STUDIA MATHEMATICA
Online First version

A remark on non-commutative L^p-spaces

by

Shinya Kato and Yoshimichi Ueda (Nagoya)

Abstract. We describe the Haagerup and the Kosaki non-commutative L^p-spaces associated with a tensor product von Neumann algebra $M_1 \bar{\otimes} M_2$ in terms of ones associated with M_i and the usual tensor products of unbounded operators. The descriptions are then shown to be useful in the quantum information theory based on operator algebras.

1. Introduction. Quantum information theory (QIT for short) can be developed in the infinite-dimensional (even non-type I) setup with the help of operator algebras (such a general framework is necessary for quantum field theory for example), although QIT is usually discussed in the finite-dimensional setup. In the finite-dimensional setup, the primary objects in QIT are density matrices, which no longer make sense in the non-type I setup. However, Haagerup's theory of non-commutative L^p-spaces (see [18]) allows us to have a certain counterpart of density matrices; actually, the so-called Haagerup correspondence $\varphi \mapsto h_\varphi$ (the operator h_φ is sometimes denoted by φ itself) between the normal functionals and a class of τ-measurable operators gives the right counterpart of density matrices in the non-type I setting.

In QIT, tensor products of systems (i.e., systems consisting of independent subsystems) naturally emerges, and hence it is desirable to clarify how Haagerup non-commutative L^p-spaces behave under von Neumann algebra tensor products. In the commutative setup, the answer is simply $L^p(\mu_1 \otimes \mu_2) = L^p(\mu_1, L^p(\mu_2)) = L^p(\mu_2, L^p(\mu_1))$ with natural identifications, by utilizing the concept of vector-valued L^p-spaces. However, that concept has not been established yet in full generality in the non-commutative setting.

The purpose of this short note is to give some descriptions of the Haagerup and the Kosaki non-commutative L^p-spaces associated with a tensor product

2020 *Mathematics Subject Classification*: Primary 46L51; Secondary 81R15.
Key words and phrases: non-commutative L^p-space, tensor product, α-z-Rényi divergence.
Received 24 July 2023; revised 10 September 2023.
Published online 27 March 2024.

DOI: 10.4064/sm230724-11-10

© Instytut Matematyczny PAN, 2024

von Neumann algebra; see Theorem 3.3 and Corollary 3.4. Those descriptions are rather natural but, to the best of our knowledge, have not been given so far. We remark that a similar but abstract result based on the interpolation method was given by Junge [10]. On the other hand, the descriptions we will give depend on the so-called Takesaki duality [17] and are provided by means of tensor products of unbounded operators. Consequently, our descriptions are really concrete, with multiplicativity of the norm. An immediate consequence of those descriptions is a natural proof of the additivity of sandwiched Rényi divergences in the non-type I setup, due to Berta et al. [3] and Jenčová [8, 9]. We remark that the additivity was claimed by Berta et al. in a different approach to non-commutative L^p-spaces (see [3, p. 1860]) and also confirmed by Hiai and Mosonyi [6, (3.16)] in the injective or AFD von Neumann algebra case.

2. Preliminaries. The basic references of this short note are [16] (on modular theory), [18] (on Haagerup non-commutative L^p-spaces), and [13] (on Kosaki non-commutative L^p-spaces), but the reader can find concise expositions of those topics in [4, Appendix A] and its expansion [5].

Let M be a von Neumann algebra. Choose a faithful semifinite normal weight φ on M. The *continuous core* of M is the crossed product $\widetilde{M} := M \bar{\rtimes}_{\sigma^\varphi} \mathbb{R}$. Let $\theta^M : \mathbb{R} \curvearrowright \widetilde{M}$ be the *dual action*, which is characterized by

$$\theta^M_s \circ \pi_\varphi = \pi_\varphi, \quad \theta^M_s(\lambda^\varphi(t)) = e^{-its}\lambda^\varphi(t) \tag{2.1}$$

for all $s, t \in \mathbb{R}$, where $\pi_\varphi : M \to M \bar{\rtimes}_{\sigma^\varphi} \mathbb{R}$ and $\lambda^\varphi : \mathbb{R} \to \widetilde{M}$ denote the canonical injective normal $*$-homomorphism from M and the canonical unitary representation of $\mathbb{R}$ into $\widetilde{M}$ that is generated by the $\pi_\varphi(a)$ and $\lambda^\varphi(t)$ as a von Neumann algebra. In what follows, we will identify $a = \pi_\varphi(a)$ and $M = \pi_\varphi(M)$ when no confusion is possible. Note the covariance relation

$$\lambda^\varphi(t)a = \sigma^\varphi_t(a)\lambda^\varphi(t), \qquad t \in \mathbb{R},\, a \in M. \tag{2.2}$$

We remark that $(\widetilde{M}, \theta^M)$ is known to be independent of the choice of φ up to conjugacy.

The canonical trace τ_M on $\widetilde{M}$ is a faithful semifinite normal tracial weight uniquely determined by

$$[D\widetilde{\varphi} : D\tau_M]_t = \lambda^\varphi(t), \qquad t \in \mathbb{R}, \tag{2.3}$$

where $[D\widetilde{\varphi} : D\tau_M]_t$ is Connes' Radon–Nikodym cocycle of $\widetilde{\varphi}$ with respect to τ_M. Here, $\widetilde{\varphi}$ is the *dual weight* of φ defined by

$$\widetilde{\varphi} := \widehat{\varphi} \circ T_M, \tag{2.4}$$

where $\widehat{\varphi}$ is the canonical extension of φ to the *extended positive part* $\widehat{M}_+$

(see, e.g., [16, §11]) and $T_M : \widetilde{M}_+ \to \widehat{M}_+$ is the operator-valued weight

$$T_M(a) := \int_{\mathbb{R}} \theta_t^M(a)\,dt, \qquad a \in \widetilde{M}_+. \tag{2.5}$$

In what follows, we denote by $s(\psi)$ the *support projection* of a semifinite normal weight ψ. We also use Connes' Radon–Nikodym cocycle with a general (not necessarily faithful) semifinite normal weight on the left-hand side; see [16, §3].

The next lemma immediately follows from the construction of Connes' Radon–Nikodym derivatives (see [16, §3]) together with [16, §2.22, (1)].

Lemma 2.1. *Let ψ be a semifinite normal weight on M and ψ' be another semifinite normal weight on M such that $s(\psi') = 1 - s(\psi)$. Then $\chi := \psi + \psi'$ is a faithful semifinite normal weight on M and*

$$[D\psi : D\varphi]_t = s(\psi)[D\chi : D\varphi]_t, \qquad t \in \mathbb{R}.$$

The *Haagerup correspondence* $\psi \mapsto h_\psi$ is a bijection from the set of all semifinite normal weights on M onto the positive self-adjoint operators h affiliated with $\widetilde{M}$, satisfying $\theta_s^M(h) = e^{-s}h$ for every $s \in \mathbb{R}$. Its construction will appear in the proof of Lemma 2.2 below.

