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Iterative roots of multifunctions

by

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Abstract. Some easily verifiable sufficient conditions for the nonexistence of iterative roots for multifunctions on arbitrary nonempty sets are presented. Typically if the graph of the multifunction has a distinguished point with a relatively large number of paths leading to it then such a multifunction does not admit any iterative root. These results can be applied to single-valued maps by considering their pullbacks as multifunctions. This is illustrated by showing the nonexistence of iterative roots of some specified orders for certain complex polynomials.

1. Introduction. Given a map $F : X \to X$ on a nonempty set X and an integer $n \geq 1$, the *iterative root problem* is to find a map $G : X \to X$ such that the functional equation

$$G^n = F \tag{1.1}$$

is true on X, where G^n is the nth order iterate of G defined recursively by $G^n = G \circ G^{n-1}$ and $G^0 = \mathrm{id}$, the identity map on X. We call G an nth *order iterative root* of F. Since the initial works of Babbage [3], Abel [1] and Königs [18], the iterative root problem (1.1), which is a weak version of the embedding flow problem [10] of dynamical systems and is applicable to informatics [17] and neural networks [13], has been extensively studied in various aspects. Many of the results are included in the monographs [19, 20], the book [40], and the survey papers [4, 41, 43]. For some of the most recent findings, see [5, 6, 9, 14, 23–28, 30, 42].

Many researchers [6, 7, 12, 38] have highlighted the difficulty in solving equation (1.1), even in the class of continuous self-maps of an interval, necessitating the extension of the iterative root concept to multifunctions. By a *multifunction* (or *multivalued function*) on a nonempty set X, we simply

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mean a function from X to the power set 2^X. Given two multifunctions F and G on X, their composition $F \circ G$ is defined by $(F \circ G)(x) = F(G(x))$, where the *image* $F(A)$ of a set $A \subseteq X$ is defined by $F(A) = \bigcup_{x \in A} F(x)$. Then, as shown in [16], the nth order iterate F^n of F is defined recursively by

$$F^n(x) = \bigcup_{y \in F^{n-1}(x)} F(y) \quad \text{and} \quad F^0(x) = \{x\},$$

the iterates satisfy $F^n(F^m(A)) = F^{n+m}(A) = F^m(F^n(A))$. As in [11], a point $x_0 \in X$ is said to be a *set-value point* of F if $\#F(x_0) \geq 2$, where $\#A$ denotes the cardinality of a set $A \subseteq X$. Powierża [33–35] and Jarczyk and Powierża [15] discussed the existence of the smallest set-valued iterative roots G of bijections F in the *inclusion sense*, where $F(x) \in G^n(x)$ for all $x \in X$. Another approach (see [16, 21, 29, 31, 44] for example) is to look for solutions $G : X \to 2^X$ of (1.1) for a multifunction F, which is known as the *identity sense*. In [16, 21], some sufficient conditions for the nonexistence of iterative square roots of multifunctions F on X with exactly one set-value point were obtained, and in [31], some of these results were improved and generalized to any order n. The special case where X is a compact interval in the real line $\mathbb{R}$ and F an increasing upper semicontinuous multifunction was also studied, and results on the construction of iterative roots were presented in [21, 22, 31, 32, 44]. However, it appears that no results have been found for multifunctions with multiple set-value points, with the exception of [29] for increasing upper semicontinuous multifunctions on compact intervals with finitely many set-value points.

In Section 2 we investigate equation (1.1) in the identity sense for multifuctions F on a general nonempty set X. The crucial observation we make is that if there is a point $x_0 \in X$ with a large number of paths ending at it in the graph of F, it creates a kind of bottleneck in the system $F : X \to 2^X$ (see Figure 1 for example) and thus the multifunction F will admit no roots of any order. This is heavily inspired by a similar phenomenon observed for single-valued maps in [5, 6]. We do impose a strong constraint that the number of 2-paths ending at x_0 is very *large* in comparison to the number of 1-paths (edges) beginning or ending at other points, in addition to some mild conditions such as F having domain X and that x_0 is not a fixed point of F. Some of the results also demand that F has image X. All of this is made precise in Theorem 2.1 by quantifying the notion of largeness in various ways. Notably, we put no constraints on the number of set-value points for F. We present several examples to show that these results can be applied in some situations where the known results in [16, 21, 31] fail.

Multifunctions appear naturally in topology, measure theory and other fields when considering inverse images of single-valued maps. Keeping this in mind, in Section 3, we introduce a fundamental class of multifunctions F called *pullback multifunctions*. An important example of such a multifunction is the *kth root function* $z := re^{i\theta} \mapsto z^{1/k} := \{r^{1/k}e^{i(\theta+2j\pi)/k} : j = 0, 1, \ldots, k-1\}$ on the complex plane $\mathbb{C}$. The iterative root problem for such multifunctions can be reduced to the corresponding problem for single-valued maps. We present several examples to demonstrate concrete applications of our general results, particularly in determining the nonexistence of iterative roots for certain complex polynomials.

2. General multifunctions. In this section we present several sufficient conditions for the nonexistence of iterative roots for multifunctions on arbitrary nonempty sets. First, we present some notions and notation required for our discussion. Recall that by a multifunction on a nonempty set X, we mean a function from X to the power set 2^X. Let X be a nonempty set and let $\mathcal{F}(X)$ consist of all multifunctions F on X. For each set $A \subseteq X$, $F \in \mathcal{F}(X)$, and $k \geq 1$, let $F^{-k}(A)$ denote the kth order *inverse image* of A by F defined by

$$F^{-k}(A) = \{x \in X : F^k(x) \cap A \neq \emptyset\}.$$

As in [2, p. 34], the *domain* and *image* of $F \in \mathcal{F}(X)$ are defined by

$$\mathrm{Dom}(F) = \{x \in X : \#F(x) \geq 1\} \quad \text{and} \quad \mathrm{Im}(F) = F(X).$$

As in [2, p. 34], we define the *graph* of $F \in \mathcal{F}(X)$ as the directed graph $\mathcal{G}_F = (X, E_F)$, with vertex set X and edge set $E_F = \{(x, y) \in X \times X : y \in F(x)\}$. For $x, y \in X$ and $k \in \mathbb{N}$, by a *k-path* (or *length k path*) in $\mathcal{G}_F$ from x to y, we mean a sequence $((x, u_1), (u_1, u_2), \ldots, (u_{k-2}, u_{k-1}), (u_{k-1}, y))$, where $u_1, \ldots, u_{k-1} \in X$ and $u_1 \in F(x)$, $u_{i+1} \in F(u_i)$ for $i = 1, \ldots, k-2$, and $y \in F(u_{k-1})$, of edges (not necessarily distinct) that joins a sequence of vertices $(x, u_1, \ldots, u_{k-1}, y)$. For all $F, G \in \mathcal{F}(X)$, $x, y, z \in X$, $A, B \subseteq X$ and $k, l \in \mathbb{N}$, let

$$\begin{aligned}
P_F(x, y; k) &:= \{p : p \text{ is a } k\text{-path in } \mathcal{G}_F \text{ from } x \text{ to } y\},\\
P_F(A, y; k) &:= \{p : p \text{ is a } k\text{-path in } \mathcal{G}_F \text{ that}\\
&\qquad\qquad \text{begins at a point in } A \text{ and ends at } y\},\\
P_F(x, A; k) &:= \{p : p \text{ is a } k\text{-path in } \mathcal{G}_F \text{ that}\\
&\qquad\qquad \text{begins at } x \text{ and ends at a point in } A\},\\
P_F(A, B; k) &:= \{p : p \text{ is a } k\text{-path in } \mathcal{G}_F \text{ that}\\
&\qquad\qquad \text{begins at a point in } A \text{ and ends at a point in } B\},
\end{aligned}$$

and

$$P_F(x, y; k) \vee P_G(y, z; l) := \{p \vee q : p \in P_F(x, y; k) \text{ and } q \in P_G(y, z; l)\},$$

where for any two paths

$$p := ((x,u_1),(u_1,u_2),\dots,(u_{k-2},u_{k-1}),(u_{k-1},y)) \in P_F(x,y;k),$$
$$q := ((y,v_1),(v_1,v_2),\dots,(v_{l-2},v_{l-1}),(v_{l-1},z)) \in P_G(y,z;l),$$

$p \vee q$ is the $(k+l)$-path in $\mathcal{G}_F \cup \mathcal{G}_G$ obtained by concatenating them, defined by

$$p \vee q = ((x,u_1),(u_1,u_2),\dots,(u_{k-2},u_{k-1}),(u_{k-1},y),$$
$$(y,v_1),(v_1,v_2),\dots,(v_{l-2},v_{l-1}),(v_{l-1},z)).$$

