STUDIA MATHEMATICA
Online First version

Approximation of polynomials from Walsh tail spaces

by

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Abstract. We derive various bounds for the L_p-distance of polynomials on the hypercube from Walsh tail spaces, extending some of Oleszkiewicz's results (2017) for Rademacher sums.

1. Introduction. Given $n \in \mathbb{N} = \mathbb{Z}_{\geq 1}$, every function $f : \{-1,1\}^n \to \mathbb{R}$ admits a unique Fourier–Walsh expansion

$$\forall x \in \{-1,1\}^n, \qquad f(x) = \sum_{S \subseteq \{1,\dots,n\}} \widehat{f}(S) w_S(x), \tag{1.1}$$

where the Walsh function w_S is given by $w_S(x) = \prod_{i\in S} x_i$ for $x = (x_1,\dots,x_n)$ in $\{-1,1\}^n$. We shall say that f is *of degree at most* $k \in \{1,\dots,n\}$ if $\widehat{f}(S) = 0$ for every subset S of $\{1,\dots,n\}$ with $|S| > k$. Similarly, we say that f belongs *in the kth tail space*, where $k \in \{1,\dots,n\}$, if $\widehat{f}(S) = 0$ for every subset S with $|S| \leq k$. More generally, given a nonempty subset $I \subseteq \{0,1,\dots,n\}$, we denote

$$\mathcal{P}^n_I := \{f : \{-1,1\}^n \to \mathbb{R} : \widehat{f}(S) = 0 \text{ for every } S \text{ with } |S| \notin I\}. \tag{1.2}$$

We shall also adopt the natural notations $\mathcal{P}^n_{>k} = \mathcal{P}^n_{\{k+1,\dots,n\}}$, $\mathcal{P}^n_{\leq k} = \mathcal{P}^n_{\{0,1,\dots,k\}}$, $\mathcal{P}^n_{=k} = \mathcal{P}^n_{\{k\}}$ and so on.

Many modern developments in discrete analysis (see [18]) are centered around quantitative properties of functions with spectrum bounded above or below, in analogy with estimates established for polynomials on the torus $\mathbb{T}^n$ or on $\mathbb{R}^n$ in classical approximation theory. One of the first results of this nature, going back at least to [3, 4], is the important fact that all finite moments of low-degree Walsh polynomials are equivalent to each other up to

2020 *Mathematics Subject Classification*: Primary 42C10; Secondary 41A17, 60E15.
Key words and phrases: discrete hypercube, polynomial inequalities, Rademacher sum, tail space.
Received 23 November 2023; revised 4 January 2024.
Published online 11 April 2024.

dimension-free factors. Namely, given any $1 \le p \le q < \infty$ and $k \in \mathbb{N}$, there exists a (sharp) constant $\mathsf{M}_{p,q}(k)$ such that for any $n \ge k$, every polynomial $f : \{-1,1\}^n \to \mathbb{R}$ of degree at most k satisfies

$$\|f\|_q \le \mathsf{M}_{p,q}(k)\|f\|_p, \tag{1.3}$$

where $\|\cdot\|_r$ always denotes the L_r-norm on $\{-1,1\}^n$ with respect to the uniform probability measure. Note that the reverse of (1.3) holds trivially with constant 1 by Hölder's inequality. We refer to [8, 13, 16] for the best known bounds on the implicit constant $\mathsf{M}_{p,q}(k)$. In the special case $k = 1$, (1.3) is the celebrated Khinchin inequality [15] for Rademacher sums.

Our starting point is the simple observation that the moment comparison estimates (1.3) have the following (equivalent) dual formulation in terms of distances from tail spaces.

Proposition 1.1. *For every $1 \le p \le q < \infty$ and $k \in \mathbb{N}$, the constant $\mathsf{M}_{p,q}(k)$ in inequality* (1.3) *is also the least constant for which every function $f : \{-1,1\}^n \to \mathbb{R}$, where $n \ge k$, satisfies*

$$\inf_{g \in \mathcal{P}^n_{>k}} \|f-g\|_{p^*} \le \mathsf{M}_{p,q}(k) \inf_{g \in \mathcal{P}^n_{>k}} \|f-g\|_{q^*}, \tag{1.4}$$

where the conjugate exponent r^* of $r \in [1,\infty]$ satisfies $\frac{1}{r^*} + \frac{1}{r} = 1$.

