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Connections between matrix summability ideals and nonpathological analytic P-ideals

by

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Abstract. We present a new characterization of nonpathological analytic P-ideals that uses intersections of matrix summability ideals. Moreover, we show that the matrix summability ideals are exactly the nonpathological generalized density ideals.

1. Introduction. The class of analytic P-ideals is one of the most popular objects in the study of ideals thanks to Solecki's famous theorem [24], which connects this class of ideals with lower semicontinuous submeasures. Among the analytic P-ideals, one can distinguish four types of ideals that have been the subject of numerous studies (e.g. [1, 5, 8, 9, 10, 15, 18]): Erdős–Ulam, density, matrix summability and generalized density ideals. The reader is referred to [26] for a detailed comparison of these classes and a revised version of Farah's characterization of those density ideals which are Erdős–Ulam ideals.

Out of the four aforementioned classes of ideals, the class of matrix summability ideals is probably the least known one. However, matrix summability itself (recall that a sequence x_n is A-*summable* to L for an infinite matrix $A = (a_{i,k})$ if $\lim_{i\to\infty} \sum_{k=1}^{\infty} a_{i,k}x_k = L$) has been researched at least since Toeplitz's characterization of regular matrix summability methods in [25]. This topic was especially popular in the 1930s, when, for example, Mazur showed (see [2, pp. 71–72] or [3, pp. 44–45]) that for a separable linear subspace $V \subseteq \ell^\infty$, every continuous linear functional $\phi : V \to \mathbb{R}$ is a *matrix summability method*, i.e., there exists an infinite matrix $(a_{i,k})$ such that, for $x \in V$, $\lim_{i\to\infty} \sum_{k=1}^{\infty} a_{i,k}x_k$ makes sense and is equal to $\phi(x)$.

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The issue of relations between matrix summability and ideal convergence (recall that, for an ideal $\mathcal{I}$ on $\mathbb{N}$, a real sequence x_n is *$\mathcal{I}$-convergent* to L if $\{n \in \mathbb{N} : |x_n - L| > \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$) can be traced back to the question posed by Mazur in the Scottish Book (see [22, Problem 5, p. 55]) whether $\mathcal{I}_d$-convergence (see Definition 2.9) is equal to some matrix summability method in the realm of bounded sequences. This problem was solved many years later by Khan and Orhan, who showed in [19, Theorem 2.2] that for any matrix summability ideal $\mathcal{I}$ (thus, in particular, for $\mathcal{I}_d$), $\mathcal{I}$-convergence is some matrix summability method for all bounded sequences (see [11] for more information and an easy observation that this result does not hold for any other ideals).

Before Khan and Orhan proved the above-mentioned result, Fridy and Miller [13, Theorem 4] proved that for each matrix ideal $\mathcal{I}$, $\mathcal{I}$-convergence on ℓ^∞ is the intersection of some matrix summability methods, i.e. there exists a family of matrices $\mathcal{M}$ such that a sequence $x \in \ell^\infty$ is $\mathcal{I}$-convergent to L if and only if it is A-summable to L for each $A \in \mathcal{M}$. Later, Gogola, Mačaj and Visnyai [14, Theorem 4.4] proved that a similar result holds for another family of ideals and they asked [14, Problem 4.6] whether the same holds for every ideal $\mathcal{I}$. This problem was answered negatively by Filipów and Tryba [11] who later [12] characterized when $\mathcal{I}$-convergence coincides with a single summability method, a union or an intersection of such methods, both in the realm of bounded sequences and the realm of all sequences. The most intriguing of these characterizations was [12, Theorem 5.5], which stated that $\mathcal{I}$-convergence is the intersection of some matrix summability methods for all bounded sequences if and only if $\mathcal{I}$ is the intersection of some matrix summability ideals. In [12] one can also find a number of examples of ideals that are intersections of matrix summability ideals and the claim that all nonpathological analytic P-ideals share this property [12, Theorem 5.14]. The authors also ask [12, Question 1] about examples of pathological F_σ or analytic P-ideals that have this property.

In the current paper, we present two main results. The first is that an analytic P-ideal is the intersection of some matrix summability ideals if and only if it is nonpathological. The second is that a generalized density ideal is a matrix summability ideal if and only if it is nonpathological.

Our work on the first main result is a continuation of the line of research in [13, 14, 12]. The results obtained improve [12, Theorem 5.14] and the theorem of Laczkovich and Recław [21, Lemma 11] about extendability of nonpathological analytic P-ideals to matrix summability ideals. This allows us to answer the question posed in [12] about the existence of pathological analytic P-ideals that are intersections of matrix summability ideals and find a new characterization of nonpathological analytic P-ideals. This new characterization may be slightly easier to use than Hrušák's [16] as it might be easier to check whether an ideal can be extended to some Erdős–Ulam ideal than whether it is below some Erdős–Ulam ideal in the Katětov order. Moreover, by using our new results, we may be able to shed some light on the rather unexplored topic of nonpathological F_σ ideals. While the current article is devoted to analytic P-ideals, we can point out that since every nonpathological F_σ ideal can be represented as the intersection of some matrix summability ideals by [12, Theorem 5.7], it follows from the second remark following Corollary 3.9 that all nonpathological F_σ ideals have to be below $\mathcal{I}_d$ in the Katětov order.

Our second main result answers Question 1 from [26] and gives a full characterization of generalized density ideals which are matrix summability ideals. Thus we enrich and almost complete the results presented in [26] about relations between the four classes of ideals mentioned in the first paragraph of this introduction. One open issue remains: to characterize those matrix summability ideals which are density ideals.