Then it is easy to see that

$$P_F(A,y;k) = \bigcup_{x\in A} P_F(x,y;k), \qquad P_F(x,A;k) = \bigcup_{y\in A} P_F(x,y;k),$$
$$P_F(A,B;k) = \bigcup_{y\in B} P_F(A,y;k) = \bigcup_{x\in A} P_F(x,B;k)$$

and

$$P_F(x,y;k) \vee P_G(y,z;l) = P_{G\circ F}(x,z;k+l)$$

for all $F,G \in \mathcal{F}(X)$, $x,y,z \in X$, $A,B \subseteq X$ and $k,l \in \mathbb{N}$. Let

$$\mathcal{F}_M(X) := \{F \in \mathcal{F}(X) : \#P_F(x,X;1) \le M \text{ for all } x \in X\} \text{ for each } M \in \mathbb{N},$$
$$\mathcal{F}_{\mathsf{f}}(X) := \{F \in \mathcal{F}(X) : P_F(x,X;1) \text{ is finite for all } x \in X\},$$
$$\mathcal{F}_{\mathsf{c}}(X) := \{F \in \mathcal{F}(X) : P_F(x,X;1) \text{ is countable for all } x \in X\}.$$

In other words, $\mathcal{F}_M(X)$ is the set of all multifunctions on X that have a set-value of cardinality at most M at each point of X. Similarly, $\mathcal{F}_{\mathsf{f}}(X)$ (resp. $\mathcal{F}_{\mathsf{c}}(X)$) is the set of all multifunctions on X that have only finite (resp. countable) sets as a value at each point of X. Further, note that $\#P_F(X,x;1) = \#F^{-1}(\{x\})$ and $\#P_F(X,x;k) \ge \#F^{-k}(\{x\})$, and similarly $\#P_F(x,X;1) = \#F(x)$ and $\#P_F(x,X;k) \ge \#F^k(x)$ for all $x \in X$ and $k \ge 2$.

Now we present our main result. It is motivated by analogous results for single-valued maps in [5, 6]. We observe that, generally, if a multifunction F has a special point x_0 with a relatively large number of paths leading to it, then F does not admit any iterative root. Actually, we only compare the number of paths of length 2 reaching x_0 with the number of edges at other points. The comparison is made precise in the following theorem.

THEOREM 2.1. *Let F be a multifunction on X such that* $\mathrm{Dom}(F) = X$ *and $x_0 \notin F(x_0)$ for some $x_0 \in X$.*

(1) (Large finite sets vs small finite sets) *Suppose that for some $M,N \in \mathbb{N}$,*

- **(1.a)** $\#P_F(X,x_0;2) > MN^3$,
- **(1.b)** $\#P_F(X,x;1) \le N$ *for all $x \ne x_0$ in X.*

Then F has no iterative roots of order $n \geq 2$ in $\mathcal{F}_M(X)$. If, in addition, $F \in \mathcal{F}_M(X)$ and $\mathrm{Im}(F) = X$, then F has no iterative roots of order $n \geq 2$ at all.

(2) (Infinite sets vs finite sets) *Suppose that*

(2.a) *$P_F(X, x_0; 2)$ is infinite,*
(2.b) *$P_F(X, x; 1)$ is finite for all $x \neq x_0$ in X.*

Then F has no iterative roots of order $n \geq 2$ in $\mathcal{F}_{\mathsf{f}}(X)$. If, in addition, $F \in \mathcal{F}_{\mathsf{f}}(X)$ and $\mathrm{Im}(F) = X$, then F has no iterative roots of order $n \geq 2$ at all.

(3) (Uncountable sets vs countable sets) *Suppose that*

(3.a) *$P_F(X, x_0; 2)$ is uncountable,*
(3.b) *$P_F(X, x; 1)$ is countable for all $x \neq x_0$ in X.*

Then F has no iterative roots of order $n \geq 2$ in $\mathcal{F}_{\mathsf{c}}(X)$. If, in addition, $F \in \mathcal{F}_{\mathsf{c}}(X)$ and $\mathrm{Im}(F) = X$, then F has no iterative roots of order $n \geq 2$ at all.

Proof. Suppose, on the contrary, that F has an iterative root G of some order $n \geq 2$ on X. The key to the proof is to estimate $P_G(G^{n-1}(X), x_0; n+1)$ in two different ways by observing that $G^{n+1} = F \circ G = G \circ F$. We begin by establishing relations (2.1)–(2.7) below, which are all true regardless of whether we are proving **(1)**, **(2)** or **(3)**.

Since $x_0 \notin F(x_0)$, it is clear that $x_0 \notin G(x_0)$. Since $P_G(X, x_0; n-1) \neq \emptyset$ by **(j.a)**, where $\mathbf{j} = \mathbf{1}$, $\mathbf{2}$ or $\mathbf{3}$ depending on whether we are proving **(1)**, **(2)** or **(3)**, and $P_G(x_0, X; 1) \neq \emptyset$ as $\mathrm{Dom}(F) = X$, we have

$$P_G(X, x_0; n-1) \vee P_G(x_0, X; 1) \neq \emptyset. \tag{2.1}$$

Also, if $n > 2$, we have

$$P_G(X, x_0; n-1) = P_G(X, G^{n-2}(X); n-2) \vee P_G(G^{n-2}(X), x_0; 1),$$

implying that

$$\#P_G(X, x_0; n-1) \geq \#P_G(G^{n-2}(X), x_0; 1). \tag{2.2}$$

Note that (2.2) is a triviality if $n = 2$. In particular, we see from (2.2) that

$$\#\big(P_G(X, x_0; n-1) \vee P_G(x_0, X; 1)\big) \geq \#P_G(G^{n-2}(X), x_0; 1). \tag{2.3}$$

Further, we have

(2.4)

$$\begin{aligned} P_G(X, x_0; n-1) \vee P_G(x_0, X; 1) &= P_G(X, x_0; n-1) \vee P_G(x_0, G(x_0); 1) \\ &\subseteq P_G(X, G(x_0); n) = P_F(X, G(x_0); 1) \\ &= \bigcup_{y \in G(x_0)} P_F(X, y; 1) \end{aligned}$$

and

$$P_F(X,x;1) = \bigcup_{y\in G^{n-1}(X)\cap G^{-1}(\{x\})} P_G(X,y;n-1)\vee P_G(y,x;1) \tag{2.5}$$

for all $x \neq x_0$ with $F^{-1}(\{x\}) \neq \emptyset$. Moreover, since $G^{n+1} = G\circ F$, we have

$$P_G(X,x_0;n+1) = \bigcup_{y\in F(X)\cap G^{-1}(\{x_0\})} P_F(X,y;1)\vee P_G(y,x_0;1), \tag{2.6}$$

and since $G^{n+1} = F\circ G$, we have

$$\begin{aligned} &P_F(X,x_0;2) = P_F(X,F(X);1)\vee P_F(F(X),x_0;1) \\ &= \big(P_G(X,G^{n-1}(X);n-1)\vee P_G(G^{n-1}(X),F(X);1)\big)\vee P_F(F(X),x_0;1) \\ &= P_G(X,G^{n-1}(X);n-1)\vee \big(P_G(G^{n-1}(X),F(X);1)\vee P_F(F(X),x_0;1)\big) \\ &= \bigcup_{x\in F^{-1}(\{x_0\})}\ \bigcup_{y\in G^{n-1}(X)\cap G^{-1}(\{x\})} P_G(X,y;n-1)\vee \big(P_G(y,x;1)\vee P_F(x,x_0;1)\big). \end{aligned} \tag{2.7}$$

To prove the first part of **(1)**, suppose that $G\in\mathcal{F}_M(X)$. We proceed by showing the following two inequalities, which we use to estimate $P_G(G^{n-1}(X),x_0;n+1)$:

$$\#P_G(G^{n-2}(X),x_0;1)\le MN, \tag{2.8}$$

$$\#P_G(X,y;n-1)\le N \text{ if } y\in G^{n-1}(X)\cap G^{-1}(\{x\}) \text{ and } x\neq x_0. \tag{2.9}$$

Since $\#G(x_0)\le M$ as $G\in\mathcal{F}_M(X)$, by using (2.3), (2.4) and **(1.b)** we have

$$\#P_G(G^{n-2}(X),x_0;1)\le \sum_{y\in G(x_0)} \#P_F(X,y;1)\le N\cdot\#G(x_0)\le NM,$$

proving (2.8). Further, by using (2.5) and **(1.b)** we see that

$$\#P_G(X,y;n-1)\le \#\big(P_G(X,y;n-1)\vee P_G(y,x;1)\big)\le \#P_F(X,x;1)\le N$$

whenever $y\in G^{n-1}(X)\cap G^{-1}(\{x\})$ and $x\neq x_0$, proving (2.9).

