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Collatz map as a non-singular transformation

by

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Abstract. Let T be the map defined on $\mathbb{N} = \{1, 2, \dots\}$ by $T(n) = n/2$ if n is even and $T(n) = (3n + 1)/2$ if n is odd. Consider the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ where μ is the counting measure. This dynamical system has the following properties:

1. There exists an invariant finite measure γ such that $\gamma(A) \le \mu(A)$ for all $A \subset \mathbb{N}$.
2. For each function $f \in L^1(\mu)$ the averages $\frac{1}{N}\sum_{n=1}^{N} f(T^n x)$ converge for every $x \in \mathbb{N}$ to $f^*(x)$ where $f^* \in L^1(\mu)$.

We also show that the Collatz conjecture is equivalent to the existence of a finite measure ν on $(\mathbb{N}, 2^{\mathbb{N}})$ making the operator $Vf = f \circ T$ power bounded in $L^1(\nu)$ with conservative part $\{1, 2\}$.

1. Introduction. The original Collatz conjecture states that if S is the map defined on $\mathbb{N}$ by $Sn = n/2$ if n is even and $Sn = 3n + 1$ if n is odd, then for each natural number n there exists $k \in \mathbb{N}$ such that $T^k n = 1$. This conjecture has been extensively studied. See the nice survey and analysis of this subject by J. Lagarias [12, 13]; see also Y. Sinai [16] and E. Akin [1]. As noted in [12], S. Kakutani, S. Ulam and P. Erdős had been interested in this problem. Several attempts have been made after Lagarias' survey [13], and some offered equivalent formulations; see [4, 10, 6, 3, 9]. An operator-theoretic approach was presented in [14].

An equivalent and more convenient map to study this conjecture is defined by

$$Tn = \tfrac{n}{2} \text{ if } n \text{ is even} \quad \text{and} \quad Tn = \tfrac{3n+1}{2} \text{ if } n \text{ is odd.}$$

A *cycle* for T is a sequence $a, Ta, \dots, T^{i-1}a$ where $a \in \mathbb{N}$, $T^j a \ne a$ for $1 \le j \le i - 1$ and $T^i a = a$. Since $T1 = 2$ and $T2 = 1$, $\{1, 2\}$ is a cycle.

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The Collatz conjecture is equivalent to the combination of the following conjectures:

1. The only cycle for T is $\{1, 2\}$.
2. The orbit of every $n \in \mathbb{N}$ under T is bounded.

The Collatz conjecture as stated is a property of recurrence to the set $\{1, 2\}$. More precisely, it states that from any point $n \in \mathbb{N} \setminus \{1, 2\}$ the iterates $T^k(n)$ will go to the set $\{1, 2\}$. Such recurrence properties have been studied quite extensively in ergodic theory in the context of measure preserving transformations but also in the context of non-singular transformations, which in our view fits better the Collatz map.

We recall that if S is a measurable map from the σ-finite measure space $(X, \mathcal{A}, \nu)$ to itself then the set $(X, \mathcal{A}, \nu, S)$ is called a *dynamical system*. The map S is said to be *non-singular* if for all $A \in \mathcal{A}$ we have $\nu(S^{-1}(A)) = 0$ whenever $\nu(A) = 0$. Such systems are called null preserving in [11, p. 3]. In this note we will focus on the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$ where $2^{\mathbb{N}}$ denotes the power set of $\mathbb{N} = \{1, 2, \dots\}$. The Collatz map T is clearly measurable with respect to this measurable space.

Definition 1.1. The set $(\mathbb{N}, 2^{\mathbb{N}}, T, \nu)$ with ν a σ-finite measure for which T is non-singular is called the *Collatz dynamical system* with measure ν.

A non-negative measure ν on $(\mathbb{N}, 2^{\mathbb{N}})$ is defined by its values on $\mathbb{N}$, that is, by the sequence of non-negative numbers $\nu(n)$ where $n \in \mathbb{N}$. We assume that $\nu(n) < \infty$ for each n. The measure ν is σ-finite if $\sum_{n=1}^{\infty} \nu(n) = \infty$ and finite if $\sum_{k=1}^{\infty} \nu(n) < \infty$. A natural σ-finite measure on $(\mathbb{N}, 2^{\mathbb{N}})$ is the *counting measure* μ where $\mu(n) = 1$ for each $n \in \mathbb{N}$. Since this counting measure has only the empty set as nullset, the map T is non-singular with respect to this measure. Another measure θ on $(\mathbb{N}, 2^{\mathbb{N}})$ is equivalent to the counting measure if $\theta(n) > 0$ for each $n \in \mathbb{N}$.