Our second result may also have some application to the question when an ideal $\mathcal{I}$ is *representable* in a Banach space X, i.e., when $\mathcal{I}$ equals $\{A \subseteq \mathbb{N} : \sum_{n\in A} f(n)$ is unconditionally convergent in $X\}$ for some function $f : \mathbb{N} \to X$. By [6, Theorem 4.4] an ideal is representable in some Banach space if and only if it is a nonpathological analytic P-ideal. A large part of [6] is devoted to representation of ideals in the Banach space c_0 (see [6, Question 5.10]). The authors show that nonpathological generalized density ideals are representable in c_0 (see [6, Example 4.2]) and that every ideal representable in c_0 that cannot be extended to any summable ideal has to be a generalized density ideal [6, Proposition 5.9]. It follows from the present paper that ideals representable in c_0 which are not contained in any summable ideal have to be matrix summability ideals.

The article is organized as follows. In Section 2 we present the definitions of various properties and classes of ideals used throughout the paper. Section 3 is devoted to showing that an analytic P-ideal can be represented as the intersection of some matrix summability ideals if and only if it is nonpathological, thus partially answering Question 1 from [12]. A number of statements equivalent to that result are also obtained; they are summarized in Corollary 3.12. In Section 4 we answer Question 1 from [26] by showing that an ideal is a matrix summability ideal if and only if it is a nonpathological generalized density ideal.

2. Preliminaries

Definition 2.1. An *ideal on* $\mathbb{N} := \{1, 2, \ldots\}$ (for short, an *ideal*) is a family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ that satisfies the following properties:

(1) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$,
(2) if $A \subseteq B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$,
(3) $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$,
(4) $\mathbb{N} \notin \mathcal{I}$.

We denote by Fin the ideal of all finite subsets of $\mathbb{N}$.

Remark 2.2. We also consider ideals on any infinite countable set by identifying this set with $\mathbb{N}$ via a fixed bijection.

Definition 2.3. For an ideal $\mathcal{I}$ and a set $A \notin \mathcal{I}$ we define the *restriction of the ideal* $\mathcal{I}$ to A by $\mathcal{I}\restriction A = \{B \subseteq \mathbb{N} : B \cap A \in \mathcal{I}\}$.

Definition 2.4. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on X and Y respectively.

We say that $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic* (written $\mathcal{I} \approx \mathcal{J}$) if there exists a bijection $f : X \to Y$ such that $A \in \mathcal{I} \Leftrightarrow f[A] \in \mathcal{J}$ for every $A \subseteq X$.

We say that $\mathcal{I}$ *is below* $\mathcal{J}$ *in the Rudin–Blass order* (written $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$) if there is finite-to-one function $f : Y \to X$ such that for every $A \subseteq X$ we have $A \in \mathcal{I}$ if and only if $f^{-1}[A] \in \mathcal{J}$.

We say that $\mathcal{I}$ and $\mathcal{J}$ are *Rudin–Blass equivalent* (for short, $\leq_{\mathrm{RB}}$-equivalent) if $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ and $\mathcal{J} \leq_{\mathrm{RB}} \mathcal{I}$.

We say that $\mathcal{I}$ *is below* $\mathcal{J}$ *in the Katětov order* (written $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$) if there is a function $f : Y \to X$ such that for every $A \subseteq X$ we have $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$.

We say that $\mathcal{I}$ and $\mathcal{J}$ are *Katětov equivalent* (for short, $\leq_{\mathrm{K}}$-equivalent) if $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ and $\mathcal{J} \leq_{\mathrm{K}} \mathcal{I}$.

Definition 2.5. An ideal $\mathcal{I}$ is *dense* (or tall) if for every infinite $A \subseteq \mathbb{N}$ there is an infinite $B \in \mathcal{I}$ such that $B \subseteq A$.

Definition 2.6. A map $\phi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a *submeasure* on $\mathbb{N}$ if

(1) $\phi(\emptyset) = 0$,
(2) if $A \subseteq B$ then $\phi(A) \leq \phi(B)$,
(3) $\phi(A \cup B) \leq \phi(A) + \phi(B)$.

We say that a (sub)measure ϕ is *concentrated on a set* $A \subseteq \mathbb{N}$ if $\{n \in \mathbb{N} : \phi(\{n\}) > 0\} \subseteq A$ and *lower semicontinuous* if $\phi(A) = \lim_{n\to\infty} \phi(A \cap [1, n])$ for each $A \subseteq \mathbb{N}$. We consider a lower semicontinuous submeasure ϕ to be *nonpathological* if $\phi(A) = \sup\{\mu(A) : \mu \leq \phi,\ \mu \text{ is a measure}\}$ for each $A \subseteq \mathbb{N}$.

Definition 2.7. An ideal $\mathcal{I}$ on X is a *P-ideal* if for every countable family $\mathcal{A} \subseteq \mathcal{I}$ there is $B \in \mathcal{I}$ such that $A \setminus B$ is finite for every $A \in \mathcal{A}$.

Most of the results in this paper concern analytic P-ideals. Therefore, let us recall a well-known characterization of analytic P-ideals.

Theorem 2.8 (Solecki [24]). *The following conditions are equivalent:*

- *$\mathcal{I}$ is an analytic P-ideal.*

- $\mathcal{I} = \text{Exh}(\phi) := \{A \subseteq \mathbb{N} : \lim_{n\to\infty} \phi(A \setminus [1,n]) = 0\}$ *for some lower semicontinuous submeasure* ϕ *such that* $\lim_{n\to\infty} \phi(\mathbb{N} \setminus \{1,\dots,n\}) \neq 0$.

We say that an analytic P-ideal $\mathcal{I}$ is *nonpathological* if there exists a nonpathological, lower semicontinuous submeasure such that $\mathcal{I} = \text{Exh}(\phi)$.

