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Projective Fraïssé limits and generalized Ważewski dendrites

by

Alessandro Codenotti (Münster) and
Aleksandra Kwiatkowska (Münster and Wrocław)

Abstract. We continue the study of projective Fraïssé limits of trees initiated by Charatonik and Roe and further continued by Charatonik, Kwiatkowska, Roe and Yang. We construct many generalized Ważewski dendrites as topological realizations of projective Fraïssé limits of families of finite trees with (weakly) coherent epimorphisms. Moreover, we use the categorical approach to Fraïssé limits developed by Kubiś to construct all generalized Ważewski dendrites as topological realizations of Fraïssé limits of suitable categories of finite structures. As an application, we recover a homogeneity result for countable dense sets of endpoints in generalized Ważewski dendrites.

1. Introduction. Projective Fraïssé limits were introduced by Irwin and Solecki in [13] by dualizing the classical construction of Fraïssé limits from model theory, as a tool to study the pseudo-arc. Panagiotopolous and Solecki generalised their construction in [22] by allowing strict subcollections of epimorphisms, instead of considering all of them. They used this approach to study the Menger curve by approximating it with finite graphs with monotone epimorphisms. Projective Fraïssé limits have since been used to study various compact spaces, for example the Lelek fan by Bartošová and Kwiatkowska [3], the Poulsen simplex by Kubiś and Kwiatkowska [17], P-adic pseudo-solenoids (for a set P of primes) by Bartoš and Kubiś [2], the universal Knaster continuum by Iyer [14] (which has independently been constructed from a projective Fraïssé limit in Wickman's PhD thesis [23]) and the generic object for a new class of compact metric spaces named *fences* by Basso and Camerlo [4]. Recently, Charatonik, Kwiatkowska, Roe and Yang studied projective Fraïssé limits of trees with monotone and confluent epi-

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morphisms, which are adaptations to topological graphs of corresponding well-established notions from continuum theory; see [6].

Dendrites are one-dimensional continua which have been widely studied in topological dynamics [9, 10, 1, 20, 11]. By a result of Charatonik and Dilks [5], for any $P \subseteq \{3, \dots, \omega\}$, there exists a unique dendrite W_P such that:

(1) every ramification point of W_P has order in P;
(2) for every $p \in P$ the set of ramification points of W_P of order p is arcwise dense in W_P.

The spaces W_P are known as the (generalized) Ważewski dendrites, they are universal for appropriate classes of dendrites and enjoy good homogeneity properties. Because of these regularity properties they are of particular interest among dendrites. Their homeomorphism groups have been studied by Duchesne [8], while the universal minimal flows of those groups were computed by Kwiatkowska [19].

Among the spaces constructed in [6] there is the Ważewski dendrite W_3 (denoted by D_3 in that paper), obtained from the projective Fraïssé limit of the family of finite trees with ramification points of order 3 and all monotone epimorphisms; we build on that work to realize many generalized Ważewski dendrites W_P for $P \subseteq \{3, \dots, \omega\}$ as the topological realizations of projective Fraïssé limits of appropriately chosen families of finite trees and epimorphisms between them.

In particular, we obtain the following theorem. See Definition 4.14, the remarks following it, and Theorems 4.19 and 4.20 for details.

THEOREM 1.1. *Let $P \subseteq \{3, \dots, \omega\}$ be coinfinite. Then the Ważewski dendrite W_P can be constructed as the topological realization of the projective Fraïssé limit of finite trees with monotone maps with additional properties.*

By moving to the more general setting of Fraïssé categories developed in [16] we remove the coinfiniteness assumption from the previous result and obtain the following general statement; see Theorems 5.10 and 5.23 for details.

THEOREM 1.2. *Let $P \subseteq \{3, \dots, \omega\}$ and let $E \subseteq W_P$ be a countable dense set of endpoints. Then the pair (W_P, E) can be constructed as the topological realization of the limit of a Fraïssé category $\ddagger\mathcal{F}_P$ of pairs (T, F), where T is a finite tree and F is a set of endpoints in T, with projection-embedding pairs of maps.*

Moreover, we show in Theorem 5.22 that any sequence in the category $\ddagger\mathcal{F}_P$ whose limit has as topological realization W_P must be a Fraïssé sequence for $\ddagger\mathcal{F}_P$.

A Fraïssé-theoretic approach to the study of Ważewski dendrites has already been employed successfully by Kwiatkowska in [19], where a Fraïssé structure A_P with $\mathrm{Aut}(A_P) \simeq \mathrm{Homeo}(W_P)$ is constructed. While this approach is well-suited to study properties of $\mathrm{Homeo}(W_P)$, it cannot be used to study the endpoints of W_P, since the injective Fraïssé limit construction produces a countable structure corresponding to the ramification points of W_P. In contrast, we use the uniqueness of Fraïssé sequences in the last section to recover a homogeneity result for the endpoints of W_P from [5], see Theorem 5.11.

The paper is structured as follows. In Sections 2 and 3 we recall some background notions concerning projective Fraïssé limits and topological graphs. In Section 4 we introduce a new class of maps between finite trees and use it to prove Theorem 1.1. Finally, in Section 5 we describe projection-embedding Fraïssé categories of finite trees that can be used to prove Theorem 1.2.

2. Projective Fraïssé limits. In this section we recall the necessary background on projective Fraïssé limits. Most definitions and results are from either [13] or [22].

DEFINITION 2.1. Let L be a first-order language consisting of a set $\{R_i\}_{i\in I}$ of relation symbols, with R_i of arity $n_i \in \mathbb{N}$, and a set $\{f_j\}_{j\in J}$ of function symbols, with f_j of arity $m_j \in \mathbb{N}$. A *topological L-structure* is a zero-dimensional, compact, metrizable topological space A together with closed subsets $R_i^A \subseteq A^{n_i}$ for every $i \in I$ and continuous functions $f_j^A \colon A^{m_j} \to A$ for every $j \in J$.

DEFINITION 2.2. Given two topological L-structures A and B we say that $g \colon B \to A$ is an *epimorphism* if g is a continuous surjection satisfying:

(1) For all $j \in J$, $f_j^A(g(b_1), \dots, g(b_{m_j})) = g(f_j^B(b_1, \dots, b_{m_j}))$.
(2) For all $i \in I$, we have $R_i^A(a_1, \dots, a_{n_i})$ if and only if there are $b_1 \in g^{-1}(a_1), \dots, b_{n_i} \in g^{-1}(a_{n_i})$ such that $R_i^B(b_1, \dots, b_{n_i})$ holds.

We can now give the definition of a projective Fraïssé family and its projective Fraïssé limit. We follow the construction from [22]. The main difference compared to the original construction from [13] is allowing a restricted class of epimorphisms between the structures under consideration.

DEFINITION 2.3. Let $\mathcal{F}$ be a class of finite topological L-structures with a fixed class of epimorphisms between them. We say that $\mathcal{F}$ is a *projective Fraïssé family* if it satisfies the following conditions:

(1) There are countably many L-structures in $\mathcal{F}$ up to isomorphism.
(2) The epimorphisms in $\mathcal{F}$ are closed under composition and the identity epimorphisms are in $\mathcal{F}$.

(3) For all $A, B \in \mathcal{F}$ there are $C \in \mathcal{F}$ and epimorphisms $f\colon C \to A$ and $g\colon C \to B$ in $\mathcal{F}$. In analogy with the classical Fraïssé limits this property is known as the *joint projection property.*
(4) For all $A, B, C \in \mathcal{F}$ and epimorphisms $f\colon B \to A$ and $g\colon C \to A$ in $\mathcal{F}$, there exist $D \in \mathcal{F}$ and epimorphisms $h\colon D \to B$ and $k\colon D \to C$ in $\mathcal{F}$ such that $f \circ h = g \circ k$. This property is known as the *projective amalgamation property.*

Following [22] we introduce $\mathcal{F}^\omega$ as the class of structures that can be approximated in $\mathcal{F}$. The projective Fraïssé limit will be an element of $\mathcal{F}^\omega$ satisfying appropriate universality and homogeneity properties.

Definition 2.4. Let $\mathcal{F}$ be a class of finite topological L-structures. We define $\mathcal{F}^\omega$ to be the class of all topological L-structures in $\mathcal{F}$ that are realized as the inverse limit of an ω-indexed inverse system of topological L-structures with bonding epimorphisms in $\mathcal{F}$. More explicitly, $F \in \mathcal{F}^\omega$ iff there is an inverse system of topological L-structures $\langle F_i, f^n_m\colon F_n \to F_m \mid i, n, m < \omega\rangle$ with $F_i, f^n_m \in \mathcal{F}$, such that $F = \varprojlim F_i$ (in particular, $\mathcal{F} \subseteq \mathcal{F}^\omega$ by picking constant sequences).

Note that, given an inverse system $\langle F_i, f^n_m\colon F_n \to F_m \mid i, n, m < \omega\rangle$ in $\mathcal{F}$ as above, $F = \varprojlim F_i$ exists. Its underlying space is the inverse limit of the F_i in the category of topological spaces, while for $f^n\colon F \to F_n$, the continuous maps obtained by construction of the limit of topological spaces, we have

$$R^F_i(a_1, \dots, a_{n_i}) \iff \forall n\ R^{F_n}_i(f^n(a_1), \dots, f^n(a_{n_i}))$$

and

$$f^F_j(a_1, \dots, a_{m_j}) = a \iff \forall n\ f^{F_n}_j(f^n(a_1), \dots, f^n(a_{m_j})) = f^n(a).$$

If $F = \varprojlim F_i \in \mathcal{F}^\omega$ and $E \in \mathcal{F}$, an epimorphism $f\colon F \to E$ is in $\mathcal{F}^\omega$ if and only if there exist $i \in \mathbb{N}$ and an epimorphism $f'\colon F_i \to E$ with $f = f' \circ f^i$. If G is another object in $\mathcal{F}^\omega$ an epimorphism $g\colon G \to F$ is in $\mathcal{F}^\omega$ iff $f^i \circ g$ is in $\mathcal{F}^\omega$ for all i.

Theorem 2.5 ([22, Theorem 3.1]). *Let $\mathcal{F}$ be a projective Fraïssé family of topological L-structures. Then there exists a unique topological L-structure $\mathbb{F} \in \mathcal{F}^\omega$ satisfying the following properties:*

(1) *For every $A \in \mathcal{F}$ there is an epimorphism $f\colon \mathbb{F} \to A$ in $\mathcal{F}^\omega$.*
(2) *For all $A, B \in \mathcal{F}^\omega$ and all epimorphisms $f\colon \mathbb{F} \to A$ and $g\colon B \to A$ in $\mathcal{F}^\omega$, there exists an epimorphism $h\colon \mathbb{F} \to B$ in $\mathcal{F}^\omega$ such that $f = g \circ h$.*

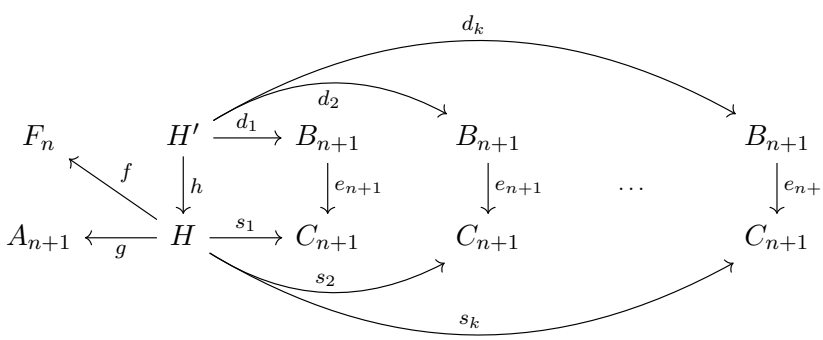
Moreover, as pointed out in [13, Lemma 2.2] *in terms of covers, we have:*

(3) *For every $\varepsilon > 0$ there is $A \in \mathcal{F}$ and an epimorphism $f\colon \mathbb{F} \to A$ in $\mathcal{F}^\omega$ such that* $\operatorname{diam}(f^{-1}(a)) \leq \varepsilon$ *for every $a \in A$.*

Proof. We give a sketch of the argument; the details can be found in [13, 22]. The idea is to obtain $\mathbb{F}$ as the inverse limit of a sequence $\langle F_i \mid i \in \mathbb{N}\rangle$ with maps $f^n_m\colon F_n \to F_m$ of elements and epimorphisms of $\mathcal{F}$. We will call this sequence a *Fraïssé sequence* for $\mathcal{F}$. Since there are only countably many finite structures in $\mathcal{F}$ up to isomorphism, we can build two countable lists $\langle A_i \mid i \in \mathbb{N}\rangle$ and $\langle e_n\colon B_n \to C_n\rangle$ such that:

- Every structure in $\mathcal{F}$ is isomorphic to A_i for some i.
- Every epimorphism type $e\colon B \to C$ in $\mathcal{F}$ appears infinitely many times in $\langle e_n \mid n \in \mathbb{N}\rangle$.

Now set $F_0 = A_0$. Inductively assume that F_n together with the epimorphisms $f^n_m\colon F_n \to F_m$ for all $m < n$ have been defined. By the joint projection property we can find $H \in \mathcal{F}$ with epimorphisms $g\colon H \to A_{n+1}$ and $f\colon H \to F_n$. Since H is finite there are only finitely many epimorphism types $s_i\colon H \to C_{n+1}$, $i = 1, \dots, k$. Using the projective amalgamation property, we can find H'_1 and epimorphisms $h_1\colon H'_1 \to H$, $c_1\colon H'_1 \to B_{n+1}$ such that $s_1 \circ h_1 = e_{n+1} \circ c_1$. Iterating this construction, for all $n \leq k$ we can find H'_n with epimorphisms $h_n\colon H'_n \to H'_{n-1}$ and $c_n\colon H'_n \to B_{n+1}$ such that $s_n \circ h_1 \circ \dots \circ h_n = e_{n+1} \circ c_n$. Setting now $H' = H'_k \in \mathcal{F}$, $h = h_1 \circ \dots \circ h_k\colon H' \to H$, $d_i = c_i \circ h_{i+1} \circ \dots \circ h_k\colon H' \to B_{n+1}$, $i = 1, \dots, k$, we find that the following diagram commutes, that is, $s_i \circ h = e_{n+1} \circ d_i$ for all i.



Once this is done, set $F_{n+1} = H'$ and set $f^{n+1}_m = f^n_m \circ f \circ h$ for all $m < n$. It is easy to verify that $\mathbb{F} = \varprojlim F_i$ satisfies the first two properties of a projective Fraïssé limit of $\mathcal{F}$. The third one follows from the definition of the product metric on $\prod F_i$. ∎

From the proof of Theorem 2.5 we extract the following important notion.

Definition 2.6. Let $\mathcal{F}$ be a projective Fraïssé family of topological L-structures. A sequence $\langle F_n \mid n \in \mathbb{N}\rangle$ with epimorphisms $f^m_n\colon F_m \to F_n$ is called a *Fraïssé sequence* for $\mathcal{F}$ if the following conditions hold:

- For every $A \in \mathcal{F}$ there exists $n \in \mathbb{N}$ such that there is an epimorphism $f\colon F_n \to A$ in $\mathcal{F}$.
- Whenever $f\colon A \to F_m$ is an epimorphism in $\mathcal{F}$, there exists $n \in \mathbb{N}$ such that there is an epimorphism $g\colon F_n \to A$ in $\mathcal{F}$ with $f \circ g = f^n_m$.

3. Topological graphs and monotone epimorphisms. In this section we reproduce for convenience definitions and results about topological graphs that will be used in Sections 4 and 5. They are mostly taken from [6].

Theorems 3.7, 3.14–3.17 first appeared in the preprint of Charatonik–Roe [7], but since that preprint is unpublished, we include the proofs for completeness with the permission of the authors. Those results will appear in [6].