Our main result in this short paper is the following.

Theorem 1.2. *Let $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ be the Collatz dynamical system with μ counting measure. The following are equivalent:*

(1) *There exists a finite measure α equivalent to μ for which the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \alpha)$ is power bounded in $L^1(\alpha)$ with conservative part $\{1, 2\}$.*
(2) *For each $n \in \mathbb{N}$ there exists k such that $T^k(n) \in \{1, 2\}$.*

Our main approach for the proof will come from ergodic theory. More precisely, we will consider the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ where μ is the counting measure, and study recurrence properties of the Collatz map T on the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$. The paper is organized as follows.

- In the second section we recall some tools from ergodic theory that we will use later. These tools include Hopf's decomposition, power bounded non-singular transformations and asymptotically mean bounded non-singular transformations.
- In the third section we apply these tools to the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ and obtain a partition of $\mathbb{N}$ into three sets C, D_1 and D_2. The set C is an at most countable union of cycles. Elements in D_1 after finitely iterates of T enter C. Finally, the invariant set D_2 under T (i.e. $T^{-1}(D_2) = D_2$) is the set of elements with unbounded trajectories. We establish ergodic properties of the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$. This decomposition is actually valid for all maps $V : \mathbb{N} \to \mathbb{N}$.
- In the fourth section we prove that the Collatz conjecture is equivalent to the existence of a finite measure ν on $(\mathbb{N}, 2^{\mathbb{N}})$ making T power bounded and non-singular.

2. Tools from ergodic theory

2.1. Hopf decomposition. In this subsection we recall some tools and definitions used in ergodic theory that we will apply in the next section to the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$.

Definition 2.1. Let $(X, \mathcal{A}, \rho)$ be a σ-finite measure space and V a measurable map from X to X. The map V is said to be *non-singular* with respect to ρ if for each measurable subset of $\mathcal{A}$, we have $\rho(V^{-1}(A)) = 0$ whenever $\rho(A) = 0$.

Definition 2.2. A set C is said to be V*-absorbing* if $C \subset V^{-1}(C)$.

It is clear that T is non-singular with respect to the counting measure μ, and to any other measure ν which is equivalent to μ. Actually for the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ the only null set is the empty set.

Definition 2.3. A measurable subset $W \in \mathcal{A}$ is said to be *wandering* if $V^{-i}(W) \cap V^{-j}(W)$ is empty whenever $i, j \in \mathbb{N}$ and $i \neq j$.

Almost every point of a wandering set W never returns to W under V. This means that for $x \in W$, $V^n x \notin W$ for each $n \in \mathbb{N}$.

The next theorem is the "Hopf decomposition" applied to a dynamical system $(X, \mathcal{A}, \rho, V)$ where V is non-singular with respect to the σ-finite measure ρ. A proof of the "Hopf Decomposition Theorem" can be found in Krengel's book [11, p. 17].

Theorem 2.4 (Hopf decomposition). *Let $(X, \mathcal{A}, V, \rho)$ be a non-singular dynamical system. The space X can be decomposed into two disjoint measurable subsets C and D with the following properties:*

(1) *C is V-absorbing.*
(2) *The restriction of V to C is conservative (the map V returns (a.e.) to every subset of C with positive measure infinitely often). In particular, if $\rho(m) > 0$ for every $m \in C$ then there exists $k(m) \in \mathbb{N}$ such that $V^{k(m)}(m) = m$.*
(3) *The set $D = C^c$ is called the dissipative part. It is an at most countable union of wandering sets.*

One can remark that "Hopf decomposition" is the same for each measure equivalent to ρ.

2.2. L^1 power bounded non-singular transformations. The second tool from ergodic theory is power bounded non-singular transformations.

DEFINITION 2.5. Let $(X, \mathcal{A}, V, \rho)$ be a non-singular dynamical system. It is *power bounded* in $L^1(\rho)$ if there exists a finite constant M such that $\rho(V^{-n}(A)) \leq M\rho(A)$ for all $A \in \mathcal{A}$ and all $n \in \mathbb{N}$.

