 10^9}{\sqrt{10^9}}\right)\right)$$
$$\leq \xi\left(1 + \frac{0.8302}{\log^3 \xi}\right) \leq \xi\left(1 + \frac{0.8302}{\log^3 10^9}\right) \leq 1.000094\xi,$$

which proves the upper bound of (3.23). From (3.22), (3.11), and (3.8), we deduce that $L \geq \log N_{\varepsilon_{i-1}} \geq \log N - 2\log \xi \geq \theta(\xi) + \theta(\xi_2) - 2\log \xi$. But, from (3.20), $\xi_2 \geq 44023$, so that from (2.6) and (3.12),

$$\theta(\xi_2) \geq \xi_2\left(1 - \frac{0.0806}{\log \xi_2}\right) \geq \xi_2\left(1 - \frac{0.0806}{\log 44023}\right) \geq 0.9924\xi_2 \geq 0.9924\xi^{1/2},$$

which is $> 2\log \xi$ for $\xi > 10^9$. Therefore, from (2.7),

$$(3.31) \qquad L \geq \theta(\xi) \geq \xi\left(1 - \frac{0.42065}{\log^3 \xi}\right) \geq \xi\left(1 - \frac{0.42065}{\log^3 10^9}\right) \geq 0.999952\xi,$$

which completes the proof of (3.23).

The proof of (3.24) follows from (3.23).

From (3.17), we obtain $\log L = \log\log X \geq \log\log N^{(0)} > 20.72$. Now we can use (3.24) to see that

$$\log \xi \leq \log 1.000049 + \log L \leq (\log L)(1 + 0.000049/20.72) < 1.0000024\lambda$$

and, similarly,

$$\log \xi \geq (\log L)(1 + \log(0.9999)/20.72) > 0.999995\lambda,$$

which proves (3.25).

In order to prove (3.26), we utilize (3.23), (3.24), and (3.25) to see that

$$|\xi - L| \leq \frac{0.8302\xi}{\log^3 \xi} \leq \frac{0.8302 \times 1.000049\, L}{(0.999995)^3\, \lambda^3} < \frac{0.8303\, L}{\lambda^3}.$$

Next, we show that the inequalities given in (3.27) hold. For this purpose, it is convenient to introduce

$$\rho = \min(\xi, L) \geq \min(\xi^{(0)}, \log N^{(0)}) > 10^9.$$

From (3.24) and (3.25), it follows that

$$L \ge \rho \ge L/1.000094, \qquad \log\rho \ge 0.999995\lambda, \qquad L \ge \rho \ge 10^9. \tag{3.32}$$

One may write

$$\left|\frac{1}{\sqrt{L}\,\lambda} - \frac{1}{\sqrt{\xi}\log\xi}\right| = \left|\int_\xi^L \frac{\log t + 2}{2t^{3/2}\log^2 t}\,\mathrm{d}t\right| \le \frac{|L-\xi|}{2\rho^{3/2}\log\rho}\left(1+\frac{2}{\log\rho}\right)$$

and, from (3.26) and (3.32), this is

$$\le \frac{0.8303L}{\log^3 L}\,\frac{1.000094^{3/2}}{2L^{3/2}\times 0.999995\lambda}\left(1+\frac{2}{\log 10^9}\right) \le \frac{0.456}{\sqrt{L}\,\lambda^4} \le \frac{0.0011}{\sqrt{L}\,\lambda^2},$$

which proves (3.27).

The proof of (3.28) is similar to that of (3.27). One has

$$\begin{aligned}\left|\frac{1}{\sqrt{L}\,\lambda^2} - \frac{1}{\sqrt{\xi}\log^2\xi}\right| &= \left|\int_\xi^L \frac{\log t + 4}{2t^{3/2}\log^3 t}\,\mathrm{d}t\right| \\ &\le \frac{0.8303\,L}{\log^3 L}\,\frac{1.000094^{3/2}}{2L^{3/2}\times 0.999995^2\,\lambda^2}\left(1+\frac{4}{\log 10^9}\right) \\ &\le \frac{0.496}{\sqrt{L}\,\lambda^5} \le \frac{0.000056}{\sqrt{L}\,\lambda^2},\end{aligned}$$

which completes the proof of Lemma 3.2. ∎

LEMMA 3.3.

(i) *The only numbers that are both SA and primorial are* 2 *and* 6.
(ii) *There is at most one primorial between two consecutive CA numbers* > 6.

Proof. Let $p \ge 7$ be a prime and M_p its primorial. We set $n = 2M_p/p < M_p$. Then

$$\frac{\sigma(n)/n}{\sigma(M_p)/M_p} = \frac{\sigma(4)/4}{\sigma(2p)/(2p)} = \frac{7/4}{3(p+1)/(2p)} = \frac{7/6}{1+1/p} > 1.$$

So, M_p is not an SA number. If $p = 5$, then $M_p = 30$ is also not an SA number because $\sigma(24)/24 = 5/2 > \sigma(30)/30 = 12/5$, which proves (i).