Let us now define several classes of ideals that will be used throughout the paper. Each of these classes has been extensively studied (e.g. [4, 6, 8, 9, 10, 15, 18, 20]), and the relations between them were described in detail in [26]. We will present them in strictly increasing order, i.e. each class of ideals is contained in all of the classes that appear below it.

DEFINITION 2.9 ([18]). Let $f = (f_n)$ be a sequence of nonnegative reals such that $f_1 > 0$, $\sum_{n=1}^{\infty} f_n = \infty$ and $\lim_{i\to\infty} f_n / \sum_{i\le n} f_i = 0$. The family

$$\mathcal{EU}_f = \left\{ A \subseteq \mathbb{N} \ : \lim_{n\to\infty} \frac{\sum_{i\in A, i\le n} f_i}{\sum_{i\le n} f_i} = 0 \right\}$$

is called the *Erdős–Ulam ideal generated by* f. The ideal

$$\mathcal{I}_d = \left\{ A \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{|A \cap \{1,\dots,n\}|}{n} = 0 \right\}$$

of all sets of *asymptotic density zero* is the Erdős–Ulam ideal $\mathcal{EU}_f$ for any constant sequence f.

DEFINITION 2.10 ([10]). Let (I_n) be a sequence of finite, pairwise disjoint intervals in $\mathbb{N}$ and let (μ_n) be a sequence of measures such that each μ_n is concentrated on I_n. Then $\mathcal{I} = \{A \subseteq \mathbb{N} : \lim_{n\to\infty} \mu_n(A) = 0\}$ is called a *density ideal on intervals.*

DEFINITION 2.11 ([9]). Let (A_n) be a sequence of finite, pairwise disjoint sets in $\mathbb{N}$ and let (μ_n) be a sequence of measures such that each μ_n is concentrated on A_n. Then $\mathcal{I} = \{A \subseteq \mathbb{N} : \lim_{n\to\infty} \mu_n(A) = 0\}$ is called a *density ideal on disjoint sets.*

These two classes are not exactly the same, but they are clearly identical up to isomorphism (i.e. every ideal of either class is isomorphic to some ideal from the other class). By *density ideals* one can mean either of these classes, but we will generally assume it is the latter as it is the larger one. However, all results in this paper will be correct and their proofs will stay the same regardless of which of these definitions is used.

DEFINITION 2.12 ([7, 26]). We say that a nonnegative matrix $A = (a_{i,k})$ is *regular* if

(1) $\lim_{i\to\infty} a_{i,k} = 0$ for every $k \in \mathbb{N}$,
(2) $\sup_i \sum_{k\in\mathbb{N}} a_{i,k} < \infty$,
(3) $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = 1$.

Definition 2.13 ([8]). For a nonnegative matrix $A = (a_{i,k})$ we define the family

$$\mathcal{I}(A) = \Big\{B \subseteq \mathbb{N} : \lim_{i\to\infty} \sum_{k\in B} a_{i,k} = 0\Big\}.$$

A set $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is called a *matrix summability ideal* if $\mathcal{I} = \mathcal{I}(A)$ for some nonnegative, regular matrix A. (It is easy to see that $\mathcal{I}$ is indeed an ideal in that case.)

Observe that matrix summability ideals are nonpathological analytic P-ideals as by [4, Proposition 12] every $\mathcal{I}(A)$ is equal to $\mathrm{Exh}(\phi)$, where ϕ is given by $\phi(B) = \sup_{n\in\mathbb{N}} \sum_{k\in B} a_{n,k}$ for every $B \subseteq \mathbb{N}$. There are plenty of nonpathological analytic P-ideals that are not matrix summability ideals, though, as can be seen by e.g. [20, Theorem 4.24].

Definition 2.14 ([10]). Let (I_n) be a sequence of finite, pairwise disjoint intervals in $\mathbb{N}$ and let (ϕ_n) be a sequence of submeasures such that each ϕ_n is concentrated on I_n. Then $\mathcal{I} = \{A \subseteq \mathbb{N} : \lim_{n\to\infty} \phi_n(A) = 0\}$ is called a *generalized density ideal.*

3. Intersections of matrix ideals

Proposition 3.1. *Every matrix summability ideal can be represented as the intersection of some density ideals.*

Proof. Take a matrix summability ideal $\mathcal{I}(A)$. By [11, Lemma 2.28] we can assume that for every $n \in \mathbb{N}$ we have $\sum_{k=1}^{\infty} a_{n,k} = 1$ and there are only finitely many k with $a_{n,k} \neq 0$. Let $B \notin \mathcal{I}(A)$. It is enough to construct a density ideal $\mathcal{I}_B$ with $B \notin \mathcal{I}_B$ and $\mathcal{I}(A) \subseteq \mathcal{I}_B$, because then

$$\mathcal{I}(A) = \bigcap_{B\notin\mathcal{I}(A)} \mathcal{I}_B.$$

Since $B \notin \mathcal{I}(A)$, there is some $\alpha > 0$ and an infinite set $Z \subseteq \mathbb{N}$ such that for every $n \in Z$ we have $\sum_{k\in B} a_{n,k} > \alpha$. Now, we will construct inductively two sequences, (n_j) and (k_j). Let n_1 be the smallest element of Z and let k_1 be the largest $k \in \mathbb{N}$ with $a_{n_1,k} \neq 0$. Suppose we have defined n_j and k_j for some $j \in \mathbb{N}$. Then we take as n_{j+1} the smallest $n \in Z$ greater than n_j such that for every $i \geq n$ we have $\sum_{k\leq k_j} a_{i,k} < \alpha/2$. As before, let k_{j+1} be the largest $k \in \mathbb{N}$ with $a_{n_{j+1},k} \neq 0$.