Power bounded non-singular dynamical systems on finite measure spaces have nice recurrence properties expressed in the next theorem.

THEOREM 2.6. *Let $(X, \mathcal{A}, \rho, V)$ be a non-singular dynamical system. Assume that $\rho(X) < \infty$ and that this dynamical system is power bounded in $L^1(\rho)$. Then:*

(1) *There exists $v_0^* \in L^1_+(\rho)$ such that $\int \mathbb{1}_A v_0^* \, d\rho = \int \mathbb{1}_A \circ V v_0^* \, d\rho$.*
(2) *The conservative part C of $(X, \mathcal{A}, \rho, V)$ is equal to the set where $v_0^* > 0$.*
(3) *For ρ-a.e. $x \in D$ there exists $m(x) \in \mathbb{N}$ such that $V^{m(x)}(x) \in C$.*

The proof of this theorem can be found in [5, Theorem 12, p. 683] and [2, Theorem III.1].

The function v_0^* is obtained by considering the adjoint U^* of the operator U defined by $Uf = f \circ V$. As a consequence of the mean ergodic theorem (see [8]) the averages $M_N(f) = \frac{1}{N}\sum_{n=1}^N f \circ V^n$ converge for each $f \in L^1(\rho)$. We also have, for each $g \in L^\infty(\rho)$, the mean convergence of the averages $M_N^*(g) = \frac{1}{N}\sum_{n=1}^N (U^*)^n(g)$ where U^* is the adjoint operator of U. As a consequence we have the convergence of the averages $M_N^*(\mathbb{1}_X)$ to the U^*-invariant function v_0^*, and for each function $f \in L^1(\rho)$ we obtain

$$\lim_N \int M_N(f) \, d\rho = \lim_N \int f.M_N^*(\mathbb{1}_X) \, d\rho = \int f.v_0^* \, d\rho. \tag{2.1}$$

One can remark that the map $\gamma : \mathcal{A} \ni A \mapsto \gamma(A) = \int \mathbb{1}_A v_0^* \, d\rho$ is invariant with respect to V (i.e. $\gamma(A) = \gamma(V^{-1}(A))$ and $\gamma(X) = \rho(X) > 0$.

For an L^1 power bounded dynamical system defined on a finite measure space, almost every point x in D (the dissipative part given by Hopf's decomposition) eventually enters its conservative part C. Indeed, assume that this is not the case. Then we could find a set $B \in \mathcal{A}$ such that $\rho(B) > 0$ and

$V^m(B) \subset D$ for each $m \in \mathbb{N}$. Then we would have $B \subset V^{-m}(D)$ for each $m \in \mathbb{N}$ and using (2.1) we would obtain the following contradiction:

$$0 < \rho(B) \le \frac{1}{N} \sum_{n=1}^{N} \rho(V^{-m}(D)) \to_N \int \mathbb{1}_D v_0^* \, d\rho = 0.$$

2.3. Asymptotically mean bounded non-singular dynamical system. In this section we introduce one more tool from ergodic theory when the measure ρ is σ-finite and $\rho(X) = \infty$.

Definition 2.7. Let $(X, \mathcal{A}, \rho, V)$ be a non-singular dynamical system. We say that this system is *asymptotically mean bounded* in $L^1(\rho)$ if there exists a finite constant M such that

$$\limsup_N \frac{1}{N} \sum_{n=1}^{N} \int_Y \mathbb{1}_A \circ V^n \, d\rho \le M\rho(A)$$

for each measurable set $A \in \mathcal{A}$ and for each Y such that $\rho(Y) < \infty$.

It was shown in [15] that the system $(X, \mathcal{A}, \rho, V)$ is asymptotically mean bounded in $L^1(\rho)$ iff for each $f \in L^1(\rho)$ the averages

$$\frac{1}{N} \sum_{n=1}^{N} f(V^n x) \text{ converge a.e. to } f^*(x) \text{ where } f^* \in L^1(\rho).$$

We present one more result.