Lemma 2.2. *Let ψ be a semifinite normal weight on M and let*

$$\widetilde{\psi} := \widehat{\psi} \circ T_M$$

be its dual weight. Then

$$[D\widetilde{\psi} : D\tau_M]_t = [D\psi : D\varphi]_t\,\lambda^\varphi(t) \quad \text{for every } t \in \mathbb{R},$$

and the Haagerup correspondence h_ψ is uniquely determined by

$$h_\psi^{it} = [D\psi : D\varphi]_t\,\lambda^\varphi(t), \qquad t \in \mathbb{R}, \tag{2.6}$$

where h_ψ^{it} is the operator $f_t(h_\psi)$ with

$$f_t(\lambda) := \begin{cases} \lambda^{it} = e^{it\log\lambda} & (\lambda > 0), \\ 0 & (\lambda = 0). \end{cases}$$

Proof. Let $\chi = \psi + \psi'$ be as in Lemma 2.1. Then $\widetilde{\chi} = \widetilde{\psi} + \widetilde{\psi'}$ and moreover $s(\widetilde{\psi}) = s(\psi)$ and $s(\widetilde{\psi'}) = s(\psi')$ by [18, Lemma 1(2)(c)]. By Lemma 2.1 we observe that

$$\begin{aligned} [D\psi : D\varphi]_t &= s(\psi)[D\chi : D\varphi]_t, \\ [D\widetilde{\psi} : D\widetilde{\varphi}]_t &= s(\psi)[D\widetilde{\chi} : D\widetilde{\varphi}]_t, \\ [D\widetilde{\psi} : D\tau_M]_t &= s(\psi)[D\widetilde{\chi} : D\tau_M]_t, \end{aligned}$$

for every $t \in \mathbb{R}$. By the chain rule for Connes' Radon–Nikodym cocycles, we have

$$\begin{aligned}[D\widetilde{\psi} : D\tau_M]_t &= s(\psi)[D\widetilde{\chi} : D\tau_M]_t = s(\psi)[D\widetilde{\chi} : D\widetilde{\varphi}]_t\,[D\widetilde{\varphi} : D\tau_M]_t \\ &= [D\widetilde{\psi} : D\widetilde{\varphi}]_t\,\lambda^{\varphi}(t)\end{aligned}$$

for every $t \in \mathbb{R}$. By [16, Theorem 11.9],

$$[D\widetilde{\psi} : D\widetilde{\varphi}]_t = s(\psi)[D\widetilde{\chi} : D\widetilde{\varphi}]_t = s(\psi)[D\chi : D\varphi]_t = [D\psi : D\varphi]_t$$

for every $t \in \mathbb{R}$. Consequently,

$$[D\widetilde{\psi} : D\tau_M]_t = [D\psi : D\varphi]_t\lambda^{\varphi}(t)$$

for every $t \in \mathbb{R}$.

The h_ψ is defined to be the Radon–Nikodym derivative of $\widetilde{\psi}$ with respect to the canonical trace τ_M, that is, $\widetilde{\psi} = \tau_M(h_\psi\,\cdot\,)$ in the sense of [18, Lemma 2] or [16, §4.4]. By [16, Corollary 4.8] we have $[D\widetilde{\psi} : D\tau_M]_t = h_\psi^{it}$. Hence we get (2.6), and it is obvious that equation (2.6) characterizes h_φ thanks to Stone's theorem. ∎

The Haagerup non-commutative L^p-space $L^p(M)$, $0 < p \le \infty$, is defined to be all τ_M-measurable operators h affiliated with $\widetilde{M}$ such that $\theta_t^M(h) = e^{-t/p}h$ for all $t \in \mathbb{R}$. The Haagerup correspondence $\psi \mapsto h_\psi$ induces a bijective linear isomorphism between the predual M_* and $L^1(M)$, and the 'trace functional' $\mathrm{tr} : L^1(M) \to \mathbb{C}$ is defined by $\mathrm{tr}(h_\psi) := \psi(1)$ for any $\psi \in M_*$. The (quasi-)norm $\|x\|_p$ of an $x \in L^p(M)$ is defined to be $\mathrm{tr}(|x|^p)^{1/p}$, where $|x|^p$ can be shown to lie in $L^1(M)$. With the 1-norm $\|\cdot\|_1$ the linear isomorphism $M_* \cong L^1(M)$ clearly becomes an isometric isomorphism. For the details of Haagerup's theory we refer to [18].

Here is another lemma, which is probably a known fact, but we give its proof for the sake of completeness.

Lemma 2.3. *Assume that $p \ge 1$ and φ is a faithful normal positive linear functional so that M must be σ-finite. Let $\mathfrak{A} \subset M$ be a σ-weakly dense $*$-subalgebra. Then $\mathfrak{A}h_\varphi^{1/p}$ is dense in $L^p(M)$.*

Proof. We will use the (left) Kosaki non-commutative L^p-space $L^p(M,\varphi)$ with norm $\|\cdot\|_{p,\varphi}$. It is the complex interpolation space $C_{1/p}(Mh_\varphi, L^1(M))$, where the embedding $M \ni a \mapsto ah_\varphi \in L^1(M)$ gives a compatible pair with norm $Mh_\varphi \ni ah_\varphi \mapsto \|ah_\varphi\|_\infty := \|a\|_M$ (operator norm) for $a \in M$. By [13, Theorem 9.1] we have $L^p(M,\varphi) = L^p(M)h_\varphi^{1/q} \subset L^1(M)$ with $1/p + 1/q = 1$.

For a given $a \in M$ the Kaplansky density theorem enables us to choose a net $a_\lambda \in \mathfrak{A}$ in such a way that $\|a_\lambda\|_M \le \|a\|_M$ for all λ and $a_\lambda \to a$ in the

σ-weak topology. By complex interpolation theory, we obtain

$$\begin{aligned}\|a_\lambda h_\varphi - a h_\varphi\|_{p,\varphi} &= \|(a_\lambda - a)h_\varphi\|_{p,\varphi} \le \|(a_\lambda - a)h_\varphi\|_\infty^{1/q}\|(a_\lambda - a)h_\varphi\|_1^{1/p} \\ &= \|a_\lambda - a\|_M^{1/q}\|h_{(a_\lambda - a)\varphi}\|_1^{1/p} = \|a_\lambda - a\|_M^{1/q}\|(a_\lambda - a)\varphi\|_{M_*}^{1/p} \\ &\le (2\|a\|_M)^{1/q}\|\varphi\|^{1/(2p)}\varphi((a_\lambda - a)^*(a_\lambda - a))^{1/(2p)} \to 0.\end{aligned}$$

Therefore, $\mathfrak{A}h_\varphi$ is dense in $L^p(M,\varphi) = L^p(M)h_\varphi^{1/q}$, because so is Mh_φ thanks to a general fact on complex interpolation spaces. Hence, for each $x \in L^p(M)$ there exists a sequence $a_n \in \mathfrak{A}$ such that $\|a_n h_\varphi - x h_\varphi^{1/q}\|_{p,\varphi} \to 0$ as $n \to \infty$. Since $a_n h_\varphi = (a_n h_\varphi^{1/p})h_\varphi^{1/q}$ and using [13, (21)] (with $\eta = 0$ there), we conclude that $\|a_n h_\varphi^{1/p} - x\|_p \to 0$ as $n \to \infty$ so that $\mathfrak{A}h_\varphi^{1/p}$ is dense in $L^p(M)$. ∎

3. Main results. Let M_i, $i = 1, 2$, be von Neumann algebras. For each $i = 1, 2$, we choose a faithful semifinite normal weight φ_i on M_i. Let

$$\widetilde{M_i} := M_i \bar{\rtimes}_{\sigma^{\varphi_i}} \mathbb{R}, \quad \widetilde{M_1 \bar{\otimes} M_2} := (M_1 \bar{\otimes} M_2) \bar{\rtimes}_{\sigma^{\varphi_1 \bar{\otimes} \varphi_2}} \mathbb{R}$$

be the continuous cores of M_i, $i = 1, 2$, and $M_1 \bar{\otimes} M_2$ together with the dual actions $\theta^{(i)} := \theta^{M_i} : \mathbb{R} \curvearrowright \widetilde{M_i}$, $i = 1, 2$, and $\theta := \theta^{M_1 \bar{\otimes} M_2} : \mathbb{R} \curvearrowright \widetilde{M_1 \bar{\otimes} M_2}$.

