ACTA ARITHMETICA
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Diophantine equations for Littlewood polynomials

by

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Dedicated to the memory of Andrzej Schinzel

1. Introduction. There are many papers in the literature concerning polynomials with coefficients belonging to the set $\{-1, 0, 1\}$. For a short survey, we refer to the introduction of the paper [4] and the references there. If the coefficients are only ± 1, the polynomials are called *Littlewood polynomials*. In [4], under certain necessary assumptions, an effective bound for $\max(|x|, |y|, m)$ in the equation

$$f(x) = y^m$$

is given in case f is a Littlewood polynomial and x, y, m are integral unknowns with $m \geq 2$. In the present paper we give effective upper bounds for the solutions of the more general equation

$$f(x) = ay^m + b$$

where $a, b \in \mathbb{Q}$. Further, we describe all cases where a Littlewood polynomial can have infinitely many common values with another polynomial. In particular, we show that for any $g(x) \in \mathbb{Q}[x]$, the equation

$$f(x) = g(y)$$

can have only finitely many solutions in integers x, y, except for certain explicitly given cases.

2. The theorems

Theorem 2.1. *Let $f(x)$ be a Littlewood polynomial of degree n with $n \geq 4$ and $a, b \in \mathbb{Q}$ with $a \neq 0$. Then all solutions $x, y, m \in \mathbb{Z}$ of the equation*

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$$f(x) = ay^m + b \tag{1}$$

with $m \geq 2$ *satisfy*

$$\max(|x|, |y|, m) \leq C_1,$$

except when $m = 2$ *and*

$$f(x) \in \{f^*(x),\ f^*(x) - 2f^*(0),\ xf^*(x) \pm 1\} \tag{2}$$

with $b = 0, -2f^*(0), \pm 1$*, respectively, where*

$$f^*(x) = \pm(x^{2\ell+1} + x^{2\ell} + \cdots + x^{\ell+1} - x^\ell - \cdots - 1)$$

or

$$f^*(x) = \pm((-x)^{2\ell+1} + (-x)^{2\ell} + \cdots + (-x)^{\ell+1} - (-x)^\ell + \cdots - 1)$$

with $\ell = \lfloor (n-1)/2 \rfloor$*, and the solutions are given by* $y = Q(x)$ *with* $Q(\pm x) = \pm(x^k + \cdots + x + 1)$*. Here* C_1 *is an effectively computable constant depending only on* n, a, b*, and we use the convention that* $m \leq 3$ *if* $|y| \leq 1$*.*

THEOREM 2.2. *Let* $f(x)$ *be a Littlewood polynomial of degree* n *with* $n \geq 4$ *and* $g(x) \in \mathbb{Z}[x]$*. Then the equation*

$$f(x) = g(y) \tag{3}$$

has only finitely many solutions in integers x, y*, except when* $g(y) = f(T(y))$ *with some polynomial* $T(y)$ *of degree* ≥ 1 *having rational coefficients, or if* $f(x)$ *is of the shape* (2) *and* $g(y) = a(cy + d)^2 + b$ *for* a, b *as in Theorem* 2.1 *and* $c, d \in \mathbb{Q}$*,* $c \neq 0$*.*

REMARK 1. In both theorems the assumption $\deg(f) \geq 4$ is necessary. The case $\deg(f) = 1$ is trivial. It is easy to construct infinitely many f, a, b with $\deg(f) = 2$, and $g(y) = ay^2 + b$ such that equation (1) becomes a Pell equation having infinitely many integer solutions x, y. Finally, also for $\deg(f) = 3$ there exist cases not fitting in the families described in the theorems. For example, taking

$$f(x) = x^3 + x^2 - x + 1, \quad a = \frac{1}{27}, \quad b = \frac{22}{27},$$

in view of

$$f(x) - b = a(3x + 5)(3x - 1)^2$$

we see that equation (1) has infinitely many integer solutions x, y.