3.1. Topological graphs and epimorphisms

Definition 3.1. A *graph* G is a pair $(V(G), E(G))$ consisting of a set of *vertices* $V(G)$ and a set of *edges* $E(G) \subseteq V(G)^2$ such that:

- For all $a, b \in V(G)$, $\langle a, b\rangle \in E(G) \Rightarrow \langle b, a\rangle \in E(G)$.
- For all $a \in V(G)$, $\langle a, a\rangle \in E(G)$.

A finite graph T is a *tree* if there are no pairwise distinct vertices $a_1, \dots, a_n \in T$, $n \geq 3$, such that $\langle a_i, a_{i+1}\rangle \in E(T)$ for $i = 1, \dots, n-1$, and $\langle a_n, a_1\rangle \in E(T)$. We will interchangeably write $a \in V(A)$ and $a \in A$ and we call the edges $\langle a, b\rangle \in E(A)$ with $a \neq b$ *nontrivial*. A *topological graph* is a graph G equipped with a compact, Hausdorff, zero-dimensional, metrizable topology on $V(G)$ such that $E(G)$ is a closed subspace of $V(G)^2$. Every topological graph is a topological L-structure for the language $L = \{R\}$ consisting of a single binary relation in an obvious way. We will keep using the notation $\langle a, b\rangle \in E(G)$ rather than aRb even though we will often think about topological graphs as topological L-structures. An epimorphism $f\colon A \to B$ of topological graphs is an epimorphism in the sense of Definition 2.2, so in particular it maps edges to edges.

Let us point out that in this context requiring topological graphs to be zero-dimensional is a natural property, since we are interested in inverse limits of finite discrete structures. We warn the reader that in the literature there also exists different definitions of topological graphs, where edges are embedded as arcs into the topological structure.

If A is a topological graph and $B \subseteq A$, $A \setminus B$ denotes the topological graph with $V(A \setminus B) = V(A) \setminus V(B)$ and for all $a_1, a_2 \in V(A \setminus B)$,

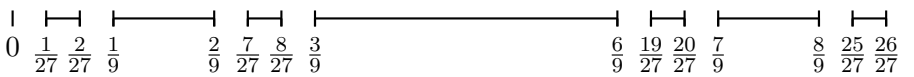
$$\langle a_1, a_2\rangle \in E(A \setminus B) \iff \langle a_1, a_2\rangle \in E(A).$$

A topological graph G is said to have a *transitive set of edges* if

$$\langle a, b\rangle, \langle b, c\rangle \in E(G) \implies \langle a, c\rangle \in E(G).$$

It is called a *prespace* if the edge relation $aRb \Leftrightarrow \langle a, b\rangle \in E(G)$ is an equivalence relation, equivalently a prespace is a topological graph with a transitive set of edges. In that case we call the quotient topological space $|G| = V(G)/R$ the *topological realization* of G.

EXAMPLE 3.2. Let $\mathcal{C} \subseteq [0,1]$ be the standard middle-thirds Cantor set. Consider the topological graph G with $V(G) = \mathcal{C}$ and for distinct $x, y \in \mathcal{C}$ we have $\langle x, y\rangle \in E(G)$ if and only if x, y are the endpoints of one of the intervals removed from $[0,1]$ in the construction of $\mathcal{C}$ (see the picture below for the first few stages of the construction, with the edges represented by solid black lines).



Then G is a prespace, in particular $|G| \cong [0,1]$.

DEFINITION 3.3. A topological graph G is called *disconnected* if one can partition G into two nonempty disjoint closed sets A, B such that whenever $a \in A$, $b \in B$, we have $\langle a, b\rangle \notin E(G)$. A topological graph is called *connected* if it is not disconnected. A topological graph A is called an *arc* if it is connected, but for all $a \in A$, except at most two vertices called the endpoints, $A \setminus \{a\}$ is disconnected.

EXAMPLE 3.4. The Cantor graph of Example 3.2 is a connected topological graph. In fact, it is an arc. Note that while there is a unique topological arc, namely the interval $[0,1]$, there are many nonisomorphic topological graphs which are arcs. Indeed, there are finite, countable and uncountable topological graphs which are arcs.

The notion of a monotone map from continuum theory, where a continuous map $f\colon X \to Y$ between continua is called monotone if $f^{-1}(y)$ is connected for every $y \in Y$, was adapted to topological graphs by Panagiotopoulos and Solecki in [22] (where those maps were called connected epimorphisms instead) and studied in detail by Charatonik, Kwiatkowska, Roe, and Yang [6]. The epimorphisms we will consider in the following sections will always be monotone, so we will use various results from [6], whose statements and proofs we include here for completeness.

DEFINITION 3.5. An epimorphism $f\colon G \to H$ between topological graphs is called *monotone* if $f^{-1}(h)$ is connected in G for every $h \in H$.

By [22, Lemma 1.1] we can equivalently require that $f^{-1}(C)$ is connected in G whenever $C \subseteq H$ is closed connected in H. From this equivalent definition it is clear that if $f\colon G \to H$ is monotone and H is connected then G

is too. The converse actually holds for arbitrary epimorphisms, an observation repeatedly used in both [22] and [6]:

REMARK 3.6. Let $f\colon G \to H$ be an epimorphism between topological graphs. If G is connected then H is connected as well.

THEOREM 3.7 ([6, Theorem 2.17]). *Let $\mathcal{G}$ be a projective Fraïssé family of graphs such that for every $G \in \mathcal{G}$ and $a, b, c \in G$ pairwise distinct with $\langle a, b\rangle, \langle b, c\rangle \in E(G)$, there is a graph $H \in \mathcal{G}$ and an epimorphism $f_G^H : H \to G$ for which whenever $p, q, r \in H$ are such that $f_G^H(p) = a$, $f_G^H(q) = b$ and $f_G^H(r) = c$, we have $\langle p, q\rangle \notin E(H)$ or $\langle q, r\rangle \notin E(H)$. If $\mathbb{G}$ denotes the projective Fraïssé limit of $\mathcal{G}$, then for every $a \in \mathbb{G}$ there is at most one $b \in \mathbb{G} \setminus \{a\}$ with $\langle a, b\rangle \in E(\mathbb{G})$. In particular, the edge relation of $\mathbb{G}$ is an equivalence relation, that is, $\mathbb{G}$ is a prespace.*

Proof. Suppose on the contrary that there are distinct vertices $a, b, c \in \mathbb{G}$ such that $\langle a, b\rangle, \langle b, c\rangle \in E(\mathbb{G})$. Let a graph $G \in \mathcal{G}$ and an epimorphism $f_G\colon \mathbb{G} \to G$ be such that $f_G(a), f_G(b)$, and $f_G(c)$ are three distinct vertices of G. Then $\langle f_G(a), f_G(b)\rangle, \langle f_G(b), f_G(c)\rangle \in E(G)$, and thus, by our assumption, there is a graph H and an epimorphism $f_G^H\colon H \to G$ such that for an epimorphism $f_H\colon \mathbb{G} \to H$ satisfying $f_G = f_G^H \circ f_H$ we have $\langle f_H(a), f_H(b)\rangle \notin E(H)$ or $\langle f_H(b), f_H(c)\rangle \notin E(H)$. This contradicts the fact that f_H maps edges to edges. ∎

DEFINITION 3.8. Let $\mathcal{T}$ be a projective Fraïssé family of finite graphs. We say that $\mathcal{T}$ *allows splitting edges* if for all $G \in \mathcal{T}$ and all distinct $a, b \in G$ with $\langle a, b\rangle \in E(G)$ the graph H defined by $V(H) = V(G) \sqcup \{*\}$ and

$$E(H) = (E(G) \setminus \{\langle a, b\rangle\}) \cup \{\langle a, *\rangle, \langle *, b\rangle\}$$

and the two morphisms $H \to G$ that map $*$ to either a or b and are the identity otherwise, are in $\mathcal{T}$.

Allowing splitting edges is a basic closure property that will be satisfied by all the projective Fraïssé families considered in what follows. Despite its simplicity it has important consequences, such as the following lemma, and it will be used for some arguments in the next sections.

LEMMA 3.9. *Suppose that $\mathcal{T}$ is a projective Fraïssé family of finite graphs that allows splitting edges. If $\mathbb{G}$ is the projective Fraïssé limit of $\mathcal{T}$, then $\mathbb{G}$ is a prespace.*

Proof. We will show that $\mathcal{T}$ satisfies the hypothesis of Theorem 3.7, from which the conclusion follows immediately. Fix $G \in \mathcal{T}$ and pairwise distinct $a, b, c \in G$ with $\langle a, b\rangle, \langle b, c\rangle \in E(G)$ and let H be the graph obtained from G by splitting the edge $\langle a, b\rangle$ twice. More formally, we have $V(H) = V(G) \sqcup \{a', b'\}$ and

$$E(H) = (E(G) \setminus \{\langle a, b\rangle\}) \cup \{\langle a, a'\rangle, \langle a', b'\rangle, \langle b', b\rangle\},$$

with the epimorphism f_G^H such that $f_G^H(a') = a$, $f_G^H(b') = b$ and f_G^H is the identity otherwise. We find that H and f_G^H are in $\mathcal{T}$ since $\mathcal{T}$ allows splitting edges, but if $p, q, r \in H$ are such that $f_G^H(p) = a$, $f_G^H(q) = b$ and $f_G^H(r) = c$, then either $\langle p, q\rangle \notin E(H)$, or $\langle q, r\rangle \notin E(H)$. ∎

3.2. Dendrites. We begin this section by recalling some facts concerning dendrites. A compact connected metrizable space is called a *dendrite* if it is locally connected and contains no simple closed curve [21, Definition 10.1]. Equivalently dendrites are locally connected dendroids [12, p. 218]. For more characterizations of dendrites see [21, Section 10.1]. Therefore a dendrite X is uniquely arcwise connected, that is between any $x \neq y \in X$ there is a unique arc. In a dendrite X there are three types of points $x \in X$, based on how many connected components $X \setminus \{x\}$ has:

- If $X \setminus \{x\}$ is connected, then x is called an *endpoint* of X.
- If $X \setminus \{x\}$ has two connected components, then x is called a *regular point* of X.
- If $X \setminus \{x\}$ has more than two connected components, then x is called a *ramification point* of X.

In all of those cases the (potentially infinite) number of connected components of $X \setminus \{x\}$ is called the *order* or *degree* of x, denoted by $\operatorname{ord}(x)$, and it coincides with the *Menger–Urysohn order* of x in X, that is the maximum number of arcs in X meeting only in x [18, §46, I]. For $P \subseteq \{3, \dots, \omega\}$ the *generalized Ważewski dendrite* W_P is a dendrite satisfying the following two properties:

- If $x \in W_P$ is a ramification point, $\operatorname{ord}(x) \in P$.
- If $p \in P$, the set of ramification points in W_P of order p is *arcwise dense*. This means that for all $x, y \in W_P$ distinct, there is a ramification point of order p in the unique arc joining x and y.

For a fixed P those two properties characterize a unique dendrite up to homeomorphism [5, Theorem 6.2], so we can actually talk about *the* generalized Ważewski dendrite W_P.

These notions were adapted to topological graphs in [6], from which the remaining definitions and results of this section are taken.

Definition 3.10. A topological graph A is called *arcwise connected* if for all $a, b \in A$, there is a subgraph of A containing a and b which is an arc. A topological graph A is called *locally connected* if every $a \in A$ has an open neighbourhood which is connected. A topological graph G is *hereditarily unicoherent* if for any two nonempty closed connected topological graphs, P and Q, with $V(P), V(Q) \subseteq V(G)$ and $E(P), E(Q) \subseteq E(G)$, the intersection

$P \cap Q = (V(P) \cap V(Q), E(P) \cap E(Q))$ is connected. The topological graph G is *unicoherent* if the above holds for all P and Q such that $V(P) \cup V(Q) = V(G)$.

DEFINITION 3.11. A topological graph G is called a *graph-dendroid* if it is hereditarily unicoherent and arcwise connected. A graph-dendroid G is called a *graph-dendrite* if it is locally connected.

DEFINITION 3.12. Let G be a topological graph. A vertex $x \in G$ is called an *endpoint* of G if whenever H is a topological graph which is an arc and $f\colon H \to G$ is an embedding with $x \in f(H)$, then x is in the image of an endpoint of H. Note that, when G is an arc, this notion agrees with the one in Definition 3.3.

DEFINITION 3.13. Let X be a graph-dendrite. We say that $x \in X$ is a *ramification point* if $X \setminus \{x\}$ has at least three components, in which case we call the (possibly infinite) number of components of $X \setminus \{x\}$ the *order* of x in X.

THEOREM 3.14 ([6, Lemma 3.7]). *If $f\colon G \to H$ is a monotone epimorphism between topological graphs and G is an arc, then H is an arc and the images of endpoints of G are endpoints of H.*

Proof. Denote the endpoints of G by a and b. We need to show that every vertex in $H \setminus \{f(a), f(b)\}$ disconnects H. Let $y \in H \setminus \{f(a), f(b)\}$; then, since G is an arc, the graph $G \setminus f^{-1}(y)$ is disconnected. Let $G \setminus f^{-1}(y)$ be the union of two disjoint graphs, $G \setminus f^{-1}(y) = U \cup V$. Thus $H \setminus \{y\} = f(U) \cup f(V)$ and $f(U) \cap f(V) = \varnothing$, so $H \setminus \{y\}$ is disconnected as needed. ∎

Our goal is to construct a large class of dendrites from projective Fraïssé limits of finite graphs. The following results from [6] will be crucial. Lemma 3.15 is only stated in [6], below we supplement a proof.

LEMMA 3.15 ([6, Lemma 3.8]). *Let $\mathbb{T}$ be the inverse limit of finite arcs with monotone epimorphisms. Then $\mathbb{T}$ is an arc.*

Proof. Let $\langle I_n \mid n < \omega \rangle$ with monotone epimorphisms $f_n^m\colon I_m \to I_n$ be such that $\mathbb{T} = \varprojlim I_n$, and let $f_n\colon \mathbb{T} \to I_n$ be the induced projections. Note that, since monotone epimorphisms map endpoints to endpoints, there are two points $a = (a_n)$ and $b = (b_n)$ in $\mathbb{T}$ such that a_n, b_n are endpoints of I_n for every n. In particular, for any $x \in \mathbb{T} \setminus \{a, b\}$ there is $N < \omega$ such that $f_k(x) \notin \{a_k, b_k\}$ for all $k \geq N$. We want to show that every $x \in \mathbb{T} \setminus \{a, b\}$ disconnects $\mathbb{T}$. Let N be as above for x and for $k \geq N$ let U_a^k and U_b^k be the connected components of $I_k \setminus \{f_k(x)\}$ containing a_k and b_k respectively. Since f_k is monotone, $f_k^{-1}(U_a^k)$ and $f_k^{-1}(U_b^k)$ are connected. Moreover, for

$k' \geq k$ we have $f_k^{-1}(U_a^k) \subseteq f_{k'}^{-1}(U_a^{k'})$ (and analogously for b), so that the set

$$\mathbb{T} \setminus \{x\} = \bigcup_{k \geq N} f_k^{-1}(U_a^k) \sqcup \bigcup_{k \geq N} f_k^{-1}(U_b^k)$$

is disconnected. ∎

THEOREM 3.16 ([6, Theorem 2.18]). *Let $\mathbb{G}$ be the inverse limit of trees with monotone epimorphisms. Then $\mathbb{G}$ is hereditarily unicoherent.*

The proof below is an adaptation of the proof in Nadler [21, Theorem 10.36] of the fact that the inverse limit of dendrites is hereditarily unicoherent. It is different from the proof given in [6].