The next fact is known in the structure analysis of type III factors. The fact is especially known among specialists on type III factors as a key tool to compute invariants such as flows of weights for tensor product type III factors.

LEMMA 3.1 (Joint flow). *We have an identification*

$$\begin{aligned}\widetilde{M_1 \bar{\otimes} M_2} &= (\widetilde{M_1} \bar{\otimes} \widetilde{M_2})^{(\theta^{(1)}_{-t} \bar{\otimes} \theta^{(2)}_t, \mathbb{R})} \\ &= \{x \in \widetilde{M_1} \bar{\otimes} \widetilde{M_2}\,;\, \theta^{(1)}_{-t} \bar{\otimes} \theta^{(2)}_t(x) = x \text{ for all } t \in \mathbb{R}\}\end{aligned}$$

by

$$\begin{aligned}\pi_{\varphi_1 \bar{\otimes} \varphi_2}(a \otimes b) &= \pi_{\varphi_1}(a) \otimes \pi_{\varphi_2}(b), \quad a \in M_1,\ b \in M_2, \\ \lambda^{\varphi_1 \bar{\otimes} \varphi_2}(t) &= \lambda^{\varphi_1}(t) \otimes \lambda^{\varphi_2}(t), \quad t \in \mathbb{R}.\end{aligned}$$

Via this identification,

$$\theta_t = (\theta^{(1)}_t \bar{\otimes} \mathrm{id})\restriction_{\widetilde{M_1 \bar{\otimes} M_2}} = (\mathrm{id} \bar{\otimes} \theta^{(2)}_t)\restriction_{\widetilde{M_1 \bar{\otimes} M_2}}, \quad t \in \mathbb{R}.$$

Proof. This follows from the formula $\sigma_t^{\varphi_1 \bar{\otimes} \varphi_2} = \sigma_t^{\varphi_1} \bar{\otimes} \sigma_t^{\varphi_2}$ and [16, Theorem 21.8] that originates in [17]. Let us explain how to apply [16, Theorem 21.8] to our problem.

Let $G := \mathbb{R}^2 > H := \{(t,t);\, t \in \mathbb{R}\}$, a closed subgroup, and define $\sigma_g := \sigma_{t_1}^{\varphi_1} \bar{\otimes} \sigma_{t_2}^{\varphi_2}$ for $g = (t_1, t_2) \in G$. Then we have an action $\sigma : G \curvearrowright M_1 \bar{\otimes} M_2$, and its restriction to H is the modular action $\sigma_t^{\varphi_1\bar\otimes\varphi_2} = \sigma_t^{\varphi_1} \bar\otimes \sigma_t^{\varphi_2}$. Thus,

$$\widetilde{M_1} \bar\otimes \widetilde{M_2} = (M_1 \bar\otimes M_2) \bar\rtimes_\sigma G \supset (M_1 \bar\otimes M_2) \bar\rtimes_\sigma H = \widetilde{M_1 \bar\otimes M_2},$$

where

$$(\pi_{\varphi_1}(a) \otimes \pi_{\varphi_2}(b))(\lambda^{\varphi_1}(t_1) \otimes \lambda^{\varphi_2}(t_2)) = \pi_\sigma(a_1 \otimes a_2)\lambda^\sigma(t_1, t_2),$$
$$a \in M_1,\ b \in M_2,\ (t_1, t_2) \in G,$$

in the first identity, the inclusion is the natural one, and

$$\pi_\sigma(a_1 \otimes a_2)\lambda^\sigma(t, t) = \pi_{\varphi_1\bar\otimes\varphi_2}(a_1 \otimes a_2)\lambda^{\varphi_1\bar\otimes\varphi_2}(t),$$
$$a \in M_1,\ b \in M_2,\ (t, t) \in H,$$

in the second identity. Here, $\pi_\sigma : M_1 \bar\otimes M_2 \to (M_1 \bar\otimes M_2) \bar\rtimes_\sigma G$ and $\lambda^\sigma : G \to (M_1 \bar\otimes M_2) \bar\rtimes_\sigma G$ denote the canonical injective normal $*$-homomorphism and the canonical unitary representation, respectively. We have $\widehat{G} = G$ with the dual pairing $\langle (t_1, t_2), (t_1', t_2')\rangle := e^{i(t_1t_1'+t_2t_2')}$ between G and its copy, and $\widehat{H}$ becomes $\{(-t, t);\ t \in \mathbb{R}\}$ in G. Moreover, the dual action $\hat\sigma_g$ with $g = (t_1, t_2) \in G$ is given by $\theta^{(1)}_{t_1} \bar\otimes \theta^{(2)}_{t_2}$ via the above identification. Hence, the first assertion immediately follows by [16, Theorem 21.8]. Then the desired identity for the dual action θ_t can be easily confirmed by investigating its behavior on the canonical generators. ■

In what follows, we use the description in Lemma 3.1 of the continuous core of $M_1 \bar\otimes M_2$ equipped with the dual action θ.

We remark that $\tau_{M_1\bar\otimes M_2}$ cannot be identified with a restriction of the tensor product trace $\tau_{M_1} \bar\otimes \tau_{M_2}$. However, $\tau_{M_1\bar\otimes M_2}$ is characterized by

$$\big[D\widetilde{\varphi_1 \bar\otimes \varphi_2} : D\tau_{M_1\bar\otimes M_2}\big]_t = \lambda^{\varphi_1}(t) \otimes \lambda^{\varphi_2}(t), \quad t \in \mathbb{R}, \tag{3.1}$$

in the description of Lemma 3.1. This is indeed a key fact in the discussion below.

Fix a $p \in (0, \infty]$. Choose a pair $(x_1, x_2) \in L^p(M_1) \times L^p(M_2)$, whose entries can be regarded as unbounded operators on Hilbert spaces $\mathcal{H}_i$, $i = 1, 2$, on which $\widetilde{M_i}$ are constructed. Let $x_i = v_i|x_i|$, $i = 1, 2$, be their polar decompositions. Then $v_i \in M_i$ and $|x_i|^p \in L^1(M_i)$ for each $i = 1, 2$. Then, for each $i = 1, 2$, there is a unique $\psi_i \in (M_i)_*^+$ such that $h_{\psi_i} = |x_i|^p$.