Theorem 2.8. *Let $(X, \mathcal{A}, \rho, V)$ be an asymptotically mean bounded non-singular dynamical system with ρ being a σ-finite measure.*

(1) *There exists an invariant σ-finite measure Δ such that $\Delta(A) \le M\rho(A)$ for all $A \in \mathcal{A}$.*
(2) *For each function $f \in L^1(\rho)$ the averages $\frac{1}{N}\sum_{n=1}^{N} f(V^n x)$ converge a.e. to $f^*(x)$ where $f^* \in L^1(\rho)$.*
(3) *If the system $(X, \mathcal{A}, \rho, V)$ satisfies conditions* (1)–(3) *of Theorem* 2.6 *then the averages $\frac{1}{N}\sum_{n=1}^{N} f(V^n x)$ converge a.e. for every $f \in L^\infty(\rho)$.*

Proof. The first two statements are consequences of results in [15]. The third statement can be derived from [7, Theorem 1]. ■

3. Ergodic properties of the Collatz map. In this section we apply the ergodic tools gathered in the previous section to the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$, T being the Collatz map and μ the counting measure. We start with the Hopf decomposition.

3.1. Hopf decomposition for the Collatz map

Theorem 3.1. *Consider the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$. There exists a partition of $\mathbb{N}$ into three sets C, D_1 and D_2 such that:*

(1) *C is the conservative part given by the Hopf decomposition (Theorem 2.4). It is composed of an at most countable number of cycles C_i, $1 \leq i < \infty$. The complement of C, the dissipative part, is partitioned into two subsets D_1 and D_2.*
(2) *D_1 is equal to $\bigcup_{k=1}^{\infty} T^{-k}(C) \setminus C$. It is the set of elements of D which enter C after finitely many iterates of T, into one of the cycles C_i.*
(3) *$T^{-1}(C \cup D_1) = C \cup D_1$.*
(4) *The set D_2 is the complement of $C \cup D_1$ into $\mathbb{N}$. It is the set of elements of $\mathbb{N}$ having an unbounded trajectory. The set D_2 is invariant, i.e. $T^{-1}(D_2) = D_2$.*

Proof. The fact that C, the conservative part, is a countable union of cycles follows from the fact that on C the map T is recurrent, meaning that it returns infinitely often to any set of positive measure in C. Since each point has positive measure, this recurrence property creates cycles in C. Their number is clearly at most countable since $\mathbb{N}$ itself is countable.

The set C is T-absorbing in the sense that $C \subset T^{-1}(C)$. The set D_1 is equal to $\bigcup_j T^{-j}(T^{-1}(C) \setminus C)$ which can be written $[\bigcup_{k=1}^{\infty} T^{-k}(C)] \setminus C$. Therefore, D_1 is the subset of $D = C^c$ composed of points not in C which enter C after finitely many iterates of T.

Since $\bigcup_{j=0}^{\infty} T^{-j}(C) = C \cup D_1$ and the sequence $T^{-j}(C)$ is increasing, we have

$$T^{-1}(C \cup D_1) = T^{-1}\Big[\bigcup_{j=0}^{\infty} T^{-j}(C)\Big] = \bigcup_{j=1}^{\infty} T^{-j}(C) = \bigcup_{j=0}^{\infty} T^{-j}(C) = C \cup D_1.$$

As a consequence, $T^{-1}(D_2) = D_2$ if we denote by D_2 the complement of $C \cup D_1$. ∎

Remark. The set D_1 is not empty because $T^{-1}(\{1,2\}) = \{1,2,4\}$ and 4 is in the dissipative part of T.

3.2. Pointwise convergence: Existence of an invariant measure. In this section we check that the tools listed in the previous section apply to the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$.

Theorem 3.2. *The system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ is asymptotically mean bounded in $L^1(\mu)$.*

Proof. We need to show that

$$\limsup_N \frac{1}{N} \sum_{n=1}^{N} \int_Y \mathbb{1}_A \circ T^n \, d\mu \leq M\mu(A). \tag{3.1}$$

for each measurable set $A \in \mathcal{A}$ and for each Y such that $\mu(Y) < \infty$. The condition $\mu(Y) < \infty$ implies that the set Y is finite. Therefore, it is enough to show that (3.1) holds for $Y = \{y\}$ for each $y \in \mathbb{N}$. We distinguish two cases.

If $\mu(A) = \infty$ then (3.1) is clearly true. We can assume that $\mu(A) < \infty$, or in other words A is a finite subset of $\mathbb{N}$. This observation allows us to reduce the proof of (3.1) to the case where $Y = \{y\}$ and $A = \{a\}$.