It is also necessary that $f(x)$ is not of the shape (2). We demonstrate this only for one case. The other cases can be checked similarly. Take

$$\begin{aligned} f(x) &= x(x^{2\ell+1} + \cdots + x^{\ell+1} - x^\ell - \cdots - 1) + 1 \\ &= x^n + \cdots + x^{n/2+1} - x^{n/2} - \cdots - x + 1. \end{aligned}$$

One can readily check that

$$f(x) - 1 = x(x - 1)(x^{n/2-1} + \cdots + x + 1)^2.$$

As the Pell equation $x(x-1) = 2y^2$ has infinitely many solutions, equation (1) has infinitely many solutions in integers x, y when taking $m = 2$, $a = 2$, $b = 1$.

REMARK 2. Let $f(x)$ be a Littlewood polynomial and write

$$f(x) = \varepsilon_0 x^n + \varepsilon_1 x^{n-1} + \varepsilon_2 x^{n-2} + \cdots + \varepsilon_{n-1} x + \varepsilon_n$$

with $\varepsilon_i \in \{-1, 1\}$ $(i = 0, 1, \ldots, n)$. Applying the transformation $x \mapsto -x$ if necessary, we may assume that $\varepsilon_0 = \varepsilon_1$. Then, taking out a factor -1 if necessary, we may suppose that $\varepsilon_0 = \varepsilon_1 = 1$. Since our statements concern the root structure of $f(x)$ and $f'(x)$, and equations involving $f(x)$, we can clearly do this in our arguments without loss of generality. So from this point on, we shall assume that $f(x)$ is of the shape

$$f(x) = x^n + x^{n-1} + \varepsilon_2 x^{n-2} + \cdots + \varepsilon_{n-1} x + \varepsilon_n. \tag{4}$$

3. Auxiliary results. We present some lemmas which we shall use in the proofs of the theorems. By the *height* $H(F(x))$ of a polynomial $F(x)$ with integer coefficients we mean the maximum of the absolute values of its coefficients.

LEMMA 3.1. *Let $F(x) \in \mathbb{Z}[x]$ of degree D and height H have two distinct (complex) roots, and let B be a non-zero rational number. Then the equation*

$$F(x) = By^m$$

with $x, y \in \mathbb{Z}$, $|y| > 1$, implies that $m < C_2$, where C_2 is effectively computable and depends only on B, D and H.

Proof. The statement follows from the Schinzel–Tijdeman theorem [6]. ∎

The following lemma is a theorem of Brindza [2]. For any finite set S of primes, write $\mathbb{Q}_S$ for those rationals whose denominators (in their primitive forms) are composed exclusively from the primes in S. By the *height* $h(s)$ of a rational number s we mean the height of its minimal defining polynomial.

LEMMA 3.2. *Let $F(x) \in \mathbb{Z}[x]$ be of degree D and height H, and write*

$$F(x) = A \prod_{i=1}^{\ell} (x - \gamma_i)^{r_i},$$

where A is the leading coefficient of F, and $\gamma_1, \ldots, \gamma_\ell$ are the distinct complex roots of $F(x)$, with multiplicities $r_1, \ldots, r_\ell$, respectively. Further, let m be an integer with $m \geq 2$, and put

$$q_i = \frac{m}{(m, r_i)} \qquad (i = 1, \ldots, \ell).$$

Suppose that $(q_1, \ldots, q_\ell)$ is not a permutation of any of the ℓ-tuples

$$(q, 1, \ldots, 1) \quad (q \geq 1), \qquad (2, 2, 1, \ldots, 1).$$

Then for any finite set S of primes and non-zero rational B, the solutions $x, y \in \mathbb{Q}_S$ of the equation

$$F(x) = By^m$$

satisfy

$$\max(h(x), h(y)) < C_3,$$

where C_3 is effectively computable and depends only on B, m, D, H, S.