Lemma 3.2. *The following hold true:*

(1) $x_1\bar\otimes x_2 = (v_1\otimes v_2)(|x_1|\bar\otimes|x_2|)$ *is the polar decomposition, where the tensor product of τ-measurable operators is understood as that on $\mathcal{H}_1 \bar\otimes \mathcal{H}_2$.*
(2) $|x_1 \bar\otimes x_2|^p = |x_1|^p \bar\otimes |x_2|^p$.
(3) $h_{\psi_1} \bar\otimes h_{\psi_2} = h_{\psi_1\bar\otimes\psi_2}$ *and* $(h_{\psi_1} \bar\otimes h_{\psi_2})^{it} = h^{it}_{\psi_1\bar\otimes\psi_2}$.

Proof. Items (1) and (2) can easily be confirmed within theory of unbounded operators; see Appendix A.

Item (3): Observe that

$$\begin{aligned} h^{it}_{\psi_1\bar\otimes\psi_2} &= [D\psi_1 \bar\otimes \psi_2 : D\varphi_1 \bar\otimes \varphi_2]_t(\lambda^{\varphi_1}(t)\otimes\lambda^{\varphi_2}(t)) \\ &= ([D\psi_1 : D\varphi_2]_t \otimes [D\psi_2 : D\varphi_2]_t)\,(\lambda^{\varphi_1}(t)\otimes\lambda^{\varphi_2}(t)) \\ &= ([D\psi_1 : D\varphi_1]_t\,\lambda^{\varphi_1}(t)) \otimes ([D\psi_2 : D\varphi_2]_t\,\lambda^{\varphi_2}(t)) \\ &= h^{it}_{\psi_1}\otimes h^{it}_{\psi_2} \end{aligned}$$

by (2.6), (3.1) and [16, Corollary 8.6]. Since $(h_{\psi_1}\bar\otimes h_{\psi_2})^{it} = h^{it}_{\psi_1}\otimes h^{it}_{\psi_2}$ (see Appendix A), we obtain (3) by Lemma 2.2 (the uniqueness part). ∎

By Lemma 3.2 we have

$$|x_1\bar\otimes x_2|^p = |x_1|^p\bar\otimes|x_2|^p = h_{\psi_1}\bar\otimes h_{\psi_2} = h_{\psi_1\bar\otimes\psi_2}.$$

Since $\psi_1\bar\otimes\psi_2\in(M_1\bar\otimes M_2)_*$, we have $x_1\bar\otimes x_2\in L^p(M_1\bar\otimes M_2)$ and

$$\begin{aligned} \|x_1\bar\otimes x_2\|_p^p &= \mathrm{tr}(|x_1\bar\otimes x_2|^p) = \mathrm{tr}(|x_1|^p\bar\otimes|x_2|^p) \\ &= \mathrm{tr}(h_{\psi_1}\bar\otimes h_{\psi_2}) = \mathrm{tr}(h_{\psi_1\bar\otimes\psi_2}) \\ &= (\psi_1\bar\otimes\psi_2)(1) = \psi_1(1)\psi_2(1) \\ &= \mathrm{tr}(h_{\psi_1})\mathrm{tr}(h_{\psi_2}) = \mathrm{tr}(|x_1|^p)\mathrm{tr}(|x_2|^p) = \|x_1\|_p^p\,\|x_2\|_p^p. \end{aligned}$$

Consequently, we have the first part of the following theorem:

THEOREM 3.3. *For any pair $(x_1,x_2)\in L^p(M_1)\times L^p(M_2)$ the unbounded operator tensor product $x_1\bar\otimes x_2$ affiliated with $\widetilde{M_1}\bar\otimes\widetilde{M_2}$ actually gives an element of $L^p(M_1\bar\otimes M_2)$, and we have*

$$\|x_1\bar\otimes x_2\|_p = \|x_1\|_p\,\|x_2\|_p.$$

The mapping $(x_1,x_2)\mapsto x_1\bar\otimes x_2$ is clearly bilinear, and induces a natural map from the vector space tensor product $L^p(M_1)\otimes_{\mathrm{alg}} L^p(M_2)$ into $L^p(M_1\bar\otimes M_2)$, which has dense image when both M_i are σ-finite and $p\ge 1$.

Proof. Let us prove the second part. We can assume that both φ_i are faithful normal states. Then $(M_1\otimes_{\mathrm{alg}} M_2)h^{1/p}_{\varphi_1\bar\otimes\varphi_2}$ is dense in $L^p(M_1\bar\otimes M_2)$ by Lemma 2.3. Therefore, elements $(ah^{1/p}_{\varphi_1})\bar\otimes(bh^{1/p}_{\varphi_2}) = (a\otimes b)\,h^{1/p}_{\varphi_1}\bar\otimes h^{1/p}_{\varphi_2} = (a\otimes b)h^{1/p}_{\varphi_1\bar\otimes\varphi_2}$ with $a\in M_1$ and $b\in M_2$ are total in $L^p(M_1\bar\otimes M_2)$. Hence we are done. ∎

We do not know whether or not the second assertion (the density of the induced map) in the above theorem holds without σ-finiteness. However, we think that an approximation by σ-finite projections might give the same assertion without σ-finiteness. We leave this question to the interested reader.

Here is a corollary for the Kosaki non-commutative L^p-space $L^p(M,\varphi)_\eta$ with $1\le p\le\infty$ and $0\le\eta\le 1$, which is defined as the complex interpolation space $C_{1/p}(h^\eta_\varphi M h^{1-\eta}_\varphi, L^1(M))$, where the embedding $M\ni$

$a \mapsto h_\varphi^\eta a h_\varphi^{1-\eta} \in L^1(M)$ gives a compatible pair with norm $h_\varphi^\eta M h_\varphi^{1-\eta} \ni h_\varphi^\eta a h_\varphi^{1-\eta} \mapsto \|h_\varphi^\eta a h_\varphi^{1-\eta}\|_\infty := \|a\|_M$.

Corollary 3.4. *Assume that both M_i are σ-finite, and both φ_i are faithful normal positive linear functionals. For each $1 \le p \le \infty$ and $0 \le \eta \le 1$, the mapping $L^1(M_1) \times L^1(M_2) \ni (x_1, x_2) \mapsto x_1 \bar{\otimes} x_2 \in L^1(M_1 \bar{\otimes} M_2)$ in* Theorem 3.3 *induces a bilinear map from the vector space tensor product $L^p(M_1, \varphi_1)_\eta \otimes_{\mathrm{alg}} L^p(M_2, \varphi_2)_\eta$ into $L^p(M_1 \bar{\otimes} M_2, \varphi_1 \bar{\otimes} \varphi_2)_\eta$ with dense image, and we have*

$$\|x_1 \bar{\otimes} x_2\|_{p,\varphi_1 \bar{\otimes} \varphi_2, \eta} = \|x_1\|_{p,\varphi_1,\eta} \|x_2\|_{p,\varphi_2,\eta}.$$

Proof. Note that the Kosaki non-commutative L^p-spaces $L^p(M_i, \varphi_i)_\eta$ and $L^p(M_1 \bar{\otimes} M_2, \varphi_1 \bar{\otimes} \varphi_2)_\eta$ are

$$h_{\varphi_i}^{\eta/q} L^p(M_i) h_{\varphi_i}^{(1-\eta)/q} \subset L^1(M_i),$$
$$h_{\varphi_1 \bar{\otimes} \varphi_2}^{\eta/q} L^p(M_1 \bar{\otimes} M_2) h_{\varphi_1 \bar{\otimes} \varphi_2}^{(1-\eta)/q} \subset L^1(M_1 \bar{\otimes} M_2),$$

respectively, where q is the dual exponent of p, that is, $1/p + 1/q = 1$.