Again, the reverse of (1.4) holds with constant 1. In the special case $k = 0$, inequality (1.4) becomes trivial with $\mathsf{M}_{p,q}(0) = 1$ as both sides are equal to $|\mathbb{E}f|$. When $k = 1$, which corresponds to the dual of the classical Khinchin inequality, we derive the following more precise formula for the distance from the tail space $\mathcal{P}^n_{>1}$.

Theorem 1.2. *For every $1 < r \le \infty$ and $n \in \mathbb{N}$, every $f : \{-1,1\}^n \to \mathbb{R}$ satisfies* ([1])

$$\inf_{g \in \mathcal{P}^n_{>1}} \|f-g\|_r \asymp |\mathbb{E}f| + \max_{i \in \{1,\dots,n\}} \big|\widehat{f}(\{i\})\big| + \sqrt{\frac{r-1}{r}}\Big(\sum_{i=1}^n \widehat{f}(\{i\})^2\Big)^{1/2}. \tag{1.5}$$

This is the dual to a well-known result of Hitczenko [10] (see also [17, 11]), providing p-independent upper and lower bounds for the L_p-norms of Rademacher sums, where $p \in [1,\infty)$.

At this point, we should point out that in both Proposition 1.1 and Theorem 1.2, the exponents of the norms are always strictly greater than 1. For instance, choosing $f_1(x) = \sum_{i=1}^n x_i$, (1.4) gives

$$\forall r \in (1,\infty], \qquad \inf_{g \in \mathcal{P}^n_{>k}} \|f_1-g\|_r \asymp_{r,k} \inf_{g \in \mathcal{P}^n_{>k}} \|f_1-g\|_2 = \sqrt{n}. \tag{1.6}$$

([1]) Throughout the paper we shall use standard asymptotic notation. For instance, $\xi \lesssim \eta$ (or $\eta \gtrsim \xi$) means that there exists a universal constant $c > 0$ such that $\xi \le c\eta$ and $\xi \asymp \eta$ stands for $(\xi \lesssim \eta) \wedge (\eta \lesssim \xi)$. We shall use subscripts of the form $\lesssim_t, \gtrsim_t, \asymp_t$ when the implicit constant c depends on some prespecified parameter t.

On the other hand, it follows from a result of Oleszkiewicz [19], which is the main precursor to this work, that the L_1-distance of f_1 from the kth tail space satisfies

$$\inf_{g\in\mathcal{P}^n_{>k}} \|f_1 - g\|_1 \asymp \min\{k, \sqrt{n}\}, \tag{1.7}$$

and thus exhibits a starkly different behavior as $n \to \infty$ from the L_r-norms with $r > 1$.

More generally, it is shown in [19] that for every $a_1 \geq \cdots \geq a_n \geq 0$, we have

$$\inf_{g\in\mathcal{P}^n_{>k}} \|f_{\boldsymbol{a}} - g\|_1 \asymp \min_{r\in\{0,1,\ldots,n\}} \Big\{ \Big(\sum_{i=1}^r a_i^2 \Big)^{1/2} + k a_{r+1} \Big\}, \tag{1.8}$$

where for $\boldsymbol{a} = (a_1, \ldots, a_n)$ we denote $f_{\boldsymbol{a}}(x) = \sum_{i=1}^n a_i x_i$ and we make the convention that $a_{n+1} = 0$. The quantity appearing on the right-hand side of (1.8) can be rephrased in terms of the K-functional of real interpolation (see [1, Chapter 3]). Recall that if (A_0, A_1) is an interpolation pair, then the Lions–Peetre K-functional is defined for every $t \geq 0$ and $a \in A_0 + A_1$ as

$$\mathsf{K}(a, t; A_0, A_1) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}. \tag{1.9}$$

It is elementary to check (see [12]) that if $a_1 \geq \cdots \geq a_n \geq 0$ and $k \in \mathbb{N}$, then

$$\min_{r\in\{0,1,\ldots,n\}} \Big\{ \Big(\sum_{i=1}^r a_i^2 \Big)^{1/2} + k a_{r+1} \Big\} \asymp \mathsf{K}(\boldsymbol{a}, k; \ell_2^n, \ell_\infty^n). \tag{1.10}$$

Note that the right-hand side is invariant under permutations of the entries of $\boldsymbol{a}$. The main result of this work is an appropriate extension of the upper bound in Oleszkiewicz's result (1.8) to polynomials of arbitrary degree on the discrete hypercube.