Proof. Let P, Q be closed connected subgraphs with $V(P), V(Q) \subseteq V(\mathbb{G})$ and $E(P), E(Q) \subseteq E(\mathbb{G})$. We want to show that $C = P \cap Q$ is connected. By assumption $\mathbb{G} = \varprojlim G_n$ is an inverse limit of trees with monotone epimorphisms $f_n^{n+1} \colon G_{n+1} \to G_n$. Let $P_n = f_n(P)$, $Q_n = f_n(Q)$ and $C_n = P_n \cap Q_n$, where $f_n \colon \mathbb{G} \to G_n$ is the canonical projection. If $C_n = \varnothing$ for some n there is nothing to prove, so we can assume that $C_n \neq \varnothing$ for all n. Since every G_n is a tree, hence hereditarily unicoherent, C_n is connected and nonempty for every n. It is now not hard to check that $P \cap Q = \varprojlim f_n(P \cap Q) = \varprojlim (f_n(P) \cap f_n(Q))$, i.e. $C = \varprojlim C_n$. This implies that C is connected, since it is the inverse limit of closed connected sets. ∎

THEOREM 3.17 ([6, Corollary 3.11 and Observation 2.16]). *Let $\mathcal{T}$ be a projective Fraïssé family of trees with monotone epimorphisms and let $\mathbb{G}$ be its projective Fraïssé limit. Then $\mathbb{G}$ is a graph-dendrite. Moreover, if $\mathbb{G}$ has a transitive set of edges, then $|\mathbb{G}|$ is a dendrite.*

Proof. It is already proved in [22, Proposition 2.1] that $\mathbb{G}$ is connected and locally connected, and it is proved in [22, Theorem 2.1] that $|\mathbb{G}|$ is a Peano continuum (i.e. it is a locally connected compact connected space).

Peano continua are arcwise connected [21, Theorem 8.23]. It follows from Lemma 3.15 that $\mathbb{G}$ is arcwise connected (see also [6, Proposition 3.10]), and it follows from Theorem 3.16 that $\mathbb{G}$ is hereditarily unicoherent. An argument by contradiction can be used to show that $|\mathbb{G}|$ is hereditarily unicoherent whenever $\mathbb{G}$ is, which concludes the proof. ∎

Note that combining Lemma 3.9 and Theorem 3.17 we find that all projective Fraïssé families of trees that allow splitting edges, with monotone maps, have as their limit a prespace whose topological realization is a dendrite. In particular, this will be true for all projective Fraïssé families considered in the following sections. The result below, which is contained in [22, Proposition 2.1], will also be important.

LEMMA 3.18. *Let $\mathcal{T}$ be a projective Fraïssé family of graphs with monotone epimorphisms. Let $\langle T_i \mid i < \omega \rangle$ with maps $f_i^{i+1} \colon T_{i+1} \to T_i$ be a Fraïssé sequence for $\mathcal{T}$ and let $\mathbb{T}$ be its projective Fraïssé limit. Then for $G \in \mathcal{T}$ all epimorphisms $\mathbb{T} \to G$ in $\mathcal{T}^\omega$ are monotone.*

Proof. Since the epimorphisms f_i^{i+1} are monotone, it is not hard to see that the projection epimorphisms $f^i \colon \mathbb{T} \to T_i$ are monotone. Any epimorphism $\mathbb{T} \to G$ is a composition of a projection epimorphism f_i, for some i, and a monotone epimorphism from T_i to G. ∎

4. Weakly coherent epimorphisms and generalized Ważewski dendrites. In this section, we introduce a new class of maps between trees, which we call (*weakly*) *coherent*, and study the relationship between weakly coherent points in a prespace and ramification points of its topological realization. Once this is done we will show how to construct many generalized Ważewski dendrites as the topological realization of a projective Fraïssé limit.

4.1. Weakly coherent epimorphisms

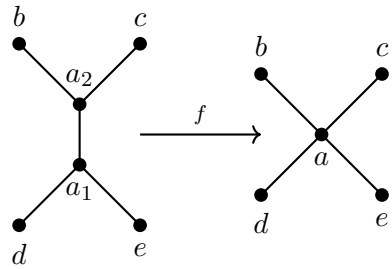
DEFINITION 4.1. Let A, B be finite trees and $f \colon B \to A$ a monotone epimorphism. Fix $a \in A$ with $\operatorname{ord}(a) = n \geq 3$ and enumerate the connected components of $A \setminus \{a\}$ as $A_0, \dots, A_{n-1}$. We say that a is a *point of weak coherence* for f if there exists $b \in f^{-1}(a)$ such that $m = \operatorname{ord}(b) \geq n$ and an injection $p \colon n \to m$ such that if $B_0, \dots, B_{m-1}$ are the connected components of $B \setminus \{b\}$, then $f^{-1}(A_i) \subseteq B_{p(i)}$ for all $0 \leq i \leq n-1$. In this case we call b the *witness* for the weak coherence of f at a. We say that f is *weakly coherent* if every $a \in A$ with $\operatorname{ord}(a) \geq 3$ is a point of weak coherence for f.

This notion can be strengthened by requiring $\operatorname{ord}(b) = \operatorname{ord}(a)$ and p to be a bijection. In that case we say that a is a *point of coherence of* f and that b is a *witness* for the coherence of f at a. We say that f is *coherent* if every $a \in A$ with $\operatorname{ord}(a) \geq 3$ is a point of coherence for f.

REMARK 4.2. The definitions of a weakly coherent and of a coherent epimorphism can be extended (in the obvious way) to epimorphisms between inverse limits of trees.

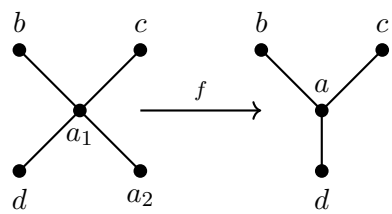
REMARK 4.3. Let A, B be inverse limits of trees, $f \colon B \to A$ a monotone epimorphism and $a \in A$ a ramification point. There exists at most one $b \in B$ witnessing the weak coherence of f at a. Consider now a third tree C with a monotone epimorphism $g \colon C \to B$. If $f \circ g$ is weakly coherent at a with witness c, then f is weakly coherent at a. Then $b = g(c)$ witnesses the weak coherence of f at a, and c witnesses the weak coherence of g at b. If $f \circ g$ and g are coherent, then f is coherent.

Example 4.4.



The map f defined by $f(a_1) = f(a_2) = a$ and mapping any other node to the node with the same name is a monotone epimorphism which is not weakly coherent at a.

Example 4.5.



The map f defined by $f(a_1) = f(a_2) = a$ and mapping any other node to the node with the same name is a weakly coherent epimorphism which is not coherent at a.

Example 4.6. Let A, B be finite trees in which every vertex has degree at most 3. Then any monotone epimorphism $f\colon A \to B$ is coherent. Indeed, fix $b \in B$ of order 3 and let b_1, b_2 and b_3 be its distinct neighbours. Let $b_1' \in f^{-1}(b_1), b_2' \in f^{-1}(b_2)$ and let $P = (p_0 = b_1', p_1, \dots, p_n = b_2')$ be the unique arc from b_1' to b_2'. By Remark 3.6, $f(P)$ is connected and since it contains b_1, b_2 and B is uniquely arcwise connected, it must also contain b. Hence $P \cap f^{-1}(b) \neq \varnothing$. Let $b_3' \in f^{-1}(b_3)$ and let Q be the shortest arc from b_3' to a point in P, let $\{p_i\} = P \cap Q$ and note that, by another application of Remark 3.6, similar to the one above, $p_i \in f^{-1}(b)$. It is now easy to check that p_i witnesses the coherence of f at b.

Remark 4.7. Note that if A, B, C are finite trees and $f\colon A \to B$, $g : B \to C$ are (weakly) coherent epimorphisms, then $g \circ f\colon A \to C$ is also (weakly) coherent. Indeed, if $c \in C$ with $\operatorname{ord}(c) \geq 3$ then there is some $b \in g^{-1}(c)$ witnessing the (weak) coherence of g at c and some $a \in f^{-1}(b)$ witnessing the (weak) coherence of f at b. It is easy to check that a also witnesses the (weak) coherence of $g \circ f$ at c.

REMARK 4.8. Note also that if A is a finite tree with at least two vertices and B with $V(B) = V(A) \sqcup \{*\}$ is obtained from A by splitting an edge as in Definition 3.8, then the two maps $B \to A$ mapping $*$ to either endpoint of the split edge are coherent.

DEFINITION 4.9. Given an inverse system $\langle A_i \mid i < \omega \rangle$ of graphs and monotone epimorphisms $f^i_j \colon A_i \to A_j$ we say that $a = (a_i)_i \in \varprojlim A_i \subseteq \prod A_i$ is a *point of weak coherence* or a *weakly coherent point* if there is a $k \in \omega$ such that for all $l > k$, a_l is a point of weak coherence of f^{l+1}_l and a_{l+1} witnesses this. We say that a is a *point of coherence* or a *coherent point* if there is a $k \in \omega$ such that for all $l > k$, a_l is a point of coherence of f^{l+1}_l, and a_{l+1} witnesses this.

LEMMA 4.10. *Let $\mathcal{T}$ be a projective Fraïssé family of finite trees that allows splitting edges with monotone epimorphisms, let $\mathbb{G}$ be its projective Fraïssé limit and let $\pi \colon \mathbb{G} \to |\mathbb{G}|$ be its topological realization, which is a dendrite by Theorem* 3.17. *Moreover, let $p \in |\mathbb{G}|$ be a ramification point. Then there is a unique $p' \in \mathbb{G}$ with $\pi(p') = p$.*

Proof. Suppose for a contradiction that $|\pi^{-1}(p)| > 1$ and, using the fact that equivalence classes in $\mathbb{G}$ have at most two elements by Theorem 3.7, write $\pi^{-1}(p) = \{p_1, p_2\}$. Let $\{A'_i \mid i \leq n\}$ be the connected components of $|\mathbb{G}| \setminus \{p\}$, where $\operatorname{ord}(p) = n \in \mathbb{N} \cup \{\omega\}$. Let $A_i = A'_i \cup \{p\}$ for $i = 1, \dots, n$ and note that $\{\pi^{-1}(A_i) \mid i \leq n\}$ are connected sets in $\mathbb{G}$ intersecting only in $\{p_1, p_2\}$. Find a finite tree G and a monotone epimorphism $f_G \colon \mathbb{G} \to G$ such that $f_G(\pi^{-1}(A_i))$ is nontrivial (meaning not contained in $\{f_G(p_1), f_G(p_2)\}$) for $i \in \{1, 2, 3\}$ and such that $f_G(p_1) \neq f_G(p_2)$. This is possible because epimorphisms in $\mathcal{T}^\omega$ are monotone by Lemma 3.18. We claim that

$$f_G(\pi^{-1}(A_i)) \cap f_G(\pi^{-1}(A_j)) \subseteq \{f_G(p_1), f_G(p_2)\},$$

for all distinct $i, j \in \{1, 2, 3\}$. Suppose for a contradiction that

$$r \in f_G(\pi^{-1}(A_1)) \cap f_G(\pi^{-1}(A_2)) \setminus \{f_G(p_1), f_G(p_2)\}.$$

Then $f_G^{-1}(r)$ and $\{p_1, p_2\}$ are disjoint, nonempty, closed, connected subsets of $\mathbb{G}$. However $f_G^{-1}(r)$ meets both $\pi^{-1}(A_1) \setminus \{p_1, p_2\}$ and $\pi^{-1}(A_2) \setminus \{p_1, p_2\}$, a contradiction since $f_G^{-1}(r)$ is connected and $(\pi^{-1}(A_1) \cup \pi^{-1}(A_2)) \setminus \{p_1, p_2\}$ is not.

The idea now is that we can always move to a bigger graph H obtained from G by splitting an edge in order to force $f_H(\pi^{-1}(A_1)) \cap f_H(\pi^{-1}(A_3)) \not\subseteq \{f_H(p_1), f_H(p_2)\}$. Indeed, at least two of $f_G(\pi^{-1}(A_1))$, $f_G(\pi^{-1}(A_2))$, $f_G(\pi^{-1}(A_3))$ are connected to the same vertex, $f_G(p_1)$ or $f_G(p_2)$; suppose without loss of generality that $f_G(\pi^{-1}(A_1))$ and $f_G(\pi^{-1}(A_3))$ are both connected to $f_G(p_2)$. Now construct a graph H by taking $H = G \sqcup \{r\}$, where

r is a vertex not in G and

$$E(H) = (E(G) \setminus \{\langle f_G(p_1), f_G(p_2)\rangle\}) \cup \{\langle f_G(p_1), r\rangle, \langle r, f_G(p_2)\rangle\},$$

together with the epimorphism $f_G^H \colon H \to G$ defined by $f_G^H(r) = f_G(p_2)$ and the identity otherwise. Denote $a = (f_G^H)^{-1}(f_G(p_1))$ and let b be the vertex in $(f_G^H)^{-1}(f_G(p_2))$ distinct from r. Note that $H, f_G^H \in \mathcal{T}$ since $\mathcal{T}$ allows splitting edges. By definition of projective Fraïssé limit there must be an epimorphism $f_H \colon \mathbb{G} \to H$ such that $f_G^H \circ f_H = f_G$. The last identity combined with the fact that f_H maps edges to edges forces $f_H(p_1) = a$ and $f_H(p_2) = r$. Now we can use the same argument as above to show that

$$f_H(\pi^{-1}(A_1)) \cap f_H(\pi^{-1}(A_3)) \subseteq \{f_H(p_1), f_H(p_2)\}.$$

This is a contradiction, since by construction those sets also meet in b. ∎

THEOREM 4.11. *Let $\mathcal{T}$ be a projective Fraïssé family of trees with monotone epimorphisms that allows splitting edges, let $\langle G_i \mid i < \omega\rangle$ with epimorphisms $f_n^m \colon G_m \to G_n$ be a Fraïssé sequence for $\mathcal{T}$, let $\mathbb{G} = \varprojlim G_i$ be the projective Fraïssé limit of $\mathcal{T}$ and let $\pi \colon \mathbb{G} \to |\mathbb{G}|$ be its topological realization, which is a dendrite by Theorem* 3.17. *Then π maps points of weak coherence of $\mathbb{G} \subseteq \prod G_i$ to ramification points of $|\mathbb{G}|$.*

Proof. Suppose that $p = (p_i)$ is a point of weak coherence of $\mathbb{G}$; we want to show that $\pi(p)$ is a ramification point of $|\mathbb{G}|$. Let $f_k \colon \mathbb{G} \to G_k$ be the projection epimorphisms obtained by construction of the inverse limit, and let $A_k^1, \ldots, A_k^{n_k}$ be the connected components of $G_k \setminus \{p_k\}$, enumerated so that $(f_k^{k+1})^{-1}(A_k^i) \subseteq A_{k+1}^i$ for all i. Let $k(i)$ be the least k such that $G_k \setminus \{p_k\}$ has at least i connected components, if such an i exists, and $k(i) = \omega$ otherwise. Note that $\{p\} = \bigcap f_k^{-1}(p_k)$ and that

$$A^i = \bigcup_{k \geq k(i)} \{f_k^{-1}(A_k^i) \mid 1 \leq i < \omega\}$$

are the connected components of $\mathbb{G} \setminus \{p\}$ (with A^i being empty if $k(i) = \omega$). We will show that the equivalence class of p in $\mathbb{G}$ is a singleton, from which we can conclude that $\pi(A^i)$ are the connected components of $|\mathbb{G}| \setminus \{\pi(p)\}$, so that $\pi(p)$ is a ramification point of $|\mathbb{G}|$ of order equal to the least n such that $k(n+1) = \omega$ or to ω if there is no such n. Suppose for a contradiction that there exists $q = (q_i) \in \mathbb{G}$ with $\langle p, q\rangle \in E(\mathbb{G})$, so in particular $\langle p_i, q_i\rangle \in E(G_i)$ for all i. Let k be large enough so that $p_k \neq q_k$ and let A_k^j be the component of $G_k \setminus \{p_k\}$ containing q_k. Note that by weak coherence $q_m \in A_m^j$ for all $m \geq k$. Consider the tree H obtained from G_k by splitting the $\langle p_k, q_k\rangle$ edge into two edges $\langle p_k, p_k'\rangle$ and $\langle p_k', q_k\rangle$, with the epimorphism $f_k^H \colon H \to G_k$ such that $f_k^H(p_k') = p_k$ and the identity otherwise. Note that f_k^H is coherent at $p_k \in G_k$, as witnessed by $p_k \in H$. By definition of Fraïssé sequences we can

find l large enough and an epimorphism $f^l_H\colon G_l \to H$ such that $f^l_k = f^H_k \circ f^l_H$, which implies that $f^l_H(q_l) = q_k \in H$. On the one hand, we now must have $f^l_H(p_l) = p_k \in H$ by weak coherence, since p_l witnesses the weak coherence of f^l_k at $p_k \in G_k$ and $p_k \in H$ witnesses the weak coherence of f^H_k at $p_k \in G_k$; on the other hand we must have $f^l_H(p_l) = p'_k$, since f^l_H maps edges to edges, a contradiction. ∎

Remark 4.12. Note in particular that if $x = (x_i)$ is a point of weak coherence in $\prod G_i$ then either $\mathrm{ord}(x_i)$ stabilizes to some finite value n or it grows unboundedly. In the former case $\pi(x)$ also has order n as a ramification point of $|\mathbb{G}|$, while in the latter, $\pi(x)$ is a ramification point of $|\mathbb{G}|$ of infinite order.