Each $(x_1, x_2) \in L^p(M_1, \varphi_1)_\eta \times L^p(M_2, \varphi_2)_\eta \subset L^1(M_1) \times L^1(M_2)$ is of the form

$$(h_{\varphi_1}^{\eta/q} x_1' h_{\varphi_1}^{(1-\eta)/q}, h_{\varphi_1}^{\eta/q} x_2' h_{\varphi_2}^{(1-\eta)/q})$$

with unique $x_1' \in L^p(M_1)$ and $x_2' \in L^p(M_2)$. Then we have, by Lemmas A.1 and 3.2(2, 3),

$$x_1 \bar{\otimes} x_2 = (h_{\varphi_1}^{\eta/q} \bar{\otimes} h_{\varphi_2}^{\eta/q})(x_1' \bar{\otimes} x_2')(h_{\varphi_1}^{(1-\eta)/q} \bar{\otimes} h_{\varphi_2}^{(1-\eta)/q}) = h_{\varphi_1 \bar{\otimes} \varphi_2}^{\eta/q} (x_1' \bar{\otimes} x_2') h_{\varphi_1 \bar{\otimes} \varphi_2}^{(1-\eta)/q},$$

which clearly lies in $L^p(M_1 \bar{\otimes} M_2, \varphi_1 \bar{\otimes} \varphi_2)_\eta$ since $x_1' \bar{\otimes} x_2' \in L^p(M_1 \bar{\otimes} M_2)$ by Theorem 3.3. Moreover, we observe that

$$\begin{aligned}
\|x_1 \bar{\otimes} x_2\|_{p,\varphi_1 \bar{\otimes} \varphi_2, \eta} &= \|h_{\varphi_1 \bar{\otimes} \varphi_2}^{\eta/q} (x_1' \bar{\otimes} x_2') h_{\varphi_1 \bar{\otimes} \varphi_2}^{(1-\eta)/q}\|_{p,\varphi_1 \bar{\otimes} \varphi_2, \eta} \\
&= \|x_1' \bar{\otimes} x_2'\|_p = \|x_1'\|_p \|x_2'\|_p \quad \text{(by Theorem 3.3 again)} \\
&= \|h_{\varphi_1}^{\eta/q} x_1' h_{\varphi_1}^{(1-\eta)/q}\|_{p,\varphi_1,\eta} \|h_{\varphi_1}^{\eta/q} x_2' h_{\varphi_2}^{(1-\eta)/q}\|_{p,\varphi_2,\eta} \\
&= \|x_1\|_{p,\varphi_1,\eta} \|x_2\|_{p,\varphi_2,\eta}.
\end{aligned}$$

That the map from $L^p(M_1, \varphi_1)_\eta \otimes_{\mathrm{alg}} L^p(M_2, \varphi_2)_\eta$ into $L^p(M_1 \bar{\otimes} M_1, \varphi_1 \bar{\otimes} \varphi_2)_\eta$ has dense image follows from Theorem 3.3 together with the definition of norm $\| \cdot \|_{p,\varphi_1 \bar{\otimes} \varphi_2, \eta}$. ∎

Here is a question. Let $(x_1, x_2) \in L^1(M_1, \varphi_1)_\eta \times L^1(M_2, \varphi_2)_\eta$ be arbitrarily given with $x_i \ne 0$. Does $x_1 \bar{\otimes} x_2 \in L^p(M_1 \bar{\otimes} M_2, \varphi_1 \bar{\otimes} \varphi_2)_\eta$ imply that $x_i \in L^p(M_i, \varphi_i)_\eta$ for both $i = 1, 2$?

4. A sample application in QIT. We illustrate how our description of the non-commutative L^p-space $L^p(M_1 \bar{\otimes} M_2)$ is useful.

Let $\alpha \in [1/2, \infty) \setminus \{1\}$ be given. The *sandwiched α-Rényi divergence* $\widetilde{D}_\alpha(\psi \,\|\, \varphi)$ for $\psi, \varphi \in M_*^+$ with $\psi \neq 0$ has several definitions, one of which is

$$\widetilde{D}_\alpha(\psi \,\|\, \varphi) := \frac{1}{\alpha - 1} \log \frac{\widetilde{Q}_\alpha(\psi \,\|\, \varphi)}{\psi(1)},$$

where

$$\widetilde{Q}_\alpha(\psi \,\|\, \varphi) := \begin{cases} \mathrm{tr}[(h_\varphi^{(1-\alpha)/(2\alpha)} h_\psi h_\varphi^{(1-\alpha)/(2\alpha)})^\alpha] & (1/2 \le \alpha < 1), \\ \|h_\psi\|_{\alpha,\varphi,1/2}^\alpha & (\alpha > 1,\ s(\psi) \le s(\varphi) \\ & \quad \text{and } h_\psi \in L^\alpha(M, \varphi)_{1/2}), \\ +\infty & (\text{otherwise}). \end{cases}$$

This formulation is mainly due to Jenčová [4, 3.3].

The sandwiched α-Rényi divergence $\widetilde{Q}_\alpha(\psi \,\|\, \varphi)$ admits a two-parameter extension, called the *α-z-Rényi divergence*, in the finite-dimensional or more generally the infinite-dimensional type I setup. See [7, 2, 14] in historical order. Here we propose a possible definition of its non-type I extension, for which we want to show that the present description of non-commutative L^p-spaces associated with tensor product von Neumann algebras works. Let $\alpha, z > 0$ with $\alpha \neq 1$ be arbitrarily given. For each pair $\varphi, \psi \in M_*^+$ with $\psi \neq 0$ we define $\widetilde{Q}_{\alpha,z}(\psi \,\|\, \varphi)$ to be

$$\begin{cases} \mathrm{tr}[(h_\varphi^{(1-\alpha)/(2z)} h_\psi^{\alpha/z} h_\varphi^{(1-\alpha)/(2z)})^z] & (\alpha < 1), \\ \|x\|_z^z & (\alpha > 1 \text{ and } (\spadesuit) \text{ holds} \\ & \quad \text{with } x \in s(\varphi) L^z(M) s(\varphi)), \\ +\infty & (\text{otherwise}), \end{cases}$$

where

$$h_\psi^{\alpha/z} = h_\varphi^{(\alpha-1)/2z} x h_\varphi^{(\alpha-1)/2z}. \tag{$\spadesuit$}$$

LEMMA 4.1. *Identity ($\spadesuit$) uniquely determines $x \in s(\varphi) L^z(M) s(\varphi)$ (if it exists). Hence $\widetilde{Q}_{\alpha,z}(\psi \,\|\, \varphi)$ is well defined.*

Proof. Choose another $y \in s(\varphi) L^z(M) s(\varphi)$ with

$$h_\psi^{\alpha/z} = h_\varphi^{(\alpha-1)/(2z)} y h_\varphi^{(\alpha-1)/(2z)}.$$