- If $(y, a) \in D \times D$ then $\lim_N \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_a(T^n y) = 0$ since $\{y\}$ and $\{a\}$ are wandering sets.
- If $(y, a) \in D_2 \times C$ then $\lim_N \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_a(T^n y) = 0$ because the orbit $\{T^n y : n \in \mathbb{N}\}$ is contained in the invariant set D_2.
- If $(y, a) \in D_1 \times C_j$ where C_j is one of the cycles in C, then
$$\lim_N \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_a(T^n y) = \frac{1}{\#C_j} \leq \mu(\{a\}).$$
- If $(y, a) \in C_l \times C_j$ where C_l and C_j are cycles in C, then
$$\lim_N \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_a(T^n y) = \frac{1}{\#C_j} \leq \mu(\{a\}) \quad \text{if } l = j$$
and $\lim_N \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_a(T^n y) = 0$ if $l \neq j$.

In summary, we have shown that
$$\lim_N \frac{1}{N} \sum_{n=1}^{N} \int_{\{y\}} \mathbb{1}_A \circ T^n \, d\mu = \lim_N \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_a(T^n y) \leq \mu(\{a\}).$$
By linearity we can conclude that
$$\lim_N \frac{1}{N} \sum_{n=1}^{N} \int_{Y} \mathbb{1}_A \circ T^n \, d\mu \leq \mu(A)$$
for each measurable set $A \in \mathcal{A}$ and for each Y such that $\mu(Y) < \infty$. In other words, the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ is asymptotically mean bounded in $L^1(\mu)$. The constant M is equal to 1. ■

As a consequence of Theorem 2.8 we obtain for the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ the properties listed in the previous section. The next theorem is part of the abstract.

THEOREM 3.3. *The dynamical system* $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ *has the following properties:*

(1) *There exists an invariant finite measure* γ *such that* $\gamma(A) \leq \mu(A)$ *for all* $A \subset \mathbb{N}$.
(2) *For each function* $f \in L^1(\mu)$ *the averages* $\frac{1}{N} \sum_{n=1}^{N} f(T^n x)$ *converge for every* $x \in \mathbb{N}$ *to* $f^*(x)$ *where* $f^* \in L^1(\mu)$.

Proof. Compared to Theorem 2.8, we can eliminate the "almost everywhere condition" in the second part of the theorem because the only nullset for $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ is the empty set.

(1) One can make the value of f^* more explicit by using the computations made for the various cases for (y,a). For $f_A = \mathbb{1}_A$ with $\mu(A) < \infty$, we have

$$f_A^* = \sum_{i=1}^{\infty} \mathbb{1}_{\bigcup_{j=0}^{\infty} T^{-j}(C_i)} \frac{\#(A \cap C_i)}{\# C_i}. \tag{3.2}$$

(2) As a consequence of (3.2) one can find a finite invariant measure equivalent to Δ. It suffices to take a finite measure β equivalent to μ and integrate f_A^* with respect to β. We obtain

$$\gamma(A) = \int f_A^* \, d\beta = \sum_{i=1}^{\infty} \beta\Big(\bigcup_{j=0}^{\infty} T^{-j}(C_i)\Big) \frac{\#(A \cap C_i)}{\# C_i}.$$

We have $\gamma(A) = \gamma(T^{-1}(A))$ for all $A \in 2^{\mathbb{N}}$ with $\mu(A) < \infty$, since $f_A^* \circ T = f_A^*$. The measure being finite, this last equality extends by continuity to $2^{\mathbb{N}}$. One can observe that $\gamma(A) = 0$ if $A \subset D$ since in this case $\#(A \cap C_i)/\# C_i = 0$, the sets C_i being in the conservative part C. In other words, the measure γ is supported on C. The measure γ is finite because the sets $\bigcup_{j=0}^{\infty} T^{-j}(C_i)$ are disjoint and thus

$$\sum_{i=1}^{\infty} \beta\Big(\bigcup_{j=0}^{\infty} T^{-j}(C_i)\Big) \le \beta(\mathbb{N}) < \infty.$$

One can choose β in such a way that $\beta(\bigcup_{j=0}^{\infty} T^{-j}(C_i)) = 2^{-i}$. Then

$$\gamma(A) = \sum_{i=1}^{\infty} 2^{-i} \frac{\nu_i(A)}{p_i},$$

where ν_i is the finite invariant measure with support C_i and defined by $\nu_i(A) = \#(A \cap C_i)$ and $p_i = \#(C_i)$ is the period of the cycle C_i. ∎

REMARK. 1. If the Collatz conjecture is true then the invariant measure γ is uniform on the cycle $\{1, 2\}$.