Let N and N' be two consecutive CA numbers with $6 < N < N'$. Further, let p the largest prime factor of N and p' the prime following p. Then $N' \le p'N$. Let us assume that there are two primes $q < q'$ such that

$$N < M_q < M_{q'} < N' \le p'N.$$

Then

$$q' \le M_{q'}/M_q < Np'/N = p'$$

so that $q' \le p$. But all primes $\le p$ divide N, whence $M_{q'}$ divides N and $M_{q'} \le N$, a contradiction, which completes the proof of Lemma 3.3. ∎

3.1. An algorithm to compute SA numbers. Let ε be a positive real number and let N be a CA number of parameter ε. For a positive integer n, the *benefit* of n is defined by

$$\operatorname{ben}_\varepsilon(n) = \log\left(\frac{\sigma(N)}{N^{1+\varepsilon}}\right) - \log\left(\frac{\sigma(n)}{n^{1+\varepsilon}}\right) = \log\left(\frac{\sigma(N)/N}{\sigma(n)/n}\right) + \varepsilon\log\left(\frac{n}{N}\right). \tag{3.33}$$

If B is a given positive real number, then the set of integers n satisfying $\operatorname{ben}_\varepsilon(n) \leq B$ is finite [24, Proposition 4.14]. In [24, Sect. 4.6], an algorithm is described to compute all integers n such that $\operatorname{ben}_\varepsilon(n) \leq B$.

A *Hardy–Ramanujan number* (*HR number* for short) is an integer n such that, if $p < p'$ are two primes, then $v_p(n) \geq v_{p'}(n)$. In [1, Theorem 1], it is proved that every SA number is an HR number. Let N' be the CA number following N. It is easy to adapt the above algorithm to compute all HR numbers $n_1, \ldots, n_r$ satisfying $N \leq n_1 < \cdots < n_r \leq N'$ and $\operatorname{ben}_\varepsilon \leq B$, and to prune them. The number n_i should be pruned if there exists $j < i$ such that $\sigma(n_j)/n_j \geq \sigma(n_i)/n_i$. Let us denote by $S_1 < \cdots < S_s$ the pruned list and let us show that these s numbers S_i are SA. Ad absurdum, assume that S_i is not SA. Let S be the largest SA number $< S_i$. Since N is SA, it follows that $S \geq N$ and we would have $\sigma(S_i)/S_i < \sigma(S)/S$, which would imply

$$\operatorname{ben}_\varepsilon(S) = \log\left(\frac{\sigma(N)/N}{\sigma(S)/S}\right) + \varepsilon\log\left(\frac{S}{N}\right) < \log\left(\frac{\sigma(N)/N}{\sigma(S_i)/S_i}\right) + \varepsilon\log\left(\frac{S_i}{N}\right) = \operatorname{ben}_\varepsilon(S_i) \leq B.$$

So, S would be equal to some $n_j < S_i$ and S_i would have been pruned. Therefore, all the elements of the pruned list are SA. But, do we get all the SA numbers between N and N'? If there exists an SA number S between S_i and S_{i+1} with $1 \leq i \leq s-1$, then $\sigma(S_i)/S_i < \sigma(S)/S$ and

$$\begin{aligned} \operatorname{ben}_\varepsilon(S) &= \log\left(\frac{\sigma(N)/N}{\sigma(S)/S}\right) + \varepsilon\log\left(\frac{S}{N}\right) \\ &< \log\left(\frac{\sigma(N)/N}{\sigma(S_i)/S_i}\right) + \varepsilon\log\left(\frac{S_{i+1}}{N}\right) \\ &= \operatorname{ben}_\varepsilon(S_i) + \varepsilon\log\left(\frac{S_{i+1}}{S_i}\right). \end{aligned} \tag{3.34}$$

Let us set

$$B' = \max_{1\leq i\leq s-1} \operatorname{ben}_\varepsilon(S_i) + \varepsilon\log(S_{i+1}/S_i).$$

If $B' \leq B$, then (3.34) implies that S would belong to the pruned list, contrary to our hypothesis. If $B' > B$, we start the algorithm again with B'

instead of B and (3.34) proves that S would belong to the new pruned list.

In the proof of Theorem 1.1, this algorithm has been used with $B = 2\varepsilon$ and B' was always smaller than B.

Let us say that an integer n is *largely superabundant* if n not an SA number and if m is the largest SA number not exceeding n, we have $\sigma(n)/n = \sigma(m)/m$. Between 1 and 10^{50} there is only one such number, namely $n = 360360$ with $\sigma(n)/n = 48/11$. The preceding SA number is $m = 332640$ with $\sigma(m)/m = 48/11$.

4. Proof of Theorem 1.1. First, we observe that for $X \geq 4$, one has

$$\frac{\Phi(X)}{\Sigma(X)} > 1. \tag{4.1}$$

Indeed, $\Sigma(X)$ is equal to $\sigma(n)/n$ where n is the largest SA number not exceeding X. So, n is an HR number (see Sect. 3.1). If P denotes the largest prime factor of n, then n can be written as

$$n = \prod_{p \leq P} p^{a_p} \quad \text{with} \quad a_p \geq 1 \text{ for all } p \leq P,$$

so that $M_P \leq n \leq X$. Now we can use (1.1) to see that

$$\Sigma(X) = \frac{\sigma(n)}{n} = \prod_{p \leq P} \left(\frac{\sigma(p^{a_p})}{p^{a_p}} \right) < \prod_{p \leq P} \left(\frac{p}{p-1} \right) = \frac{M_P}{\varphi(M_P)} \leq \Phi(X),$$

which proves (4.1). It is convenient to set

$$g(X) = \frac{\Phi(X)}{\Sigma(X)} - 1 > 0 \tag{4.2}$$

and define $\rho(X)$ via

$$g(X) = \frac{2\sqrt{2}}{\sqrt{\log X} \log\log X} + \frac{\rho(X)}{\sqrt{\log X} (\log\log X)^2}.$$