The converse to Theorem 4.11 is false in general, but it is true when monotone epimorphisms are replaced with weakly coherent ones, as shown in the following theorem.

Theorem 4.13. *Let $\mathcal{T}$ be a projective Fraïssé family of trees with weakly coherent epimorphisms, let $\langle G_i \mid i < \omega\rangle$ with epimorphisms $f^m_n\colon G_m \to G_n$ be a Fraïssé sequence for $\mathcal{T}$ and let $\pi\colon \mathbb{G} \to |\mathbb{G}|$ be its topological realization, which is a dendrite by Theorem* 3.17. *Then π^{-1} maps ramification points of $|\mathbb{G}|$ to points of weak coherence of $\mathbb{G} \subseteq \prod G_i$.*

Proof. As above let $f_k\colon \mathbb{G} \to G_k$ be the projection epimorphisms obtained by construction of the limit and let $x' \in |\mathbb{G}|$ be a ramification point. By Lemma 4.10 there is a unique $x = (x_i) \in \pi^{-1}(x')$ and we want to show that it is a point of weak coherence in $\prod G_i$. Let $\{A^i \mid i < \mathrm{ord}(x)\}$ with $\mathrm{ord}(x) \in \mathbb{N} \cup \{\omega\}$ be the connected components of $\mathbb{G} \setminus \{x\}$ and let k be large enough so that $f_k(A^1), f_k(A^2), f_k(A^3)$ are not singletons in G_k. Let A^1_k, A^2_k and A^3_k be the connected components of $G_k \setminus \{x_k\}$, numbered so that $f_k^{-1}(A^i_k) \subseteq A^i$ for $i = 1, 2, 3$, which is always possible since $f_k^{-1}(A^i_k)$ is connected by Lemma 3.18 and does not meet $f_k^{-1}(x_k)$ by construction. By assumption $f^{k+1}_k\colon G_{k+1} \to G_k$ is weakly coherent at x_k, as witnessed by some vertex $a \in G_{k+1}$. Suppose for a contradiction that $x_{k+1} \neq a$. Note that if A^i_{k+1} is a connected component of $G_{k+1} \setminus \{x_{k+1}\}$, then there is a unique j such that $f^{-1}_{k+1}(A^i_{k+1}) \subseteq A^j$, once again because $f^{-1}_{k+1}(A^i_{k+1})$ is connected and does not meet $f^{-1}_{k+1}(x_{k+1})$. Since $x_{k+1} \neq a$ and G_{k+1} is a tree, there is a connected component A^n_{k+1} of $G \setminus \{x_{k+1}\}$ that contains all connected components of $G \setminus \{a\}$ except at most one, so it also contains $(f^{k+1}_k)^{-1}(A^i_k)$ for at least two distinct i. Assume without loss of generality that

$$(f^{k+1}_k)^{-1}(A^1_k) \subseteq A^n_{k+1},$$
$$(f^{k+1}_k)^{-1}(A^2_k) \subseteq A^n_{k+1},$$

and let j be the unique index such that $f_{k+1}^{-1}(A_{k+1}^n) \subseteq A^j$. But now we have both $f_k^{k+1} \circ f_{k+1} = f_k$ and

$$(f_{k+1}^{-1} \circ (f_k^{k+1})^{-1})(A_k^1) \subseteq A^j, \quad f_k^{-1}(A_k^1) \subseteq A^1,$$
$$(f_{k+1}^{-1} \circ (f_k^{k+1})^{-1})(A_k^2) \subseteq A^j, \quad f_k^{-1}(A_k^2) \subseteq A^2,$$

which cannot all hold at the same time regardless of the value of j, a contradiction. ∎

4.2. Generalized Ważewski dendrites from projective Fraïsse families. We now introduce some families of finite trees. The remainder of this section will be dedicated to showing that these are projective Fraïssé families and that the topological realizations of their projective Fraïssé limits are generalized Ważewski dendrites.

Definition 4.14. Let $P \subseteq \{3, \dots, \omega\}$. We consider two cases. If $\omega \in P$ we consider the family $\mathcal{F}_P$ whose elements are finite trees with no vertices of order 2. Given $A, B \in \mathcal{F}_P$, an epimorphism of graphs $f\colon B \to A$ is in $\mathcal{F}_P$ if:

(1) f is monotone.
(2) If $a \in A$ is such that $\operatorname{ord}(a) \in P$, then f is coherent at a.
(3) If $a \in A$ is such that $\operatorname{ord}(a) \notin P$ and $\operatorname{ord}(a) \geq 3$ then f is weakly coherent at a, and if $b \in f^{-1}(a)$ is the witness for the weak coherence of f at a, then $\operatorname{ord}(b) \notin P$.

On the other hand, if $\omega \notin P$, we consider the family $\mathcal{G}_P$ whose elements are finite trees all of whose vertices either are endpoints or have order in P, with coherent monotone epimorphisms.

Remark 4.15. The families $\mathcal{F}_P$ and $\mathcal{G}_P$ do not allow splitting edges as defined earlier, since the trees in those families do not have any vertices of degree 2. However, if T is a tree in $\mathcal{F}_P$ or $\mathcal{G}_P$ and $a, b \in T$ with $\langle a, b\rangle \in E(T)$, it is still possible to split the edge $\langle a, b\rangle$ by removing it and then adding a new point x connected to a and b. We have to also add enough new neighbours x_i to x until $\operatorname{ord}(x) \in P$. This is enough for all the arguments in the previous sections that used the edge splitting property.

We will now verify that $\mathcal{F}_P$ and $\mathcal{G}_P$ are projective Fraïssé families and later we will identify the topological realizations of their projective Fraïssé limits. It seems natural to guess that if $\mathbb{F}_P$ and $\mathbb{G}_P$ are the projective Fraïssé limits of $\mathcal{F}_P$ and $\mathcal{G}_P$, then $|\mathbb{F}_P| \cong W_P$ and $|\mathbb{G}_P| \cong W_P$, but unfortunately this is not always the case. It is true for all families of the form $\mathcal{G}_P$, but for families of the form $\mathcal{F}_P$ it only holds when $\{3, \dots, \omega\} \setminus P$ is infinite, as we will see later. Clearly, $\mathcal{F}_P$ and $\mathcal{G}_P$ each contain countably many structures up to isomorphism, contain the identity morphisms and their morphisms are

closed under composition by Remark 4.7. We begin by verifying the joint projection property for $\mathcal{F}_P$ and $\mathcal{G}_P$, with a construction that works for both families.

LEMMA 4.16. *Let $A, B \in \mathcal{F}_P$ (respectively $A, B \in \mathcal{G}_P$). Then there exists $C \in \mathcal{F}_P$ (respectively $C \in \mathcal{G}_P$) with epimorphisms $f_1 \colon C \to A$, $f_2 \colon C \to B$ in $\mathcal{F}_P$ (respectively in $\mathcal{G}_P$).*

Proof. Let $a \in A$, $b \in B$ be two endpoints. Consider the tree C' obtained by taking the disjoint union of A and B and identifying a with b. Let C be the tree obtained from C' by adding new vertices x_i connected only to x until $\mathrm{ord}(x) \in P$, where x is the vertex a and b have been identified in. Define $f_1 \colon C \to A$ to be the identity on $V(A)$, while any other vertex is mapped to x. Analogously define $f_2 \colon C \to B$ to be the identity on $V(B)$, while any other vertex is mapped to x. It is immediate to check that C witnesses the joint projection property for A, B. ∎

It remains to verify that the projective amalgamation property is satisfied. We do so for families of the form $\mathcal{F}_P$ and note that the procedure described to produce an amalgam D from a diagram $C \to A \leftarrow B$ of structures in $\mathcal{F}_P$ can also be applied to diagrams of the same shape in $\mathcal{G}_P$, and the amalgam D produced in that case is itself an element of $\mathcal{G}_P$.

LEMMA 4.17. *The family $\mathcal{F}_P$ satisfies the projective amalgamation property, so it is a projective Fraïssé family. Explicitly, for all $A, B, C \in \mathcal{F}_P$ with epimorphisms $f \colon B \to A$ and $g \colon C \to A$ in $\mathcal{F}_P$, there exists $D \in \mathcal{F}_P$ with epimorphisms $h_1 \colon D \to B$ and $h_2 \colon D \to C$ in $\mathcal{F}_P$ such that $f \circ h_1 = g \circ h_2$.*

Proof. We proceed by induction on $|E(B)| + |E(C)|$. If there are no nontrivial edges in B and C, then A must also be a singleton and we can take D to be a singleton as well. For the inductive step, suppose now that B and C are not both singletons and that the lemma is proved for all diagrams in $\mathcal{F}_P$ of the form $F \to A \leftarrow G$ with $|E(F)|+|E(G)| < |E(B)|+|E(C)|$. If $A = B = C$ and $f = g = \mathrm{Id}_A$ we can simply take $D = A$ and $h_1 = h_2 = \mathrm{Id}_A$, so we can assume that at least one of f and g is nontrivial. Suppose without loss of generality that f is nontrivial. Find $a \in A$ such that $|f^{-1}(a)| > 1$. There are two possibilities now: a could be a ramification point or an endpoint of A, and we deal with those two cases separately.

CASE 1: *a is a ramification point.* Let $b \in B$ be the witness for the (weak) coherence of f at a and let B_1 be a component of $B \setminus \{b\}$ with $B_1 \cap f^{-1}(a) \neq \varnothing$. Let $B_2 = B_1 \cap f^{-1}(a)$. We need to distinguish two cases, based on whether $B_2 = B_1$ or not.

CASE 1a: $B_2 \neq B_1$. We start with the harder case, that is $B_2 \neq B_1$ and explain how to deal with the easier case later. Let b_1 be the only element

of B_2 with $\langle b, b_1\rangle \in E(B)$ and let b_2 be the only element of B_2 for which there exists a $b_3 \in B_1 \setminus B_2$ with $\langle b_2, b_3\rangle \in E(B)$ (uniqueness of b_1, b_2, b_3 follows from B being a tree). Consider now the graph B' obtained from B by collapsing B_2 to b, formally $V(B') = V(B) \setminus V(B_2)$ and

$$E(B') = (E(B) \setminus (E(B_2) \cup \{\langle b, b_1\rangle, \langle b_2, b_3\rangle\})) \cup \{\langle b, b_3\rangle\},$$

see Figure 1. There is a natural map $\pi\colon B \to B'$ given by $\pi(x) = b$ if $x \in B_2$ and the identity otherwise. Note that $B' \in \mathcal{F}_P$ since both b and b_3 have the same orders as $\pi(b)$ and $\pi(b_3)$, and π is a coherent epimorphism, so in particular it is in $\mathcal{F}_P$. Let $f'\colon B' \to A$ be the unique map with $f' \circ \pi = f$ and note that f' is still in $\mathcal{F}_P$ by construction. We reach the situation in Figure 1, where the important vertices and edges are drawn, while the dashed sections represent parts of the trees whose precise structure is not important.

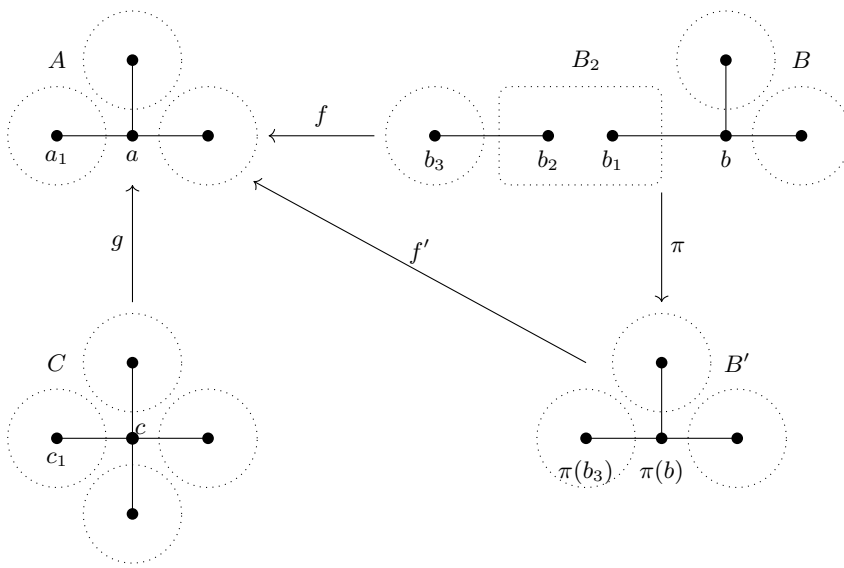


Fig. 1. The construction of B' from B

Since $|E(B')| < |E(B)|$ we can amalgamate f' and g over A by inductive hypothesis, so we can find $D' \in \mathcal{F}_P$ and epimorphisms $h'_1\colon D' \to B'$ and $h'_2\colon D' \to C$ in $\mathcal{F}_P$ such that $f' \circ h'_1 = g \circ h'_2$. Now we want to construct D together with $h_1\colon D \to B$ and $h_2\colon D \to C$ from D', h'_1 and h'_2. The intuitive idea is that we just need to paste back B_2 in D' in the right spot, in particular B_2 should be inserted on the edge of D' that corresponds to the edge $\langle \pi(b), \pi(b_3)\rangle$ in B' (analogously to how B is obtained from B' by inserting B_2 on the $\langle \pi(b), \pi(b_3)\rangle$ edge). Let a_1 be the unique vertex of A with $\langle a, a_1\rangle \in E(A)$ and $f^{-1}(a_1) \subseteq B_1$ and let A_1 be the connected component of $A \setminus \{a\}$ containing a_1. Let $c \in C$ witness the (weak) coherence

of g at a and let C_1 be the connected component of $C \setminus \{c\}$ containing $g^{-1}(a_1)$. Let $c_1 \in C_1$ be the unique vertex of C with $\langle c, c_1\rangle \in E(C)$. Let $d \in D'$ witness the (weak) coherence of h'_2 at c and let D_1 be the connected component of $D' \setminus \{d\}$ containing $(h'_2)^{-1}(c_1)$. Note that, since $f' \circ h'_1 = g \circ h'_2$ and witnesses for (weak) coherence are unique, d also witnesses the (weak) coherence of h'_1 at b and $(h'_1)^{-1}(\pi(B_1)) \subseteq D_1$ by Remark 4.3. Now let $d_1 \in \{d\} \cup (D_1 \cap (h'_1)^{-1}(\pi(b)))$ and $d_2 \in D_1 \cap (h'_1)^{-1}(\pi(b_3))$ be the unique such vertices with $\langle d_1, d_2\rangle \in E(D')$. We can now construct D by inserting B_2 between d_1 and d_2, formally $V(D) = V(D') \sqcup V(B_2)$ and

$$E(D) = (E(D') \setminus \{\langle d_1, d_2\rangle\}) \cup E(B_2) \cup \{\langle b_1, d_1\rangle, \langle b_2, d_2\rangle\}.$$