We now estimate $P_G(G^{n-1}(X),x_0;n+1)$ in two different ways to get a contradiction. Since $G^{n+1} = G\circ F$, by using (2.6), **(1.b)** and (2.8) we see that

$$\begin{aligned} \#P_G(X,x_0;n+1) &\le N \sum_{y\in F(X)\cap G^{-1}(\{x_0\})} \#P_G(y,x_0;1) \\ &= N\cdot\#P_G(F(X)\cap G^{-1}(\{x_0\}),x_0;1) \\ &\le N\cdot\#P_G(F(X),x_0;1) = N\cdot\#P_G(G^n(X),x_0;1) \\ &\le N\cdot\#P_G(G^{n-2}(X),x_0;1)\le N\cdot MN = MN^2. \end{aligned}$$

Therefore, as $G^{n-1}(X)\subseteq X$, we have

$$\#P_G(G^{n-1}(X),x_0;n+1)\le MN^2. \tag{2.10}$$

On the other hand, since $G^{n+1} = F \circ G$, by using (2.7), **(1.a)** and (2.9) we get

$$\begin{aligned} MN^3 < \#P_F(X, x_0; 2) &\le N \sum_{\substack{x \in F^{-1}(\{x_0\}) \\ y \in G^{n-1}(X) \cap G^{-1}(\{x\})}} \#\big(P_G(y, x; 1) \vee P_F(x, x_0; 1)\big) \\ &= N \sum_{y \in G^{n-1}(X) \cap G^{-(n+1)}(\{x_0\})} \#P_G(y, x_0; n+1) \\ &= N \cdot \#P_G(G^{n-1}(X) \cap G^{-(n+1)}(\{x_0\}), x_0; n+1) \\ &\le N \cdot \#P_G(G^{n-1}(X), x_0; n+1), \end{aligned}$$

implying that

$$\#P_G(G^{n-1}(X), x_0; n+1) > MN^2,$$

contrary to (2.10).

Now, to prove the second part of **(1)**, let $F \in \mathcal{F}_M(X)$ with $\mathrm{Im}(F) = X$, and suppose that $G \in \mathcal{F}(X)$. Then $\mathrm{Im}(G) = X$, and by the first part we have $\#P_G(\tilde{x}, X; 1) > M$ for some $\tilde{x} \in X$. Let $\tilde{x} \in G^{n-1}(\tilde{y})$ for some $\tilde{y} \in X$, which exists because $G^{n-1}(X) = X$. Since

$$F(\tilde{y}) = G(G^{n-1}(\tilde{y})) \supseteq G(\tilde{x}) \tag{2.11}$$

and $\#P_G(\tilde{x}, X; 1) > M$, we obtain $\#P_F(\tilde{y}, X; 1) \ge \#P_G(\tilde{x}, X; 1) > M$, which contradicts $F \in \mathcal{F}_M(X)$. This completes the proof of **(1)**.

Next, to prove the first part of **(2)**, suppose that $G \in \mathcal{F}_{\mathsf{f}}(X)$. Then, as seen above, we have $x_0 \notin G(x_0)$, and (2.1)–(2.7) are satisfied. Additionally, since $P_G(x_0, X; 1)$ is finite as $G \in \mathcal{F}_{\mathsf{f}}(X)$, and $P_F(X, y; 1)$ is finite for all $y \neq x_0$ by **(2.b)**, we see from (2.4) that $P_G(X, x_0; n-1) \vee P_G(x_0, X; 1)$ is finite, which implies by (2.3) that $P_G(G^{n-2}(X), x_0; 1)$ is finite. Further, it follows from (2.5) and **(2.b)** that

$$P_G(X, y; n-1) \text{ is finite} \quad \text{if } y \in G^{n-1}(X) \cap G^{-1}(\{x\}) \text{ and } x \neq x_0. \tag{2.12}$$

We now estimate $\#P_G(G^{n-1}(X), x_0; n+1)$ in two different ways, as above, to find a contradiction. In fact, since $G^{n+1} = G \circ F$, by using (2.6), **(2.b)** and the fact that $P_G(G^{n-2}(X), x_0; 1)$ is finite, we see that $P_G(G^{n-1}(X), x_0; n+1)$ is finite. On the other hand, since $G^{n+1} = F \circ G$, by using (2.7), (2.12) and **(2.a)** it follows that $P_G(G^{n-1}(X), x_0; n+1)$ is infinite.

In order to prove the second part of **(2)**, let $F \in \mathcal{F}_{\mathsf{f}}(X)$ with $\mathrm{Im}(F) = X$, and suppose that $G \in \mathcal{F}(X)$. Then $\mathrm{Im}(G) = X$, and by the first part $P_G(\tilde{x}, X; 1)$ is infinite for some $\tilde{x} \in X$. Since $G^{n-1}(X) = X$, we have $\tilde{x} \in G^{n-1}(\tilde{y})$ for some $\tilde{y} \in X$. Then, as $\#P_F(\tilde{y}, X; 1) \ge \#P_G(\tilde{x}, X; 1)$ by (2.11), and $P_G(\tilde{x}, X; 1)$ is infinite, it follows that $P_F(\tilde{y}, X; 1)$ is infinite, contradicting our assumption that $F \in \mathcal{F}_{\mathsf{f}}(X)$.

We see that this proof of **(2)** is based on the fact that a finite union of finite sets is finite. The proof of **(3)** is similar, using the fact that a countable union of countable sets is countable. In fact, a proof of **(3)** can be produced from that of **(2)** simply by replacing "**(2.a)**", "**(2.b)**", "$\mathcal{F}_{\mathsf{f}}(X)$", "finite", and "infinite" with "**(3.a)**", "**(3.b)**", "$\mathcal{F}_{\mathsf{c}}(X)$", "countable", and "uncountable", respectively. ∎

As a consequence of the above theorem, we have the following corollary. Instead of counting paths, we now count the number of points in certain sets.

Corollary 2.2. *Let F be a multifunction on X such that* $\mathrm{Dom}(F) = X$ *and $x_0 \notin F(x_0)$ for some $x_0 \in X$.*

(1) (Large finite sets vs small finite sets) *Suppose that $\#F^{-2}(\{x_0\}) > MN^3$ and $\#F^{-1}(\{x\}) \le N$ for all $x \neq x_0$ in X and for some $M, N \in \mathbb{N}$. Then F has no iterative roots of order $n \ge 2$ in $\mathcal{F}_M(X)$. If, in addition, $F \in \mathcal{F}_M(X)$ and* $\mathrm{Im}(F) = X$, *then F has no iterative roots of order $n \ge 2$ at all.*

(2) (Infinite sets vs finite sets) *Suppose that $F^{-2}(\{x_0\})$ is infinite and $F^{-1}(\{x\})$ is finite for all $x \neq x_0$ in X. Then F has no iterative roots of order $n \ge 2$ in $\mathcal{F}_{\mathsf{f}}(X)$. If, in addition, $F \in \mathcal{F}_{\mathsf{f}}(X)$ and* $\mathrm{Im}(F) = X$, *then F has no iterative roots of order $n \ge 2$ at all.*

(3) (Uncountable sets vs countable sets) *Suppose that $F^{-2}(\{x_0\})$ is uncountable and $F^{-1}(x)$ is countable for all $x \neq x_0$ in X. Then F has no iterative roots of order $n \ge 2$ in $\mathcal{F}_{\mathsf{c}}(X)$. If, in addition, $F \in \mathcal{F}_{\mathsf{c}}(X)$ and* $\mathrm{Im}(F) = X$, *then F has no iterative roots of order $n \ge 2$ at all.*

Proof. Follows from Theorem 2.1, because $\#P_F(X, x; 2) \ge \#F^{-2}(\{x\})$ and $\#P_F(X, x; 1) = \#F^{-1}(\{x\})$ for all $x \in X$ and $F \in \mathcal{F}(X)$. ∎

We stress that, unlike the known results in [16, 21, 31], the above results and those that follow do not demand any restriction on the cardinality of the set-value points of the map under consideration. Furthermore, in the special case where $M = 1$, the above corollary reduces to [6, Theorem 2], valid for single-valued maps. However, the present proof is different and simpler.