Using (2.10), we find that $\rho(X)$ can be expressed as

$$\rho(X) = h(g(X), t) = g(X) e^{t/2} t^2 - 2t\sqrt{2} \quad \text{with} \quad t = \log\log X. \tag{4.3}$$

4.1. Proof of Theorem 1.1 for $X > N^{(0)}$. We first consider the case $X > N^{(0)}$, where $N^{(0)}$ is defined by (3.16). If m is the largest SA number $\leq X$ then (1.11) and (1.9) imply that $\Sigma(X) = \sigma(m)/m$. Similarly, as the champions for $n/\varphi(n)$ are the numbers M_{p_k} (see (1.10)), $\Phi(X) = M_{p_k}/\varphi(M_{p_k})$ with M_{p_k} defined by $M_{p_k} \leq X < M_{p_{k+1}}$. We determine the two CA numbers N' and N that are chosen such that $N' = N_{\varepsilon_{i-1}} \leq X < N = N_{\varepsilon_i}$. Then, we have $N > N^{(0)}$, $\xi = \xi(N) \geq \xi^{(0)} > 10^9$, and $\xi_k \geq \xi_k^{(0)}$.

Since every CA number is an SA number, we obtain

$$\frac{\sigma(N')}{N'} \leq \Sigma(X) < \frac{\sigma(N)}{N}. \tag{4.4}$$

Let p_r be the largest prime factor of N. From (3.8), it follows that

$$p_r \leq \xi < p_{r+1} \tag{4.5}$$

and $N \geq M_{p_r}$. We define u to be the smallest positive integer such that

$$M_{p_{r+u-1}} < N \leq M_{p_{r+u}} \tag{4.6}$$

and let $M = M_{p_{r+u}}$. Next, we give some effective estimates for u. In order to do this, we first note that (3.8) implies

$$\log N = \log N_{\varepsilon_i} = \theta(\xi) + E \quad \text{with} \quad E = \sum_{k=2}^{K} \theta(\xi_k). \tag{4.7}$$

This notion of excess has already been used in [12, Sect. 3.5]. Since $\xi_2 \geq \xi_2^{(0)}$, we can utilize (3.20) to get $\xi_2 \geq 44023$. If we apply successively (3.8), (2.6), (3.12), and (3.13), it follows that

$$\begin{aligned} E &\geq \theta(\xi_2) \geq \xi_2\left(1 - \frac{0.0806}{\log \xi_2}\right) \geq \xi_2\left(1 - \frac{0.0806}{\log \sqrt{\xi}}\right) \\ &\geq \sqrt{2\xi}\left(1 - \frac{0.347}{\log \xi}\right)\left(1 - \frac{0.1612}{\log \xi}\right) \geq \sqrt{2\xi}\left(1 - \frac{0.5082}{\log \xi}\right). \end{aligned} \tag{4.8}$$

On the other hand, we can apply successively (2.5), (3.21), (2.6), (3.12), and (3.14) to see that

$$\begin{aligned} E &\leq \theta(\xi_2) + (1 + 1.93378 \times 10^{-8}) \sum_{k=3}^{K} \xi_k \leq \theta(\xi_2) + 1.977\xi^{1/3} \\ &\leq \xi_2\left(1 + \frac{0.0806}{\log \xi_2}\right) + 1.977\xi^{1/3} \\ &\leq \sqrt{2\xi}\left(1 - \frac{0.323}{\log \xi}\right)\left(1 + \frac{0.1612}{\log \xi}\right) + \frac{1.977 \log \xi}{\sqrt{2}\,\xi^{1/6}} \frac{\sqrt{2\xi}}{\log \xi} \\ &\leq \sqrt{2\xi}\left(1 - \frac{0.1618}{\log \xi}\right) + \frac{1.977 \log(10^9)}{\sqrt{2} \cdot 10^{9/6}} \frac{\sqrt{2\xi}}{\log \xi} \leq \sqrt{2\xi}\left(1 + \frac{0.755}{\log \xi}\right). \end{aligned} \tag{4.9}$$

From (4.5)–(4.7), it follows that

$$\log M = \theta(\xi) + \sum_{i=1}^{u} \log p_{r+i} \geq \log N = \theta(\xi) + E$$

and

$$\log M/p_{r+u} = \theta(\xi) + \sum_{i=1}^{u-1} \log p_{r+i} \leq \log N = \theta(\xi) + E.$$

Using (4.5), these inequalities imply that

$$(u-1)\log\xi \le E \le u\log p_{r+u}. \tag{4.10}$$

If we combine (4.9) with the left-hand inequality of (4.10), it follows that

$$u \le 1+\frac{\sqrt{2\xi}}{\log\xi}\left(1+\frac{0.755}{\log\xi}\right) \le \frac{\sqrt{2\xi}}{\log\xi}\left(1+\frac{0.765}{\log\xi}\right) \le 1.467\frac{\sqrt{\xi}}{\log\xi}. \tag{4.11}$$

Note that Lemma 2.5 and (4.11) yield

$$\pi\left(\xi\left(1+\frac{0.069}{\log^2\xi}\right)\right)-\pi(\xi) \ge 2.083\frac{\sqrt{\xi}}{\log\xi} \ge u. \tag{4.12}$$

Since $\pi(\xi)=r$ (see (4.5)), (4.12) implies

$$\xi \le p_{r+u} \le \xi\left(1+\frac{0.069}{\log^2\xi}\right) \le \xi\left(1+\frac{0.0034}{\log\xi}\right) \le 1.00017\xi. \tag{4.13}$$