In the proof of Theorem 2.2, the decomposability of polynomials will play an important role. We call $F(x) \in \mathbb{Q}[x]$ *decomposable* over $\mathbb{Q}$ if there exist $G(x), H(x) \in \mathbb{Q}[x]$ with $\deg(G) > 1$, $\deg(H) > 1$ such that $F = G(H)$, and otherwise *indecomposable*.

LEMMA 3.3. *Let $F(x) \in \mathbb{Z}[x]$ be of the form*

$$F(x) = x^n + u_1 x^{n-1} + \cdots + u_{n-1}x + u_n.$$

If $\gcd(u_1, n) = 1$ then $F(x)$ is indecomposable over $\mathbb{Q}$.

Proof. The statement is a simple consequence of [3, Theorems 2 and 3]. ∎

We further apply a deep result of Bilu and Tichy. Let δ be a non-zero rational number and μ be a positive integer. Then the *μth Dickson polynomial* is defined by

$$D_\mu(x, \delta) := \sum_{i=0}^{\lfloor \mu/2 \rfloor} d_{\mu,i} x^{\mu-2i} \quad \text{where} \quad d_{\mu,i} = \frac{\mu}{\mu - i}\binom{\mu - i}{i}(-\delta)^i.$$

For properties of Dickson polynomials see e.g. [5]. The polynomials $F, G \in \mathbb{Q}[x]$ form a *standard pair* over $\mathbb{Q}$ if either $(F(x), G(x))$ or $(G(x), F(x))$ appears in Table 1.

Table 1. Standard pairs. Here α, β are non-zero rational numbers, μ, ν, q are positive integers, p is a non-negative integer, $v(x) \in \mathbb{Q}[x]$ is a non-zero but possibly constant polynomial.

Kind	Standard pair (unordered)	Parameter restrictions
First	$(x^q, \alpha x^p v(x)^q)$	$0 \le p < q$, $(p, q) = 1$, $p + \deg(v) > 0$
Second	$(x^2, (\alpha x^2 + \beta)v(x)^2)$	—
Third	$(D_\mu(x, \alpha^\nu), D_\nu(x, \alpha^\mu))$	$\gcd(\mu, \nu) = 1$
Fourth	$(\alpha^{-\mu/2} D_\mu(x, \alpha), -\beta^{-\nu/2} D_\nu(x, \beta))$	$\gcd(\mu, \nu) = 2$
Fifth	$((\alpha x^2 - 1)^3, 3x^4 - 4x^3)$	—

LEMMA 3.4 (Bilu, Tichy [1, Theorem 1.1]). *Let $f(x), g(x) \in \mathbb{Q}[x]$ be non-constant polynomials. Then the following two statements are equivalent:*

(I) *The equation $f(x) = g(y)$ has infinitely many rational solutions x, y with a bounded denominator.*

(II) *We have $f = \varphi(F(\kappa))$ and $g = \varphi(G(\lambda))$, where $\kappa(x), \lambda(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$, and $F(x), G(x)$ form a standard pair over $\mathbb{Q}$ such that the equation $F(x) = G(y)$ has infinitely many rational solutions with a bounded denominator.*

LEMMA 3.5. *Let $f(x)$ be a Littlewood polynomial and $b \in \mathbb{Q}$. If $f(x) - b$ has a root of multiplicity ≥ 3, or has at least two roots of multiplicities ≥ 2, then $b \in \mathbb{Z}$. Further, in both cases the multiple roots of $f(x) - b$ are units.*