Since all the τ-measurable operators form a $*$-algebra, we have

$$\begin{aligned} 0 &= h_\varphi^{(\alpha-1)/(2z)} x h_\varphi^{(\alpha-1)/(2z)} - h_\varphi^{(\alpha-1)/(2z)} y h_\varphi^{(\alpha-1)/(2z)} \\ &= h_\varphi^{(\alpha-1)/(2z)} (x - y) h_\varphi^{(\alpha-1)/(2z)}. \end{aligned}$$

Moreover, h_φ is a τ-measurable operator and non-singular affiliated with $s(\varphi)\widetilde{M}s(\varphi)$. In addition, $f(t) = t^{(\alpha-1)/(2z)}$ is a continuous strictly increasing function on $[0,\infty)$ with $f(0)=0$ if $\alpha>1$. Hence, $h_\varphi^{(\alpha-1)/(2z)}$ is also τ-measurable (see, e.g., [5, Proposition 4.19]; but this fact is implicitly utilized in the general theory of Haagerup non-commutative L^p-spaces that we have employed) and non-singular. Thus, $x-y=0$ by [12, Lemma 2.1], and hence $x=y$. ∎

Then

$$\widetilde{D}_{\alpha,z}(\psi\,\|\,\varphi) := \frac{1}{\alpha-1}\log\frac{\widetilde{Q}_{\alpha,z}(\psi\,\|\,\varphi)}{\psi(1)}$$

is called the *α-z-Rényi divergence*.

Lemma 4.2. *$\widetilde{Q}_{\alpha,\alpha}(\psi\,\|\,\varphi)=\widetilde{Q}_\alpha(\psi\,\|\,\varphi)$ for every $\alpha\ge 1/2$ with $\alpha\neq 1$.*

Proof. When $\alpha\in[1/2,1)$, the identity clearly holds by the definitions of $\widetilde{Q}_{\alpha,\alpha}(\psi\,\|\,\varphi)$ and $\widetilde{Q}_\alpha(\psi\,\|\,\varphi)$.

We then consider the case $\alpha=z>1$. Then identity (♠) holds with $x\in s(\varphi)L^\alpha(M)s(\varphi)$ if and only if $h_\psi\in h_\varphi^{(\alpha-1)/2\alpha}L^\alpha(M)h_\varphi^{(\alpha-1)/2\alpha}=L^\alpha(M,\varphi)_{1/2}$, where the dual exponent of α is $\alpha/(\alpha-1)$. Moreover, in this case, we have $\|h_\psi\|_{\alpha,\varphi,1/2}^\alpha=\|x\|_\alpha^\alpha$. ∎

The next fact was claimed for the sandwiched α-Rényi divergence $\widetilde{Q}_\alpha(\psi\,\|\,\varphi)$ in [3, p. 1860] without a detailed proof. Its detailed proof when both M_i are injective or AFD was given by Hiai and Mosonyi [6, (3.16)] by using the finite-dimensional result and also the martingale convergence property that they established. We believe that the proof below is more natural than those.

Proposition 4.3. *For any $\psi_i,\varphi_i\in(M_i)_*^+$ with $\psi_i\neq 0$, $i=1,2$, we have*

$$\widetilde{Q}_{\alpha,z}(\psi_1\mathbin{\bar\otimes}\psi_2\,\|\,\varphi_1\mathbin{\bar\otimes}\varphi_2)=\widetilde{Q}_{\alpha,z}(\psi_1\,\|\,\varphi_1)\widetilde{Q}_{\alpha,z}(\psi_2\,\|\,\varphi_2),$$
$$\widetilde{D}_{\alpha,z}(\psi_1\mathbin{\bar\otimes}\psi_2\,\|\,\varphi_1\mathbin{\bar\otimes}\varphi_2)=\widetilde{D}_{\alpha,z}(\psi_1\,\|\,\varphi_1)+\widetilde{D}_{\alpha,z}(\psi_2\,\|\,\varphi_2),$$

when $\alpha<1$ or both $\widetilde{Q}_{\alpha,z}(\psi_i\,\|\,\varphi_i)$ are finite.

When $\alpha=z\in[1/2,1)\cup(1,\infty)$, the identities hold without any assumptions.

Proof. We first consider the case when $\alpha<1$. By Lemma 3.2(2,3) together with Lemma A.1, we have

$$\begin{aligned}
&(h_{\varphi_1\bar\otimes\varphi_2}^{(1-\alpha)/(2z)}\,h_{\psi_1\bar\otimes\psi_2}^{\alpha/z}\,h_{\varphi_1\bar\otimes\varphi_2}^{(1-\alpha)/(2z)})^z\\
&\quad=((h_{\varphi_1}^{(1-\alpha)/(2z)}\mathbin{\bar\otimes}h_{\varphi_2}^{(1-\alpha)/(2z)})(h_{\psi_1}^{\alpha/z}\mathbin{\bar\otimes}h_{\psi_2}^{\alpha/z})(h_{\varphi_1}^{(1-\alpha)/(2z)}\mathbin{\bar\otimes}h_{\varphi_2}^{(1-\alpha)/(2z)}))^z\\
&\quad=((h_{\varphi_1}^{(1-\alpha)/(2z)}h_{\psi_1}^{\alpha/z}h_{\varphi_1}^{(1-\alpha)/(2z)})\mathbin{\bar\otimes}(h_{\varphi_2}^{(1-\alpha)/(2z)}h_{\psi_2}^{\alpha/z}h_{\varphi_2}^{(1-\alpha)/(2z)}))^z\\
&\quad=(h_{\varphi_1}^{(1-\alpha)/(2z)}h_{\psi_1}^{\alpha/z}h_{\varphi_1}^{(1-\alpha)/(2z)})^z\mathbin{\bar\otimes}(h_{\varphi_2}^{(1-\alpha)/(2z)}h_{\psi_2}^{\alpha/z}h_{\varphi_2}^{(1-\alpha)/(2z)})^z
\end{aligned}$$

as unbounded operators on $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$, on which $\widetilde{M}_1 \bar{\otimes} \widetilde{M}_2$ naturally acts. Since both the tensor components of the above right-most side fall into $L^1(M_i)$, $i = 1, 2$, respectively, we conclude, by Theorem 3.3, that the desired multiplicativity of $\widetilde{Q}_{\alpha,z}$ holds true.