Hence $\log p_{r+u} \le \log\xi + 0.00017$. Together with (4.8) and the right-hand inequality of (4.10), we obtain

$$\begin{aligned} u &\ge \frac{E}{\log p_{r+u}} \ge \frac{\sqrt{2\xi}}{\log\xi}\left(1-\frac{0.522}{\log\xi}\right)\left(\frac{1}{1+0.00017/\log\xi}\right) \\ &\ge \frac{\sqrt{2\xi}}{\log\xi}\left(1-\frac{0.522}{\log\xi}\right)\left(1-\frac{0.00017}{\log\xi}\right) \\ &\ge \frac{\sqrt{2\xi}}{\log\xi}\left(1-\frac{0.523}{\log\xi}\right) \ge 1.377\frac{\sqrt{\xi}}{\log\xi}. \end{aligned} \tag{4.14}$$

Hence $u \ge 3$ and we set $M' = M/(p_{r+u}p_{r+u-1}p_{r+u-2})$. From (4.5), (4.6), and (3.11), it follows that

$$\begin{aligned} M' &< \frac{M}{p_{r+u}p_{r+1}^2} < \frac{M}{p_{r+u}\xi^2} < \frac{N}{\xi^2} \\ &\le N_{\varepsilon_{i-1}} = N' \le X < N = N_{\varepsilon_i} \le M, \end{aligned} \tag{4.15}$$

and consequently $M'/\varphi(M') \le \Phi(X) < M/\varphi(M)$. Combined with (4.4), this yields

$$\frac{M'N}{\varphi(M')\sigma(N)} < \frac{\Phi(X)}{\Sigma(X)} < \frac{MN'}{\varphi(M)\sigma(N')}. \tag{4.16}$$

Using the definition of M', one gets

$$\begin{aligned} \frac{M'}{\varphi(M')} &= \frac{M}{\varphi(M)}\left(1-\frac{1}{p_{r+u}}\right)\left(1-\frac{1}{p_{r+u-1}}\right)\left(1-\frac{1}{p_{r+u-2}}\right) \\ &> \frac{M}{\varphi(M)}\left(1-\frac{1}{\xi}\right)^3 > \frac{M}{\varphi(M)}\left(1-\frac{3}{\xi}\right). \end{aligned}$$

If ε_i is ordinary, we can use (3.9) and (3.12) to get

$$\frac{\sigma(N')}{N'} > \frac{\sigma(N)}{N}\left(1-\frac{1}{\xi_{k_i}^{k_i}}\right) \geq \left(1-\frac{1}{\xi}\right)\frac{\sigma(N)}{N}$$

while when ε_i is extraordinary, (3.10) and (3.12) can be used to see that

$$\frac{\sigma(N')}{N'} > \left(1-\frac{1}{\xi}\right)^2\frac{\sigma(N)}{N} > \left(1-\frac{2}{\xi}\right)\frac{\sigma(N)}{N}.$$

Therefore, (4.16) yields

$$\left(1-\frac{3}{\xi}\right)\frac{M}{\varphi(M)}\,\frac{N}{\sigma(N)} < \frac{\Phi(X)}{\Sigma(X)} < \left(\frac{1}{1-2/\xi}\right)\frac{M}{\varphi(M)}\,\frac{N}{\sigma(N)}. \tag{4.17}$$

4.1.1. *Estimates of* $\log\left(\frac{M}{\varphi(M)}\frac{N}{\sigma(N)}\right)$. From (3.8), it follows that

$$\begin{aligned}\log\left(\frac{\sigma(N)}{N}\right) &= \log\left(\prod_{k=1}^{K}\prod_{\xi_{k+1}<p\leq\xi_k}\frac{1-1/p^{k+1}}{1-1/p}\right) \\ &= U_1+U_2+U_3-U_4-U_5\end{aligned} \tag{4.18}$$

with

$$U_1 = \sum_{k=3}^{K}\sum_{\xi_{k+1}<p\leq\xi_k}\log\left(1-\frac{1}{p^{k+1}}\right), \qquad U_2 = \sum_{\xi_3<p\leq\xi_2}\log\left(1-\frac{1}{p^3}\right),$$

$$U_3 = \sum_{p>\xi_2}\log\left(1-\frac{1}{p^2}\right), \qquad U_4 = \sum_{p>\xi}\log\left(1-\frac{1}{p^2}\right),$$

$$U_5 = \sum_{p\leq\xi}\log\left(1-\frac{1}{p}\right).$$

Without the assumption that the Riemann hypothesis is true, the following estimates were found in [24, (5.5)–(5.8)]:

$$-\frac{0.251}{\xi^{2/3}} \leq U_1 \leq 0, \qquad -\frac{0.087}{\xi^{2/3}} \leq U_2 \leq 0, \tag{4.19}$$

and

$$-\frac{\sqrt{2}}{\sqrt{\xi}\log\xi}+\frac{2.5627}{\sqrt{\xi}\log^2\xi} \leq U_3-U_4 \leq -\frac{\sqrt{2}}{\sqrt{\xi}\log\xi}+\frac{3.353}{\sqrt{\xi}\log^2\xi}. \tag{4.20}$$

We can utilize (4.6) and (4.5) to get

$$\log\left(\frac{M}{\varphi(M)}\right) = -U_5+U_6 \quad\text{where}\quad U_6 = -\sum_{i=1}^{u}\log\left(1-\frac{1}{p_{r+i}}\right). \tag{4.21}$$