Now that we have constructed the sequences (n_j) and (k_j), let $I_1 = [1, k_1]$ and $I_j = (k_{j-1}, k_j]$ for $j > 1$. Define the measure μ_j by putting $\mu_j(\{k\}) = a_{n_j,k}$ for $k \in I_j$ and $\mu_j(\{k\}) = 0$ otherwise. Let $\mathcal{I}_B = \{C \subseteq \mathbb{N} : \lim_{j\to\infty} \mu_j(C) = 0\}$. Clearly, $\mathcal{I}_B$ is a density ideal and it remains to show that it has the required properties.

First, notice that $B \notin \mathcal{I}_B$. Indeed, since for every $j \in \mathbb{N}$ we have $\sum_{k \in B} a_{n_j,k} > \alpha$, we obtain

$$\mu_j(B) = \sum_{k \in B \cap I_j} a_{n_j,k} \geq \sum_{k \in B} a_{n_j,k} - \sum_{k \leq k_{j-1}} a_{n_j,k} > \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}.$$

Therefore, $B \notin \mathcal{I}_B$.

To see that $\mathcal{I}(A) \subseteq \mathcal{I}_B$, take $C \in \mathcal{I}(A)$. Then for every $\varepsilon > 0$ there is some $J \in \mathbb{N}$ such that for every $n \geq n_J$ we have $\sum_{k \in C} a_{n,k} < \varepsilon$. It follows that for every $j \geq J$ we get

$$\mu_j(C) \leq \sum_{k \in C} a_{n_j,k} < \varepsilon,$$

thus $C \in \mathcal{I}_B$. ∎

PROPOSITION 3.2. *Every dense density ideal can be represented as the intersection of some Erdős–Ulam ideals.*

Proof. Let $\mathcal{I}$ be a dense density ideal and let $B \notin \mathcal{I}$. Once again, it is enough to construct an Erdős–Ulam ideal $\mathcal{I}_B$ with $B \notin \mathcal{I}_B$ and $\mathcal{I} \subseteq \mathcal{I}_B$. It is easy to see that $\mathcal{I}{\restriction}B$ is a dense density ideal. Therefore, by [26, Proposition 3.18] there exists $C \notin \mathcal{I}{\restriction}B$ such that $\mathcal{I}{\restriction}C$ is an Erdős–Ulam ideal. Since $\mathbb{N} \setminus B \in \mathcal{I}{\restriction}B$, we can assume that $C \subseteq B$. Let $\mathcal{I}_B = \mathcal{I}{\restriction}C$. Obviously, $C \notin \mathcal{I}{\restriction}C$, thus $B \notin \mathcal{I}_B$. It is also easy to see that $\mathcal{I} \subseteq \mathcal{I}_B$ because $\mathcal{I}_B$ is a restriction of $\mathcal{I}$. ∎

PROPOSITION 3.3. *Every density ideal can be represented as the intersection of some Erdős–Ulam ideals.*

Proof. Let $\mathcal{I}$ be a density ideal and take $B \notin \mathcal{I}$. As before, it is enough to find an Erdős–Ulam ideal $\mathcal{I}_B$ with $B \notin \mathcal{I}_B$ and $\mathcal{I} \subseteq \mathcal{I}_B$. If $\mathcal{I}{\restriction}B$ is dense, we can find such an $\mathcal{I}_B$ by Proposition 3.2. If $\mathcal{I}{\restriction}B$ is not dense then we can find an infinite set $A \subseteq B$ such that $\mathcal{I}{\restriction}A = \mathrm{Fin}{\restriction}A$. In this case, simply define the sequence (f_n) by $f_n = 1$ for $n \in A \cup \{1\}$ and $f_n = 0$ otherwise. Take $\mathcal{I}_B = \mathcal{EU}_f$. It is easy to see that $\mathcal{I} \subseteq \mathcal{I}_B$ and $B \supseteq A \notin \mathcal{I}_B$, thus $B \notin \mathcal{I}_B$. ∎

COROLLARY 3.4. *An ideal can be represented as the intersection of some Erdős–Ulam ideals if and only if it can be represented as the intersection of some matrix summability ideals.*

Proof. The 'only if' part follows from the fact that every Erdős–Ulam ideal is a matrix summability ideal. The 'if' part follows from Propositions 3.1 and 3.3. ∎

THEOREM 3.5. *If $\mathcal{I}$ is a nonpathological analytic P-ideal then it can be represented as the intersection of some Erdős–Ulam ideals.*

Proof. By [12, Theorem 5.14] every nonpathological analytic P-ideal can be represented as the intersection of some matrix summability ideals. Thus, the theorem follows from Corollary 3.4. ∎

To show that the implication in the above theorem can be reversed for analytic P-ideals, we will use the following result of Hrušák and Meza-Alcántara.

THEOREM 3.6 ([16, Corollary 5.26], [23, Theorem 3.7.5] or [17, Corollary 4.6]). *Let $\mathcal{I}$ be an analytic P-ideal. Then $\mathcal{I}$ is nonpathological if and only if for every $A \notin \mathcal{I}$ we have $\mathcal{I} \restriction A \leq_{\mathrm{K}} \mathcal{I}_d$.*

LEMMA 3.7. *Let $\mathcal{I}$ be an ideal and let $A \notin \mathcal{I}$. If there exists an Erdős–Ulam ideal $\mathcal{EU}_f$ with $\mathcal{I} \subseteq \mathcal{EU}_f$ and $A \notin \mathcal{EU}_f$ then there exists an Erdős–Ulam ideal $\mathcal{EU}_g$ with $\mathcal{I} \restriction A \subseteq \mathcal{EU}_g$.*

Proof. Let (I_n) be pairwise disjoint intervals in $\mathbb{N}$ and (p_n) be probability measures concentrated on I_n with $\lim_{n\to\infty} \max_{k\in\mathbb{N}} p_n(\{k\}) = 0$, such that $\mathcal{EU}_f = \{B \subseteq \mathbb{N} : \lim_{n\to\infty} p_n(B) = 0\}$ (there are such by [9, Theorem 1.13.3(a)]). Since $A \notin \mathcal{EU}_f$, there exists a sequence (i_n) and some $\alpha > 0$ such that $\lim_{n\to\infty} p_{i_n}(A) = \alpha$.