Proof. Let $f(x)$ be given by (4) as in Remark 2. For any root α of $f(x)-b$ let $v_\alpha(x)$ denote the monic minimal defining polynomial of α over $\mathbb{Q}$. If α is a triple (or higher multiplicity) root of $f(x)-b$, then let $v(x) = v_\alpha(x)$. Similarly, if α is a double root of $f(x)-b$ with $\deg(v_\alpha) \geq 2$, then let $v(x) = v_\alpha(x)$. Finally, if $\deg(v_\alpha) = 1$ in the case of at least two roots of multiplicities ≥ 2, then take any other multiple root β of $f(x) - b$ and let $v(x) = v_\alpha(x)v_\beta(x)$. Observe that in each case, we can write

$$f(x) - b = g(x)(v(x))^\ell \tag{5}$$

with a monic $g \in \mathbb{Q}[x]$ and $\ell \geq 2$, and either $k := \deg(v) \geq 2$ or $\ell \geq 3$. Write $b = q_1/q_2$ with coprime integers q_1, q_2 ($q_2 > 0$), and $v(x) = v^*(x)/v_0$, $g(x) = g^*(x)/g_0$ with $v^*, g^* \in \mathbb{Z}[x]$ primitive polynomials, v_0, g_0 positive integers. (Since v and g are monic, such v^*, g^*, v_0, g_0 exist.) Rewrite (5) as

$$q_2 f(x) - q_1 = \frac{q_2}{g_0 v_0^\ell} g^*(x)(v^*(x))^\ell. \tag{6}$$

Since $q_2 f(x) - q_1$ and $g^*(x)(v^*(x))^\ell$ are primitive polynomials in $\mathbb{Z}[x]$ (the latter one by the Gauss lemma), we see that $q_2/g_0 v_0^\ell = 1$ in (6). Suppose that $q_2 \neq 1$. Let p be any prime with $p \mid q_2$. Then taking (6) modulo p, we see that

$$v^*(x) \equiv c \pmod{p} \tag{7}$$

for some integer c with $p \nmid c$. Taking now derivatives in (5) we obtain

$$f'(x) = (v(x))^{\ell-1} h(x) \tag{8}$$

with

$$h(x) = g'(x)v(x) + \ell g(x)v'(x).$$

Note that $\deg(f') = n - 1$, $\deg(h) = n - 1 - k(\ell - 1)$. There exist coprime positive integers h_0, h_1 and a primitive polynomial $h^*(x) \in \mathbb{Z}[x]$ such that $h(x) = h_1 h^*(x)/h_0$. Thus we can rewrite (8) as

$$f'(x) = \frac{h_1}{v_0^{\ell-1} h_0} v^*(x)^{\ell-1} h^*(x). \tag{9}$$

Recall Remark 2. Since

$$f'(x) = nx^{n-1} + (n-1)x^{n-2} + \cdots + 2\varepsilon_{n-2}x + \varepsilon_{n-1}$$

as well as $v^*(x)^{\ell-1}h^*(x)$ are primitive polynomials in $\mathbb{Z}[x]$, we see that $h_1/v_0^{\ell-1}h_0 = 1$. Taking (9) modulo p with the above prime $p \,|\, q_2$, we deduce by (7) that

$$\deg(f'(x) \pmod p) \le n - 1 - k(\ell - 1).$$

However, since the coefficients of the first two terms of $f'(x)$ are n and $n-1$, which are coprime, we see that

$$\deg(f'(x) \pmod p) \ge n - 2.$$

As either $k, \ell \ge 2$ or $\ell \ge 3$, this is a contradiction. Hence we conclude that $q_2 = 1$, hence $b \in \mathbb{Z}$.

Next we show that under the assumptions of the lemma, the multiple roots of $f(x) - b$ are units. Let α be any such root. Then, since $b \in \mathbb{Z}$, α is an algebraic integer. Thus $v_\alpha(x) \in \mathbb{Z}[x]$ and $(v_\alpha(x))^2 \,|\, f(x) - b$ over $\mathbb{Z}$, whence $v_\alpha(x) \,|\, f'(x)$ over $\mathbb{Z}$. As $f'(0) = \pm 1$, our claim follows. ∎

We shall also apply the following information concerning the roots of shifted Littlewood polynomials.