Now we apply (4.14) and (4.13) to the definition of U_6 to see that

$$U_6 \geq \sum_{i=1}^{u} \frac{1}{p_{r+i}} \geq \frac{u}{p_{r+u}} \geq \frac{u}{\xi(1+0.0034/\log\xi)} \tag{4.22}$$
$$\geq \frac{\sqrt{2}}{\sqrt{\xi}\log\xi}\left(1-\frac{0.523}{\log\xi}\right)\left(1-\frac{0.0034}{\log\xi}\right)$$
$$\geq \frac{\sqrt{2}}{\sqrt{\xi}\log\xi}\left(1-\frac{0.5264}{\log\xi}\right) \geq \frac{\sqrt{2}}{\sqrt{\xi}\log\xi} - \frac{0.745}{\sqrt{\xi}\log^2\xi}.$$

On the other hand, we can use inequality (2.2) with $w = 1/\xi$ and $w_0 = 10^{-9}$ to obtain

$$U_6 \leq -u\log\left(1-\frac{1}{\xi}\right) \leq \frac{u}{\xi}\left(1+\frac{1}{2\xi(1-10^{-9})}\right) \leq \frac{u}{\xi}\left(1+\frac{\log 10^9}{2(\log\xi)(10^9-1)}\right).$$

Substituting (4.11) into the last inequality, we get

$$U_6 \leq \frac{\sqrt{2}}{\sqrt{\xi}\log\xi}\left(1+\frac{0.779}{\log\xi}\right)\left(1+\frac{0.0001}{\log\xi}\right) \tag{4.23}$$
$$\leq \frac{\sqrt{2}}{\sqrt{\xi}\log\xi}\left(1+\frac{0.77911}{\log\xi}\right) \leq \frac{\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{1.102}{\sqrt{\xi}\log^2\xi}.$$

If we combine (4.18) with (4.21), it follows that (note that U_5 disappears)

$$\log\left(\frac{M}{\varphi(M)}\frac{N}{\sigma(N)}\right) = -(U_1+U_2)-(U_3-U_4)+U_6. \tag{4.24}$$

The inequalities given in (4.19) imply that

$$0 \leq -(U_1+U_2) \leq \frac{0.338}{\xi^{(2/3)}} \leq \frac{0.338\log^2 10^9}{10^{9/6}\sqrt{\xi}\log^2\xi} \leq \frac{4.6}{\sqrt{\xi}\log^2\xi}. \tag{4.25}$$

Now we can substitute (4.25), (4.20), and (4.23) into (4.24) to see that

$$\log\left(\frac{M}{\varphi(M)}\frac{N}{\sigma(N)}\right) \tag{4.26}$$
$$\leq \frac{2\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{4.6-2.5627+1.102}{\sqrt{\xi}\log^2\xi} = \frac{2\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{3.1393}{\sqrt{\xi}\log^2\xi}.$$

On the other hand, if we substitute (4.25), (4.20), and (4.22) into (4.24), we get

$$\log\left(\frac{M}{\varphi(M)}\frac{N}{\sigma(N)}\right) \tag{4.27}$$
$$\geq \frac{2\sqrt{2}}{\sqrt{\xi}\log\xi} - \frac{3.353+0.745}{\sqrt{\xi}\log^2\xi} = \frac{2\sqrt{2}}{\sqrt{\xi}\log\xi} - \frac{4.098}{\sqrt{\xi}\log^2\xi}.$$

Applying (2.2) with $w = 2/\xi$ and $w_0 = 2 \times 10^{-9}$, we obtain

$$-\log\left(1 - \frac{2}{\xi}\right) \le \frac{2}{\xi(1 - 2 \times 10^{-9})} \le \frac{2.0001}{\xi} = \frac{2.0001}{\sqrt{\xi}\log^2 \xi}\frac{\log^2 \xi}{\sqrt{\xi}} \le \frac{2.0001}{\sqrt{\xi}\log^2 \xi}\frac{\log^2 10^9}{\sqrt{10^9}} \le \frac{0.0272}{\sqrt{\xi}\log^2 \xi}, \tag{4.28}$$

and similarly with $w = 3/\xi$ and $w_0 = 3 \times 10^{-9}$,

$$\log\left(1 - \frac{3}{\xi}\right) \ge -\frac{3}{\xi(1 - 3 \times 10^{-9})} \ge -\frac{0.0408}{\sqrt{\xi}\log^2 \xi}. \tag{4.29}$$

Consequently, (4.17), (4.26), (4.28), (4.27), and (4.29) allow us to write

$$\frac{2\sqrt{2}}{\sqrt{\xi}\log \xi} - \frac{4.1388}{\sqrt{\xi}\log^2 \xi} \le \log\left(\frac{\Phi(X)}{\Sigma(X)}\right) \le \frac{2\sqrt{2}}{\sqrt{\xi}\log \xi} + \frac{3.1665}{\sqrt{\xi}\log^2 \xi}. \tag{4.30}$$

By using (3.27), (3.28), and the notation $L = \log X$ and $\lambda = \log\log X$, (4.30) yields

$$\frac{2\sqrt{2}}{\sqrt{L}\,\lambda} - \frac{4.143}{\sqrt{L}\,\lambda^2} \le \log\left(\frac{\Phi(X)}{\Sigma(X)}\right) \le \frac{2\sqrt{2}}{\sqrt{L}\,\lambda} + \frac{3.1698}{\sqrt{L}\,\lambda^2} \le \frac{2.9814}{\sqrt{L}\,\lambda} \le 0.000005. \tag{4.31}$$