THEOREM 1.3. *For every $d \in \mathbb{N}$, there exists $\mathsf{C}_d \in (0, \infty)$ such that for any $n \geq k \geq d$, every polynomial $f : \{-1,1\}^n \to \mathbb{R}$ of degree at most d satisfies*

$$\inf_{g\in\mathcal{P}^n_{>k}} \|f - g\|_1 \leq \mathsf{K}(\widehat{f}, \mathsf{C}_d k^d; \ell_2^m, \ell_{\frac{2d}{d-1}}^m), \tag{1.11}$$

where $\widehat{f}$ is the vector of Fourier coefficients of f, viewed as an element of $\mathbb{R}^m$ with $m = \binom{n}{0} + \cdots + \binom{n}{d}$.

As was already pointed out by Oleszkiewicz, the method of [19] does not appear to extend beyond Rademacher sums. Instead, in our proof we shall employ the discrete *Bohnenblust–Hille* inequality from approximation theory (see [2, 7, 6, 5]) along with a classical bound of Figiel on the Rademacher projection of polynomials. A discussion concerning the size of the implicit constant C_d appearing in (1.11) is postponed to Section 2 (see Remark 2.3 there).

Unlike the two-sided inequality (1.8), our bound (1.11) is only one-sided and as a matter of fact there are examples in which it is far from optimal. In particular, for functions which are permutationally symmetric, we obtain a more accurate estimate. In what follows, we shall denote by $T_k(x)=\sum_{\ell=0}^k c(k,\ell)x^\ell$ the kth Chebyshev polynomial of the first kind characterized by the property $T_k(\cos\theta)=\cos(k\theta)$, where $\theta\in\mathbb{R}$. Moreover, we shall use the ad hoc notation

$$\tilde{c}(k,\ell) := \begin{cases} c(k,\ell) & \text{if } k-\ell \text{ is even,} \\ c(k-1,\ell) & \text{if } k-\ell \text{ is odd.} \end{cases} \tag{1.12}$$

For $\ell\in\{1,\ldots,n\}$, let f_ℓ be the ℓth elementary symmetric multilinear polynomial

$$\forall x\in\{-1,1\}^n, \qquad f_\ell(x) := \sum_{\substack{S\subseteq\{1,\ldots,n\} \\ |S|=\ell}} w_S(x). \tag{1.13}$$

We have the following bound on the distance of symmetric polynomials from tail spaces.

Theorem 1.4. *Let $n,k,d\in\mathbb{N}$ with $n\geq k\geq d$. Then every symmetric polynomial*

$$f=\sum_{\ell=0}^d \alpha_\ell f_\ell \tag{1.14}$$

of degree at most d on $\{-1,1\}^n$ satisfies

$$\inf_{g\in\mathcal{P}^n_{>k}} \|f-g\|_1 \leq \sum_{\ell=0}^d |\alpha_\ell|\,|\tilde{c}(k,\ell)|. \tag{1.15}$$

This bound can sometimes be reversed and, in particular, it gives a sharp estimate as $n\to\infty$ for the L_1-distance of the elementary symmetric polynomial f_d from the kth tail space.

Corollary 1.5. *For any $n,k,d\in\mathbb{N}$ with $n\geq k\geq d$, there exists $\varepsilon_n(k,d)>0$ such that*

$$|\tilde{c}(k,d)|-\varepsilon_n(k,d) \leq \inf_{g\in\mathcal{P}^n_{>k}} \|f_d-g\|_1 \leq |\tilde{c}(k,d)| \tag{1.16}$$

and $\lim_{n\to\infty}\varepsilon_n(k,d)=0$.

The main motivation behind [19] was a question of Bogucki, Nayar and Wojciechowski, asking to estimate the L_1-distance of the symmetric Rademacher sum f_1 from the kth tail space. Corollary 1.5 extends (at least asymptotically in n) the answer given by Oleszkiewicz to all symmetric homogeneous polynomials. We point out though that for $k=1$, (1.16) is sharper than

Oleszkiewicz's bound (1.7) as $n \to \infty$, as (1.7) is tight only up to a multiplicative constant.

2. Proofs. We start with the simple duality argument leading to Proposition 1.1, variants of which will be used throughout the paper.