LEMMA 3.6. *Let $f(x)$ be a Littlewood polynomial of degree n and let $b \in \mathbb{Z}$. Then for any root α of $f(x) - b$ with $|\alpha| > 2$ we have*

$$\frac{|\alpha| - 2}{|\alpha| - 1}|\alpha|^n < |b|.$$

Proof. We have

$$|\alpha|^n \le |\alpha|^{n-1} + |\alpha|^{n-2} + \cdots + |\alpha| + 1 + |b| = \frac{|\alpha|^n - 1}{|\alpha| - 1} + |b| < \frac{|\alpha|^n}{|\alpha| - 1} + |b|.$$

From this the statement follows. ∎

Finally, we shall also use the following result from [4].

LEMMA 3.7. *Let $Q(x) \in \mathbb{Z}[x]$ be a non-constant polynomial and r, t be integers with $0 \le r < t$, $t \ge 2$. If all the coefficients of the polynomial $(x-1)^r(Q(x))^t$ belong to $\{-1, 1\}$, then $t = 2$, $r = 1$ and $Q(x)$ is of the form*

$$Q(x) = \pm(x^k + \cdots + x + 1) \tag{10}$$

with some $k \ge 1$. If all the coefficients of the polynomial $(x+1)^r(Q(x))^t$ belong to $\{-1, 1\}$, then $t = 2$, $r = 1$ and $Q(x)$ is of the form

$$Q(-x) = \pm(x^k + \cdots + x + 1) \tag{11}$$

with some $k \ge 1$.

Proof. The first statement is [4, Lemma 3.6]. The second statement follows by the substitution $x \mapsto -x$. ∎

4. Proofs of the theorems

Proof of Theorem 2.1. Let $f(x)$ be given by (4). The bound for m follows from Lemma 3.1, unless $f(x) - b$ is of the shape $f(x) = (x - s)^n$ with $s \in \mathbb{Q}$. Since Lemma 3.5 implies $b \in \mathbb{Z}$, we have $s \in \mathbb{Z}$. However, we get a contradiction with the fact that the coefficient of x^{n-1} is 1 in $f(x) - b$.

Thus, by Lemma 3.1, we may assume that m is fixed. Now our claim follows from Lemma 3.2, except for the following two cases:

(i) $m \geq 2$ is arbitrary and $f(x) - b = (P(x))^r (Q(x))^t$ with $0 \leq r < t$, $t \geq 2$ and $P, Q \in \mathbb{Q}[x]$, $\deg(P) \leq 1$;
(ii) $m = 2$ and $f(x) - b = P(x)(Q(x))^2$ with $P, Q \in \mathbb{Q}[x]$, $\deg(P) = 2$.

Throughout the proof we suppose without loss of generality that P, Q are both monic.

For $n = 4$ a simple computer calculation shows that (i) is impossible, while (ii) can only occur when we have

$$(f(x), b)) = (x^4 + x^3 - x^2 - x \pm 1, \pm 1).$$

Since this possibility is among the exceptional cases (2), we may assume that $n \geq 5$. Lemma 3.5 implies $b \in \mathbb{Z}$, so we infer that $P, Q \in \mathbb{Z}[x]$. We consider cases (i) and (ii) in turn.

Assume first that (i) holds. If $r = 0$ or $P(x)$ is constant then, since the coefficient of x^{n-1} is 1 on the left-hand side, while it is divisible by t on the right-hand side, we get a contradiction. So $r \geq 1$ and $P(x)$ is linear. We write $P(x) = x - s$ with $s \in \mathbb{Z}$. Since either $t \geq 3$ or $\deg(Q) \geq 2$, and the roots of Q are multiple roots of $f(x) - b$, by Lemma 3.5 we obtain $Q(0) = \pm 1$. Further, the same lemma implies that for $r \geq 2$ we have $s = \pm 1$. We apply Lemma 3.7 and obtain a contradiction. We conclude that the statement of Theorem 2.1 holds if $r \geq 2$.