The lower bound of (1.12) follows from (2.3) while the upper bound is obtained by applying (2.4) with $u = 2\sqrt{2}/(\sqrt{L}\,\lambda) + 3.1698/(\sqrt{L}\,\lambda^2)$, $u_0 = 0.000005$, and

$$\frac{u^2}{2(1-u_0)} \le \frac{2.9814^2}{1.99999(\sqrt{L}\,\lambda)^2} \le \frac{2.9814^2}{1.99999\sqrt{10^9 L}\,\lambda^2} \le \frac{0.000141}{\sqrt{L}\,\lambda^2}.$$

This completes the proof of Theorem 1.1 for every real $X > N^{(0)}$.

4.2. Proof of Theorem 1.1 for $X \le N^{(0)}$

4.2.1. *Proof of Theorem 1.1 for* $4 \le X \le 10^{49}$. We can use the algorithm described in [24, Sect. 4.1] (see also [23, Sect. 3.4]), to compute all SA numbers up to 10^{50}. We also compute the primorials in the range $[4, 10^{50}]$. We arrange these 351 numbers in ascending order into a sequence $n_1 = 4$, $n_2 = 6$, up to n_{351}. The largest one is the SA number

$$n_{351} = 2^8 3^5 5^3 7^3 11^2 13^2 \prod_{17 \le p \le 107} p = 5.12\ldots \times 10^{49}.$$

For $X \in [n_j, n_{j+1})$, $g(X)$ is constant and equal to $a_j = m_j s_j/(\varphi(m_j)\sigma(s_j)) - 1$, where s_j (resp. m_j) is the largest SA number (resp. primorial) $\le n_j$ and $g(X)$ is given by (4.2). Let us assume $1 \le j \le 350$ and $n_j \le X < n_{j+1}$. Then $n_j \ge 4 > e$ and $\rho(X)$ is given by (4.3). Now, we have to find the maximum and the minimum of $\rho(X)$. In order to do this, we make use of the convexity of h (see Lemma 2.6; note that $\log\log X \ge \log\log 4 > 0 > 2\sqrt{2}-4$). We set $t_j = \log\log n_j$, $t_{j+1} = \log\log n_{j+1}$ and consider the following cases:

- If $h'(g(X),t_j) \ge 0$ then $\rho(X) = h(g(X),t)$ is increasing on $[t_j, t_{j+1})$ (Case 1).
- If $h'(g(X),t_j) < 0$ and $h'(g(X),t_{j+1}) \le 0$ then $\rho(X) = h(g(X),t)$ is decreasing on $[t_j, t_{j+1})$ (Case 2).
- If $h'(g(X),t_j) < 0$ and $h'(g(X),t_{j+1}) > 0$ then $\rho(X) = h(g(X),t)$ has its minimum on (t_j, t_{j+1}) (Case 3).

Note that Case 2 occurs for X in $[4,30)$ and $[120,210)$ while Case 3 occurs just once, namely for X in $[60,120)$. The smallest value of $\rho(X)$ is $\rho(27720) = -3.3308\ldots$ and the largest one is $1.5566\ldots$ in the interval $[M_{127}, N^{(1)})$ when X tends to

$$N^{(1)} = 2^9 3^4 5^3 7^2 11^2 13^2 \prod_{17\le p\le 103} p = 4.14\ldots \times 10^{48}$$

(see [34]).

4.2.2. *Proof of Theorem 1.1 for* $10^{49} \le X \le 10^{7648}$. Let $12 \le N' < N$ be two consecutive CA numbers. By the algorithm described above in Sect. 3.1, one determines all SA numbers n satisfying $N' \le n \le N$ and also, from a precomputed table of primorials, the largest primorial $M' < N'$ and the largest primorial $M < N$. From Lemma 3.3, we have either $M = M'$ or M is the primorial following M'. We order these SA numbers n and M (if $M > N'$) in a sequence

$$M' < N' = n_1 < \cdots < n_r = N.$$

For $X \in [n_j, n_{j+1})$ with $1 \le j \le r-1$, $g(X)$ is equal to $mn/(\varphi(m)\sigma(n)) - 1$ where m is the largest of M' and M satisfying $m \le n_j$ and n is the largest SA number $\le n_j$. Then we apply the algorithm of Sect. 4.2.1. As explained in [24, Sect. 4.2], from a precomputed table of CA2 numbers, one generates the CA numbers $> 5.98 \times 10^{44}$ up to $1.19\ldots \times 10^{7648}$. For $X < 10^{49}$, we have checked that the results coincide with those of Sect. 4.2.1 and for $X \ge 10^{49}$ all intervals are of Case 1. The minimal value $-0.977\ldots$ of $\rho(X)$ is obtained for (see [34])

$$N^{(2)} = 2^8 3^5 5^3 7^2 11^2 13^2 \prod_{17\le p\le 113} p = 8.201\ldots \times 10^{54}.$$

The maximal value $2.419\ldots$ is obtained for $X < N^{(3)}$ and tending to

$$N^{(3)} = 2^{12} 3^7 5^5 7^4 11^3 13^3 \prod_{17\le p\le 43} p^2 \prod_{47\le p\le 1091} p = 1.036\ldots \times 10^{485}.$$