Proof of Proposition 1.1. Consider the identity operator acting as $\mathsf{id}(h) = h$ on a function of the form $h : \{-1,1\}^n \to \mathbb{R}$. Then, the optimal constant $\mathsf{M}_{p,q}(k)$ can be expressed as

$$\begin{aligned}(2.1)\quad &\mathsf{M}_{p,q}(k)\\ &= \big\|\mathsf{id} : (\mathcal{P}^n_{\le k}, \|\cdot\|_p) \to (\mathcal{P}^n_{\le k}, \|\cdot\|_q)\big\| = \big\|\mathsf{id}^* : (\mathcal{P}^n_{\le k}, \|\cdot\|_q)^* \to (\mathcal{P}^n_{\le k}, \|\cdot\|_p)^*\big\|\end{aligned}$$

by duality. Moreover, observe that since $(\mathcal{P}^n_{\le k}, \|\cdot\|_r)$ is a subspace of L_r, its dual is isometric to

$$(\mathcal{P}^n_{\le k}, \|\cdot\|_r)^* = L_{r^*}/(\mathcal{P}^n_{\le k})^{\perp} = L_{r^*}/\mathcal{P}^n_{>k}, \tag{2.2}$$

where $A^{\perp}$ is the annihilator of A. Since it is also clear that $\mathsf{id}^* = \mathsf{id}$, (2.1) concludes the proof. ∎

Using a theorem of Hitczenko [10] as input and the same duality, we deduce Theorem 1.2.

Proof of Theorem 1.2. The result of [10] asserts that if $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$ and $f_{\boldsymbol{a}}(x) = a_0 + \sum_{i=1}^n a_i x_i$, then

$$\|f_{\boldsymbol{a}}\|_{r^*} = \Big(\mathbb{E}\Big|\sum_{i=0}^n a_i x_i\Big|^{r^*}\Big)^{1/r^*} \asymp \mathsf{K}(\boldsymbol{a}, \sqrt{r^*}; \ell_1^{n+1}, \ell_2^{n+1}), \tag{2.3}$$

where $x_0, x_1, \ldots, x_n$ are independent Bernoulli random variables, and the first equality holds due to symmetry. In other words, the linear operator

$$T : (\mathbb{R}^{n+1}, \mathsf{K}(\cdot, \sqrt{r^*}; \ell_1^{n+1}, \ell_2^{n+1})) \to (\mathcal{P}^n_{\le 1}, \|\cdot\|_{r^*}) \tag{2.4}$$

given by $T\boldsymbol{a} = f_{\boldsymbol{a}}$ is an isomorphism, and thus the same holds for its adjoint. Recalling that

$$\mathsf{K}(\boldsymbol{a}, \sqrt{r^*}; \ell_1^{n+1}, \ell_2^{n+1}) = \inf\{\|\boldsymbol{b}\|_{\ell_1^{n+1}} + \sqrt{r^*}\|\boldsymbol{c}\|_{\ell_2^{n+1}} : \boldsymbol{a} = \boldsymbol{b} + \boldsymbol{c}\} \tag{2.5}$$

and the duality between sums and intersections of normed spaces [1, Theorem 2.7.1], we see that the norm of the dual of $(\mathbb{R}^{n+1}, \mathsf{K}(\cdot, \sqrt{r^*}; \ell_1^{n+1}, \ell_2^{n+1}))$ can be given by

$$\forall y \in \mathbb{R}^{n+1}, \quad \|y\|_{(\mathbb{R}^{n+1}, \mathsf{K}(\cdot, \sqrt{r^*}; \ell_1^{n+1}, \ell_2^{n+1}))^*} = \max\Big\{\|y\|_{\ell_\infty^{n+1}}, \frac{\|y\|_{\ell_2^{n+1}}}{\sqrt{r^*}}\Big\}. \tag{2.6}$$

By Parseval's identity, the action of the adjoint

$$T^* : L_r/\mathcal{P}^n_{>1} \to (\mathbb{R}^{n+1}, \mathsf{K}(\cdot, \sqrt{r^*}; \ell_1^{n+1}, \ell_2^{n+1}))^* \tag{2.7}$$

is given by

$$T^*(f + \mathcal{P}^n_{>1}) = (\mathbb{E}f, \widehat{f}(\{1\}), \dots, \widehat{f}(\{n\})) \tag{2.8}$$

and thus the conclusion is equivalent to fact that T^* is an isomorphism. ∎

We now proceed to the proof of the general upper bound for polynomials given in Theorem 1.3. The first ingredient for the proof is a discrete version of the classical Bohnenblust–Hille inequality from approximation theory (see the survey [7]) proven in [2, 6]. This asserts that for every $d \in \mathbb{N}$, there exists a (sharp) constant $\mathsf{B}_d \in (0, \infty)$ such that for any $n \geq d$, every polynomial $f : \{-1,1\}^n \to \mathbb{R}$ of degree at most d satisfies

$$\Big(\sum_{S \subseteq \{1,\dots,n\}} |\widehat{f}(S)|^{\frac{2d}{d+1}} \Big)^{\frac{d+1}{2d}} \leq \mathsf{B}_d \|f\|_\infty. \tag{2.9}$$

Moreover, $\frac{2d}{d+1}$ is the least exponent for which the implicit constant becomes independent of the ambient dimension n. The best known upper bound

$$B_d \leq \exp(C\sqrt{d \log d})$$

for the constant B_d is due to Defant, Mastyło and Pérez [6].