2. If the conjecture is false in the sense that there are additional cycles then there are many invariant finite measures, e.g. barycentric averages of the uniform invariant measures defined on these cycles.

3. If the conjecture is false and there is an unbounded orbit then we can construct on D_2 many σ-finite infinite invariant measures. Such a construction is made in the following theorem.

We will need the following lemma.

LEMMA 3.4. *Let* $\mathbb{N}_2 = \{k \in \mathbb{N} : 2k = 1 \bmod 3\}$ *and let* $a \in D_2$. *There exist an infinite number of* $k \in \mathbb{N}$ *such* $T^k a \in \mathbb{N}_2$.

Proof. The set $\mathbb{N}_2$ is also the set of natural numbers of the form $3p + 2$. We denote $\mathbb{N}_1 = \{3p + 1 : p \in \mathbb{N}\}$ and $\mathbb{N}_0 = \{3p : p \in \mathbb{N}\}$. It is enough to show that if we take an element in the orbit of a, say $q = T^m a \in \mathbb{N}_0 \cup N_1$, then there exists a natural number s such that $T^{s+m}a \in \mathbb{N}_2$. We distinguish two cases:

(1) If $q = 3^k 2^h n$ where $n \notin \mathbb{N}_0 \cup 2\mathbb{N}$ then $T^h q = 3^k n$, which is odd. Thus $T^{h+1} q = \frac{1}{2}(3^{k+1} n + 1)$, which belongs to $\mathbb{N}_2$.
(2) If $q = 6p + r$ where $r = 1, 4$ (the case $r = 3$ implies $q \in \mathbb{N}_0$ treated in (1)) then $T(6p+1) = 9p + 2 \in \mathbb{N}_2$ and $T(6p+4) = 3p + 2 \in \mathbb{N}_2$.

These estimates prove the lemma. ∎

THEOREM 3.5. *If the Collatz conjecture is false then there exist σ-finite infinite invariant measures with support in D_2. Furthermore these measures do not satisfy condition* (1) *of Theorem* 2.8.

Proof. Denote again $\mathbb{N}_2 = \{k \in \mathbb{N} : 2k = 1 \bmod 3\}$. These are the only natural numbers k such that $\#\{T^{-1}(k)\} = 2$. Take $a \in D_2$ and consider the invariant set $\mathcal{F} = \bigcup_{j=0}^{\infty} T^{-j} \bigcup_{k=0}^{\infty} \{T^k a\}$. To check that a measure θ is invariant it is enough to verify that $\theta(T^{-1}(x)) = \theta(x)$ at each $x \in \mathcal{F}$.

We start by setting $\theta(a) = 1$. In the subtree generated by $\{a\}$, that is, the set $\mathcal{T}_1 = \bigcup_{j=0}^{\infty} T^{-j}(a)$, we distinguish two cases:

- If $a \notin \mathbb{N}_2$ then $\#\{T^{-1}(a)\} = 1$ and we define $\theta(T^{-1}(a)) = 1$.
- If $a \in \mathbb{N}_2$ then $\{T^{-1}(a)\} = \{b_1, b_2\}$ and we define $\theta(b_1) = \theta(b_2) = 1/2$.

We have $\theta(T^{-1}(a)) = \theta(a)$. We proceed in a similar way along the subtree $\bigcup_{j=0}^{\infty} T^{-j}(a)$. To preserve the invariance property of θ along this subtree we consider $T^{-1}(b_1)$ and $T^{-1}(b_2)$, again distinguishing the cases where the cardinality of these sets is 1 or 2. For instance, if $T^{-1}(b_1) = \{c_1, c_2\}$ then we set $\theta(c_1) = \theta(c_2) = \frac{1}{2}\theta(b_1)$. If $T^{-1}(b_2) = \{d\}$ then $\theta(d) = \theta(b_2)$.

Proceeding inductively along the subtree we define θ for each node of this subtree in such a way that $\theta(T^{-1}(n)) = \theta(n)$ for each n in the subtree.

Now we can define θ on the subtree $\mathcal{T}_2 = \bigcup_{j=0}^{\infty} T^{-j}(Ta)$. Here again we distinguish two cases:

- If $Ta \notin \mathbb{N}_2$ then we set $\theta(Ta) = \theta(a) = 1$.
- If $Ta \in \mathbb{N}_2$ then $T^{-1}(Ta) = \{a, e\}$, and we set $\theta(e) = 1$ and $\theta(Ta) = 2$.

