ACTA ARITHMETICA
Online First version

On sums of fractional parts

by

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Dedicated to Henryk Iwaniec with gratitude

1. Introduction. For $\alpha \in \mathbb{R}$ let $\{a\} = a - \lfloor a \rfloor$. For α irrational and a nonnegative integer n define

$$S(\alpha, n) = \sum_{k=1}^{n} \left(\{k\alpha\} - \frac{1}{2} \right).$$

This deceptively simple looking sum has been thoroughly studied for a long time (1), but it still presents attractive unsolved problems. It is well known (see e.g. [10, p.104]) that $|S(\alpha, n)|$ is unbounded in n for a fixed irrational α. Since it is obvious that $S(\alpha, n) + S(\beta, n) = 0$ if $\alpha + \beta \in \mathbb{Z}$, a natural question arises: is it possible for

$$|S(\alpha, n) + S(\beta, n)|$$

to be bounded when α is irrational and $\alpha + \beta \in \mathbb{Q}$ is not an integer?

THEOREM 1.1. *Suppose that α is irrational and that $\alpha + \beta$ is rational. Then the values of $|S(\alpha, n) + S(\beta, n)|$ are unbounded in n unless $\alpha + \beta \in \mathbb{Z}$, in which case the value is zero.*

As a consequence, given any (irrational) real quadratic α, we find that $|S(\alpha, n) + S(\alpha', n)|$ is bounded if and only if $\alpha + \alpha' \in \mathbb{Z}$, where α' is the conjugate of α. Under the additional assumption that $\alpha\alpha' = 1$, this consequence was conjectured in [3, Conj. 6.17] in relation to some interesting

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(1) See [10, IX, §2] for a summary of the classical literature on S up until about 1935. A more recent source is [11], especially Chapter 2. An elegant elementary approach was given in [15] (see the Math. Review MR0006753 for some corrections). The book [1] contains a striking central limit theorem for $S(\alpha, n)$ when α is real quadratic.

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problems in symplectic geometry about symplectic embeddings of ellipsoids (see also [12]).

The sum S arises in the problem of counting lattice points in a right triangle whose sides are on the positive axes. This connection is also behind its appearance in [3]. Suppose that $\alpha, \beta > 0$. Consider the counting function of lattice points inside the closed triangle Δ with vertices at $(0,0)$, $(0,\alpha)$ and $(0,\beta)$, when it is scaled by $t > 0$:

$$F(t) = \#(t\Delta \cap \mathbb{Z}^2).$$

A very special case of a well-known result of Ehrhart (see [2]) implies that for integers α, β and *integral* ℓ the function $F(\ell)$ is a quadratic polynomial in ℓ. Explicitly, when $\gcd(\alpha, \beta) = 1$, we have

$$F(\ell) = \frac{\alpha\beta}{2}\ell^2 + \frac{\alpha+\beta+1}{2}\ell + 1. \tag{1.1}$$

For general rational α, β, Ehrhart's result is that (1.1) still holds provided we replace 1 and the coefficient of ℓ by certain periodic functions of ℓ having integral periods. Although they need not be constant, these periodic coefficients are clearly still bounded.

Suppose now that α/β is irrational and define, for any $t > 0$,

$$C(t) = F(t) - \left(\frac{\alpha\beta}{2}t^2 + \frac{1}{2}(\alpha+\beta)t\right). \tag{1.2}$$

By [7, Theorem A1] we have $C(t) = o(t)$. When is $C(\ell)$ bounded for integers ℓ? For certain α, β, the answer follows easily from Theorem 1.1.

COROLLARY 1.2. *Suppose that* $\alpha, \beta = \alpha'$ *are the* (*real quadratic*) *solutions to*

$$ax^2 - bx + b = 0,$$

where $a, b \in \mathbb{Z}^+$ *are such that* $b^2 - 4ab > 0$ *is not a square and* $\gcd(a,b) = 1$. *Then* $|C(\ell)|$ *is bounded if and only if* α, β *are real quadratic integers, in which case* $C(\ell) = 1$.

Proof. For general $\alpha, \beta > 0$ and $m, n \in \mathbb{Z}^+$ we have the identity

$$C\left(\frac{m}{\alpha} + \frac{n}{\beta}\right) = 1 - S\left(\frac{\alpha}{\beta}, n\right) - S\left(\frac{\beta}{\alpha}, m\right).$$

For a proof see [16, Theorem I]. Our assumptions imply that $1/\alpha + 1/\beta = 1$, so

$$C(\ell) = 1 - S\left(\frac{\alpha}{\beta}, \ell\right) - S\left(\frac{\beta}{\alpha}, \ell\right).$$

The result now follows from Theorem 1.1 after noting that α, β are real quadratic integers exactly when $a = 1$, while

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(\alpha+\beta)^2}{\alpha\beta} - 2 = \frac{b}{a} - 2. \ \blacksquare$$

REMARK. Under the assumptions of Corollary 1.2, if α, β are real quadratic integers then for $\ell \in \mathbb{Z}^+$ we have

$$F(\ell) = \frac{b}{2}\ell^2 + \frac{b}{2}\ell + 1,$$

which is an example of the Ehrhart function of a pseudo-integral triangle (see [4]).