The ℓ*-Rademacher projection* of a function $f : \{-1,1\}^n \to \mathbb{R}$ is defined as

$$\forall x \in \{-1,1\}^n, \qquad \mathrm{Rad}_\ell f(x) := \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S| = \ell}} \widehat{f}(S) w_S(x). \tag{2.10}$$

Moreover, we write $\mathrm{Rad}_{\leq d} = \sum_{\ell \leq d} \mathrm{Rad}_\ell$. Apart from the discrete version of the Bohnenblust–Hille inequality (2.9), we will also use a standard bound on the norm of the ℓ-Rademacher projections which is usually attributed to Figiel (see also [9, Section 3] for a short proof).

PROPOSITION 2.1. *Let $n \geq k \geq d$. Then every function $f : \{-1,1\}^n \to \mathbb{R}$ of degree at most k satisfies*

$$\forall\, 0 \leq \ell \leq d, \qquad \|\mathrm{Rad}_\ell f\|_\infty \leq |\tilde{c}(k,\ell)|\, \|f\|_\infty, \tag{2.11}$$

and thus

$$\|\mathrm{Rad}_{\leq d} f\|_\infty \leq \sum_{\ell=0}^{d} \|\mathrm{Rad}_\ell f\|_\infty \leq \sum_{\ell=0}^{d} |\tilde{c}(k,\ell)|\, \|f\|_\infty, \tag{2.12}$$

where $\tilde{c}(k,\ell)$ is given by (1.12). *It is moreover known that $|\tilde{c}(k,\ell)| \leq k^\ell/\ell!$.*

Combining the above with Parseval's identity, we deduce the following bound.

LEMMA 2.2. *Let $n \geq k \geq d$. Then every function $f : \{-1,1\}^n \to \mathbb{R}$ of degree at most k satisfies*

$$\max\{\|(\mathrm{Rad}_{\leq d} f)^\wedge\|_{\ell_2^m}, \sigma(k,d)^{-1}\|(\mathrm{Rad}_{\leq d} f)^\wedge\|_{\ell^m_{\frac{2d}{d+1}}}\} \leq \inf_{g\in \mathcal{P}^n_{>d}\cap\mathcal{P}^n_{\leq k}} \|f-g\|_\infty, \tag{2.13}$$

where $m = \binom{n}{0} + \cdots + \binom{n}{d}$ and $\sigma(k,d) = \mathsf{B}_d \sum_{\ell=0}^d |\tilde{c}(k,\ell)|$.

Proof. Fix a function $g \in \mathcal{P}^n_{>d} \cap \mathcal{P}^n_{\leq k}$. Then

$$\|(\mathrm{Rad}_{\leq d} f)^\wedge\|_{\ell_2^m} \leq \|\widehat{f} - \widehat{g}\|_{\ell_2^M} = \|f-g\|_2 \leq \|f-g\|_\infty, \tag{2.14}$$

where $M = \binom{n}{0} + \cdots + \binom{n}{k}$. Moreover, we have

$$\begin{aligned}\|(\mathrm{Rad}_{\leq d} f)^\wedge\|_{\ell^m_{\frac{2d}{d+1}}} &\overset{(2.9)}{\leq} \mathsf{B}_d\|\mathrm{Rad}_{\leq d}(f)\|_\infty = \mathsf{B}_d\|\mathrm{Rad}_{\leq d}(f-g)\|_\infty \\ &\overset{(2.12)}{\leq} \mathsf{B}_d \sum_{\ell=0}^d |\tilde{c}(k,\ell)|\,\|f-g\|_\infty. \quad \blacksquare\end{aligned}$$