So we may assume that $r = 1$. Comparing the constant terms, we see that $b = \varepsilon_n \pm s$. Lemma 3.6 with $n \geq 5$, $|b| \leq 3$ yields $|s| < 3$.

If $|s| = 1$ then Lemma 3.7 implies that $Q(x)$ is of the form (10) or (11). This leads to the first two options of (2).

If $s = 0$ then comparing the coefficient of x^{n-1} on both sides we get a contradiction: it is 1 on the left-hand side, while it is a multiple of t on the right-hand side.

Hence we are left with $s = \pm 2$. Since s is a root of $f(x) - b$, we have (recall Remark 2)

$$f(s) - b = s^n + s^{n-1} + \varepsilon_2 s^{n-2} + \cdots + \varepsilon_{n-1} s + \varepsilon_n - b = 0. \tag{12}$$

In view of $|s^n + s^{n-1}| \geq 2^{n-1}$, and as by $\varepsilon_n - b = \pm 2$ we have

$$|\varepsilon_2 s^{n-2} + \cdots + \varepsilon_{n-1} s + \varepsilon_n - b| \leq 2^{n-2} + 2^{n-3} + \cdots + 2^1 + 2 = 2^{n-1},$$

the equality (12) is only possible if $s = -2$ and all other terms in (12) have signs opposite to that of s^n. Thus we conclude

$$f(x) - b = x^n + x^{n-1} - x^{n-2} + \cdots + (-1)^{n-2}x + (-1)^{n-1} \cdot 2. \tag{13}$$

Hence we easily get

$$(Q(x))^t = x^{n-1} - x^{n-2} + \cdots .$$

However, this is not possible, since the coefficient of x^{n-2} is not divisible by t. Thus the theorem is true in case (i).

Suppose that (ii) holds. Write $P(x) = x^2 + ux + w$. Recall that $n = \deg(f) \geq 5$; thus now in fact $n \geq 6$. First we clarify the parity of u and w. Taking the equation in (ii) modulo 2 we obtain

$$\begin{aligned} x^n + x^{n-1} + x^{n-2} + x^{n-3} + \cdots \\ \equiv (x^2 + ux + w)(x^{n-2} + \delta_1 x^{n-4} + \delta_2 x^{n-6} + \cdots) \pmod 2. \end{aligned}$$

Here a priori $\delta_1, \delta_2 \in \{0, 1\}$. Comparing the coefficients of $x^{n-1}, x^{n-3}, x^{n-2}$ (in this order) on both sides, we successively find that u is odd, $\delta_1 = 1$ and w is even.

Since $n \geq 6$, Lemma 3.5 implies (as in the case $r \geq 2$ of (i)) that $Q(0) = \pm 1$, and consequently $f(0) - b = w$. Observe that $P(x)$ has a root α with $|\alpha| \geq \sqrt{|w|}$. Since $b = -w \pm 1$, for $n \geq 6$ Lemma 3.6 yields

$$\frac{\sqrt{|w|} - 2}{\sqrt{|w|} - 1}|w|^3 < |w| + 1.$$

This implies $|w| \leq 4$. Hence by the parity condition above, we obtain $w \in \{0, \pm 2, \pm 4\}$. Assume first that $w = 0$. Then $f(0) - b = 0$, and on taking out a factor x the equality in (ii) simplifies to

$$\frac{f(x) - b}{x} = (x + u)(Q(x))^2.$$

Observe that the polynomial on the left-hand side is a Littlewood polynomial. So $u = \pm 1$, and by Lemma 3.7 we obtain (similarly to the case $r = 1$, $s = \pm 1$ of (i)) that $Q(x)$ is of the form (10) or (11). This yields the third option of (2) and

$$\frac{f(x) - b}{x} = (x - 1)(x^k + \cdots + x + 1)^2.$$

From this our claim follows in the case $w = 0$. For the remaining values of w, Lemma 3.6 implies