We now illustrate the above results with some examples.

Example 2.3. Let

$$X = \{x_i : i \ge 0\} \cup \{x_{-1}^{(j)} : 1 \le j \le 4\} \cup \{x_{-i}^{(j)} : i \ge 2 \text{ and } j = 1, 2\}$$

and $F_1 : X \to 2^X$ be defined by

$$\begin{aligned} F_1(x_i) &= \{x_{i+1}\} && \text{for } i \ge 0,\\ F_1(x_{-1}^{(j)}) &= \{x_0\} && \text{for } 1 \le j \le 4,\\ F_1(x_{-2}^{(j)}) &= \{x_{-1}^{(2j-1)}, x_{-1}^{(2j)}\} && \text{for } j = 1, 2,\\ F_1(x_{-i}^{(j)}) &= \{x_{-(i-1)}^{(j)}\} && \text{for } i \ge 3 \text{ and } j = 1, 2 \end{aligned}$$

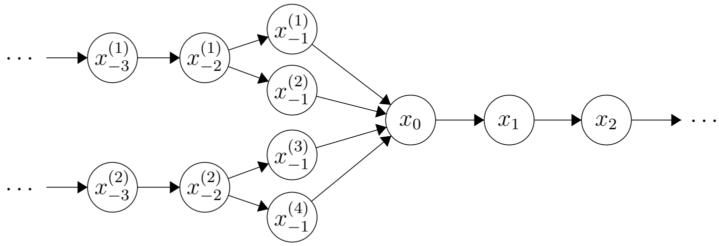

Fig. 1. F_1

(see Figure 1). Then F_1 has no iterative roots of order $n \geq 2$ by Theorem 2.1**(1)** with $M = 2$ and $N = 1$. Further, it is clear that Corollary 2.2 is not applicable, because $\#F^{-2}(\{x_0\}) = 2 = MN^3$.

Example 2.4. Consider

$$X = \{x_0\} \cup \{x_i^{(j)} : i \geq 1 \text{ and } j = 1, 2\} \cup \{x_{-i}^{(j)} : j \geq 1 \text{ and } j = 1, 2, 3\}$$

and let $F_2 : X \to 2^X$ be defined by

$$\begin{aligned} F_2(x_0) &= \{x_1^{(1)}, x_1^{(2)}\}, \\ F_2(x_i^{(j)}) &= \{x_{i+1}^{(j)}\} && \text{for } i \geq 1 \text{ and } j = 1, 2, \\ F_2(x_{-1}^{(j)}) &= \{x_0\} && \text{for } j = 1, 2, 3, \\ F_2(x_{-i}^{(j)}) &= \{x_{-(i-1)}^{(j)}\} && \text{for } i \geq 2 \text{ and } j = 1, 2, 3 \end{aligned}$$

(see Figure 2). Then F_2 has no iterative roots of order $n \geq 2$ by Corollary 2.2**(1)** (or Theorem 2.1**(1)**) with $M = 2$ and $N = 1$.

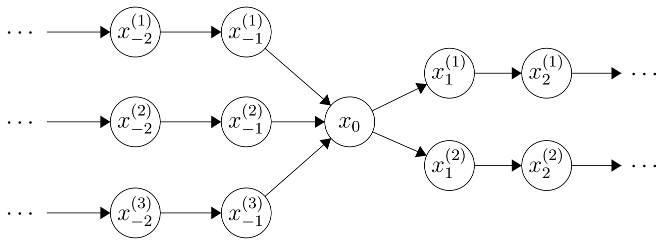

Fig. 2. F_2

Example 2.5. Consider the multifunction $F_3 : [0, 1] \to 2^{[0,1]}$ given by

$$F_3(x) = \begin{cases} \{0, 1\} & \text{if } x = 0, \\ \{1/4 - x/2\} & \text{if } x \in (0, 1/4], \\ \{1/8\} & \text{if } x \in [1/4, 1/2], \\ \{2x - 1\} & \text{if } x \in (1/2, 1] \end{cases}$$

with exactly one set-value point $c = 0$ (see Figure 3). Since $\{c\}$ is not a value of F_3, not only Theorems 1 and 2 of [16] and Theorem 1 of [21] for iterative square roots, but also Theorem 2 of [31] for iterative nth roots, do not work. Furthermore, since $F_3(c) = \{0, 1\}$ and $F_3(1) = \{1\}$, Theorem 2 of [21] is not applicable either. However, F_3 has no iterative roots of order $n \geq 2$ by Theorem 2.1**(2)** (or Corollary 2.2**(2)**) with $x_0 = 1/8$.

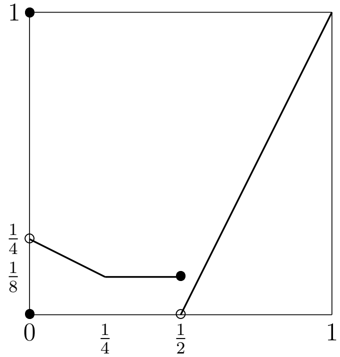

Fig. 3. F_3

EXAMPLE 2.6. Consider the multifunction $F_4 : [0, 1] \to 2^{[0,1]}$ given by

$$F_4(x) = \begin{cases} \{3/4, 1\} & \text{if } x = 0, \\ \{1/4 - x\} & \text{if } x \in (0, 1/4], \\ \{0\} & \text{if } x \in [1/4, 1/2), \\ \{0\} \cup ([3/4, 1] \cap \mathbb{Q}) & \text{if } x = 1/2, \\ \{2x - 1\} & \text{if } x \in (1/2, 1] \end{cases}$$

with the set-value points 0 and 1/2 (see Figure 4). Since F_4 has more than

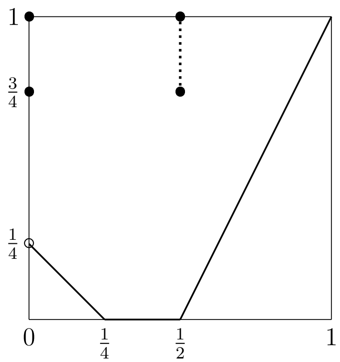

Fig. 4. F_4

one set-value point, none of the known results mentioned in Example 2.5 apply. However, F_4 has no iterative roots of order $n \geq 2$ by Theorem 2.1**(3)** (or Corollary 2.2**(3)**) with $x_0 = 0$.

When $M = 1$, the bound MN^3 in Theorem 2.1**(1)** is optimal, as demonstrated by the examples in [6, Section 4, **(iv)**]. Additionally, assuming the axiom of choice, Theorem 2.1**(2)** and Theorem 2.1**(3)** can be further generalized to the context of infinite cardinal numbers to provide many more similar results on nonexistence of iterative roots. More precisely, we have the following result, where $\aleph_\alpha$'s are precisely the infinite cardinal numbers indexed by the ordinal numbers α, $\leq$ is the order among cardinal numbers, and

$$\mathcal{F}_{\aleph_\alpha}(X) := \{F \in \mathcal{F}(X) : \#P_F(x, X; 1) < \aleph_\alpha \text{ for all } x \in X\}$$

for each ordinal number α.

Theorem 2.7 (Sets of cardinality at least $\aleph_\alpha$ vs those less than $\aleph_\alpha$). *Let F be a multifunction on X such that* $\mathrm{Dom}(F) = X$ *and $x_0 \notin F(x_0)$ for some $x_0 \in X$. Further, suppose that $\#P_F(X, x_0; 2) \geq \aleph_\alpha$ and $\#P_F(X, x; 1) < \aleph_\alpha$ for all $x \neq x_0$ in X and for some infinite cardinal number $\aleph_\alpha$. Then F has no iterative roots of order $n \geq 2$ in $\mathcal{F}_{\aleph_\alpha}(X)$. If, in addition, $F \in \mathcal{F}_{\aleph_\alpha}(X)$ and* $\mathrm{Im}(F) = X$, *then F has no iterative roots of order $n \geq 2$ at all.*

Proof. The proof of Theorem 2.1**(2)** is based on the fact that a finite union of finite sets is finite. The proof of this result is similar, using the result that a union of a collection of cardinality $\aleph_\alpha$ of sets of cardinality $\aleph_\alpha$ has cardinality $\aleph_\alpha$. ∎

As a consequence of the above theorem, we have the following result, with a proof similar to that of Corollary 2.2.