4.2.3. *Proof of Theorem 1.1 for* $10^{7648} < X \le N^{(0)}$. For X very large, the algorithm of Sect. 4.2.2 would take too long to carry out. So, from our precomputed table of CA2 numbers, we generate all the CA numbers up to $N^{(0)}$. If $N' < N$ are two consecutive CA numbers and $N' \le X \le N$ then $\sigma(N')/N' \le \Sigma(X) \le \sigma(N)/N$. If M' is the largest primorial $< N'$ and M the largest primorial $< N$, we see that $M'/\varphi(M') \le \Phi(X) \le M/\varphi(M)$. Note that N' and N are chosen in the same way in terms of X as in Sect. 4.1. Therefore, for $N' \le X \le N$, we obtain

$$a_1 = \frac{M'N}{\varphi(M')\sigma(N)} - 1 \le g(X) = \frac{\Phi(X)}{\Sigma(X)} - 1 \le a_2 = \frac{MN'}{\varphi(M)\sigma(N')} - 1, \tag{4.32}$$

and from (4.3), we get

$$h(a_1, t) \le \rho(X) = h(g(X), t) \le h(a_2, t) \quad \text{with} \quad t = \log\log X. \tag{4.33}$$

In view of applying Lemma 2.6(ii), one checks that $h(a_1, \log\log N') > -5$. This implies that $h(a_2, \log\log N') > -5$. The two mappings $t \mapsto h(a_1, t)$ and $t \mapsto h(a_2, t)$ are increasing on the interval $[\log\log N', \log\log N]$, which provides

$$h(a_1, \log\log N') \le \rho(X) \le h(a_2, \log\log N).$$

For $10^{7648} < X \le N^{(0)}$ this yields $-1.91 \le \rho(X) < 2.39$ (cf. [34]).

Conclusion. By gathering the results of Sections 4.2.1–4.2.3, we see that

$$-3.34 \le \rho(X) \le 2.42 \qquad \text{for } 4 \le X \le N^{(0)},$$

which completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2. In order to prove that inequality (1.21) holds for every $n \ge X^{(0)}$ (defined by (1.20)), we introduce the following notation. Let $M^{(0)}$ be defined by (1.16) and let $i = 564\,397\,542$. Then $p_i = 12\,530\,577\,161$ and

$$M_{p_{i-1}} < X^{(0)} < M_{p_i} = \exp(12\,530\,479\,278.847331\ldots).$$

Similarly, let $m = 1\,469\,923\,277$. Then $p_m = 34\,110\,324\,851$ and

$$M_{p_m} = \exp(34\,110\,069\,410.651478\ldots).$$

5.1. Proof of Theorem 1.2 for $n \ge M^{(0)}$. For every integer $n \ge M^{(0)}$, the required inequality (1.21) was already proven in [3] (see (1.18)). So it suffices to deal with $X^{(0)} \le n < M^{(0)}$.

5.2. Proof of Theorem 1.2 for $M_{p_m} \le n < M^{(0)}$. In this case, we can utilize Theorem 1.1 to see that

$$\frac{n}{\varphi(n)} \le \Sigma(n)\left(1 + \frac{2\sqrt{2}}{\sqrt{\log n}\log\log n} + \frac{3.17}{\sqrt{\log n}\,(\log\log n)^2}\right). \tag{5.1}$$

Let N be the largest superabundant number not exceeding n. Then $5040 < N < M^{(0)}$ and $\Sigma(n) = \sigma(N)/N$. Applying (1.17), we get

$$\Sigma(n) = \frac{\sigma(N)}{N} < e^{\gamma} \log\log N \leq e^{\gamma} \log\log n.$$

Now we substitute this inequality into (5.1) to obtain

$$\frac{n}{\varphi(n)} \leq e^{\gamma} \log\log n + \frac{2\sqrt{2}\, e^{\gamma}}{\sqrt{\log n}} + \frac{3.17 e^{\gamma}}{\sqrt{\log n} \log\log n}. \tag{5.2}$$

Note that

$$\frac{2\sqrt{2}}{\sqrt{\log x}} + \frac{3.17}{\sqrt{\log x} \log\log x} < \frac{0.0094243}{(\log\log x)^2} \tag{5.3}$$

for every $x \geq M_{p_m}$. If we combine (5.2) with (5.3), it follows that the inequality (1.21) holds for every integer n with $M_{p_m} \leq n < M^{(0)}$.

5.3. Proof of Theorem 1.2 for $M_{p_i} \leq n < M_{p_m}$. For the sake of readability, we introduce the function

$$H(x) = e^{\gamma} \log\log x + \frac{\alpha_0}{(\log\log x)^2}. \tag{5.4}$$

Note that H is increasing on the interval $[3.69, \infty)$. We need to show that $n/\varphi(n) \leq H(n)$ for every integer n with $M_{p_i} \leq n < M_{p_m}$. For this purpose, we check with a computer that the inequality

$$\prod_{p \leq p_r} \frac{p}{p-1} \leq H(\exp(\theta(p_r))) \tag{5.5}$$

holds for every integer r with $i \leq r \leq m$ but it fails for $r = i - 1$. Let n be an integer with $n \in [M_{p_i}, M_{p_m})$ and let k be the unique positive integer with $M_{p_k} \leq n < M_{p_{k+1}}$. Then $i \leq k < m$. Now we can use (1.10), (5.5), and the identity $\log M_{p_k} = \theta(p_k)$ to get

$$\frac{n}{\varphi(n)} \leq \frac{M_{p_k}}{\varphi(M_{p_k})} = \prod_{p \leq p_k} \frac{p}{p-1} \leq H(M_{p_k}) \leq H(n).$$

Hence, the required inequality (1.21) holds for every integer n with $M_{p_i} \leq n < M_{p_m}$.