Consider the sequence (μ_n) of measures defined by $\mu_n(\{k\}) = p_{i_n}(\{k\})$ for $k \in A \cap I_{i_n}$ and $\mu_n(\{k\}) = 0$ otherwise. Then $\lim_{n\to\infty} \mu_n(\mathbb{N}) = \alpha$, μ_n are concentrated on pairwise disjoint intervals I_{i_n} and $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \mu_n(\{k\}) = 0$. Therefore, the measures μ_n fulfill all five conditions of [26, Theorem 3.7], sufficient for $\mathcal{EU}_g = \{B \subseteq \mathbb{N} : \lim_{n\to\infty} \mu_n(B) = 0\}$ to be an Erdős–Ulam ideal.

Notice that since $\mu_n(B) \leq p_{i_n}(B)$ for all $n \in \mathbb{N}$ and $B \subseteq \mathbb{N}$, we have $\mathcal{I} \subseteq \mathcal{EU}_f \subseteq \mathcal{EU}_g$. Moreover, $\mu_n(\mathbb{N} \setminus A) = 0$ for all $n \in \mathbb{N}$, thus $\mathbb{N} \setminus A \in \mathcal{EU}_g$. By combining these two facts we obtain $\mathcal{I} \restriction A \subseteq \mathcal{EU}_g$. ∎

THEOREM 3.8. *Let $\mathcal{I}$ be an analytic P-ideal. Then it can be represented as the intersection of some Erdős–Ulam ideals if and only if $\mathcal{I}$ is nonpathological.*

Proof. The 'if' part is Theorem 3.5. To prove the 'only if' part, take a pathological analytic P-ideal $\mathcal{I}$. By Theorem 3.6, there exists $A \notin \mathcal{I}$ such that $\mathcal{I} \restriction A \not\leq_{\mathrm{K}} \mathcal{I}_d$. Since all Erdős–Ulam ideals are $\leq_{\mathrm{RB}}$-equivalent (and therefore $\leq_{\mathrm{K}}$-equivalent) by [9, Lemma 1.13.10], we find that $\mathcal{I} \restriction A \not\leq_{\mathrm{K}} \mathcal{EU}_f$ for any Erdős–Ulam ideal $\mathcal{EU}_f$. It follows that $\mathcal{I} \restriction A$ is not contained in any Erdős–Ulam ideal. Therefore, by Lemma 3.7 there is no Erdős–Ulam ideal $\mathcal{EU}_f$ containing $\mathcal{I}$ with $A \notin \mathcal{EU}_f$. Hence $\mathcal{I}$ cannot be represented as an intersection of Erdős–Ulam ideals. ∎

Thus, we can partially answer Question 1 from [12] in the following way.

COROLLARY 3.9. *If $\mathcal{I}$ is a pathological analytic P-ideal then it cannot be represented as an intersection of matrix summability ideals.*

Remark 3.10. If an ideal is the intersection of some matrix summability ideals then it does not have to be a nonpathological analytic P-ideal, because, for example, in [12, Theorem 5.23] we can find an ideal that is not a P-ideal, but can be represented as an intersection of matrix summability ideals.

Remark 3.11. By combining Corollary 3.4 with the proof of the 'only if' part of Theorem 3.8 we find that if an ideal $\mathcal{I}$ can be represented as an intersection of matrix summability ideals then $\mathcal{I}\restriction A \leq_{\mathrm{K}} \mathcal{I}_d$ for every $A \notin \mathcal{I}$.

Let us summarize all the results regarding analytic P-ideals mentioned in this section.

Corollary 3.12. *Let $\mathcal{I}$ be an analytic P-ideal. The following are equivalent:*

(1) *$\mathcal{I}$ is nonpathological.*
(2) *For every $A \notin \mathcal{I}$, $\mathcal{I}\restriction A \leq_{\mathrm{K}} \mathcal{I}_d$.*
(3) *For every $A \notin \mathcal{I}$, $\mathcal{I}\restriction A$ is contained in some Erdős–Ulam ideal.*
(4) *$\mathcal{I}$ can be represented as the intersection of some Erdős–Ulam ideals.*
(5) *$\mathcal{I}$ can be represented as the intersection of some density ideals.*
(6) *$\mathcal{I}$ can be represented as the intersection of some matrix summability ideals.*
(7) *$\mathcal{I}$ can be represented as the intersection of some nonpathological analytic P-ideals.*
(8) *$\mathcal{I}$-convergence is the intersection of some matrix summability methods in the realm of all bounded sequences.*

4. Generalized density ideals. The following useful result was essentially proved in [7], but since it was scattered among several theorems and proofs, we present it here with a full proof for completeness.

Proposition 4.1. *Let $A = (a_{i,k})$ be a nonnegative matrix such that*

- $\forall_k \ \lim_{i\to\infty} a_{i,k} = 0$;
- $\limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{i,k} > 0$;
- $\exists_N \ \forall_{i\geq N} \ \sum_{k=1}^{\infty} a_{i,k} < \infty$.

Then there exists a nonnegative regular matrix B such that $\mathcal{I}(A) = \mathcal{I}(B)$, i.e. $\mathcal{I}(A)$ is a matrix summability ideal.