$$\frac{|\alpha_{1,2}| - 2}{|\alpha_{1,2}| - 1}|\alpha_{1,2}|^6 < |w| + 1 \leq 5$$

for any of

$$\alpha_{1,2} = \frac{-u \pm \sqrt{u^2 - 4w}}{2}$$

with absolute value > 2. A simple calculation shows that $|\alpha_{1,2}| < 2.1$. By a further calculation, both roots are below this bound in absolute value only if $u = 0$ for $w = -4$; $|u| \le 4$ for $w = 4$; $|u| \le 1$ for $w = -2$; $|u| \le 3$ for $w = 2$. Since u must be odd, we are left with the following polynomials:

$$P(x) = x^2 \pm 3x + 4,\ x^2 \pm x + 4,\ x^2 \pm x - 2,\ x^2 \pm 3x + 2,\ x^2 \pm x + 2.$$

We handle these possibilities in turn.

Let α be a root of any of the polynomials $P(x) = x^2 \pm 3x + 4$, $x^2 \pm x + 4$. Then $|\alpha| = 2$, and α is a root of $f(x) - b$. Since the constant term of $f(x) - b$ is 4, we obtain

$$2^n = |\alpha|^n \le |\alpha|^{n-1} + \cdots + |\alpha|^4 + M = 2^n - 16 + M,$$

where

$$M = \max_{\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{-1,1\}} |\varepsilon_3\alpha^3 + \varepsilon_2\alpha^2 + \varepsilon_1\alpha + 4|.$$

However, a computer calculation shows that $M < 16$ for these choices of $P(x)$. Hence these cases cannot occur.

Consider now the polynomials $P(x) = x^2 \pm x - 2$, $x^2 \pm 3x + 2$. Observe that -2 or 2 is a root of these polynomials. Further, the constant term of $f(x) - b$ equals ± 2 in these cases. Thus we get (similar to (13), recall that $f(x)$ is of the form (4), and that n is even)

$$f(x) - b = x^n + x^{n-1} - x^{n-2} + x^{n-3} - x^{n-4} + \cdots - x^2 + x - 2$$

with a root -2. This, in view of the signs of the constant terms, rules out the polynomials $P(x) = x^2 \pm 3x + 2$. In case $P(x) = x^2 \pm x - 2$ we get, since f is a Littlewood polynomial,

$$(Q(x))^2 = x^{n-2} + x^{n-4} + \cdots + x^2 + 1.$$

Then, writing

$$Q(x) = x^{(n-2)/2} + q_1 x^{(n-4)/2} + q_2 x^{(n-6)/2} + \cdots,$$

from the coefficients of x^{n-3} we see that $q_1 = 0$, and then from the coefficients of x^{n-4} that $2q_2 = 1$. This contradicts $Q(x) \in \mathbb{Z}[x]$. So these cases are not possible either.

Thus we are left with $P(x) = x^2 \pm x + 2$.

$$Q(x) = x^k + q_1 x^{k-1} + \cdots + q_{k-1}x + q_k$$

with $n = 2k + 2$. Recall that $q_1, \ldots, q_k \in \mathbb{Z}$ with $q_k = \pm 1$. First we argue that the equality

$$f(x) - b = (x^2 \pm x + 2)(Q(x))^2 \tag{14}$$

implies that $q_1, \dots, q_k$ are all odd. Indeed, if i is the smallest index with q_i even, then the coefficients of x^{2i-1}, x^{2i}, x^{2i+1} would all be even in $(Q(x))^2$, so the coefficient of x^{2i+1} would be even in $f(x)-b$, a contradiction. Expanding the first few coefficients on the right-hand side of (14) we get

$$x^n + x^{n-1} + \varepsilon_2 x^{n-2} + \cdots = x^n + (2q_1 \pm 1)x^{n-1} + (2q_2 + q_1^2 \pm 2q_1 + 2)x^{n-2} + \cdots .$$