Note that $B_2 \in \mathcal{F}_P$ and we did not change the degrees of d_1 and d_2, so $D \in \mathcal{F}_P$ as well. Now consider the projection $\pi' \colon D \to D'$ collapsing B_2 to d_1, so $\pi'(x) = d_1$ if $x \in B_2$ and $\pi'(x) = x$ otherwise, which is also an epimorphism in $\mathcal{F}_P$. We are now in a situation that looks like Figure 2. Define $h_2 = h'_2 \circ \pi'$ and $h_1 \colon D \to B$ by

$$h_1(x) = \begin{cases} x & \text{if } x \in B_2, \\ h'_1(x) & \text{otherwise,} \end{cases}$$

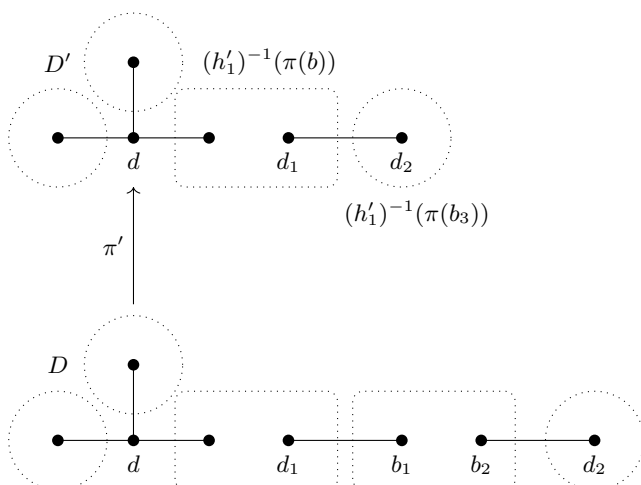
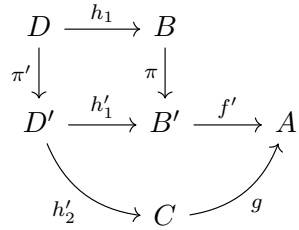


Fig. 2. The construction of D' from D

where we identify points of $B \setminus B_2$ with their images in B' through π. Note that h_1 is an epimorphism in $\mathcal{F}_P$, indeed it is (weakly) coherent at points of B_2 because they are witnesses for their own (weak) coherence and it is (weakly) coherent elsewhere because h'_1 is. Moreover, we have $h'_1 \circ \pi' = \pi \circ h_1$, so that the following diagram commutes:



It only remains to check that $(f \circ h_1)(x) = (g \circ h_2)(x)$ for all $x \in D$, for which there are two cases:

- If $x \in B_2$ then $h_1(x) = x$ and $f(h_1(x)) = a$, while $h_2(x) = h'_2(\pi'(x)) = h'_2(d_1)$, but $g \circ h'_2 = f' \circ h'_1$ and $f'(h'_1(d_1)) = a$, so $g(h_2(x)) = a$.
- Otherwise, $h_1(x)$ and $h_2(x)$ agree with $h'_1(x)$ and $h'_2(x)$ respectively, so we immediately obtain $f(h_1(x)) = g(h_2(x))$ by construction.

This concludes the argument when $B_2 \neq B_1$.

Case 1b: $B_2 = B_1$. Let $b \in B$ be the witness for the (weak) coherence of f at a and let $c \in C$ be the witness for the (weak) coherence of g at a. Suppose now that for every component $\hat{B}$ of $B \setminus \{b\}$ that meets $f^{-1}(a)$, we have $\hat{B} \subseteq f^{-1}(a)$, otherwise by choosing a different component B_1 of $B \setminus \{b\}$ we would be in the previous case. Similarly, we can assume that for every component $\hat{C}$ of $C \setminus \{c\}$ that meets $g^{-1}(a)$, we have $\hat{C} \subseteq g^{-1}(a)$, since otherwise we would be in the previous case after possibly renaming B and C. Let $a^A_1, \dots, a^A_n$ enumerate the connected components of $A \setminus \{a\}$. By the weak coherence of f we can find components $a^B_1, \dots, a^B_n$ of $B \setminus \{b\}$ such that $f^{-1}(a^A_i) \subseteq a^B_i$ for all i. We define $a^C_1, \dots, a^C_n$ analogously. Now let $B_1 = f^{-1}(a)$ and $C_1 = g^{-1}(a)$.

The idea now is to amalgamate the diagram

$$a^B_i \cup \{b\} \to \{a\} \cup \{a^A_i\} \leftarrow a^C_i \cup \{c\} \tag{1}$$

for every i, then to obtain D_1 from B_1 and C_1 by the joint projection property, and to glue everything back together at the end. By inductive hypothesis we can amalgamate diagrams of the form (1) (note that the trees involved are still members of $\mathcal{F}_P$ and so are the maps) obtaining an amalgam a^D_i with maps $h^i_1 \colon a^D_i \to a^B_i \cup \{b\}$ and $h^i_2 \colon a^D_i \to a^C_i \cup \{c\}$. Note that, since b and c are the only preimages of a in $a^B_i \cup \{b\}$ and $a^C_i \cup \{c\}$ respectively, there must be an endpoint $x_i \in a^D_i$ such that $h^i_1(x_i) = b$ and $h^i_2(x_i) = c$ for every i.

We now need to deal with B_1 and C_1. Enumerate the components of $C_1 \setminus \{c\}$ as $\widehat{c_1}, \dots, \widehat{c_r}$ and those of $B_1 \setminus \{b\}$ as $\widehat{b_1}, \dots, \widehat{b_s}$. Assume without loss of generality $s \leq r$. For $i \leq r$, let c_i be the only vertex of $\widehat{c_i}$ such that $\langle c_i, c\rangle \in E(C)$ and let b_i, for $i \leq s$, be defined analogously. Consider now the

tree D_1 built from

$$\Big(\bigsqcup_{i\leq r}\widehat{c_i}\Big)\sqcup\Big(\bigsqcup_{i\leq s}\widehat{b_i}\Big)$$

by adding $r+1$ new vertices $z, z_1, \dots, z_r$ and edges $\langle z_i, c_i\rangle$ for all $i \leq r$, $\langle z_i, b_i\rangle$ for all $i \leq s$ and $\langle z, z_i\rangle$ for all $i \leq r$.

Finally, build a tree D by identifying $z \in D_1$ and $x_i \in a_i^D$ for every i, and denote by d the image of all those points in D. Then add neighbours to z_i, b_i and c_i until they have orders in P. Define a map $h_1\colon D \to B$ by

$$h_1(y) = \begin{cases} h_1^i(y) & \text{if } y \in a_i^D, \\ y & \text{if } y \in \widehat{b_i}, \\ b & \text{otherwise.} \end{cases}$$

Similarly, define a map $h_2\colon D \to C$ by

$$h_2(y) = \begin{cases} h_2^i(y) & \text{if } y \in a_i^D, \\ y & \text{if } y \in \widehat{c_i}, \\ c & \text{otherwise.} \end{cases}$$

It is easy to check that D together with the maps h_1, h_2 is an amalgam of B and C over A.

This concludes the argument for a ramification point $a \in A$.

Case 2: *a is an endpoint.* Suppose now that $a \in A$ is an endpoint. The construction is essentially an easier version of the one we just carried out for the case $B_1 \neq B_2$. Since $B \setminus f^{-1}(a)$ is connected we can let $b \in f^{-1}(a)$ be the unique vertex for which there is a $b' \in B \setminus f^{-1}(a)$ with $\langle b, b'\rangle \in E(B)$ and let B_2 be $f^{-1}(a)$. As in the previous case, consider $\pi\colon B \to B'$ collapsing B_2 to b and let $f'\colon B' \to A$ be the unique map with $f' \circ \pi = f$. Moreover, let D' be an amalgam of f' and g over A with maps $h_1'\colon D' \to B'$ and $h_2'\colon D' \to C$ such that $g \circ h_2' = f' \circ h_1'$. Once again we build D from D' by glueing B_2 to D' in the right spot. Let $\alpha \in (h_1')^{-1}(b)$ be an endpoint and take the disjoint union of B_2 and D', identifying $\alpha \in D'$ and $b \in B_2$. Let d be the image of α and b through this identification.

As in the previous case, we obtain a map $\pi'\colon D \to D'$ by mapping the whole of B_2 to d, and a map $h_1\colon D \to B$ defined by

$$h_1(x) = \begin{cases} x & \text{if } x \in B_2, \\ h_1'(x) & \text{otherwise.} \end{cases}$$

Now, by considering $h_2 = h_2' \circ \pi'$ we see that D with the maps $h_2\colon D \to C$ and $h_1\colon D \to B$ defined above gives an amalgam of f and g over A, as desired. ∎

As mentioned before Lemma 4.17, the construction in its proof also shows the following:

LEMMA 4.18. *The family $\mathcal{G}_P$ satisfies the projective amalgamation property, so it is a projective Fraïssé family.*

Now that we know that $\mathcal{F}_P$ and $\mathcal{G}_P$ are projective Fraïssé families, let $\mathbb{F}_P$ and $\mathbb{G}_P$ denote their projective Fraïssé limits. We know from Lemma 3.9 and Theorem 3.17 that $|\mathbb{F}_P|$ and $|\mathbb{G}_P|$ are dendrites, and in the rest of this section we will identify which dendrites they are exactly. Let us start with the easiest case, namely $\omega \notin P$, so that we are looking at the family $\mathcal{G}_P$ with projective Fraïssé limit $|\mathbb{G}_P|$.

THEOREM 4.19. *Fix $P \subseteq \{3, 4, \ldots\}$. Then $|\mathbb{G}_P| \cong W_P$.*

Proof. For the remainder of this proof we fix a Fraïssé sequence G_i for $\mathcal{G}_P$, with epimorphisms $f^i_j \colon G_i \to G_j$. We consider $\mathbb{G}_P$ as a subspace of $\prod G_i$, and call $f_i \colon \mathbb{G}_P \to G_i$ the canonical projections. We know that $|\mathbb{G}_P|$ is a dendrite, so we need to check that all ramification points of $|\mathbb{G}_P|$ have orders in P, and that for all $p \in P$ the set

$$\{x \in |\mathbb{G}_P| \mid \operatorname{ord}(x) = p\}$$

is arcwise dense in $|\mathbb{G}_P|$. The first part follows quickly from previous results: if $x \in |\mathbb{G}_P|$ is a ramification point, then $\pi^{-1}(x) = (x_i)$ is a coherent point in $\mathbb{G}_P$ by Theorem 4.13, where $\pi \colon \mathbb{G}_P \to |\mathbb{G}_P|$ is the topological realization. Note that $\operatorname{ord}(x_i) \in P$ for large enough i, so by Remark 4.12 we know that, for k large enough, $\operatorname{ord}(x) = \operatorname{ord}(x_k) \in P$.

It remains to show that for every $p \in P$ the set of ramification points in $|\mathbb{G}_P|$ is arcwise dense in $|\mathbb{G}_P|$. Fix $p \in P$ and $X \subseteq |\mathbb{G}|$ an arc. Let $a', b' \in X$ be distinct and fix $a \in \pi^{-1}(a'), b \in \pi^{-1}(b')$. Let i be large enough to have $f_i(a) \neq f_i(b)$ and let $\langle w, y\rangle$ be any edge on the unique arc in G_i joining $f_i(a)$ and $f_i(b)$. Consider the graph H obtained from G_i by splitting the edge $\langle w, y\rangle$ into two edges $\langle w, x^i_p\rangle$,$\langle x^i_p, y\rangle$ and by adding $p-2$ new vertices $z_1, \ldots, z_{p-2}$ with edges $\langle x^i_p, z_j\rangle$ for all $1 \leq j \leq p$. Note that there is an epimorphism $f^H_i \colon H \to G_i$ in $\mathcal{G}_P$ with $f^H_i(v) = w$ for all $v \in V(H) \setminus V(G_i)$ and the identity otherwise, and in H there is a point of order p on the arc connecting $f_i(a)$ to $f_i(b)$. We can now find k large enough so that there is an epimorphism $f^k_H \colon G_k \to H$ in $\mathcal{G}_P$ with $f^H_i \circ f^k_H = f^k_i$ and note that in G_k there is a point of order p on the arc connecting $f_k(a)$ and $f_k(b)$, namely the point $x^k_p \in (f^k_H)^{-1}(x^i_p)$ witnessing the coherence of f^k_H at x^i_p. This is because $f_k(a)$, $f_k(b)$ are in different components of $G_k \setminus \{x^k_p\}$ by coherence, and G_k is uniquely arcwise connected. Now for all $m > k$ let $x^m_p \in G_m$ be a point witnessing the coherence of $f^m_{m-1} \colon G_m \to G_{m-1}$ at x^{m-1}_p and note that for all $m > k$, x^m_p is a point of order p on the unique arc in G_m joining $f_m(a)$

and $f_m(b)$. Since $x = (x_p^m)_{m\in\mathbb{N}} \in \mathbb{G}_P$ is a coherent sequence, it determines a ramification point of order p in $|\mathbb{G}_P|$ by Theorem 4.11 and Remark 4.12. By construction, a and b belong to different connected components of $\mathbb{G}_P \setminus \{x\}$, and since $\mathbb{G}_P$ is hereditarily unicoherent this implies that any connected set in $\mathbb{G}_P$ containing both a and b must also contain x. In particular, $x \in \pi^{-1}(X)$ and so $\pi(x) \in X$. ∎

We can now analyze what happens in the case $\omega \in P$, so that we are looking at the families $\mathcal{F}_P$ with projective Fraïssé limit $\mathbb{F}_P$ instead. The situation is more subtle.

THEOREM 4.20. *Fix $P \subseteq \{3, \dots, \omega\}$. If $\{3, \dots, \omega\} \setminus P$ is infinite and $\omega \in P$, then $|\mathbb{F}_P| \cong W_P$.*

Proof. The proof is very similar to that of Theorem 4.19. For the remainder of this proof we fix a Fraïssé sequence F_i for $\mathcal{F}_P$, with epimorphisms $f_j^i \colon F_i \to F_j$. We consider $\mathbb{F}_P$ as a subspace of $\prod F_i$, and call $f_i \colon \mathbb{F}_P \to F_i$ the canonical projections. We already know that $|\mathbb{F}_P|$ is a dendrite so we need to check that all ramification points of $|\mathbb{F}_P|$ have orders in P, and that for all $p \in P$ the set

$$\{x \in |\mathbb{F}_P| \mid \operatorname{ord}(x) = p\}$$

is arcwise dense in $|\mathbb{F}_P|$. The first part is very similar to the argument in the proof of Theorem 4.19: if $x \in |\mathbb{F}_P|$ is a ramification point, then $\pi^{-1}(x) = (x_i)$ is a weakly coherent point of $\mathbb{F}_P$ by Theorem 4.11, where $\pi \colon \mathbb{F}_P \to |\mathbb{F}_P|$ is the topological realization. Now the way the family $\mathcal{F}_P$ is constructed allows two options: either $\operatorname{ord}(x_i)$ becomes constant for large enough i, so that (x_i) is actually a coherent point of $\mathbb{F}_P$ and we are in the same case as in the proof of Theorem 4.19, or $\operatorname{ord}(x_i) \to \infty$ as $i \to \infty$, which by Remark 4.12 implies that x is a ramification point of $|\mathbb{F}_P|$ of infinite order. In either case $\operatorname{ord}(x) \in P$.