2. Proof of Theorem 1.1. The theorem follows without difficulty from a result of Schoißengeier [17], which is a reformulation of one of Oren [13]. These papers give very useful developments of earlier work on local discrepancies of the sequence $\{k\alpha\}$, especially [6, 8, 9, 14]. Here "local" refers to the estimation of the discrepancy from uniform distribution of the sequence when measured with respect to a fixed interval or, more generally, with respect to integration of a fixed function.

Set $D(\alpha,\gamma,n) = S(\alpha+\gamma,n) - S(\alpha,n)$ where α is irrational and γ is rational. We want to show that $|D(\alpha,\gamma,n)|$ is unbounded unless $\gamma \in \mathbb{Z}$. Let $\gamma = p/q \in \mathbb{Q}$ be in reduced form with $q > 1$. We have

$$D(\alpha,\gamma,n) = \sum_{1\le\ell\le n} \left\{\ell\left(\alpha + \frac{p}{q}\right)\right\} - \{\ell\alpha\}.$$

Write $\ell = qk - r$ and $\delta = q\alpha$. We may assume that $1 \le r \le q-1$ and $1 \le k \le m$ if $n = mq - 1$. Note that the terms with $r = 0$ are zero and can be omitted. By splitting into arithmetic progressions modulo q, it follows that

$$D(\alpha,\gamma,n) = \sum_{1\le r<q} \sum_{1\le k\le m} \left\{k\delta - r\alpha - \frac{pr}{q}\right\} - \{k\delta - r\alpha\}.$$

Next apply the following elementary identity for $x \in \mathbb{R}$:

$$\{k\delta + x\} = \{k\delta\} + \chi_{[0,\{-x\})}(k\delta) - \{-x\},$$

where χ is the usual characteristic function made $\mathbb{Z}$-periodic. Thus

$$\begin{aligned} D(\alpha,\gamma,n) &= \sum_{1\le k\le m} \Big(\sum_{1\le r<q} \chi_{[0,\{r\alpha+pr/q\})}(k\delta) - \chi_{[0,\{r\alpha\})}(k\delta)\Big) \\ &\quad - m \sum_{1\le r<q} \left\{r\alpha + \frac{pr}{q}\right\} - \{r\alpha\} \\ &= \sum_{k\le m} f(k\delta) - m\int_0^1 f(x)\,dx, \end{aligned}$$

where f is a periodic step function. It follows from [17, Cor. 3] that $D(\alpha, \gamma, n)$ is bounded if and only if f is in the space of periodic step functions generated by functions of the form $\chi_{I+\mathbb{Z}}(x)$, where $I \subset [0, 1)$ is an interval whose length is in $\mathbb{Z} + q\alpha\mathbb{Z}$. Since α is irrational and $1 \leq r < q$, we see that f is not in this space, proving Theorem 1.1. See Figure 1 for an illustration of a step function f that arises.

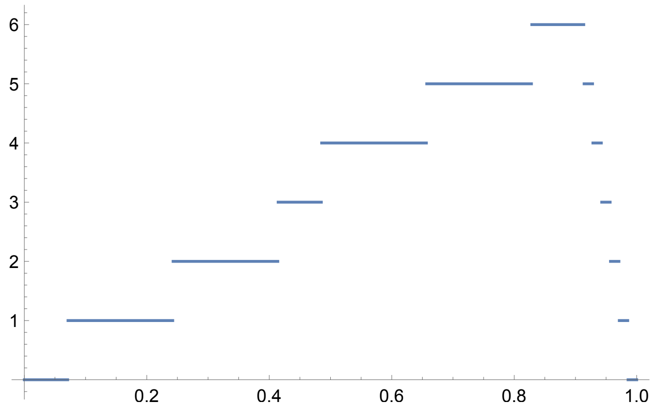

Fig. 1. The step function f when $\alpha = \sqrt{2}$ and $p/q = 4/7$

In case $q = 2$ the above calculation is quite transparent and the result is a consequence of [9] or [6]. For this, assume that $p/q = 1/2$ and that $0 < \alpha < 1/2$. Then

$$D\left(\alpha, \frac{1}{2}, 2m - 1\right) = \sum_{1 \leq k \leq m} \chi_{[\alpha, \alpha+1/2)}(2k\alpha) - \frac{m}{2},$$

which is the local discrepancy of the sequence $\{2k\alpha\}$ for $1 \leq k \leq m$ in $[\alpha, \alpha + 1/2)$. By [9] this is unbounded since $1/2 \notin \mathbb{Z} + 2\alpha\mathbb{Z}$. ∎

Remark. An apparently difficult problem is to give criteria for the one-sided boundedness of $D(\alpha, \gamma, n)$. In particular, the possible one-sided boundedness of C from (1.2) is of interest for the problems of [3] mentioned above. This issue does not seem to have been extensively treated for general local discrepancies. Even simple local discrepancies like that of $\{2k\alpha\}$ in the interval $[\alpha, \alpha + 1/2)$ remain mysterious. Some results are given in [5].

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