Proof. First, since modifying finitely many rows do not change $\mathcal{I}(A)$, we can suppose that $\sum_{k=1}^{\infty} a_{i,k} < \infty$ for every $i \in \mathbb{N}$. It is also easy to see that $\mathcal{I}(A) = \mathcal{I}(Z)$ for $Z = (z_{i,k})$ given by $z_{i,k} = \min\{a_{i,k}, 1\}$ for all $i, k \in \mathbb{N}$, so we can suppose that $a_{i,k} \leq 1$ for all $i, k, \in \mathbb{N}$. Now, two cases are possible:

(1) $\exists_{M>0} \ \exists_N \ \forall_{i\geq N} \ \sum_{k=1}^{\infty} a_{i,k} \in [1/M, M]$;
(2) $\liminf_{i\to\infty} \sum_{k=1}^{\infty} a_{i,k} = 0$ or $\limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{i,k} = \infty$.

In the first case, we can put $b_{i,k} = a_{i,k}/\sum_{l\in\mathbb{N}} a_{i,l}$. It is easy to see that $B = (b_{i,k})$ is a regular matrix such that $\mathcal{I}(A) = \mathcal{I}(B)$.

In the second case, if $\liminf_{i\to\infty}\sum_{k=1}^{\infty} a_{i,k} > 0$ then we define a matrix $C = (c_{i,k})$ by simply putting $C = A$. If, however, $\liminf_{i\to\infty}\sum_{k=1}^{\infty} a_{i,k}$ is zero, we find a $\delta > 0$ such that the sets $A^- = \{i \in \mathbb{N} : \sum_{k\in\mathbb{N}} a_{i,k} < \delta\}$ and $A^+ = \{i \in \mathbb{N} : \sum_{k\in\mathbb{N}} a_{i,k} \geq \delta\}$ are both infinite; it exists because $\limsup_{i\to\infty}\sum_{k\in\mathbb{N}} a_{i,k} > 0$. Enumerate the elements of A^- increasingly as (s_i) and the elements of A^+ as (t_i). Define $C = (c_{i,k})$ by $c_{i,k} = a_{s_i,k} + a_{t_i,k}$.

To prove that $\mathcal{I}(C) = \mathcal{I}(A)$, take a set $D \subseteq \mathbb{N}$ with $\sum_{k\in D} a_{i,k} > \varepsilon$ for some $\varepsilon > 0$ and infinitely many $i \in \mathbb{N}$. Clearly, there are infinitely many $j \in \mathbb{N}$ with $\sum_{k\in D} c_{j,k} > \varepsilon$, thus $\mathcal{I}(C) \subseteq \mathcal{I}(A)$. On the other hand, if we take a set $D \subseteq \mathbb{N}$ with $\sum_{k\in D} c_{i,k} > \varepsilon$ for some $\varepsilon > 0$ and infinitely many $i \in \mathbb{N}$, then we can find infinitely many $i \in \mathbb{N}$ with $\sum_{k\in D} a_{s_i,k} > \varepsilon/2$ or $\sum_{k\in D} a_{t_i,k} > \varepsilon/2$. In both cases, $D \notin \mathcal{I}(A)$, hence $\mathcal{I}(A) \subseteq \mathcal{I}(C)$.

Now, we have dealt with the subcase $\liminf_{i\to\infty}\sum_{k=1}^{\infty} a_{i,k} = 0$. Observe that if there is $M > 0$ such that $\limsup_{i\to\infty}\sum_{k=1}^{\infty} a_{i,k} \leq M$ then clearly $\limsup_{i\to\infty}\sum_{k=1}^{\infty} c_{i,k} \leq 2M$. As in the first case, we can now put $b_{i,k} = c_{i,k}/\sum_{l\in\mathbb{N}} c_{i,l}$ and it is easy to see that $B = (b_{i,k})$ is a regular matrix such that $\mathcal{I}(B) = \mathcal{I}(C) = \mathcal{I}(A)$.

Thus, we now only need to deal with the subcase $\limsup_{i\to\infty}\sum_{k=1}^{\infty} a_{i,k} = \infty$, which implies that $\limsup_{i\to\infty}\sum_{k=1}^{\infty} c_{i,k} = \infty$. First, notice that since $\liminf_{i\to\infty}\sum_{k=1}^{\infty} c_{i,k} > 0$, there is an $M \in \mathbb{N}$ such that $\sum_{k=1}^{\infty} c_{i,k} \geq 1/M$ for almost all $i \in \mathbb{N}$. Since multiplying all $c_{i,k}$ by M will not change the ideal, we can suppose that $\sum_{k=1}^{\infty} c_{i,k} \geq 1$ for almost all $i \in \mathbb{N}$. Moreover, since for $Z = (z_{i,k})$ given by $z_{i,k} = \min\{c_{i,k}, 1\}$ for all $i, k \in \mathbb{N}$ we get $\mathcal{I}(Z) = \mathcal{I}(C)$, we can assume that $c_{i,k} \leq 1$ for all $i, k, \in \mathbb{N}$.

Now, for every $i \in \mathbb{N}$ let $y_i \in \mathbb{N}$ be such that $\sum_{k>y_i} c_{i,k} < 1/i$ (we can find such since $\sum_{k\in\mathbb{N}} c_{i,k} < \infty$). Then for every $i \in \mathbb{N}$ we define the set $N_i = \{D \subseteq [1, y_i] : \sum_{k\in D} c_{i,k} \in [1, 2)\}$ and take $n_i = \sum_{j\leq i} |N_j|$. Since $\sup_{k\in\mathbb{N}} c_{i,k} \leq 1$ for all $i \in \mathbb{N}$ and $\liminf_{i\to\infty}\sum_{k\in\mathbb{N}} c_{i,k} \geq 1$, N_i is nonempty for almost all i. Enumerate the elements of every N_i in any order as $(D_{n_{i-1}+1}, \ldots, D_{n_i})$, where $n_0 = 0$.