Equipped with Lemma 2.2, we can complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Consider the normed spaces $X = (\mathcal{P}^n_{\leq k}, \|\cdot\|_\infty)$ and $Y = (\mathbb{R}^m, \|\cdot\|_Y)$ with

$$\forall y \in \mathbb{R}^m, \qquad \|y\|_Y = \max\{\|y\|_{\ell_2^m}, \sigma(k,d)^{-1}\|y\|_{\ell^m_{\frac{2d}{d+1}}}\}, \tag{2.15}$$

where $m = \binom{n}{0} + \cdots + \binom{n}{d}$. Moreover, let $Z = \mathcal{P}^n_{>d} \cap \mathcal{P}^n_{\leq k} \subset X$, viewed as a normed subspace of X. Lemma 2.2 asserts that the linear operator $A : X/Z \to Y$ given by

$$\forall f \in X, \qquad A(f+Z) = (\widehat{f}(S))_{|S|\leq d}, \tag{2.16}$$

has norm $\|A\| \leq 1$. Therefore, the same holds for its adjoint $A^* : Y^* \to (X/Z)^*$.

By the usual duality between sums and intersections of normed spaces [1, Theorem 2.7.1], we see that the space Y^* is isometric to

$$\forall w \in \mathbb{R}^m, \qquad \|w\|_{Y^*} = \mathsf{K}(w, \sigma(k,d); \ell_2^m, \ell^m_{\frac{2d}{d-1}}). \tag{2.17}$$

Moreover, as X/Z is a quotient of X, its dual is the subspace of $X^* = L_1/\mathcal{P}^n_{>k}$ which is identified with the annihilator of Z inside X^*. In other words, it is the set

$$\begin{aligned}(X/Z)^* &= \{f + \mathcal{P}^n_{>k} : \mathbb{E}[fg] = 0 \text{ for every } g \in Z\} \\ &= \{f + \mathcal{P}^n_{>k} : f \in \mathcal{P}^n_{\leq d}\} \\ &= \mathrm{span}(\mathcal{P}^n_{\leq d} \cup \mathcal{P}^n_{>k})/\mathcal{P}^n_{>k}\end{aligned}$$

with the L_1 quotient norm. Finally, for a sequence $\boldsymbol{a} = (a_S)_{|S|\le d} \in Y^*$ and an equivalence class $f + Z \in X/Z$, we have

(2.18)
$$\langle \boldsymbol{a}, A(f+Z)\rangle = \sum_{\substack{S\subseteq\{1,\ldots,n\}\\ |S|\le d}} a_S \widehat{f}(S) = \Big\langle \sum_{\substack{S\subseteq\{1,\ldots,n\}\\ |S|\le d}} a_S w_S + \mathcal{P}^n_{>k}, f+Z\Big\rangle$$
$$= \langle A^*(\boldsymbol{a}), f+Z\rangle,$$

where the first brackets $\langle\cdot,\cdot\rangle$ denote the duality in Y and the following brackets denote the duality in X/Z. Therefore, we conclude that

$$\forall \boldsymbol{a}\in Y^*, \qquad A^*(\boldsymbol{a}) = \sum_{\substack{S\subseteq\{1,\ldots,n\}\\ |S|\le d}} a_S w_S + \mathcal{P}^n_{>k}, \tag{2.19}$$

and thus the condition $\|A^*\| \le 1$ means that for any $f : \{-1,1\}^n \to \mathbb{R}$ of degree at most d,

$$\inf_{g\in\mathcal{P}^n_{>k}} \|f-g\|_1 = \|A^*(\widehat{f})\|_{(X/Z)^*} \le \|\widehat{f}\|_{Y^*} = \mathsf{K}(\widehat{f}, \sigma(k,d); \ell_2^m, \ell^m_{\frac{2d}{d-1}}). \tag{2.20}$$

Finally, since

$$\sigma(k,d) \le \mathsf{B}_d \sum_{\ell=0}^{d} |\tilde{c}(k,\ell)| \le \mathsf{B}_d \sum_{\ell=0}^{d} \frac{k^\ell}{\ell!} \le e\mathsf{B}_d k^d, \tag{2.21}$$

we deduce the conclusion of the theorem with $\mathsf{C}_d = e\mathsf{B}_d$. ∎

Remark 2.3. To the best of our knowledge, there are no nonconstant lower bounds on the size of the discrete Bohnenblust–Hille constant B_d, so it is even conceivable that the constant C_d in (1.11) can be chosen to be independent of d.