Corollary 2.8 (Sets of cardinality at least $\aleph_\alpha$ vs those less than $\aleph_\alpha$). *Let F be a multifunction on X such that* $\mathrm{Dom}(F) = X$ *and $x_0 \notin F(x_0)$ for some $x_0 \in X$. Further, suppose that $\#F^{-2}(\{x_0\}) \geq \aleph_\alpha$ and $\#F^{-1}(\{x\}) < \aleph_\alpha$ for all $x \neq x_0$ in X and for some infinite cardinal number $\aleph_\alpha$. Then F has no iterative roots of order $n \geq 2$ in $\mathcal{F}_{\aleph_\alpha}(X)$. If, in addition, $F \in \mathcal{F}_{\aleph_\alpha}(X)$ and* $\mathrm{Im}(F) = X$, *then F has no iterative roots of order $n \geq 2$ at all.*

3. Inverse and pullback multifunctions. To describe the continuity or measurability of single-valued maps, we look at their inverse images. We may think of them as multivalued functions. It is clearly an important class, and we refer to them as pullback multifunctions. Unfortunately they are not covered by the results of the previous section due to some trivial reasons suggesting that we should be analyzing the inverses of general multifunctions. The graph of the inverse of a multifunction is obtained simply by reversing arrows/directions of the original graph. Almost all the results of the previous section translate to this setting, and we present them here first without proofs. Then we specialize to pullback multifunctions and see the implications of these results.

As in [2, p. 34], the *inverse* of any $F \in \mathcal{F}(X)$ is the multifunction $F^{-1} \in \mathcal{F}(X)$ defined by $x \in F^{-1}(y)$ if $y \in F(x)$. The image of F is thus the domain of F^{-1}, and symmetrically, the domain of F is the image of F^{-1}. Also, the edge set $E_{F^{-1}}$ equals $\{(y,x) : (x,y) \in E_F\}$. Furthermore, $F = G_1 \circ G_2$ if and only if $F^{-1} = G_2^{-1} \circ G_1^{-1}$ for all multifunctions $G_1, G_2 \in \mathcal{F}(X)$ as shown in [2, p. 37], implying that F has an iterative root of order n in $\mathcal{E} \subseteq \mathcal{F}(X)$ if and only if F^{-1} has an iterative root of order n in $\mathcal{E}^{-1} := \{H : H^{-1} \in \mathcal{E}\}$. Consequently, we can deduce the following corollary from Theorems 2.1 and 2.7, with

$$\begin{aligned}
&\mathcal{F}_M^{-1}(X) := \{F \in \mathcal{F}(X) : F^{-1} \in \mathcal{F}_M(X)\} \quad \text{for each } M \in \mathbb{N},\\
&\mathcal{F}_{\mathsf{f}}^{-1}(X) := \{F \in \mathcal{F}(X) : F^{-1} \in \mathcal{F}_{\mathsf{f}}(X)\},\\
&\mathcal{F}_{\mathsf{c}}^{-1}(X) := \{F \in \mathcal{F}(X) : F^{-1} \in \mathcal{F}_{\mathsf{c}}(X)\},\\
&\mathcal{F}_{\aleph_\alpha}^{-1}(X) := \{F \in \mathcal{F}(X) : F^{-1} \in \mathcal{F}_{\aleph_\alpha}(X)\} \quad \text{for each ordinal number } \alpha,
\end{aligned}$$

and we assume the axiom of choice in **(4)**.

Corollary 3.1. *Let F be a multifunction on X such that* $\mathrm{Im}(F) = X$ *and $x_0 \notin F(x_0)$ for some $x_0 \in X$.*

(1) (Large finite sets vs small finite sets) *Suppose that $\#P_F(x_0, X; 2) > MN^3$ and $\#P_F(x, X; 1) \leq N$ for all $x \neq x_0$ in X and for some $M, N \in \mathbb{N}$. Then F has no iterative roots of order $n \geq 2$ in $\mathcal{F}_M^{-1}(X)$. If, in addition, $F \in \mathcal{F}_M^{-1}(X)$ and* $\mathrm{Dom}(F) = X$*, then F has no iterative roots of order $n \geq 2$ at all.*

(2) (Infinite sets vs finite sets) *Suppose that $P_F(x_0, X; 2)$ is infinite and $P_F(x, X; 1)$ is finite for all $x \neq x_0$ in X. Then F has no iterative roots of order $n \geq 2$ in $\mathcal{F}_{\mathsf{f}}^{-1}(X)$. If, in addition, $F \in \mathcal{F}_{\mathsf{f}}^{-1}(X)$ and* $\mathrm{Dom}(F) = X$*, then F has no iterative roots of order $n \geq 2$ at all.*

(3) (Uncountable sets vs countable sets) *Suppose that $P_F(x_0, X; 2)$ is uncountable and $P_F(x, X; 1)$ is countable for all $x \neq x_0$ in X. Then F has no iterative roots of order $n \geq 2$ in $\mathcal{F}_{\mathsf{c}}^{-1}(X)$. If, in addition, $F \in \mathcal{F}_{\mathsf{c}}^{-1}(X)$ and* $\mathrm{Dom}(F) = X$*, then F has no iterative roots of order $n \geq 2$ at all.*

(4) (Sets of cardinality at least $\aleph_\alpha$ vs those less than $\aleph_\alpha$) *Suppose that $\#P_F(x_0, X; 2) \geq \aleph_\alpha$ and $\#P_F(x, X; 1) < \aleph_\alpha$ for all $x \neq x_0$ in X and for some infinite cardinal number $\aleph_\alpha$. Then F has no iterative roots of order $n \geq 2$ in $\mathcal{F}_{\aleph_\alpha}^{-1}(X)$. If, in addition, $F \in \mathcal{F}_{\aleph_\alpha}^{-1}(X)$ and* $\mathrm{Dom}(F) = X$*, then F has no iterative roots of order $n \geq 2$ at all.*

As a consequence of the above corollary, analogous to the respective Corollaries 2.2 and 2.8 of Theorems 2.1 and 2.7, we have the following.

Corollary 3.2. *Let F be a multifunction on X such that* $\mathrm{Im}(F) = X$ *and $x_0 \notin F(x_0)$ for some $x_0 \in X$.*

(1) (Large finite sets vs small finite sets) *Suppose that $\#F^2(x_0) > MN^3$ and $\#F(x) \le N$ for all $x \ne x_0$ in X and for some $M, N \in \mathbb{N}$. Then F has no iterative roots of order $n \ge 2$ in $\mathcal{F}_M^{-1}(X)$. If, in addition, $F \in \mathcal{F}_M^{-1}(X)$ and* $\mathrm{Dom}(F) = X$, *then F has no iterative roots of order $n \ge 2$ at all.*

(2) (Infinite sets vs finite sets) *Suppose that $F^2(x_0)$ is infinite and $F(x)$ is finite for all $x \ne x_0$ in X. Then F has no iterative roots of order $n \ge 2$ in $\mathcal{F}_{\mathsf{f}}^{-1}(X)$. If, in addition, $F \in \mathcal{F}_{\mathsf{f}}^{-1}(X)$ and* $\mathrm{Dom}(F) = X$, *then F has no iterative roots of order $n \ge 2$ at all.*

(3) (Uncountable sets vs countable sets) *Suppose that $F^2(x_0)$ is uncountable and $F(x)$ is countable for all $x \ne x_0$ in X. Then F has no iterative roots of order $n \ge 2$ in $\mathcal{F}_{\mathsf{c}}^{-1}(X)$. If, in addition, $F \in \mathcal{F}_{\mathsf{c}}^{-1}(X)$ and* $\mathrm{Dom}(F) = X$, *then F has no iterative roots of order $n \ge 2$ at all.*

(4) (Sets of cardinality at least $\aleph_\alpha$ vs those less than $\aleph_\alpha$) *Suppose that $\#F^2(x_0) \ge \aleph_\alpha$ and $\#F(x) < \aleph_\alpha$ for all $x \ne x_0$ in X and for some infinite cardinal number $\aleph_\alpha$. Then F has no iterative roots of order $n \ge 2$ in $\mathcal{F}_{\aleph_\alpha}^{-1}(X)$. If, in addition, $F \in \mathcal{F}_{\aleph_\alpha}^{-1}(X)$ and* $\mathrm{Dom}(F) = X$, *then F has no iterative roots of order $n \ge 2$ at all.*

The examples below illustrate the above results.