We then consider the case of $\alpha > 1$. Assume first that $\widetilde{Q}_{\alpha,z}(\psi_i \,\|\, \varphi_i) < +\infty$ for $i = 1, 2$. Then there are $x_i \in s(\varphi_i)L^z(M_i)s(\varphi_i)$, $i = 1, 2$, such that $h_{\psi_i}^{\alpha/z} = h_{\varphi_i}^{(\alpha-1)/(2z)} x_i h_{\varphi_i}^{(\alpha-1)/(2z)}$. By Lemmas 3.2(2, 3) and A.1, we have

$$
\begin{aligned}
&h_{\varphi_1 \bar{\otimes} \varphi_2}^{(\alpha-1)/(2z)}\, x_1 \bar{\otimes} x_2\, h_{\varphi_1 \bar{\otimes} \varphi_2}^{(\alpha-1)/(2z)} \\
&\qquad = (h_{\varphi_1}^{(\alpha-1)/(2z)} \bar{\otimes} h_{\varphi_2}^{(\alpha-1)/(2z)})(x_1 \bar{\otimes} x_2)(h_{\varphi_1}^{(\alpha-1)/(2z)} \bar{\otimes} h_{\varphi_2}^{(\alpha-1)/(2z)}) \\
&\qquad = (h_{\varphi_1}^{(\alpha-1)/(2z)} x_1 h_{\varphi_1}^{(\alpha-1)/(2z)}) \bar{\otimes} (h_{\varphi_2}^{(\alpha-1)/(2z)} x_2 h_{\varphi_2}^{(\alpha-1)/(2z)}) \\
&\qquad = h_{\psi_1}^{\alpha/z} \bar{\otimes} h_{\psi_2}^{\alpha/z} = h_{\psi_1 \bar{\otimes} \psi_2}^{\alpha/z}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\widetilde{Q}_{\alpha,z}(\psi_1 \bar{\otimes} \psi_2 \,\|\, \varphi_1 \bar{\otimes} \varphi_2) \\
&\qquad\qquad = \|x_1 \bar{\otimes} x_2\|_z^z = \|x_1\|_z^z \|x_2\|_z^z = \widetilde{Q}_{\alpha,z}(\psi_1 \,\|\, \varphi_1)\widetilde{Q}_{\alpha,z}(\psi_2 \,\|\, \varphi_2)
\end{aligned}
$$

by Theorem 3.3.

We finally assume that $\alpha = z > 1$. By Lemma 4.2, $\widetilde{Q}_{\alpha,z}(\psi_1 \bar{\otimes} \psi_2 \,\|\, \varphi_1 \bar{\otimes} \varphi_2) = \widetilde{Q}_\alpha(\psi_1 \bar{\otimes} \psi_2 \,\|\, \varphi_1 \bar{\otimes} \varphi_2)$ and $\widetilde{Q}_{\alpha,z}(\psi_i \,\|\, \varphi_i) = \widetilde{Q}_\alpha(\psi_i \,\|\, \varphi_i)$, $i = 1, 2$. We also assume that at least one of the $\widetilde{Q}_\alpha(\psi_i \,\|\, \varphi_i)$ is infinite, say $\widetilde{Q}_\alpha(\psi_1 \,\|\, \varphi_1) = +\infty$. Then we apply the monotonicity property (see [4, Theorem 3.16(4)]) to the unital $*$-homomorphism (i.e., unital normal positive map) $\gamma : M_1 \to M_1 \bar{\otimes} M_2$ sending $a \in M_1$ to $a \otimes 1$ and obtain

$$
\widetilde{Q}_\alpha(\psi_1 \bar{\otimes} \psi_2 \,\|\, \varphi_1 \bar{\otimes} \varphi_2) \geq \widetilde{Q}_\alpha(\psi_2(1)\psi_1 \,\|\, \varphi_2(1)\varphi_1).
$$

If $\varphi_2(1) = 0$, then $\varphi_2(1)\varphi_1 = 0$ and thus $\widetilde{Q}_\alpha(\psi_2(1)\psi_1 \,\|\, \varphi_2(1)\varphi_1) = +\infty$. Hence we may and do assume that $\varphi_2(1) \neq 0$. Then $s(\psi_2(1)\psi_1) \leq s(\varphi_2(1)\varphi_1)$ if and only if $s(\psi_1) \leq s(\varphi_1)$, and we easily see that $h_{\psi_2(1)\psi_1} = \psi_2(1)h_{\psi_1} \in L^\alpha(M_1, \varphi_2(1)\varphi_1)_{1/2}$ if and only if $h_{\psi_1} \in L^\alpha(M_1, \varphi_1)_{1/2}$. Hence

$$
\widetilde{Q}_\alpha(\psi_2(1)\psi_1 \,\|\, \varphi_2(1)\varphi_1) = +\infty,
$$

and thus the desired multiplicative property of $\widetilde{Q}_\alpha$ holds as $+\infty = +\infty$. ∎

Proving the above additivity property in full generality, i.e., without assuming $\widetilde{Q}_{\alpha,z}(\psi_i \,\|\, \varphi_i) < +\infty$, needs the monotonicity property (the so-called data processing inequality) or other similar property. It is an important question to determine the range of (α, z) for which the monotonicity property holds. (This question was completely settled by Zhang [19] in the finite-dimensional case.) Moreover, general properties of $\widetilde{Q}_{\alpha,z}(\psi \,\|\, \varphi)$ will be discussed in a subsequent work [11] by the first-named author.

Appendix. Simple tensors of (τ-measurable) unbounded operators. Let P and Q be semifinite von Neumann algebras which act on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Let τ_P and τ_Q be faithful semifinite normal tracial weights on P and Q. We are interested in the tensor product of a τ_P-measurable x and a τ_Q-measurable y. Recall that x is said to be a τ_P-measurable operator if it is a closed densely defined operator affiliated with P (denoted by $x\eta P$) such that for each $\delta > 0$ there is a projection $e \in P$ such that $e\mathcal{H} \subset \mathrm{D}(x)$ and $\tau_P(1-e) < \delta$, where $\mathrm{D}(x)$ denotes the domain of x as an operator on $\mathcal{H}$.

By, e.g., [15, Lemma 7.21] the algebraic tensor product of x and y becomes a densely defined closable operator, whose closure is denoted by $x \mathbin{\bar\otimes} y$. It is also known (see, e.g., [15, Proposition 7.26]) that $(x \mathbin{\bar\otimes} y)^* = x^* \mathbin{\bar\otimes} y^*$ in general. Let $x = u|x|$ and $y = v|y|$ be the polar decompositions (see, e.g., [15, Theorem 7.2]). Then we have the polar decomposition $x \mathbin{\bar\otimes} y = (u \otimes v)(|x| \mathbin{\bar\otimes} |y|)$, and in particular $|x \mathbin{\bar\otimes} y| = |x| \mathbin{\bar\otimes} |y|$; see [15, Exercise 7.6.13]. This is indeed Lemma 3.2(1). Again by, e.g., [15, Proposition 7.26] we observe that $|x| \mathbin{\bar\otimes} |y|$ is self-adjoint. Moreover, by, e.g., [15, Section 5.5.1 and Lemma 7.24], there is a unique spectral measure $e_{|x|,|y|}$ over $[0,\infty)^2$ such that $e_{|x|,|y|}(\varLambda_1 \times \varLambda_2) = e_{|x|}(\varLambda_1) \otimes e_{|y|}(\varLambda_2)$ for any Borel subsets $\varLambda_1, \varLambda_2 \subset [0,\infty)$, and more importantly,

$$|x| \mathbin{\bar\otimes} |y| = \int_{[0,\infty)^2} \lambda_1\lambda_2 \, e_{|x|,|y|}(d\lambda_1, d\lambda_2)$$

in the sense of spectral integrals. Here, $e_{|x|}$ and $e_{|y|}$ are the spectral measures of $|x|$ and $|y|$, respectively. Define

$$e_{|x|\bar\otimes|y|}(\varLambda) := \int_{[0,\infty)^2} \mathbf{1}_\varLambda(\lambda_1\lambda_2)\, e_{|x|,|y|}(d\lambda_1, d\lambda_2), \qquad \varLambda \subset [0,\infty).$$