Remark 2.4. A duality argument similar to that employed for Theorem 1.3 shows that for every $d \in \mathbb{N}$, the constant B_d in inequality (2.9) is also the least constant for which every function $f : \{-1,1\}^n \to \mathbb{R}$, where $n \ge d$, satisfies

$$\inf_{g\in\mathcal{P}^n_{>d}} \|f-g\|_1 \le \mathsf{B}_d \Big(\sum_{\substack{S\subseteq\{1,\ldots,n\}\\ |S|\le d}} |\widehat{f}(S)|^{\frac{2d}{d-1}}\Big)^{\frac{d-1}{2d}}. \tag{2.22}$$

Remark 2.5. It was pointed out to us by Oleszkiewicz that the main result (1.8) of [19] also admits a dual formulation. Namely, for every $\boldsymbol{a} = (a_1,\ldots,a_n) \in \mathbb{R}^n$, we have

$$\inf\{\|f_{\boldsymbol{a}} - g\|_\infty : g \in \mathcal{P}^n_{\{0\}\cup\{2,\ldots,k\}}\} \asymp \max\{\|\boldsymbol{a}\|_{\ell_2^n}, \|\boldsymbol{a}\|_{\ell_1^n}/k\}. \tag{2.23}$$

This can be proven using similar ideas as in the proof of Theorem 1.3.

A slight variant of the arguments above also yields Theorem 1.4 for symmetric functions.

Proof of Theorem 1.4. Let f be a permutationally symmetric function of the form $f=\sum_{\ell=0}^{d}\alpha_\ell f_\ell$, where f_ℓ is the ℓth elementary symmetric polynomial. Then the Hahn–Banach theorem gives

$$\inf_{g\in\mathcal{P}^n_{>k}}\|f-g\|_1=\sup_{0\neq h\in\mathcal{P}^n_{\leq k}}\frac{\mathbb{E}[fh]}{\|h\|_\infty}. \tag{2.24}$$

Observe now that we can write

$$\mathbb{E}[fh]=\sum_{\ell=0}^{d}\alpha_\ell\mathbb{E}[f_\ell h]=\sum_{\ell=0}^{d}\alpha_\ell\sum_{\substack{S\subseteq\{1,\ldots,n\}\\|S|=\ell}}\widehat{h}(S)=\sum_{\ell=0}^{d}\alpha_\ell\,\mathrm{Rad}_\ell\,h(1,\ldots,1). \tag{2.25}$$

Thus, by Figiel's bound (2.11),

$$\mathbb{E}[fh]\leq\sum_{\ell=0}^{d}|\alpha_\ell|\,\|\mathrm{Rad}_\ell h\|_\infty\leq\sum_{\ell=0}^{d}|\alpha_\ell|\,|\tilde{c}(k,\ell)|\,\|h\|_\infty \tag{2.26}$$

and the desired inequality follows from (2.24). ∎

Equipped with Theorem 1.4, we present the proof of Corollary 1.5.

Proof of Corollary 1.5. The upper bound in (1.16) follows immediately from Theorem 1.4. For the lower bound, consider the auxiliary symmetric function $H_{k,n}:\{-1,1\}^n\to\mathbb{R}$ given by

$$\forall x\in\{-1,1\}^n,\qquad H_{k,n}(x):=T_k\Big(\frac{x_1+\cdots+x_n}{n}\Big)=\sum_{\ell=0}^{k}\beta_{\ell,k,n}f_\ell(x), \tag{2.27}$$

where f_ℓ is the ℓth elementary symmetric polynomial, and notice that $H_{k,n}$ has degree at most k. As $T_k(x)$ has the same parity as k, it follows that $\beta_{\ell,k,n}=0$ if $k-\ell$ is odd. We distinguish two cases depending on the parity of $k-d$.

• Suppose that $k-d$ is even and consider the function $\varphi_{d,k,n}:\{-1,1\}^n\to\mathbb{R}$ given by

$$\forall x\in\{-1,1\}^n,\qquad \varphi_{d,k,n}(x):=\sum_{0\leq\ell\leq d:\,2|d-\ell}\mathrm{sign}(\beta_{\ell,k,n})f_\ell(x), \tag{2.28}$$

that is also symmetric and of degree at most d. Then on the one hand we know that

$$\inf_{g\in\mathcal{P}^n_{>k}}\|\varphi_{d,k,n}-g\|_1\overset{(1.15)}{\leq}\sum_{0\leq\ell\leq d:\,2|d-\ell}|\tilde{c}(k,\ell)|=\sum_{0\leq\ell\leq d:\,2|d-\ell}|c(k,\ell)|. \tag{2.29}$$