Example 3.3. Let $F = F_1^{-1}$ (see Figure 5), where F_1 is the map considered in Example 2.3. Then it is easy to see that

$$\#P_F(X, x; 1) = \begin{cases} 2 & \text{if } x \in \{x_{-2}^{(1)}, x_{-2}^{(2)}\}, \\ 1 & \text{otherwise} \end{cases}$$

and

$$\#P_F(x, X; 1) = \begin{cases} 4 & \text{if } x = x_0, \\ 1 & \text{otherwise.} \end{cases}$$

If we wish to apply Theorem 2.1**(1)** to F, then we must consider M and N

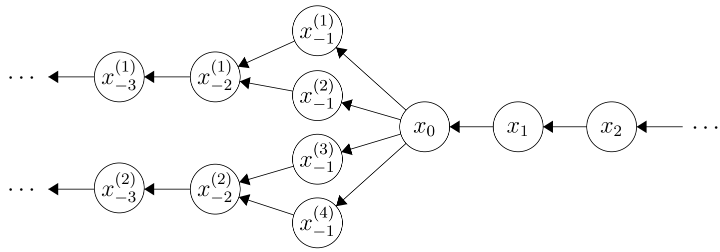

Fig. 5. F_1^{-1}

such that $M \geq 4$ and $N \geq 2$; but, in that case, there is no $x \in X$ such that condition **(1.a)** is satisfied. Therefore Theorem 2.1**(1)** does not apply to F directly. Additionally, since $P_F(X, x; 2)$ is finite for all $x \in X$, neither **(2)** nor **(3)** of Theorem 2.1 apply. Consequently, the results of Corollary 2.2 do not work either. However, F has no iterative roots of order $n \geq 2$ by Corollary 3.1**(1)** with $M = 2$ and $N = 1$.

EXAMPLE 3.4. Consider the multifunction $F : [0,1] \to 2^{[0,1]}$ defined by

$$F(x) = \begin{cases} \{4x\} & \text{if } x \in [0, 1/4), \\ [1/2, 3/4] & \text{if } x = 1/4, \\ \{(4x-1)/3\} & \text{if } x \in (1/4, 1] \end{cases}$$

with exactly one set-value point $c = 1/4$ (see Figure 6). Then F has no iterative roots of order $n \geq 2$ by Corollary 3.1**(2)** (or Corollary 3.2**(2)**) with $x_0 = 1/4$.

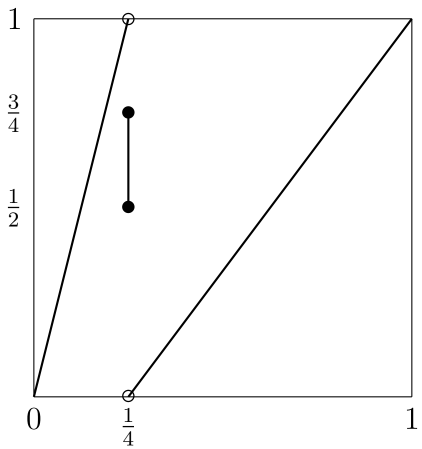

Fig. 6. F

So far, we have been looking at iterative roots of inverses of general multifunctions. We now restrict ourselves to a subclass of such multifunctions that arise from single-valued maps for further investigation. In what follows, let $\mathfrak{F}(X)$ denote the set of all maps $f : X \to X$, and for each set $A \subseteq X$, $f \in \mathfrak{F}(X)$ and $k \geq 1$, let $f^{-k}(A)$ denote the kth order *inverse image* of A by f defined by $f^{-k}(A) = \{x \in X : f^k(x) \in A\}$. An $F \in \mathcal{F}(X)$ is said to be

- **(i)** the *pullback* of a map $f \in \mathfrak{F}(X)$ if $F(x) = f^{-1}(\{x\})$ for all $x \in X$,
- **(ii)** a *pullback multifunction* if it is the pullback of some $f \in \mathfrak{F}(X)$.

Let $\mathcal{F}_{\mathsf{p}}(X)$ denote the set of all pullback multifunctions in $\mathcal{F}(X)$. The following proposition provides a characterization of such multifunctions.

PROPOSITION 3.5. *Let* $F \in \mathcal{F}(X)$ *such that* $\mathrm{Dom}(F) = X$. *Then* $F \in \mathcal{F}_{\mathsf{p}}(X)$ *if and only if the following conditions are satisfied:*

(a) $F(x) \cap F(y) = \emptyset$ *for* $x \neq y$ *in* X;
(b) $\mathrm{Im}(F) = X$.

Proof. Suppose that F is the pullback of an $f \in \mathfrak{F}(X)$. Then F clearly satisfies condition **(a)** because $f^{-1}(\{x\}) \cap f^{-1}(\{y\}) = \emptyset$ for $x \neq y$ in X. Further, since $f : X \to X$ is a map, for each $x \in X$ there exists a unique $y \in X$ such that $x \in f^{-1}(\{y\}) = F(y) \subseteq F(X)$, implying that $X \subseteq F(X)$. The reverse inclusion follows trivially. Therefore F satisfies condition **(b)**.

Conversely, assume that F satisfies conditions **(a)** and **(b)**. Define a map $f : X \to X$ by $f(x) = y$ if $x \in F(y)$. Since for each $x \in X$ there exists a unique $y \in X$ such that $x \in F(y)$, where the existence is guaranteed by **(b)** and uniqueness by **(a)**, f is clearly well-defined. Further, it is easy to check that $F(x) = f^{-1}(\{x\})$ for all $x \in X$. Therefore F is the pullback of f. ∎

We recall that Theorem 2.1 (resp. Corollary 2.2) does not directly apply to pullback multifunctions F for the following reason. In fact, for any multifunction F on X the assumption in Theorem 2.1 (resp. Corollary 2.2) that $\#P_F(X, x_0; 2)$ (resp. $\#F^{-2}(\{x_0\})$) is very large in comparison to $\#P_F(X, x; 1)$ (resp. $\#F^{-1}(\{x\})$) for all $x \neq x_0$ in X implies that there exist $x \neq y$ in X such that $x_0 \in F(x) \cap F(y)$, violating the necessary condition **(a)** of Proposition 3.5 that F must satisfy for it to be a pullback multifunction. However, we can certainly use Corollaries 3.1 and 3.2 for F. Here Corollary 3.1 is derived from Theorem 2.1 and Corollary 3.2 follows from Corollary 2.2. It is also worth noting that the class of pullback multifunctions is closed under the operation of taking iterative roots, as shown in the following.

PROPOSITION 3.6. *Let* $F \in \mathcal{F}_{\mathsf{p}}(X)$ *be such that* $F(x) = (G_1 \circ G_2)(x)$ *for all* $x \in X$ *and some multifunctions* $G_1, G_2 \in \mathcal{F}(X)$, *where* $\mathrm{Dom}(F) = \mathrm{Dom}(G_1) = \mathrm{Dom}(G_2) = X$. *Then*

(i) $\mathrm{Im}(G_1) = X$,
(ii) $G_2(x) \cap G_2(y) = \emptyset$ *for* $x \neq y$ *in* X.

Further, if $F \in \mathcal{F}_{\mathsf{p}}(X)$ *and* $F(x) = G^n(x)$ *for all* $x \in X$ *and some multifunction* $G \in \mathcal{F}(X)$ *such that* $\mathrm{Dom}(G) = X$, *where* $n \geq 2$, *then* $G \in \mathcal{F}_{\mathsf{p}}(X)$.

Proof. Since $X = F(X) = G_1(G_2(X)) \subseteq G_1(X)$, **(i)** is trivial. If $z \in G_2(x) \cap G_2(y)$ for some $x \neq y$ in X, then $\emptyset \neq G_1(z) \subseteq F(x) \cap F(y)$, which is a contradiction. Therefore **(ii)** follows. The second part follows from the first and the "if" part of Proposition 3.5, because $F(x) = (G \circ G^{n-1})(x) = (G^{n-1} \circ G)(x)$ for all $x \in X$. ∎

The discussion just before Corollary 3.1 shows that a multifunction $F \in \mathcal{F}_{\mathsf{p}}(X)$ with domain X, which is a pullback of an $f \in \mathfrak{F}(X)$, has an iterative root of order n on X if and only if f has an iterative root of order n in $\mathfrak{F}(X)$. More precisely, it is easy to check the following.