We construct the matrix $B = (b_{j,k})$ in the following way. If $j \in (n_i, n_{i+1}]$ we take $b_{j,k} = c_{i+1,k}/\sum_{k\in D_j} c_{i+1,k}$ for $k \in D_j$ and $b_{j,k} = 0$ otherwise. Then B is a regular matrix as $\sum_{k\in\mathbb{N}} b_{j,k} = 1$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} b_{j,k} = 0$ for every $k \in \mathbb{N}$ (because $\lim_{i\to\infty} c_{i,k} = 0$ for every $k \in \mathbb{N}$).

It is easy to see that $\mathcal{I}(A) = \mathcal{I}(C) \subseteq \mathcal{I}(B)$ since $\sum_{k\in D} b_{j,k} \leq \sum_{k\in D} c_{i+1,k}$ for every $D \subseteq \mathbb{N}$ and $j \in (n_i, n_{i+1}]$. To finish this proof, we need to show that $\mathcal{I}(B) \subseteq \mathcal{I}(C)$. Take $D \notin \mathcal{I}(C)$. Then there is an $\varepsilon \in (0, 1)$ and infinitely many $i \in \mathbb{N}$ such that $\sum_{k\in D} c_{i,k} > \varepsilon$. Therefore, there are also infinitely many $i \in \mathbb{N}$ such that $\sum_{k\in D\cap[1,y_i]} c_{i,k} > \varepsilon/2$. It follows that for every such i

we can find a set $K_i \subseteq D \cap [1, y_i]$ such that $\sum_{k\in K_i} c_{i,k} \in (\varepsilon/2, 2)$, hence there is a set L_i with $K_i \subseteq L_i \subseteq [1, y_i]$ and $\sum_{k\in L_i} c_{i,k} \in [1, 2)$. Since L_i belongs to N_i, there is a $j \in (n_{i-1}, n_i]$ such that $L_i = D_j$ and

$$\sum_{k\in D} b_{j,k} \geq \sum_{k\in K_i} b_{j,k} > \sum_{k\in K_i} \frac{c_{i,k}}{2} > \frac{\varepsilon}{4}.$$

Since there are infinitely many such $i \in \mathbb{N}$, it follows that $D \notin \mathcal{I}(B)$. ∎

REMARK 4.2. Note that the first condition of Proposition 4.1 is equivalent to $\mathrm{Fin} \subseteq \mathcal{I}(A)$, and the second condition is equivalent to $\mathbb{N} \notin \mathcal{I}(A)$.

If the third condition alone does not hold for A then (following [7]) A is called a *semiregular matrix of type* 2, and $\mathcal{I}(A)$ does not have to be a matrix summability ideal. For example, the summable ideal $\mathcal{I}_{1/n} = \{X \subseteq \mathbb{N} : \sum_{n\in X} 1/n < \infty\}$ is equal to $\mathcal{I}(B)$ for some semiregular matrix B of type 2 (see [7, Proposition 4.6]), but $\mathcal{I}_{1/n}$ is not a matrix summability ideal by [11, Proposition 4.11]. Further details about ideals generated by semiregular matrices can be found in [7].

We will need one more simple result that may be known but we could not find it in the literature.

LEMMA 4.3. *If $\mathcal{I}$ is a nonpathological generalized density ideal then there exists a sequence (I_n) of pairwise disjoint intervals in $\mathbb{N}$ and a sequence (ϕ_n) of nonpathological submeasures such that each ϕ_n is concentrated on I_n and $\mathcal{I} = \{A \subseteq \mathbb{N} : \lim_{n\to\infty} \phi_n(A) = 0\}$.*

Proof. It is enough to take $\phi_n(A) = \phi(A \cap I_n)$, where ϕ is a nonpathological submeasure such that $\mathcal{I} = \mathrm{Exh}(\phi)$. ∎

Finally, we can now prove the main result of this section and thus answer Question 1 from [26].

THEOREM 4.4. *If $\mathcal{I}$ is a nonpathological generalized density ideal then $\mathcal{I}$ is a matrix summability ideal.*

Proof. Let (I_n) be pairwise disjoint intervals in $\mathbb{N}$ and (ϕ_n) be submeasures concentrated on I_n such that $\mathcal{I} = \{A \subseteq \mathbb{N} : \lim_{n\to\infty} \phi_n(A) = 0\}$. Since $\mathcal{I}$ is nonpathological, we can assume by the previous lemma that each ϕ_n is nonpathological. Moreover, we can assume that $\phi_n(A) < \infty$ for every $A \subseteq \mathbb{N}$ since each ϕ_n has finite support.

For every $n \in \mathbb{N}$ take $j_n = \sum_{i=1}^{n} |\mathcal{P}(I_i) \setminus \{\emptyset\}|$ and enumerate the elements of every $\mathcal{P}(I_n) \setminus \{\emptyset\}$ in any order as $(B_{j_{n-1}+1}, \ldots, B_{j_n})$, where $j_0 = 0$. Since each ϕ_n is nonpathological, we can see that for every nonempty $B_i \subseteq I_n$ there exists a measure μ_i such that $\mu_i \leq \phi_n$ and $\mu_i(B_i) \geq \phi_n(B_i)/2$.

We can now proceed with the construction of a matrix $C = (c_{i,k})$ (not necessarily regular) such that $\mathcal{I}(C) = \mathcal{I}$. For every $i \in \mathbb{N}$ we take $c_{i,k} = \mu_i(\{k\})$ if $k \in B_i$ and put $c_{i,k} = 0$ otherwise. We will show that $\mathcal{I}(C) = \mathcal{I}$.