On the other hand, we have the following lower estimate:

$$\inf_{g\in\mathcal{P}^n_{>k}} \|\varphi_{d,k,n}-g\|_1 \overset{(2.24)}{=} \sup_{0\neq h\in\mathcal{P}^n_{\leq k}} \frac{\mathbb{E}[\varphi_{d,k,n}h]}{\|h\|_\infty} \geq \frac{|\mathbb{E}[\varphi_{d,k,n}H_{k,n}]|}{\|H_{k,n}\|_\infty}. \tag{2.30}$$

By definition, $\|H_{k,n}\|_\infty \leq \sup_{x\in[-1,1]}|T_k(x)|\leq 1$ and $H_{k,n}(1,\ldots,1)=T_k(1)=1$. Therefore,

$$\begin{aligned}\inf_{g\in\mathcal{P}^n_{>k}} \|\varphi_{d,k,n}-g\|_1 &\geq |\mathbb{E}[\varphi_{d,k,n}H_{k,n}]| \\ &= \Big|\sum_{0\leq\ell\leq d:\, 2|d-\ell} \operatorname{sign}(\beta_{\ell,k,n})\operatorname{Rad}_\ell H_{k,n}(1,\ldots,1)\Big| \\ &= \sum_{0\leq\ell\leq d:\, 2|d-\ell} |\operatorname{Rad}_\ell H_{k,n}(1,\ldots,1)|.\end{aligned} \tag{2.31}$$

To further estimate this sum, we use [14, Lemma 27], which implies that there exists a positive constant $\varepsilon_n(k,d)>0$, with $\varepsilon_n(k,d)=O_{k,d}(1/n)$ as $n\to\infty$, such that

$$\sum_{0\leq\ell\leq d:\, 2|d-\ell} |\operatorname{Rad}_\ell H_{k,n}(1,\ldots,1)| \geq \sum_{0\leq\ell\leq d:\, 2|d-\ell} |c(k,\ell)| - \varepsilon_n(k,d). \tag{2.32}$$

Hence, combining the above we conclude that

$$\inf_{g\in\mathcal{P}^n_{>k}} \|\varphi_{d,k,n}-g\|_1 \geq \sum_{0\leq\ell\leq d:\, 2|d-\ell} |c(k,\ell)| - \varepsilon_n(k,d). \tag{2.33}$$

Finally, to bound from below the L_1-distance of f_d from the tail space, we write (putting $\varphi_{0,k,n}=\varphi_{-1,k,n}\equiv 0$)

$$f_d = \operatorname{sign}(\beta_{d,k,n})(\varphi_{d,k,n}-\varphi_{d-2,k,n}) \tag{2.34}$$

and using the triangle inequality, we get

$$\begin{aligned}\inf_{g\in\mathcal{P}^n_{>k}} \|f_d-g\|_1 &\geq \inf_{g\in\mathcal{P}^n_{>k}} \|\varphi_{d,k,n}-g\|_1 - \inf_{g\in\mathcal{P}^n_{>k}} \|\varphi_{d-2,k,n}-g\|_1 \\ &\overset{(2.29)\wedge(2.33)}{\geq} \sum_{0\leq\ell\leq d:\, 2|d-\ell} |c(k,\ell)| - \varepsilon_n(k,d) - \sum_{0\leq\ell\leq d-2:\, 2|d-2-\ell} |c(k,\ell)| \\ &= |c(k,d)| - \varepsilon_n(k,d),\end{aligned} \tag{2.35}$$

thus concluding the proof of the lower bound in (1.16).

- If $k-d$ is odd, we use the identity (putting $\varphi_{0,k,n}=\varphi_{-1,k,n}=\varphi_{d,-1,n}\equiv 0$)

$$f_d = \operatorname{sign}(\beta_{d,k-1,n})(\varphi_{d,k-1,n}-\varphi_{d-2,k-1,n}). \tag{2.36}$$

The rest of the argument is identical. ∎

Remark 2.6. In this paper, we studied dual versions of moment comparison estimates on the hypercube (1.3) and investigated the endpoint case of their duals (1.4) for polynomials. By formal reasoning similar to the proof of Proposition 1.1, one can derive dual versions of various other polynomial inequalities, including Bernstein–Markov inequalities and their reverses and bounds for the action of the heat semigroup. We refer to [8] for a systematic treatment of such estimates.

Acknowledgements. We are grateful to Krzysztof Oleszkiewicz for valuable discussions. H. Z. is grateful to Institut de Mathématiques de Jussieu for the hospitality during a visit in 2023.

This material is based upon work supported by the NSF grant DMS-1929284 while the authors were in residence at ICERM for the Harmonic Analysis and Convexity program.

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