Then it is not hard to see that $e_{|x|\bar\otimes|y|}$ gives a unique spectral measure of $|x| \mathbin{\bar\otimes} |y|$. Thus, we have

$$f(|x| \mathbin{\bar\otimes} |y|) = \int_{[0,\infty)} f(\lambda)\, e_{|x|\bar\otimes|y|}(d\lambda) = \int_{[0,\infty)^2} f(\lambda_1\lambda_2)\, e_{|x|,|y|}(d\lambda_1, d\lambda_2)$$

for every non-negative Borel function f on $[0,\infty)$. If $f(\lambda_1\lambda_2) = f(\lambda_1)f(\lambda_2)$, then

$$f(|x| \mathbin{\bar\otimes} |y|) = \int_{[0,\infty)^2} f(\lambda_1)f(\lambda_2)\, e_{|x|,|y|}(d\lambda_1, d\lambda_2) = f(|x|) \mathbin{\bar\otimes} f(|y|).$$

This also holds even if the non-negativity of f is replaced with the boundedness of f. Thus, the above formula is applicable to the f_t in Lemma 2.2. Hence what we have established indeed includes Lemma 3.2(2). The properties we have explained so far are valid for just closed densely defined (unbounded) operators, without τ-measurability and even affiliation with P and Q.

Using the monotone class theorem in measure theory we can easily see that the spectral measure $e_{|x|,|y|}$ takes values in $P \bar{\otimes} Q$ since $e_{|x|,|y|}(\Lambda_1 \times \Lambda_2) = e_{|x|}(\Lambda_1) \otimes e_{|y|}(\Lambda_2) \in P \bar{\otimes} Q$ by affiliation. Therefore, $|x \bar{\otimes} y| = |x| \bar{\otimes} |y|$ is affiliated with $P \bar{\otimes} Q$. It is non-trivial whether or not $x \bar{\otimes} y$, or equivalently, $|x| \bar{\otimes} |y|$, is $\tau_P \bar{\otimes} \tau_Q$-measurable. In fact, this is not the case in general; see [1, Examples 3.14, 3.15].

Let x' and y' be τ_P- and τ_Q-measurable operators, respectively. It is known that the usual products xx' and yy' are densely defined closable, and hence the closures $\overline{xx'}$ and $\overline{yy'}$ become τ_P- and τ_Q-measurable again. By [15, Lemma 7.22] we have $(\overline{xx'}) \bar{\otimes} (\overline{yy'}) = (xx') \bar{\otimes} (yy')$, where the right-hand side coincides with the closure of $(x \bar{\otimes} y)(x' \bar{\otimes} y')$. Thus, the following holds:

LEMMA A.1. *All the linear combinations of 'simple tensors' $x \bar{\otimes} y$ with τ_P- and τ_Q-measurable $x \eta P$ and $y \eta Q$ form a $*$-algebra with strong sum and strong product. We simply understand $(x \bar{\otimes} y)(x' \bar{\otimes} y')$ as the strong product of $x \bar{\otimes} y$ and $x' \bar{\otimes} y'$ without the use of closure sign. With this notational rule,*

$$(x \bar{\otimes} y)(x' \bar{\otimes} y') = (xx') \bar{\otimes} (yy')$$

for any τ_P- and τ_Q-measurable $x, x' \eta P$ and $y, y' \eta Q$, where xx' and yy' are understood as the strong product.

Acknowledgements. We are greatly indebted to Fumio Hiai for giving a series of online lectures entitled 'Quantum Analysis and Quantum Information Theory' during Feb.–Mar. 2022 delivered via Nagoya University, supported by Grant-in-Aid for Scientific Research (B) JP18H01122. The lectures motivated us to investigate non-commutative L^p-spaces associated with tensor product von Neumann algebras. We also thank him for his comments on a draft of this note.

This work was supported in part by Grant-in-Aid for Scientific Research (B) JP18H01122.

References

[1] M. Anoussis, V. Felouzis and I.G. Todorov, *Tensor products of measurable operators*, Illinois J. Math. 59 (2015), 577–595.

[2] K. M. R. Audenaert and N. Datta, *α-z-relative Rényi entropies*, J. Math. Phys. 56 (2015), art. 022202, 16 pp.

[3] M. Berta, V. B. Scholz and M. Tomamichel, *Rényi divergences as weighted non-commutative vector-valued L^p-spaces*, Ann. Henri Poincaré 19 (2018), 1843–1867.

[4] F. Hiai, *Quantum f-Divergences in von Neumann Algebras*, Math. Phys. Stud., Springer, Singapore, 2021.

[5] F. Hiai, *Lectures on Selected Topics in von Neumann Algebras*, EMS Ser. Lect. Math., Eur. Math. Soc., Berlin, 2021.

[6] F. Hiai and M. Mosonyi, *Quantum Rényi divergences and the strong converse exponent of state discrimination in operator algebras*, Ann. Henri Poincaré 24 (2023), 1681–1724.
[7] V. Jaksic, Y. Ogata, Y. Pautrat and C.-A. Pillet, *Entropic fluctuations in quantum statistical mechanics: an introduction*, in: Quantum Theory from Small to Large Scales, Lecture Notes of the Les Houches Summer School 95, Oxford Univ. Press, 2012.
[8] A. Jenčová, *Rényi relative entropies and noncommutative L^p-spaces*, Ann. Henri Poincaré 19 (2018), 2513–2542.
[9] A. Jenčová, *Rényi relative entropies and noncommutative L^p-spaces II*, Ann. Henri Poincaré 22 (2021), 3235–3254.
[10] M. Junge, *Fubini's theorem for ultraproducts of noncommutative L_p-spaces*, Canad. J. Math. 56 (2004), 983–1021.
[11] S. Kato, *On α-z-Rényi divergence in the von Neumann algebra setting*, J. Math. Phys., to appear; arXiv:2311.01748
[12] H. Kosaki, *Positive cones associated with a von Neumann algebra*, Math. Scand. 47 (1980), 295–307.
[13] H. Kosaki, *Applications of the complex interpolation method to a von Neumann algebra: non-commutative L^p-spaces*, J. Funct. Anal. 56 (1984), 29–78.
[14] M. Mosonyi, *The strong converse exponent of discriminating infinite-dimensional quantum states*, Comm. Math. Phys. 400 (2023), 83–132.
[15] K. Schmüdgen, *Unbounded Self-Adjoint Operators on Hilbert Spaces*, Grad. Texts in Math. 265, Springer, 2012.
[16] Ş. Strătilă, *Modular Theory in Operator Algebras*, Abacus Press, Tunbridge Wells, 1981.
[17] M. Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type III*, Acta Math. 131 (1973), 249–310.
[18] M. Terp, *L^p spaces associated with von Neumann algebras*, notes, Math. Institute, Copenhagen Univ., 1981.
[19] H. Zhang, *From Wigner–Yanase–Dyson conjecture to Carlen–Frank–Lieb conjecture*, Adv. Math. 365 (2020), art. 107053, 18 pp.

Shinya Kato, Yoshimichi Ueda
Graduate School of Mathematics
Nagoya University
Nagoya 464-8602, Japan
E-mail: kato.shinya.k6@s.mail.nagoya-u.ac.jp
ueda@math.nagoya-u.ac.jp