If $A \in \mathcal{I}$ then $\lim_{n\to\infty} \phi_n(A \cap I_n) = 0$. Since for every $i \in [j_{n-1}+1, j_n]$ we have $\mu_i \le \phi_n$ and $\sum_{k\in A} c_{i,k} = \mu_i(A \cap I_n)$, we obtain $\lim_{i\to\infty} \sum_{k\in A} c_{i,k} = 0$, thus $A \in \mathcal{I}(C)$.

If $A \notin \mathcal{I}$ then there exists an $\varepsilon > 0$ and an infinite set $F \subseteq \mathbb{N}$ such that for all $n \in F$ we have $\phi_n(A) \ge \varepsilon$. Now, for every $n \in F$ we find $i \in [j_{n-1}+1, j_n]$ such that $B_i = A \cap I_n$. Then

$$\mu_i(A \cap I_n) = \mu_i(B_i) \ge \phi_n(B_i)/2 \ge \varepsilon/2.$$

Therefore, for infinitely many $i \in \mathbb{N}$ we have $\sum_{k\in A} c_{i,k} = \mu_i(A \cap I_n) \ge \varepsilon/2$, hence $\limsup_{i\to\infty} \sum_{k\in A} c_{i,k} \ge \varepsilon/2$ and $A \notin \mathcal{I}(C)$.

The matrix C fulfills the conditions of Proposition 4.1 as $\mathrm{Fin} \subseteq \mathcal{I}(C)$, $\mathbb{N} \notin \mathcal{I}(C)$ and for every $i \in \mathbb{N}$ we have $n \in \mathbb{N}$ with $\sum_{k=1}^{\infty} c_{i,k} \le \phi_n(\mathbb{N}) < \infty$. Hence there exists a regular matrix B such that $\mathcal{I}(B) = \mathcal{I}(C) = \mathcal{I}$. ■

COROLLARY 4.5. *$\mathcal{I}$ is a nonpathological generalized density ideal if and only if $\mathcal{I}$ is a matrix summability ideal.*

References

[1] M. Balcerzak, P. Das, M. Filipczak, and J. Swaczyna, *Generalized kinds of density and the associated ideals*, Acta Math. Hungar. 147 (2015), 97–115.

[2] S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne 1, Warszawa, 1932.

[3] S. Banach, *Theory of Linear Operations*, North-Holland Math. Library 38, North-Holland, 1987.

[4] A. Bartoszewicz, P. Das and Sz. Głąb, *On matrix summability of spliced sequences and A-density of points*, Linear Algebra Appl. 487 (2015), 22–42.

[5] P. Borodulin-Nadzieja and B. Farkas, *Analytic P-ideals and Banach spaces*, J. Funct. Anal. 279 (2020), art. 108702, 31 pp.

[6] P. Borodulin-Nadzieja, B. Farkas and G. Plebanek, *Representations of ideals in Polish groups and in Banach spaces*, J. Symbolic Logic 80 (2015), 1268–1289.

[7] P. Das, R. Filipów and J. Tryba, *A note on nonregular matrices and ideals associated with them*, Colloq. Math. 159 (2020), 29–45.

[8] L. Drewnowski and P. Paúl, *The Nikodým property for ideals of sets defined by matrix summability methods*, Rev. R. Acad. Cienc. Exactas Fís. Nat. 94 (2000), 485–503.

[9] I. Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. 148 (2000), no. 702, xvi+177 pp.

[10] I. Farah, *How many Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are there?* Illinois J. Math. 46 (2002), 999–1033.

[11] R. Filipów and J. Tryba, *Ideal convergence versus matrix summability*, Studia Math. 245 (2019), 101–127.

[12] R. Filipów and J. Tryba, *Representation of ideal convergence as a union and intersection of matrix summability methods*, J. Math. Anal. Appl. 484 (2020), art. 123760, 21 pp.

[13] J. A. Fridy and H. I. Miller, *A matrix characterization of statistical convergence*, Analysis 11 (1991), 59–66.

[14] J. Gogola, M. Mačaj and T. Visnyai, *On $\mathcal{I}_c^{(q)}$-convergence*, Ann. Math. Inform. 38 (2011), 27–36.
[15] F. Hernández-Hernández and M. Hrušák, *Cardinal invariants of analytic P-ideals*, Canad. J. Math. 59 (2007), 575–595.
[16] M. Hrušák, *Combinatorics of filters and ideals*, in: Set Theory and its Applications, Contemp. Math. 533, Amer. Math. Soc., Providence, RI, 2011, 29–69.
[17] M. Hrušák, *Katětov order on Borel ideals*, Arch. Math. Logic 56 (2017), 831–847.
[18] W. Just and A. Krawczyk, *On certain Boolean algebras $\mathcal{P}(\omega)/I$*, Trans. Amer. Math. Soc. 285 (1984), 411–429.
[19] M. K. Khan and C. Orhan, *Matrix characterization of A-statistical convergence*, J. Math. Anal. Appl. 335 (2007), 406–417.
[20] A. Kwela and P. Leonetti, *Density-like and generalized density ideals*, J. Symbolic Logic 87 (2022), 228–251.
[21] M. Laczkovich and I. Recław, *Ideal limits of sequences of continuous functions*, Fund. Math. 203 (2009), 39–46.
[22] R. D. Mauldin (ed.), *The Scottish Book*, 2nd ed., Birkhäuser/Springer, Cham, 2015.
[23] D. Meza-Alcántara, *Ideals and filters on countable sets*, Ph.D. thesis, Universidad Nacional Autónoma de México, 2009.
[24] S. Solecki, *Analytic ideals and their applications*, Ann. Pure Appl. Logic 99 (1999), 51–72.
[25] O. Toeplitz, *Über allgemeine lineare Mittelbildungen*, Prace Matematyczno-Fizyczne 22 (1911), 113–119.
[26] J. Tryba, *Different kinds of density ideals*, J. Math. Anal. Appl. 498 (2021), art. 124930, 18 pp.

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