To define θ on the subtree $\bigcup_{j=0}^{\infty} T^{-j}(e)$ we can proceed as we did for $\bigcup_{j=0}^{\infty} T^{-j}(a)$. One can observe that the two subtrees $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint since any node in the intersection would have two distinct images under T.

The set $\{T^k a : k \in \mathbb{N}\}$ containing no cycle because D_2 does not have any, we can proceed by induction on $T^j(a)$ and define an invariant measure on the invariant set $\mathcal{F}$ the measure is σ-finite, and infinite since

$$\sum_{k=0}^{\infty} \theta(T^k a) \geq \sum_{k=0}^{n} \theta(T^k a) \geq n + 1.$$

To complete the proof we can use Lemma 3.4. In the orbit of $\{a\}$ under T there are infinitely nodes in $\mathbb{N}_2$. Therefore, $\limsup_k \theta(T^k a) = \infty$ and this violates condition (1) in Theorem 2.8 since $M\mu(T^k a) = M$. ∎

4. Characterization of the Collatz conjecture through power bounded non-singular transformations. In this section we use some of the ideas in the proof of Theorem 3.1 to derive a characterization of the Collatz map through power bounded non-singular transformations. More precisely, we prove the following theorem by exploiting the fact that the map T is onto and that $T^{-1}(k)$ is a singleton unless $2k = 1 \bmod 3$.

THEOREM 4.1. *Let $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ be the Collatz dynamical system with μ the counting measure. The following are equivalent:*

(1) *There exists a finite measure α equivalent to μ for which the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \alpha)$ is power bounded in $L^1(\alpha)$ with conservative part $\{1, 2\}$.*
(2) *For each $n \in \mathbb{N}$ there exists k such that $T^k(n) \in \{1, 2\}$.*

Proof. (1) implies (2) since as indicated in Theorem 2.6, for a power bounded non-singular transformation, points in D enter the conservative part C after finitely many iterates. The assumption in (2) implies that $\{1, 2\}$ is the conservative part of $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ (and for any measure equivalent to μ). Simple considerations show that $T^{-1}(C) \setminus C = \{4\}$. Therefore D is just the "tree" created by the inverse map T^{-1} starting at 4. This is the set $\bigcup_{j=0}^{\infty} T^{-j}(\{4\})$.

One can construct a power bounded transformation by starting with $\delta = \alpha(4) > 0$ and then looking at the predecessors of 4. Since $8 = T^{-1}(4)$ is the only predecessor of 4, we set $\alpha(8) = \frac{1}{4}\delta$. Then 8 has two predecessors 16 and 5. We set $\alpha(16) = \frac{1}{4}\alpha(8)$ and $\alpha(5) = \frac{1}{4}\alpha(8)$. By induction we define $\alpha(n)$ for each $n \in D$ to be $\frac{1}{4}$ of the value of its predecessor in the tree. With this process for any subset A of the tree $\bigcup_{j=0}^{\infty} T^{-j}(\{4\})$ we have $\alpha(T^{-1}(A)) \le \frac{1}{2}\alpha(A)$. Furthermore, we can see that $\alpha(T^{-n}(A)) \le \frac{1}{2^n}\alpha(A)$ for every $n \in \mathbb{N}$ since $T^{-1}(D) \subset D$. Therefore, we just need to control the values of $\alpha(T^{-n}(1))$ and $\alpha(T^{-n}(2))$ for each $n \in \mathbb{N}$. But since $T^{-n}(2) = T^{-(n+1)}(1)$, it is actually enough to show that for appropriate choices of δ, $\alpha_1 = \alpha(1)$ and $\alpha_2 = \alpha(2)$ we can make the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \alpha)$ power bounded in $L^1(\alpha)$.