It remains to show that for every $p \in P$ the set of ramification points in $|\mathbb{F}_P|$ is arcwise dense in $|\mathbb{F}_P|$. If $p \neq \omega$ we can repeat verbatim the argument given in the proof of Theorem 4.19, so we need to show that the ramification points of infinite order are arcwise dense in $|\mathbb{F}_P|$. Fix an arc $X \subseteq |\mathbb{F}_P|$, distinct $a', b' \in X$, $a \in \pi^{-1}(a'), b \in \pi^{-1}(b')$ and i large enough to have $f_i(a) \neq f_i(b)$. Fix also a strictly increasing sequence $(n_j)_{j\in\mathbb{N}}$ of natural numbers such that $n_j \notin P$ for all j and $n_j \to \infty$ as $j \to \infty$. Let $\langle w, y\rangle$ be any edge on the unique arc in F_i joining $f_i(a)$ and $f_i(b)$. Consider the graph H obtained from F_i by splitting the edge $\langle w, y\rangle$ into two edges $\langle w, x_1\rangle$,$\langle x_1, y\rangle$ and by adding $n_1 - 2$ new vertices $z_1, \dots, z_{n_1-2}$ with edges $\langle x_1, z_j\rangle$ for all $1 \leq j \leq n_1 - 2$. Note that there is an epimorphism $f_i^H \colon H \to F_i$ in $\mathcal{F}_P$ with $f_i^H(v) = w$ for all $v \in V(H) \setminus V(F_i)$ and the identity otherwise, and note that in H there is a point of order exactly n_1 on the arc connecting $f_i(a)$

to $f_i(b)$. We can now find k large enough so that there is an epimorphism $f_H^k\colon F_k \to H$ in $\mathcal{F}_P$ with $f_i^H \circ f_H^k = f_i^k$ and note that in F_k there is a point of order at least n_1 on the arc connecting $f_k(a)$ and $f_k(b)$, namely the point $x_2 \in (f_H^k)^{-1}(x_1)$ witnessing the weak coherence of f_H^k at x_1.

Now we can iterate this construction, by considering the tree H_2 obtained from F_k by adding at least $n_s - n_1 - 2$ neighbours to x_2, with the map $f\colon H_2 \to F_k$ sending all of those new vertices to x_2, where s is large enough to have $\operatorname{ord}(x_2) < n_s - n_1 - 2$. Then we can find some $l > k$ and an epimorphism $f_H^l\colon F_l \to F_H$ such that $f_k^{H_2} \circ f_{H_2}^l = f_k^l$ and as above we can find a point $x_3 \in F_l$ of order at least n_s on the arc between $f_l(a)$ and $f_l(b)$. Repeating this construction we build a weakly coherent sequence (x_i) such that $\operatorname{ord}(x_i) \to \infty$ and x_i lies on the unique arc in F_i between $f_i(a)$ and $f_i(b)$. The same argument as at the end of the proof of Theorem 4.19 allows us to conclude that the ramification point of infinite order $\pi(x_i) \in |\mathbb{F}_P|$ belongs to the arc X. ∎

Remark 4.21. In Theorem 4.20 we had to assume that $\{3, \dots, \omega\} \setminus P$ is infinite, which was used to find a sequence n_j of natural numbers with $\lim n_j \to \infty$ and $n_j \notin P$, in order to construct weakly coherent points of $\mathbb{F}_P$ corresponding to ramification points of $|\mathbb{F}_P|$ of infinite order. When P is cofinite in $\{3, \dots, \omega\}$, the family $\mathcal{F}_P$ is still a projective Fraïssé family, as we showed above, but the problem is that the order of the points in weakly coherent sequences must stabilize instead of growing unboundedly. Indeed, the techniques used in the proof of Theorem 4.20 easily establish that $|\mathbb{F}_P| = W_{P'}$ where $P' = P \setminus \{\omega\}$ if $P = \{3, \dots, \omega\}$ and $P' = (P \setminus \{\omega\}) \cup \{a\}$ otherwise, where a is the smallest integer such that $n \in P$ for all $n > a$.

5. Fraïssé categories and projection-embedding pairs. In this section, we construct all generalized Ważewski dendrites as projective Fraïssé limits by moving to the more general setting of Fraïssé categories developed by Kubiś [16]. These are categories in which the amalgamation and joint embedding properties hold (when expressed in terms of appropriate diagrams). As an application, we recover a countable dense homogeneity result for $\operatorname{End}(W_P)$. We quickly recall some definitions from [16], but we stress that the approach followed there is far more general than what we need. In particular, since we are only working with countable collections of finite objects (with finitely many morphisms between any two of them) we can ignore all the issues regarding existence of Fraïssé sequences and uniqueness of Fraïssé limits that arise when looking at the uncountable case. Of particular interest to us is Section 6 of [16] in which projection-embedding pairs are introduced as a tool to produce objects that are universal both in an injective and a projective sense.

5.1. Fraïssé categories. The following definitions and results are taken from [16, Sections 2–3], where they are presented in a more general way, which works in the uncountable case as well.

As a warning, let us point out that while Kubiś's approach is phrased in terms of injective Fraïssé limits, we decided to phrase everything in terms of projective Fraïssé limits. There is no real difference between the two approaches, since it is enough to look at the opposite category $\mathfrak{K}^{\mathrm{op}}$ to translate between the two.

Given a category $\mathfrak{K}$ and two objects $a, b \in \mathfrak{K}$, let $\mathfrak{K}(a,b)$ denote the set of morphisms $a \to b$ in $\mathfrak{K}$.

Definition 5.1. Let $\mathfrak{K}$ be a category. We say that $\mathfrak{K}$ has the *amalgamation property* if for all $a, b, c \in \mathfrak{K}$ and all morphisms $f \in \mathfrak{K}(b,a), g \in \mathfrak{K}(c,a)$, there exist $d \in \mathfrak{K}$ and morphisms $f' \in \mathfrak{K}(d,b), g' \in \mathfrak{K}(d,c)$ with $f \circ f' = g \circ g'$. We say that $\mathfrak{K}$ is *directed* or that $\mathfrak{K}$ has the *joint projection property* if for all $a, b \in \mathfrak{K}$ there exists $c \in \mathfrak{K}$ such that $\mathfrak{K}(c,a) \neq \varnothing \neq \mathfrak{K}(c,b)$.

We now wish to define a category of projective sequences in $\mathfrak{K}$, which we denote by $\sigma\mathfrak{K}$, whose morphisms will be defined in a way that guarantees the following property. Suppose that $\bar{x}, \bar{y} \in \sigma\mathfrak{K}$ and $\mathfrak{K}$ is embedded in a category in which these two sequences have a limit. Given an arrow in $\sigma\mathfrak{K}(\bar{x}, \bar{y})$ we want an induced arrow $\varprojlim \bar{x} \to \varprojlim \bar{y}$. Given $\bar{x} \in \sigma\mathfrak{K}$ we denote $x(n)$ by x_n and the morphism $x(n) \to x(m)$ for $n \geq m$ by x^n_m. By definition a sequence in $\mathfrak{K}$ is a functor $\omega \to \mathfrak{K}$, where we think about the former as a poset category with the reverse order, so a transformation $\bar{x} \to \bar{y}$ is by definition a natural transformation $F\colon \bar{x} \to \bar{y} \circ \varphi$, where $\varphi\colon \omega \to \omega$ is order-preserving. To define an arrow $\bar{x} \to \bar{y}$ in $\sigma\mathfrak{K}$ we need to identify some of those natural transformations.

Definition 5.2. Let $F\colon \bar{x} \to \bar{y} \circ \varphi$ and $G\colon \bar{x} \to \bar{y} \circ \psi$ be two natural transformations. We say that F and G are *equivalent* if

(1) For every n there exists $m \geq n$ such that $\varphi(m) \geq \psi(n)$ and
$$y^{\varphi(m)}_{\psi(n)} \circ F(m) = G(n) \circ x^m_n;$$
(2) For every n there exists $m \geq n$ such that $\psi(m) \geq \varphi(n)$ and
$$y^{\psi(m)}_{\varphi(n)} \circ G(m) = F(n) \circ x^m_n.$$

An arrow $\bar{x} \to \bar{y}$ in $\sigma\mathfrak{K}$ is an equivalence class of natural transformations $\bar{x} \to \bar{y}$.

Note that the category $\sigma\mathfrak{K}$ in Kubiś's approach is the analogue of the class of structures $\mathcal{F}^\omega$ in the approach by Panagiotopolous and Solecki. We can now define a Fraïssé sequence for a category $\mathfrak{K}$, which is the analogue of the Fraïssé sequence in the classical or projective Fraïssé limit constructions.

Kubiś defined in an analogous manner a category of sequences of length κ for any cardinal κ. Similarly, the following definition is a special case of the definition of a κ-Fraïssé sequence from [16].

DEFINITION 5.3. Let $\mathfrak{K}$ be a category. A *Fraïssé sequence* in $\mathfrak{K}$ is a projective sequence $\bar{u}\colon \omega \to \mathfrak{K}$ satisfying:

(1) For every $x \in \mathfrak{K}$ there exists $n < \omega$ with $\mathfrak{K}(u_n, x) \neq \varnothing$.
(2) For every $n < \omega$ and every arrow $f \in \mathfrak{K}(y, u_n)$, with $y \in \mathfrak{K}$, there exist $m \geq n$ and $g \in \mathfrak{K}(u_m, y)$ such that $f \circ g = u_n^m$.

Note that if $\mathfrak{K}$ has the amalgamation property then $\bar{u}$ also satisfies:

(3) For all arrows $f \in \mathfrak{K}(a, b)$, $g \in \mathfrak{K}(u_n, b)$, there exist $m \geq n$ and $h \in \mathfrak{K}(u_m, a)$ with $g \circ u_n^m = f \circ h$.

The following is a special case of [16, Corollary 3.8] and shows existence of Fraïssé sequences:

LEMMA 5.4. *Let $\mathfrak{K}$ be a category with the amalgamation and joint embedding properties with countably many objects and such that between any two objects there are finitely many arrows. Then $\mathfrak{K}$ has a Fraïssé sequence.*

The following is [16, Theorem 3.15] and shows the uniqueness (up to isomorphism) of Fraïssé sequences, hence also the uniqueness of Fraïssé limits when we embed into a category in which the Fraïssé sequence has a limit:

THEOREM 5.5. *Assume $\overline{u}$ and $\overline{v}$ are Fraïssé sequences in a given category $\mathfrak{K}$. Assume further that $k, l < \omega$ and $f \in \mathfrak{K}(u_k, v_l)$. Then there exists an isomorphism $F\colon \overline{u} \to \overline{v}$ in $\sigma\mathfrak{K}$ such that the diagram*

$$\begin{array}{ccc} \overline{u} & \xrightarrow{F} & \overline{v} \\ {\scriptstyle u_k}\downarrow & & \downarrow{\scriptstyle v_l} \\ u_k & \xrightarrow[f]{} & v_l \end{array}$$

commutes. In particular $\overline{u} \approx \overline{v}$.

5.2. Generalized Ważewski dendrites from Fraïssé categories. We now have all the tools needed to talk about projection-embedding categories and construct the generalized Ważewski dendrites not obtained in previous sections.

DEFINITION 5.6. Given $P \subseteq \{3, \ldots, \omega\}$, let $\mathcal{L}'_P$ denote the language with a binary relation R and unary relations U_p for every $p \in P$. Let $\mathcal{L}_R = \{R\} \subseteq \mathcal{L}'_P$ and let $\mathcal{L}_P = \{U_p \mid p \in P\} \subseteq \mathcal{L}'_P$. Let $\ddagger\mathcal{F}_P$ denote the following category:

- $A \in \ddagger\mathcal{F}_P$ iff A is a finite tree (with R as the edge relation), A contains at least one ramification point and every vertex of A is either an endpoint or a ramification point. Moreover, for every ramification point $a \in A$ there

is exactly one $p \in P$ such that $U_p(a)$ holds, and in that case $\mathrm{ord}(a) \leq p$. When a is a ramification point we will refer to the unique $p \in P$ for which $U_p(a)$ holds as the *label* of a. If a instead is an endpoint of A, then $U_p(a)$ does not hold for any $p \in P$.

- Given $A, B \in \ddagger\mathcal{F}_P$, an arrow $f \in \ddagger\mathcal{F}_P(B, A)$ is a pair $(p(f), e(f))$ where $p(f)\colon B \to A$ is a weakly coherent epimorphism from B to A viewed as $\mathcal{L}_R$-structures with the discrete topology. Moreover if for $a \in A$, $U_p(a)$ holds and $b \in B$ witnesses the weak coherence of $p(f)$ at a, then $U_p(b)$ holds. In particular, we are not requiring that if $U_p(b)$ holds, then $U_p(p(f)(b))$ also holds. On the other hand, $e(f)\colon \mathrm{End}(A) \to \mathrm{End}(B)$ is a map associating to every $a \in \mathrm{End}(A)$ a $b \in \mathrm{End}(B)$ with $p(f) \circ e(f) = \mathrm{Id}_{\mathrm{End}(A)}$, where $\mathrm{End}(X)$ is the set of endpoints of X. In other words, $p(f)$ is a weakly coherent epimorphism between the $\mathcal{L}_R$-reducts, while $e(f)$ is a partial right inverse defined on the endpoints.

Given two arrows $f\colon B \to A$ and $g\colon C \to B$ in $\ddagger\mathcal{F}_P$ we define their composition in the obvious way: $f \circ g$ is the pair $(p(f) \circ p(g), e(g) \circ e(f))$. It is clear that composition is associative and that the pair $(\mathrm{Id}_A, \mathrm{Id}_A)$ is the identity morphism for A in $\ddagger\mathcal{F}_P$, so $\ddagger\mathcal{F}_P$ is a category.

Remark 5.7. The argument used to prove Lemma 4.16 shows that $\ddagger\mathcal{F}_P$ is directed, so in order to check that it is a Fraïssé category we only need to check that it has the amalgamation property.

Lemma 5.8. *The category $\ddagger\mathcal{F}_P$ has amalgamation. In other words, whenever we have arrows $f \in \ddagger\mathcal{F}_P(B, A)$, $g \in \ddagger\mathcal{F}_P(C, A)$, there exists $D \in \ddagger\mathcal{F}_P$ with arrows $h_1 \in \ddagger\mathcal{F}_P(D, B)$, $h_2 \in \ddagger\mathcal{F}_P(D, C)$ such that $f \circ h_1 = g \circ h_2$.*

Proof. The proof is very similar to the proof of amalgamation given in the previous section for $\mathcal{F}_P$, so we do not write all the details. Instead, we point out the extra steps that need to be taken to adapt the proof of Lemma 4.17 to the current setting.

Let A, B, C, f, g be as in the statement of the lemma. We proceed by induction on $|E(B)|+|E(C)|$. The smallest structures in $\ddagger\mathcal{F}_P$ are trees with a single ramification point labelled with any $p \in P$, that has three neighbours. Clearly if B and C are of that form, then A must be the same tree on four vertices as well, and all the ramification points in A, B, C must have the same label, so we can take $D = A = B = C$ with identity maps to be the amalgam. Our induction hypothesis now becomes that not only diagrams of the form $B \to A \leftarrow C$ with smaller $|E(B)| + |E(C)|$ can be amalgamated in $\ddagger\mathcal{F}_P$, but that the amalgamation can be carried out in a way that preserves labels and is compatible with the embedding part of the arrows. With this assumption we can use the same construction as in the proof of Lemma 4.17. Indeed, as in that lemma, suppose without loss of generality that $p(f)^{-1}(a)$

is nontrivial for some $a \in A$. Suppose first that a is a ramification point and we are in the $B_2 \neq B_1$ case. We can produce B' with $\pi \in \ddagger\mathcal{F}_P(B, B')$, $f' \in \ddagger\mathcal{F}_P(B', A)$ and D' with $h_1' \in \ddagger\mathcal{F}_P(D', B')$ and $h_2' \in \ddagger\mathcal{F}_P(D', C)$ as in the proof of Lemma 4.17. Note that in this case $p(\pi)$ is the identity map between $\mathrm{End}(B)$ and $\mathrm{End}(B')$, so that $e(f')$ can be defined to agree with $e(f)$. We then build D from D' by glueing back B_2 along an edge. Points in D are already labelled correctly, with labels coming from either B_2 or D', and since glueing B_2 into D' does not collapse any endpoints in D' we can take $e(h_2) = e(h_2')$, while $e(h_1) = e(h_1')$ on $\mathrm{End}(B')$ and is the identity on $\mathrm{End}(B_2) \setminus \{b_1, b_2\}$.