Hence, using the fact that q_1 and q_2 are odd, we obtain successively

$$q_1 = 1, \quad P(x) = x^2 - x + 2, \quad q_2 = -1, \quad \varepsilon_2 = -1.$$

Write α for a root of $x^2 - x + 2$. Since α is a root of $f(x) - b$, we obtain

$$|\alpha^n + \alpha^{n-1} - \alpha^{n-2}| \le |\alpha|^{n-3} + \cdots + |\alpha| + |f(0) - b|.$$

Note that $|\alpha| = \sqrt{2}$ and $|\alpha^2 + \alpha - 1| > 6$. Since the constant term $f(0) - b$ of $f(x) - b$ is 2, we obtain

$$6 \cdot (\sqrt{2})^{n-2} < \frac{(\sqrt{2})^{n-2} - 1}{\sqrt{2} - 1} + 1.$$

This gives a contradiction, which shows that $P(x) = x^2 - x + 2$ is also impossible. Hence the theorem is proved. ∎

Proof of Theorem 2.2. Let $f(x)$ be of the form (4). Then Lemma 3.3 implies that $f(x)$ is indecomposable over $\mathbb{Q}$. Thus, if equation (3) has infinitely many solutions in integers x, y, then by Lemma 3.4 we have only two options. Either $g(x)$ is of the form $g(x) = f(T(x))$ with some $T(x) \in \mathbb{Z}[x]$ (in which case (3) clearly has infinitely many integer solutions indeed) or $f(x)$ is of the shape

$$f(x) = AF(ux + w) + B, \tag{15}$$

with some $A, B, u, w \in \mathbb{Q}$, $Au \neq 0$, where F belongs to a standard pair from Table 1. Only the latter case needs more investigation.

Suppose first that $F(x)$ belongs to case I or II of Table 1. Since a Littlewood polynomial cannot be a perfect power of another polynomial, in these cases $G(x)$ is a perfect power of x and $F(x)$ is the other possibility in Table 1. Therefore $f(x)$ is of the shape occurring as (i) or (ii) in the proof of Theorem 2.1 and $g(x) = P(cx+d)$ for some $c, d \in \mathbb{Q}, c \neq 0$. So the statement follows from Theorem 2.1 in cases I and II.

Now assume that we are in case III or IV of Table 1. Then $F(x)$ is a constant multiple of a Dickson polynomial in (15). Clearly, (15) is equivalent to

$$f\left(\frac{x - w}{u}\right) = AD_n(x, \delta) + B \tag{16}$$

with some non-zero $\delta \in \mathbb{Q}$, where n is the degree of f. (Here in case III, A is replaced by another constant.) Recall that $n \geq 4$. Since D_n is either an odd or an even polynomial (depending on the parity of n), comparing the coefficients of x^{n-1} and x^{n-3} in (16), we get

$$-\frac{nw}{u^n} + \frac{1}{u^{n-1}} = 0$$

or

$$-\binom{n}{3}\frac{w^3}{u^n} + \binom{n}{2}\frac{w^2}{u^{n-1}} - \varepsilon_2\frac{nw}{u^{n-2}} + \varepsilon_3\frac{1}{u^{n-3}} = 0,$$

respectively. These equalities imply

$$(3\varepsilon_3 - 3\varepsilon_2 + 1)n^2 - 1 = 0.$$

Hence $n = \pm 1$, which is excluded.

Finally, suppose that $F(x)$ comes from case V of Table 1. The polynomial $(\alpha x^2 - 1)^3$ is an even polynomial of degree 6 and it can be handled and excluded in the same way as the possibilities in cases III and IV. If $F(x) = 3x^4 - 4x^3$, then (15) gives

$$f(x) = A(3(ux + w)^4 - 4(ux + w)^3) + B.$$

A simple calculation shows that $f(x)$ cannot be a Littlewood polynomial. Hence the theorem is proved. ∎

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