Remark 5.1. In order to check inequality (5.5) for every integer r with $i \leq r \leq m$, we first use PARI/GP, and independently Maple, to compute the value

$$\frac{M_{p_m}}{\varphi(M_{p_m})} = \prod_{p \leq p_m} \frac{p}{p-1} = 43.19611605653021721681\ldots \tag{5.6}$$

with an accuracy of 70 digits. Next, we successively check (5.5) with Maple for every integer r with $i \leq r \leq m$ with an accuracy of 70 digits; this requires a run time of 200 hours on a standard desktop.

Remark 5.2. It is natural to ask whether it is sufficient to utilize Rosser and Schoenfeld type bounds (see [30]) for the computation of the product in (5.6). If we use, for instance, [4, Proposition 7.1], we see that

$$43.196082\ldots \leq \frac{M_{p_m}}{\varphi(M_{p_m})} = \prod_{p \leq p_m} \frac{p}{p-1} \leq 43.196138\ldots,$$

while

$$H(M_{p_m}) = 43.196125\ldots.$$

Thus, we cannot conclude that $M_{p_m}/\varphi(M_{p_m}) \leq H(M_{p_m})$. For this reason, we need the exact value of the product given in (5.6).

5.4. Proof of Theorem 1.2 for $X^{(0)} \leq n < M_{p_i}$. Finally, let n be an integer with $X^{(0)} \leq n < M_{p_i}$. Since $X^{(0)} \in (M_{p_{i-1}}, M_{p_i})$, it follows that

$$\frac{n}{\varphi(n)} \leq \frac{M_{p_{i-1}}}{\varphi(M_{p_{i-1}})} = 41.41251143922748882925 8\ldots$$
$$< 41.412511439227488829267\ldots = H(X^{(0)}) \leq H(n),$$

and we arrive at the end of the proof of Theorem 1.2.

6. Proof of Theorem 1.3. For better readability, we write

$$Y^{(0)} = \exp(26\,318\,064\,420) \quad \text{and} \quad Y^{(1)} = \exp(35\,528\,457\,899).$$

6.1. Proof of Theorem 1.3 for $n \geq Y^{(1)}$. Let n be an integer with $n \geq Y^{(1)}$ and let

$$f(n) = 1 + \omega \quad \text{with} \quad \omega = \frac{2\sqrt{2}}{\sqrt{\log n}\log\log n} - \frac{4.143}{\sqrt{\log n}\,(\log\log n)^2}.$$

Note that $0 \leq \omega \leq 2\sqrt{2}/(\sqrt{\log n}\log\log n)$. Using Theorem 1.1, we get

$$(6.1) \qquad \frac{\sigma(n)}{n} \leq \Sigma(n) \leq \frac{\Phi(n)}{f(n)} = \frac{\Phi(n)}{1+\omega} \leq \Phi(n)(1 - \omega + \omega^2)$$
$$\leq \Phi(n)\left(1 - \frac{2\sqrt{2}}{\sqrt{\log n}\log\log n} + \frac{4.143}{\sqrt{\log n}\,(\log\log n)^2} + \frac{8}{(\log n)(\log\log n)^2}\right).$$

Define M_{p_r} to be the largest primorial not exceeding n. Then we have $\Phi(n) = M_{p_r}/\varphi(M_{p_r})$. Now we can utilize Theorem 1.2 to get

$$\Phi(n) = \frac{M_{p_r}}{\varphi(M_{p_r})} \leq H(M_{p_r}) \leq H(n),$$

where $H(x)$ is defined as in (5.4). If we substitute this inequality into (6.1), we see that

$$\frac{\sigma(n)}{n} \le e^\gamma \log\log n + \frac{\alpha_0}{(\log\log n)^2} - \frac{2\sqrt{2}\, e^\gamma}{\sqrt{\log n}} + \frac{4.143 e^\gamma}{\sqrt{\log n}\log\log n} + \frac{r(n)}{\sqrt{\log n}\log\log n},$$

where α_0 is defined as in (1.19) and

$$r(x) = -\frac{2\sqrt{2}\,\alpha_0}{(\log\log x)^2} + \frac{4.143\alpha_0}{(\log\log x)^3} + \frac{8\alpha_0}{\sqrt{\log x}\,(\log\log x)^3} + \frac{8e^\gamma}{\sqrt{\log x}}.$$

It suffices to note that $r(x) < 0$ for every $x \ge Y^{(1)}$ to conclude that inequality (1.25) holds for every integer $n \ge Y^{(1)}$.

6.2. Proof of Theorem 1.3 for $Y^{(0)} \le n < Y^{(1)}$. In order to prove (1.25) for every integer n satisfying $Y^{(0)} \le n < Y^{(1)}$, we note that

$$\frac{a_0 e^\gamma}{(\log\log x)^2} - \frac{2\sqrt{2}\, e^\gamma}{\sqrt{\log x}} + \frac{4.143 e^\gamma}{\sqrt{\log x}\log\log x} > 0$$

for every $x \ge Y^{(0)}$. If we combine this inequality with (1.17), we get the required inequality (1.25) for every integer n with $Y^{(0)} \le n < Y^{(1)}$, which completes the proof of Theorem 1.3.

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