Simple computations show $T^{-1}(1) = \{2\}$, $T^{-2}(1) = \{1, 4\}$, $T^{-3}(1) = \{T^{-1}(1), T^{-1}(4)\} = \{2, T^{-1}(4)\}$. By induction one can show that

$$T^{-n}(1) \subset \{1, 2\} \cup \bigcup_{j=0}^{n-2} T^{-j}(\{4\}).$$

Indeed, assuming this is true for n we have

$$T^{-(n+1)}(\{1,2\}) \subset T^{-1}(\{1,2\}) \cup \bigcup_{j=1}^{n-1} T^{-j}(\{4\}) \subset \{1,2\} \cup \bigcup_{j=0}^{n-1} T^{-j}(\{4\}).$$

Setting $\alpha_1 = \alpha_2 = \delta$ we see that

$$\alpha(T^{-n}(1)) \le \alpha_1 \sum_{j=0}^{n-2} \frac{1}{2^j} \le 2\alpha_1.$$

Combining this with our previous estimates one can conclude that for each $n \in \mathbb{N}$ and each $A \subset \mathbb{N}$ we have $\alpha(T^{-n}(A)) \le 2\alpha(A)$. ∎

We can also derive the following result which characterizes the boundedness of the trajectories of the Collatz map.

THEOREM 4.2. *Let $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ be the Collatz dynamical system with μ the counting measure. The following are equivalent:*

(1) *There exists a finite measure α equivalent to μ for which the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \alpha)$ is power bounded in $L^1(\alpha)$.*
(2) *The set D_2 is empty.*
(3) *The trajectory of each point $n \in \mathbb{N}$ is bounded.*
(4) *For every bounded f on $\mathbb{N}$, the averages $\frac{1}{N}\sum_{n=1}^{N} f(T^n x)$ converge for every $x \in \mathbb{N}$.*

Proof. The equivalence of (2) and (3) follows from Theorem 3.1. Statement (1) implies (2) because all points in the dissipative part eventually enter one of the cycles C_i. Therefore, their trajectories are bounded and the set D_2 must be empty. Conversely, if D_2 is empty then $\mathbb{N}$ can be partitioned into the disjoint sets

$$F_j = \bigcup_{i=1}^{\infty} T^{-i}(C_j).$$

Using the same method as in the proof of Theorem 4.1 we can define on each set F_j a measure ν_j such that for any $A_j \subset F_j$ we have $\nu_j(T^{-n}(A_j)) \le 2\nu_j(A_j)$ for each $n \in \mathbb{N}$. We now define $\alpha = \sum_{j=1}^{\infty} \frac{1}{2^j}\nu_j$. One can check that $\alpha(T^{-n}(A)) \le 2\alpha(A)$ for each subset A of $\mathbb{N}$. Thus the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \alpha)$ is power bounded in $L^1(\alpha)$. This shows that (1) and (2) are equivalent.

It remains to show that (4) is equivalent to all the other statements. It is enough to show that it is equivalent to (2). By [7], (4) implies (2). For the converse one can observe that Theorem 3.3 shows that there exists a finite invariant measure with support C. This together with (2) implies (4) by [7], completing the proof. ∎

We have the following corollary.

Corollary 4.3. *The following are equivalent for the Collatz dynamical system* $(\mathbb{N}, 2^{\mathbb{N}}, \mu, T)$*:*

(1) *For each* $n \in \mathbb{N}$ *there exists* k *such that* $T^k n \in \{1, 2\}$.
(2) *For every bounded* f *and* $x \in \mathbb{N}$ *we have*

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n x) \to \frac{1}{2}(f(1) + f(2)).$$

Proof. The first statement implies that D_2 is empty and that the only cycle is $\{1, 2\}$. By Theorem 4.2(4), for each bounded f on $\mathbb{N}$ the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n x)$$

converge for each $x \in \mathbb{N}$. It remains to identify the limit to obtain (2). But this a consequence of the fact that for any x there exists a natural number $m(x)$ such $T^{m(x)}x = 1$. For $k > m(x)$ the terms $T^k x$ alternate between 2 and 1. This implies that the limit of the averages is equal to $\frac{f(1)+f(2)}{2}$.

For the converse, (2) shows that $\{1, 2\}$ is the only cycle. If not, taking x in another cycle with period p, $C' = \{a, Ta, \dots, T^{p-1}a\}$, and applying the second statement with $f = \mathbb{1}_{C'}$ we would get

$$\frac{\sum_{k=1}^{p} f(T^k x)}{p} = 1 \neq 0 = \frac{f(1) + f(2)}{2}.$$

Since the averages for f bounded on $\mathbb{N}$ converge for each $x \in \mathbb{N}$, this implies that the set D_2 is empty by Theorem 4.2, and proves (1). ∎

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