The $B_2 = B_1$ case is easier to deal with: we follow the same construction as in the proof of Lemma 4.17 and we never run into issues with the embedding part of the morphisms, since endpoints are never collapsed during the construction.

If instead a is an endpoint, there are still no issues mimicking the construction from Lemma 4.17 as far as labels are concerned, but a little more care is needed for the embedding part of the arrows, so we will write out some of the details. Let $b \in p(f)^{-1}(a)$ be the unique vertex for which there is a $b' \in B \setminus p(f)^{-1}(a)$ with $\langle b', b\rangle \in E(B)$ and let B_2 be $p(f)^{-1}(a)$. As in the previous case, we construct B' from B by collapsing B_2 to b through the map $p(\pi)\colon B \to B'$, but now we also need to define the embedding part $e(\pi)$ in order to get a morphism in $\ddagger\mathcal{F}_P$. Let $e(\pi)$ be the identity on $\mathrm{End}(B') \setminus \{b\}$ and $e(\pi)(p(\pi)(b)) = e(f)(a)$. Similarly, we obtain a coherent epimorphism $p(f')\colon B' \to A$ as the only such morphism satisfying $p(f') \circ p(\pi) = p(f)$. There is only one choice for $e(f')$, namely

$$e(f')(x) = \begin{cases} e(f)(x) & \text{if } x \in \mathrm{End}(A) \setminus \{a\}, \\ p(\pi)(b) & \text{if } x = a. \end{cases}$$

Once again we obtain, by inductive hypothesis, an amalgam D' with $h_1' \in \ddagger\mathcal{F}_P(D', B')$ and $h_2' \in \ddagger\mathcal{F}_P(D', C)$ such that $f' \circ h_1' = g \circ h_2'$. We now have to glue back B_2 to D' in order to build an amalgam D with maps $h_1 \in \ddagger\mathcal{F}_P(D, B)$ and $h_2 \in \ddagger\mathcal{F}_P(D, C)$. We already know how to build D as a graph and how to define $p(h_1), p(h_2)$, since the construction is the same as the one in the proof of Lemma 4.17. As in the previous section, we obtain D from D' and B_2 by identifying $b \in B_2$ with an endpoint of D'. Here we take $\alpha \in \mathrm{End}(D')$ to be $e(h_1')(b)$ which is also equal to $e(h_2')(e(g)(a)))$. Points of D are already labelled correctly, with labels coming from either B_2 or D, and we define $p(h_1), p(h_2)$ exactly as above. We only need to define $e(h_1), e(h_2)$. There is nothing to be done for $e(h_2)$ which we simply define to be equal to $e(h_2')$ on all endpoints except $e(g)(a) \in B_2$, where it now takes value $e(f)(a)$. To define $e(h_1)$ there are a few cases to consider:

$$e(h_1)(x) = \begin{cases} e(h_1')(x) & \text{if } x \in \mathrm{End}(B) \setminus \mathrm{End}(B_2), \\ x & \text{if } x \in \mathrm{End}(B_2). \end{cases}$$

It is now easy to check that the resulting square diagram is commutative, as desired. ∎

Since $\ddagger\mathcal{F}_P$ has countably many objects and between any two objects of $\ddagger\mathcal{F}_P$ there are finitely many arrows, Lemma 5.4 and Theorem 5.5 immediately imply the following:

THEOREM 5.9. *The category $\ddagger\mathcal{F}_P$ has a Fraïssé sequence $\overline{u}$ and any two Fraïssé sequences are isomorphic as elements of $\sigma\ddagger\mathcal{F}_P$. In particular, whenever $\ddagger\mathcal{F}_P$ is embedded in a category in which Fraïssé sequences have colimits, the colimits arising from two Fraïssé sequences are isomorphic.*

As mentioned in the introduction to this section, we want to embed $\sigma\ddagger\mathcal{F}_P$ in a category in which those sequences actually have a limit. We do so by considering $\sigma\ddagger\mathcal{F}_P$ as a subcategory of a category $\mathcal{G}$ of *labelled topological graphs* defined as follows: The objects of $\mathcal{G}$ are pairs $G = (B, E)$ where B is a graph-dendrite, equipped with unary predicates U_n, for $n \in P$, such that if a is a ramification point of G, there is a unique n for which $U_n(a)$ holds and $n \leq \mathrm{ord}(a)$, and E is a countable subset of $\mathbb{U}$. The morphisms of $\mathcal{G}$ from (B, E) to (A, F) are pairs $f = (p(f), e(f))$ such that $e(f)\colon F \to E$ is an injection, while $p(f)\colon B \to A$ is an epimorphism of topological graphs such that $p(f) \circ e(f) = \mathrm{Id}_F$.

To a sequence $\overline{u} \in \sigma\ddagger\mathcal{F}_P$ we associate the pair $(\mathbb{U}, E)$, where $\mathbb{U}$ is the topological graph obtained as the projective limit of the projection part of $\overline{u}$, while E is the direct limit of the embedding part of $\overline{u}$. Moreover, every ramification point x of $\mathbb{U}$ is labelled with its order as label, where the order of x is the (potentially infinite) number of components of $\mathbb{U} \setminus \{x\}$. Given another $\overline{v} \in \sigma\ddagger\mathcal{F}_P$, a morphism $\psi\colon \overline{u} \to \overline{v}$ is by definition a natural transformation ψ from $\overline{u}$ to $\overline{v} \circ \varphi$, where $\varphi\colon \omega \to \omega$ is order-preserving. It is easy to check that if $(\mathbb{U}, E)$ and $(\mathbb{V}, F)$ are the objects associated to $\overline{u}$ and $\overline{v}$ as described above, then the projection part of ψ converges to a continuous surjection $p(\psi)\colon \mathbb{U} \to \mathbb{V}$, while the embedding part converges to an injection $e(\psi)\colon F \to E$ such that $p(\psi) \circ e(\psi) = \mathrm{Id}_F$. Note that, since in Definition 5.6 we required that the witness of weak coherence at a point a has the same label as a, $p(\psi)$ has the property that if $v \in \mathbb{V}$ is a ramification point with label l, then there is a ramification point $u \in p(\psi)^{-1}(v)$ with the same label. This describes the desired functor $\sigma\ddagger\mathcal{F}_P \to \mathcal{G}$.

To a sequence $\overline{u} \in \sigma\ddagger\mathcal{F}_P$ we associated a pair $(\mathbb{U}, E)$, where $\mathbb{U}$ is a topological graph. We will show that whenever $\overline{u}$ is a Fraïssé sequence for $\ddagger\mathcal{F}_P$, then $|\mathbb{U}|$ is the generalized Ważewski dendrite W_P, where $\pi\colon \mathbb{U} \to |\mathbb{U}|$ is the topological realization.

Note that the embedding part of $\overline{u}$ plays no role in the following theorem. This observation will be made precise and become relevant later.

THEOREM 5.10. *Let $\overline{u} = \langle F_i \mid i < \omega \rangle$ with maps $f_i^{i+1} \in \ddagger\mathcal{F}_P(F_{i+1}, F_i)$ be a Fraïssé sequence for $\ddagger\mathcal{F}_P$ and let $(\mathbb{F}_P, E)$ be its limit. Then $|\mathbb{F}_P| = W_P$.*

Proof. Even though $\ddagger\mathcal{F}_P$ does not allow splitting edges, we can proceed as in Remark 4.15 to use all the results that required splitting edges. Because of this, we know that $|\mathbb{F}_P|$ is a dendrite thanks to Theorem 3.17 since $p(f_i^{i+1})$ is monotone and (weakly) coherent; moreover, we still have the correspondence between (weakly) coherent points of $\mathbb{F}_P$ and ramification points of $|\mathbb{F}_P|$ established in Theorems 4.11 and 4.13. The fact that each ramification point has order in P follows from an argument similar to the one in the proof of Theorem 4.20: if $x = (x_n) \in \mathbb{F}_P$ is labelled with U_p, then x_n must be labelled with U_p from some n on. If $p < \omega$, then $\mathrm{ord}(x_n)$ must also be p from some (possibly bigger) n onward, which implies that $\mathrm{ord}(x) = p$, and so $\mathrm{ord}(\pi(x)) = p$. If $x = (x_n) \in \mathbb{F}_P$ is labelled with U_ω instead, then $\mathrm{ord}(x_n) \to \infty$ as $n \to \infty$ as in the proof of Theorem 4.20, hence $\mathrm{ord}(\pi(x)) = \omega$. The fact that the ramification points of each order in P are arcwise dense in $|\mathbb{F}_p|$ follows from exactly the same argument as the one at the end of the proof of Theorem 4.20. ∎

5.3. A countable dense homogeneity result for $\mathrm{End}(W_P)$. We now want to prove the following homogeneity result for the endpoints of W_P, first stated in [5] for W_3:

THEOREM 5.11. *Let $P \subseteq \{3, \dots, \omega\}$ and let Q_1, Q_2 be countable dense subsets of $\mathrm{End}(W_P)$. Then there is a homeomorphism $h\colon W_P \to W_P$ with $h(Q_1) = Q_2$.*

We begin with a more precise description of the endpoints of a projective Fraïssé limit of trees; in particular, we want to show that if $(\mathbb{U}, E)$ is the projective Fraïssé limit of $\ddagger\mathcal{F}_P$, then $\pi(E)$ is a dense set of endpoints in $|\mathbb{U}|$. We first show that endpoints are edge-related only to themselves.

PROPOSITION 5.12. *Let $\mathcal{F}$ be a projective Fraïssé family of trees with (weakly) coherent epimorphisms that allows splitting edges and let $\mathbb{F}$ be its projective Fraïssé limit. If $e \in \mathrm{End}(\mathbb{F})$, then there is no $e' \in \mathbb{F} \setminus \{e\}$ with $\langle e, e' \rangle \in E(\mathbb{F})$.*

Proof. Suppose for a contradiction that there is $e \neq e' \in \mathbb{F}$ with $\langle e, e' \rangle \in E(\mathbb{F})$. Let F_n with maps $f_n^m\colon F_m \to F_n$ be a Fraïssé sequence for $\mathcal{F}$ and let $g_n\colon \mathbb{F} \to F_n$ be the canonical projections. Let e_n, e_n' be $g_n(e)$ and $g_n(e')$ respectively.

First, we claim that for all but finitely many n, $e_n \in \mathrm{End}(F_n)$. Indeed, suppose for a contradiction that this is not the case and let m be large

enough so that $e_m \neq e'_m$ and e_m is not in $\mathrm{End}(F_m)$. Let a_m be a point in F_m such that e_m is on the unique arc joining a_m and e'_m. Without loss of generality we can assume that a_m is a ramification point (we can for example take $a_m = e_m$) and, for $k > m$, let a_k denote the vertex of F_k witnessing the (weak) coherence of f^k_m at a_m. By monotonicity of f^k_m and since $\langle e_k, e'_k\rangle \in E(F_k)$, we find that for all $k > m$, e_k is on the unique arc joining a_k and e'_k. Indeed, $(f^k_m)^{-1}([a_m, e'_m])$ is connected by monotonicity of f^k_m and, since F_k is a tree, there is a unique edge between $(f^k_m)^{-1}([a_m, e_m])$ and $(f^k_m)^{-1}([e_m, e'_m])$ which must be the edge between e_k and e'_k. Since the unique arc from a_k to e'_k goes from $(f^k_m)^{-1}([a_m, e_m])$ to $(f^k_m)^{-1}([e_m, e'_m])$, it must go through this edge, showing that e_k lies on the arc from a_k to e'_k. As in the proof of arcwise density in Theorem 4.19 we obtain a ramification point $a \in \mathbb{F}$ such that e is contained in the arc $[a, e']$. Since $e \neq a$, as the latter is a ramification point, and $e \neq e'$ by assumption, this contradicts the fact that e is an endpoint of $\mathbb{F}$.

We have thus shown that, if $e \in \mathrm{End}(\mathbb{F})$ and $e' \in \mathbb{F}$ is such that $\langle e, e'\rangle \in E(\mathbb{F})$, then for all but finitely many $n \in \mathbb{N}$, e_n is an endpoint of F_n.

Now let H be the tree obtained by splitting the $\langle e_m, e'_m\rangle$ edge, so that $V(H) = V(F_m) \sqcup \{x\}$ and

$$E(H) = (E(F_m) \setminus \{\langle e_m, e'_m\rangle\}) \cup \{\langle e_m, x\rangle, \langle x, e'_m\rangle\}$$

and consider the epimorphism $\varphi\colon H \to F_m$ defined by $\varphi(x) = e_m$ and φ is the identity on H otherwise (when applying this result to the $\mathcal{F} = \ddagger\mathcal{F}_P$ case we will also need to add neighbours to x until it has the correct order). By assumption, we can find $n > m$ and an epimorphism $\psi\colon F_n \to H$ such that $\varphi \circ \psi = f^n_m$. Since ψ respects the edge relation we must have $\psi(e_n) = x$, $\psi(e'_n) = e'_m$. But now, since $e_n \in \mathrm{End}(F_n)$ and ψ is monotone, it is impossible for e_m to be in the image of ψ, contradicting that the latter is an epimorphism. ∎

Lemma 5.13. *Let $\mathcal{F}$ be a projective Fraïssé family of trees with (weakly) coherent epimorphisms that allows splitting edges. Let $\langle F_i \mid i < \omega\rangle$ be a Fraïssé sequence for $\mathcal{F}$ and let $\mathbb{F} \subseteq \prod F_i$ be its projective Fraïssé limit. Let $e = (e_n)_{n\in\mathbb{N}} \in \mathbb{F}$. If there is $N \in \mathbb{N}$ such that for every $m \geq N$, e_m is an endpoint of F_m, then $e \in \mathrm{End}(\mathbb{F})$.*

Proof. Assume that $(e_n)_{n\in\mathbb{N}}$ is such that e_m is an endpoint of F_m for all $m \geq N$. Suppose for a contradiction that e is not an endpoint of $\mathbb{F}$ and let $A \subseteq \mathbb{F}$ be an arc with $e \in A$, but $e \neq a_1, a_2$, where a_1, a_2 are the endpoints of A. Let k be large enough so that e_k is an endpoint of F_k, and $f_k(a_1)$, $f_k(a_2)$, $f_k(e)$ are pairwise distinct. By Theorem 3.18 f_k is monotone, so by Lemma 3.14 $f_k(a_1)$, $f_k(a_2)$ and e_k all belong to $\mathrm{End}(f_k(A))$, a contradiction. ∎

LEMMA 5.14. *Let $\ddagger\mathcal{F}_P$ be as above, let $\overline{u} = \langle F_i \mid i < \omega\rangle$ be a Fraïssé sequence for $\ddagger\mathcal{F}_P$ with maps $f_n^m \in \ddagger\mathcal{F}_P(F_m, F_n)$ and let $(\mathbb{F}, E)$ be its projective Fraïssé limit. Then $E \subseteq \mathrm{End}(\mathbb{F})$ and E is dense in $\mathbb{F}$.*

Proof. It follows from Lemma 5.13 that $E \subseteq \mathrm{End}(\mathbb{F})$. Let now $U \subseteq \mathbb{F}$ be a nonempty open set. We want to show that $E \cap U \neq \varnothing$. Let $n \in \omega$ be sufficiently large so that there is $x \in F_n$ such that $f_n^{-1}(x) \subseteq U$. Consider any finite tree $T \in \ddagger\mathcal{F}_P$ with an epimorphism $f \in \ddagger\mathcal{F}_P(T, F_n)$ such that $p(f)^{-1}(x)$ contains an endpoint of T. By definition of a Fraïssé sequence we can find $m \geq n$ and $g \in \ddagger\mathcal{F}_P(F_m, T)$ such that $f \circ g = f_n^m$. Since the preimage of an endpoint through a (weakly) coherent map always contains an endpoint, this implies that there is an endpoint $y \in \mathrm{End}(F_m)$ with $f_m^{-1}(y) \subseteq U$. Now the point of $\mathbb{F}$ determined by the sequence $e(f_m^k)(y)$ for $k \geq m$ is an endpoint by Lemma 5.13 and is contained in both E and U by construction. ∎

As a consequence of the previous two lemmas and Proposition 5.12, we obtain the following corollary:

COROLLARY 5.15. *If $(\mathbb{U}, E)$ is the projective Fraïssé limit of $\ddagger\mathcal{F}_P$, then $\pi(E)$ is a dense set of endpoints in $|\mathbb{U}| \simeq W_P$, where $\pi\colon \mathbb{U} \to |\mathbb{U}|$ is the topological realization.*

The last piece we need before being able to prove the desired countable dense homogeneity result for the endpoints of W_P is Theorem 5.22. Intuitively, this theorem will say that the embedding part of the morphisms is not important in determining whether a sequence is Fraïssé for $\ddagger\mathcal{F}_P$. This is not surprising, since in Theorem 5.10 the embedding part of the morphisms played no role. We first need to introduce new definitions and make a few simple remarks.