THEOREM 3.7. *Let $F \in \mathcal{F}_{\mathsf{p}}(X)$ be the pullback of an $f \in \mathfrak{F}(X)$ such that* $\mathrm{Dom}(F) = X$. *Then G is an iterative root of F of order n in $\mathcal{F}_{\mathsf{p}}(X)$ if and only if g is an iterative root of f of order n in $\mathfrak{F}(X)$, where G is the pullback of g.*

Thus, pullback multifunctions provide numerous examples of multifunctions with or without iterative roots, based on the known results on iterative roots for single-valued maps. An easy example is that the 4th root multifunction $z \mapsto z^{1/4}$ has the square root multifunction $z \mapsto z^{1/2}$ as its iterative square root on $\mathbb{C}$ because the square function $z \mapsto z^2$ is an iterative square root of the 4th power function $z \mapsto z^4$. We now give a brief summary of known results on iterative roots of pullbacks of complex polynomials.

COROLLARY 3.8.

(1) *The pullback of any complex quadratic polynomial has no iterative roots of order $n \geq 2$ on $\mathbb{C}$. In particular, the square root multifunction $z \mapsto z^{1/2}$ has no iterative roots of order $n \geq 2$ on $\mathbb{C}$.*
(2) *The dth root multifunction $z \mapsto z^{1/d}$ has no iterative roots of order $n \geq 2$ on $\mathbb{C}$ provided $d^p \not\equiv d \pmod{p^2}$ for all primes $p \leq d$. The first 25 such d are* $2, 3, 6, 11, 14, 15, 34, 39, 47, 58, 59, 66, 83, 86, 87, 95, 102, 103, 106, 111, 114, 119, 123, 139$ *and* 142.
(3) *Let F be the pullback of a complex cubic polynomial f, where f is not linearly conjugate to $p(z) = z^3 - z^2 + z$ (i.e., $f = h \circ p \circ h^{-1}$ for no linear function $h(z) = \alpha z + \beta$, $\alpha \neq 0$ on $\mathbb{C}$) and has less than three distinct fixed points. Then F has no iterative roots of order $n \geq 2$ on $\mathbb{C}$.*
(4) *Let F be the pullback of a complex polynomial of degree $d \geq 2$. Then F has no iterative roots of order $n > d(d-1)$ on $\mathbb{C}$.*
(5) *Let F be the pullback of a complex polynomial of degree $d \geq 2$. If $n > d$ is a prime, then F has no iterative roots of order n on $\mathbb{C}$.*

Proof. Follows from the following known results on complex polynomials, respectively, and the discussions above.

(i) ([37, Theorem 1] or [8, Theorem 2]) Complex quadratic polynomials have no iterative roots of any order $n \geq 2$ on $\mathbb{C}$. In particular, the square function $z \mapsto z^2$ has no iterative roots of any order $n \geq 2$ on $\mathbb{C}$.
(ii) ([39]) The polynomial function $z \mapsto z^d$ has no iterative roots of order $n \geq 2$ on $\mathbb{C}$ provided $d^p \not\equiv d \pmod{p^2}$ for all primes $p \leq d$. The first 25 such d are those listed above in result **(2)**, as shown in [36].
(iii) ([8, Theorem 6]) Let f be a complex cubic polynomial which is not linearly conjugate to $p(z) = z^3 - z^2 + z$ and has less than three distinct fixed points. Then f has no iterative roots of order $n \geq 2$ on $\mathbb{C}$.

(iv) ([37, Theorem 4]) Let f be a complex polynomial of degree $d \geq 2$. Then f has no iterative roots of order $n > d(d-1)$ on $\mathbb{C}$.

(v) ([8, Theorem 1]) Let f be a complex polynomial of degree $d \geq 2$. If $n > d$ is a prime, then f has no iterative roots of order n on $\mathbb{C}$. ■

In this circle of ideas, we now make some general remarks, for which we require some additional terminology and notation. A fixed point $x \in X$ of an $f \in \mathfrak{F}(X)$ is said to be *isolated* if there is no $y \neq x$ in X such that $f(y) = x$. Let $T_x(f) := f^{-1}(\{x\}) \setminus \{x\}$ for each fixed point x of f, with $T_x(f)$ being nonempty if and only if x is nonisolated. It is important to note that Theorem 4 in [37] for complex polynomials cited above is a consequence of the general result in that paper, which is stated as follows.

THEOREM 3.9 ([37, Lemma 11]). *Suppose $f \in \mathfrak{F}(X)$ has a nonisolated fixed point $x \in X$ such that $f^{-1}(\{y\}) \neq \emptyset$ for some $y \in T_x(f)$. Then f has no iterative roots of order $n > L$, where $L := \#(\bigcup_{x'} T_{x'}(f))$ with the union taken over all nonisolated fixed points x' of f.*

On the other hand, Theorem 1 in [8], as seen in item **(v)**, is based on the degrees of the polynomials under consideration and applies only to iterative roots of prime orders. We now prove a similar but more general result for general single-valued maps based on the number of nonisolated fixed points. It can also be applied to show the nonexistence of iterative roots of nonprime orders.

THEOREM 3.10. *Suppose $f \in \mathfrak{F}(X)$ has $k \geq 2$ nonisolated fixed points $x_1, \ldots, x_k$ such that*

(a) *$\#T_{x_j}(f) \leq l$ for all $1 \leq j \leq k$,*
(b) *$f^{-1}(\{y\}) \neq \emptyset$ for all $y \in T_{x_j}(f)$ and $1 \leq j \leq k$.*

Then f has no iterative roots of order $n > l$ such that $m \nmid n$ for all $2 \leq m \leq k$.

Proof. Suppose, on the contrary, that $f = g^n$ for some $g \in \mathfrak{F}(X)$, where $n > l$ and $m \nmid n$ for all $2 \leq m \leq k$. First, we assert that $g(x_j)$ is a nonisolated fixed point of f for all $1 \leq j \leq k$.

Since $f(g(x_j)) = g(f(x_j)) = g(x_j)$, clearly $g(x_j)$ is a fixed point of f for all $1 \leq j \leq k$. Suppose that $g(x_{j_0})$ is isolated for some $1 \leq j_0 \leq k$, and let $y_{j_0}, z_{j_0} \in X$ be such that $y_{j_0} \in T_{x_{j_0}}(f)$ and $z_{j_0} \in f^{-1}(\{y_{j_0}\})$, where y_{j_0} exists by the assumption that x_{j_0} is nonisolated and z_{j_0} exists by **(b)**. Then, as $g(x_{j_0})$ is isolated and

$$f^2(g(z_{j_0})) = g(f^2(z_{j_0})) = g(f(y_{j_0})) = g(x_{j_0}),$$

we have $g(z_{j_0}) = g(x_{j_0})$, implying that

$$y_{j_0} = f(z_{j_0}) = g^{n-1}(g(z_{j_0})) = g^{n-1}(g(x_{j_0})) = f(x_{j_0}) = x_{j_0}.$$

This contradicts our assumption that $y_{j_0} \in T_{x_{j_0}}(f)$. Therefore our assertion holds.

Now, since $x_1, \ldots, x_k$ are the only nonisolated fixed points of f by hypothesis, we have

$$g(\{x_1, \ldots, x_k\}) \subseteq \{x_1, \ldots, x_k\}. \tag{3.1}$$

We discuss the following two cases.

CASE **(i)**: Suppose that $g(x_{j_0}) = x_{j_0}$ for some $1 \leq j_0 \leq k$. Without loss of generality, let $j_0 = 1$. We first prove by induction that elements of $T_{x_1}(f)$ can be arranged in a sequence $y_1, \ldots, y_{l_1}$ such a way that

$$g(y_i) \in \{x_1, y_1, \ldots, y_{i-1}\} \quad \text{for all } 1 \leq i \leq l_1, \tag{3.2}$$

where $l_1 := \#T_{x_1}(f)$.

Since $f(g(y)) = g(f(y)) = g(x_1) = x_1$ for all $y \in T_{x_1}(f)$, it is clear that

$$g(T_{x_1}(f)) \subseteq T_{x_1}(f) \cup \{x_1\}.$$

Further, since $f = g^n$, we cannot have $g(T_{x_1}(f)) \subseteq T_{x_1}(f)$ (otherwise, $g^i(T_{x_1}(f)) \subseteq T_{x_1}(f)$ for all $i \in \mathbb{N}$, and in particular $f(T_{x_1}(f)) = g^n(T_{x_1}(f)) \subseteq T_{x_1}(f)$, which contradicts the definition of $T_{x_1}(f)$). Therefore there exists a $y_1 \in T_{x_1}(f)$ such that $g(y_1) = x_1$, proving (3.2) for $i = 1$. Next, assume that we have already defined $y_1, \ldots, y_i$ for some $1 \leq i \leq l_1 - 1$. Then, using a similar argument, we see that

$$g(T_{x_1}(f) \setminus \{y_1, \ldots, y_i\}) \nsubseteq T_{x_1}(f) \setminus \{y_1, \ldots, y_i\}.$$

Therefore there exists a $y_{i+1} \in T_{x_1}(f) \setminus \{y_1, \ldots, y_i\}$ such that $g(y_{i+1}) = x_1$, proving (3.2) for $i + 1$. Thus, the existence of a desired rearrangement of $T_{x_1}(f)$ follows by induction.

Now, since $f^{-1}(\{y_1\}) \neq \emptyset$ by **(b)**, there exists a $z_1 \in X$ such that $f(z_1) = y_1$. Then

$$f(g(z_1)) = g(f(z_1)) = g(y_1) = x_1,$$

implying that $g(z_1) \in T_{x_1}(f) \cup \{x_1\}$. Also, from (3.2) we see that $g^i(y) = x_1$ for all $y \in T_{x_1}(f) \cup \{x_1\}$ and $i \geq l_1$. Therefore, as $n - 1 \geq l$ and $l \geq l_1$ by **(a)**, we obtain

$$y_1 = f(z_1) = g^{n-1}(g(z_1)) = x_1,$$

contrary to our assumption that $y_1 \in T_{x_1}(f)$.

CASE **(ii)**: Suppose that Case (i) does not hold. Then, as $g(x_j) \neq x_j$ for all $1 \leq j \leq k$, by using (3.1) we see that g has a q-periodic point x_{j_0} in $\{x_1, \ldots, x_k\}$ for some $1 \leq j_0 \leq k$ and $2 \leq q \leq k$. As in Case (i), without loss of generality we assume that $j_0 = 1$. By our assumption on x_1, we have

$$g^q(x_1) = x_1 \quad \text{and} \quad g^i(x_1) \neq x_1 \text{ for all } 1 \leq i \leq q - 1. \tag{3.3}$$

Further, since $2 \le q \le k$, and $m \nmid n$ for all $2 \le m \le k$ by hypothesis, we see from the division algorithm that $n = sq+r$ for some $s \in \mathbb{N}$ and $1 \le r \le q-1$. Therefore, by (3.3) we have

$$f(x_1) = g(x_1) = g^{sq+r}(x_1) = g^r(x_1) \neq x_1,$$

which contradicts our assumption that x_1 is a fixed point of f.

Thus, we get a contradiction in both the cases, proving that f has no iterative roots of order $n > l$ such that $m \nmid n$ for all $2 \le m \le k$. ∎

It is worth noting that the assumption on n that $n > l$ and $m \nmid n$ for all $2 \le m \le k$ in the above theorem cannot be relaxed, as shown in the following.

Example 3.11. Let

$$X = \{x_j : 1 \le j \le 4\} \cup \{y_j^{(i)} : i = 1, 2 \text{ and } 1 \le j \le 4\} \cup \{z_j^{(i)} : i = 1, 2 \text{ and } 1 \le j \le 4\}$$

and $f : X \to X$ be defined by

$$\begin{aligned} f(x_j) &= x_j && \text{for } 1 \le j \le 4, \\ f(y_j^{(i)}) &= x_j && \text{for } i = 1, 2 \text{ and } 1 \le j \le 4, \\ f(z_j^{(i)}) &= y_j^{(i)} && \text{for } i = 1, 2 \text{ and } 1 \le j \le 4 \end{aligned}$$

(see Figure 7). Then it is clear that x_1, x_2, x_3, x_4 are the only nonisolated fixed points of f, and $\#T_{x_j}(f) = 2$ and $f^{-1}(\{y\}) \neq \emptyset$ for all $y \in T_{x_j}(f)$ and

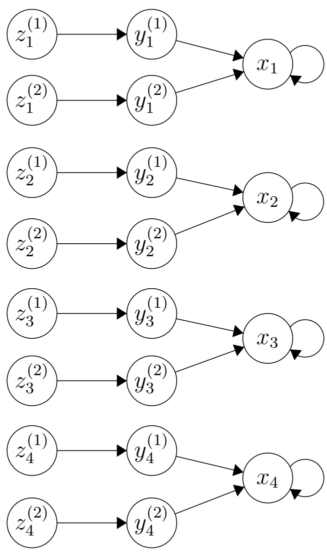

Fig. 7. f

$1 \le j \le 4$. Therefore, it follows from Theorem 3.10 with $k = 4$ and $l = 2$ that f has no iterative roots of order $n > l$ such that $m \nmid n$ for all $2 \le m \le k$. However, f has an iterative root g of order $n = 4 > l$ on X, given by

$$\begin{aligned} g(x_j) &= x_{j+1\,(\mathrm{mod}\ 4)} && \text{for } 1 \le j \le 4,\\ g(y_j^{(i)}) &= y_{j+1}^{(i)} && \text{for } i = 1,2 \text{ and } 1 \le j \le 3,\\ g(y_j^{(i)}) &= x_1 && \text{for } i = 1,2 \text{ and } j = 4,\\ g(z_j^{(i)}) &= z_{j+1}^{(i)} && \text{for } i = 1,2 \text{ and } 1 \le j \le 3,\\ g(z_j^{(i)}) &= y_1^{(i)} && \text{for } i = 1,2 \text{ and } j = 4 \end{aligned}$$

(see Figure 8), which is divisible by $m = 2 < k$.

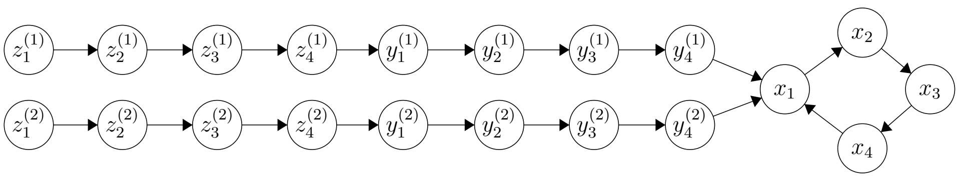

Fig. 8. g

Since every complex polynomial of degree $d \ge 2$ can have at most one isolated fixed point as seen from [8, Lemma 4], the only possible values for the pair (k, l) considered in Theorem 3.10 are $(d, d-1)$ and $(d-1, d-1)$. However, if $(k, l) = (d, d-1)$, then, in light of [37, Theorem 4], both Theorem 3.10 and [8, Theorem 1] provide the same results because every composite number n between $k-1$ and $k(k-1)$ is divisible by m for some $2 \le m \le k$. Thus, the actual contribution of our Theorem 3.10 to complex polynomials f is limited to the case $(k, l) = (d-1, d-1)$, where f has the form $\alpha(z-\beta)^d + \beta$ with $\alpha \ne 0$. In fact, Theorem 3.10 implies that *for all primes p, no pth power has pth roots* which is not guaranteed by either [37, Theorem 4] or [8, Theorem 1]. More precisely, we have the following.

Corollary 3.12. *If $d \ge 2$ is a prime, then the complex polynomial function $z \mapsto \alpha(z-\beta)^d + \beta$ with $\alpha \ne 0$ has no iterative roots of order d on $\mathbb{C}$.*

Proof. Let $f(z) = \alpha(z-\beta)^d + \beta$ for all $z \in \mathbb{C}$, where $\alpha \ne 0$ and $d \ge 2$ is a prime. Then f has $d-1$ nonisolated fixed points, which are precisely $\beta + \alpha^{-1/d-1}$, where $\alpha^{-1/d-1}$ denote any of the $d-1$ possible values. Therefore we see from Theorem 3.10 that f has no iterative roots of order $n > d-1$ on $\mathbb{C}$ with $m \nmid n$ for all $2 \le m \le d-1$. In particular, as $d \ge 2$ is a prime, it follows that f has no iterative roots of order d on $\mathbb{C}$. ■

The following example illustrates the above corollary.

Example 3.13. Consider the polynomial function $f(z) = z^5$ on $\mathbb{C}$, to which the result in [39] does not apply, as seen in item **(ii)** of the proof of Corollary 3.8. Then it follows from the above corollary that f has no iterative roots of order 5 on $\mathbb{C}$, which is not guaranteed by [8, Theorem 1].

Finally, we point out that, while Theorem 3.10 and Corollary 3.12 are concerned with single-valued maps, we can use Theorem 3.7 to obtain similar results for their pullback multifunctions. These results differ from those obtained for general multifunctions in Section 2 in the following sense: The former are based on a fixed point of the map under consideration, whereas the latter are based on a point that is not a fixed point.

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