DEFINITION 5.16. Let X be a dendrite. Given any three distinct $x, y, z \in X$, their *centre* $C(x, y, z)$ is the unique point in $[x, y] \cap [y, z] \cap [x, z]$. A set $F \subseteq X$ is called *centre-closed* if $C(f_1, f_2, f_3) \in F$ whenever $f_1, f_2, f_3 \in F$ are distinct. Given a finite set $A \subseteq X$, its *centre closure* is the smallest centre-closed $B \subseteq X$ with $A \subseteq B$. Given a finite centre-closed $F \subseteq X$ and $a \in F$ let $\hat{a}_F$ denote the set of connected components of $X \setminus \{a\}$ that contain no point of F. Given distinct $a, b \in F$ let $C_a(b)$ be the connected component of $X \setminus \{a\}$ containing b, $C_b(a)$ the connected component of $X \setminus \{b\}$ containing a, and $C_{a,b} = C_b(a) \cap C_a(b)$. The *partition associated to* F is then

$$\Omega_F = \bigcup_{a \in F} \hat{a}_F \cup \big\{C_{a,b} \;\big|\; a \neq b \in F \text{ with } [a, b] \cap F = \{a, b\}\big\},$$

which is exactly the set of connected components of $X \setminus F$.

If X is a graph-dendrite instead, we have analogous notions: for any three distinct points $x, y, z \in X$ we define their *centre* $C(x, y, z)$ to be the

unique point in $[x, y] \cap [y, z] \cap [x, z]$, where those are now arcs in the sense of topological graphs. The other notions are defined similarly.

DEFINITION 5.17. An *immersion* of a finite tree A in a graph-dendrite B is a map $f\colon A \to B$ (which will usually not be a graph homomorphism) such that for all distinct $a, b, c \in A$,

$$b \in [a, c] \iff f(b) \in [f(a), f(c)].$$

In other words, there exist a topological graph A' obtained from A by replacing every edge of A with an arc (as in Definition 3.3) and a graph embedding $f'\colon A' \to B$ with $f = f'\big|_A$.

Note that the image of an immersion $f\colon A \to B$ is centre-closed, so it determines a partition $\Omega_f = \Omega_{f(A)}$ of $B \setminus f(A)$ as above. There are two particular instances of immersions that will be relevant. They are described in the following examples.

EXAMPLE 5.18. An arrow $f \in \ddagger\mathcal{F}_P(B, A)$ determines an immersion $i(f)\colon A \to B$ by setting

$$i(f)(a) = \begin{cases} e(f)(a) & \text{if } a \in \mathrm{End}(A), \\ w(a) & \text{otherwise,} \end{cases}$$

where $w(a) \in B$ is the witness for the (weak) coherence of $p(f)$ at a.

EXAMPLE 5.19. Let $\langle F_n \mid n < \omega \rangle$ with maps $f_n^m \in \ddagger\mathcal{F}_P(F_m, F_n)$ be a sequence in $\ddagger\mathcal{F}_P$ and let $\mathbb{F}$ with maps $f_n^\infty \in \ddagger\mathcal{F}_P(\mathbb{F}, F_n)$ be its limit. We have immersions $h_n\colon F_n \to \mathbb{F}$ given by

$$x \mapsto (p(f_0^n)(x), p(f_1^n)(x), \dots, p(f_{n-1}^n)(x), x, w_{n+1}^n(x), w_{n+2}^n(x), \dots),$$

where $w_j^i(x)$ denotes the witness for the (weak) coherence of $f_i^j\colon F_j \to F_i$ at x for x a ramification point. For endpoints we follow the embeddings instead of the witnesses, that is

$$x \mapsto (p(f_0^n)(x), p(f_1^n)(x), \dots, p(f_{n-1}^n)(x), x, e(f_n^{n+1})(x), e(f_n^{n+2})(x), \dots),$$

when x is an endpoint.

REMARK 5.20. Let $i\colon A \to B$ be an immersion of finite trees. Suppose that $f\colon B \to A$ is a monotone map such that

- for every $a \in A$, $f(i(a)) = a$,
- for every $a \in A$ and every $b \in \bigcup \widehat{i(a)}_{i(A)}$, $f(b) = a$,
- for every $a \neq a' \in A$ with $\langle a, a' \rangle \in E(A)$ and every $b \in C_{i(a),i(a')}$, $f(b) \in \{a, a'\}$.

Then f is weakly coherent.

DEFINITION 5.21. We say that a topological graph G is a *W_P-prespace* if G is a prespace, G is a graph-dendrite, for every $p \in P$ the set of ramification

points of order p is arcwise dense in G and every ramification point of G has order in P. In particular if G is a W_P-prespace, then $|G| = W_P$.

We can now characterize Fraïssé sequences in $\ddagger\mathcal{F}_P$.

THEOREM 5.22. *A sequence $\langle F_n \mid n < \omega\rangle$ with maps $f_n^m \in \ddagger\mathcal{F}_P(F_m, F_n)$ is Fraïssé if and only if the inverse limit of $p(f_n^m)$ considered as a topological graph is a W_P-prespace.*

Proof. The if part of the statement was already proved in Theorem 5.10. Suppose that $\langle F_n \mid n < \omega\rangle$ with epimorphisms $f_n^m \in \ddagger\mathcal{F}_P(F_m, F_n)$ is a sequence such that

$$\mathbb{F} = \varprojlim\Big(\cdots \to F_{n+1} \xrightarrow{p(f_n^{n+1})} F_n \xrightarrow{p(f_{n-1}^{n})} \cdots \xrightarrow{p(f_0^1)} F_0\Big)$$

is a W_P-prespace. We want to show that $\langle F_n \mid< \omega\rangle$ is a Fraïssé sequence for $\ddagger\mathcal{F}_P$. In other words, given $A \in \ddagger\mathcal{F}_P$ and an epimorphism $g \in \ddagger\mathcal{F}_P(A, F_n)$, we want to find $m \geq n$ and an epimorphism $h \in \ddagger\mathcal{F}_P(F_m, A)$ such that $g \circ h = f_n^m$. As in Example 5.19, we obtain an immersion $h_n\colon F_n \to \mathbb{F}$. Moreover, we also have an immersion $j\colon F_n \to A$ as in Example 5.18. Using the fact that $\mathbb{F}$ is a W_P-prespace we can find an immersion $k\colon A \to \mathbb{F}$ such that $k \circ j = h_n$. We construct k inductively on elements of A. There is only one choice for the value of k on $j(F_n)$ such that $k \circ j = h_n$. Enumerate $A \setminus j(F_n)$ as $\{a_1, \ldots, a_r\}$. We define $k(a_1)$ to be any point of $\mathbb{F}$ with $\operatorname{ord}(a_1) = \operatorname{ord}(k(a_1))$ such that k is an isomorphism of trees with the betweenness relation from $j(F_n) \cup \{a_1\}$ to $h_n(A) \cup \{k(a_1)\}$. Such a point $k(a_1) \in \mathbb{F}$ exists because $\mathbb{F}$ is a W_P-prespace. Proceeding inductively in a similar manner, we define $k(a_i)$ for all $i \leq r$. Now we can find $m \geq n$ such that, calling $f_m^\infty \in \ddagger\mathcal{F}_P(\mathbb{F}, F_m)$ the canonical projection, all vertices in $k(A)$ have distinct images through f_m^∞. We check that $i = p(f_m^\infty) \circ k$ is an immersion of A in F_m.

If $a, b, c \in A$ are such that $b \in [a, c]$, then $k(b) \in [k(a), k(c)]$ because k is an immersion, and $i(b) \in [i(a), i(c)]$ because $p(f_m^\infty)$ is monotone, so it must map arcs to arcs. Conversely, suppose for a contradiction that $i(b) \in [i(a), i(c)]$ but $b \notin [a, c]$. Let $d \in [a, c]$ be the only point of $[a, c]$ which belongs to the shortest arc from b to $[a, c]$. If $d = a$, then $a \in [b, c]$, which, arguing as in the previous implication, implies $k(a) \in [k(b), k(c)]$ and so $i(a) \in [i(b), i(c)]$, a contradiction. If $d = c$ we reach a contradiction in the same manner, which only leaves the case $d \in (a, c)$. But then we have $k(d) \in [k(a), k(c)]$ and $k(b)$ is in a component of $\widehat{k(d)}$ that contains neither $k(a)$ nor $k(c)$, since k is an immersion. By assumption, $p(f_m^\infty)(a), p(f_m^\infty)(b), p(f_m^\infty)(c)$ and $p(f_m^\infty)(d)$ are pairwise distinct, so we must have $i(d) \in (i(a), i(c))$, since $p(f_m^\infty)$ maps arcs to arcs, but $i(b)$ must also be in a component of $\widehat{i(d)}$ containing neither $i(a)$ nor $i(c)$, a contradiction.

We can now define a weakly coherent map from F_m to A. We only need to be careful and make sure that it makes the appropriate diagram commute. Explicitly, we define $h \in \ddagger\mathcal{F}_P(F_m, A)$ as follows. Let $x \in F_m$. If $x = i(a)$ for some $a \in A$, then $p(h)(x) = a$. If $x \in \widehat{a}_{i(A)}$ (as defined in Definition 5.16) for some $a \in A$, then $p(h)(x) = a$. If $x \in C_{i(a),i(a')}$ for some $a, a' \in A$ with $\langle a, a' \rangle \in E(A)$, then we check whether $p(g)(a) = f_n^m(x)$ or $p(g)(a') = f_n^m(x)$ and define $p(h)(x)$ accordingly. Since the map we have just defined is monotone, it must also be weakly coherent by Remark 5.20. For the embedding part of h consider $a \in \mathrm{End}(A)$. If $a \in e(g)(\mathrm{End}(F_n))$ set $e(h)(a) = e(f_n^m)(b)$, where $b \in \mathrm{End}(F_n)$ is such that $e(g)(b) = a$. If instead a is not in the image of $e(g)$, then $e(h)(a)$ can be any endpoint in $p(h)^{-1}(a)$. Note that $p(h) \circ e(h)(a) = a$ for any $a \in \mathrm{End}(A)$ and $g \circ h = f_n^m$. ∎

We conclude by proving the following theorem, from which Theorem 5.11 follows immediately by the uniqueness of Fraïssé limits.

THEOREM 5.23. *Let $P \subseteq \{3, \dots, \omega\}$ and let $Q \subseteq \mathrm{End}(W_P)$ be a countable dense set of endpoints. Then $\ddagger\mathcal{F}_P$ has a Fraïssé sequence $\overline{u} = \langle F_i \mid i < \omega \rangle$ with morphisms $f_n^m \in \ddagger\mathcal{F}_P(F_m, F_n)$ such that if $(\mathbb{U}, D)$ is its Fraïssé limit, then $\pi(D) = Q$, where $\pi \colon \mathbb{U} \to |\mathbb{U}| \cong W_P$ is the topological realization.*

Proof. Enumerate Q without repetitions as $\{q_i\}_{i \in \mathbb{N}}$. Let $A_0 = \{q_1, q_2, q_3, c\}$ where $c = C(q_1, q_2, q_3)$ and, for all $0 < n < \omega$, let A_n be the centre closure of $\{q_1, \dots, q_{k_n}\}$, where $(k_n)_{n<\omega}$ is an increasing sequence chosen so that $|A_{n+1} \cap (a, a')| \geq 2$, whenever $a, a' \in A_n$ are such that $[a, a'] \cap A_n = \{a, a'\}$, which is possible since Q is dense. We now build a finite graph F_n with a vertex v_a for every $a \in A_n$, and an edge $\langle v_a, v_{a'} \rangle$ iff $[a, a'] \cap A_n = \{a, a'\}$. We label every $v_c \in F_n$ that comes from a ramification point $c \in A_n$ with the order of c in W_P. It only remains to define morphisms $f_n^{n+1} \in \ddagger\mathcal{F}_P(F_{n+1}, F_n)$. Since we have $A_n \subseteq A_{n+1}$, we also get an immersion $i \colon F_n \to F_{n+1}$ that maps $v_a \in F_n$ to $v_a \in F_{n+1}$, for every $a \in A_n$. The embedding part of f_n^{n+1} is exactly $i\big|_{\mathrm{End}(F_n)}$. For the projection part, for all pairs $a, a' \in A_n$ with $[a, a'] \cap A_n = \{a, a'\}$, we partition $C_{v_a,v_{a'}} \cup \{v_a, v_{a'}\} \subseteq F_{n+1}$ into two sets H, H' such that

(1) $v_a \in H$, $v_{a'} \in H'$,
(2) H, H' are connected,
(3) $|H|, |H'| \geq 2$.

Note that the last condition is easily satisfied since we have assumed that $|(a, a') \cap A_{n+1}| \geq 2$. We then define $p(f_n^{n+1})(H) = v_a$ and $p(f_n^{n+1})(H') = v_{a'}$. By construction, $p(f_n^{n+1})$ is monotone and satisfies the hypothesis of Remark 5.20, so it must be weakly coherent.

Thanks to Theorem 3.7 and the condition $|(a, a') \cap A_{n+1}| \geq 2$, the limit of this sequence is a prespace. Since Q is dense we know that $\bigcup_n A_n =$

$\mathrm{Br}(W_P) \cup Q$. Using this fact it is easy to check that the limit of this sequence is a W_P-prespace, so that the sequence is Fraïssé by Theorem 5.22. Denoting by $(\mathbb{U}, D)$ its limit, we now only need to identify $(|\mathbb{U}|, \pi(D))$ with (W_P, Q). To any $a \in A_{n+1} \setminus A_n$ we have associated the point

$$r_a = (p(f_0^{n+1})(v_a), \ldots, p(f_{n-1}^{n+1})(v_a), p(f_n^{n+1})(v_a), v_a, v_a, \ldots)$$

of $\mathbb{U}$, which is a ramification point by Theorem 4.11 if a is a ramification point of A_{n+1}, and which is an endpoint if a is an endpoint of A_{n+1}. We define a continuous function $h\colon \mathrm{Br}(W_P) \cup Q \to |\mathbb{U}|$ by setting $h(a) = \pi(r_a)$ for every $a \in \bigcup_n A_n$. To verify the continuity of h note that if $x_n \to x$ is a convergent sequence of points in the domain of h whose limit is also in the domain of h, then for any $m \in \mathbb{N}$ we can find $N \in \mathbb{N}$ large enough so that A_m does not separate x_k and x, for any $k \geq N$. In other words, r_{x_k} and r_x agree in at least the first m coordinates, which implies that $r_{x_k} \to r_x$, since m was arbitrary. This shows that the function $a \mapsto r_a$ is continuous, from which the continuity of h follows immediately, since h is obtained by composition of that function with π, which is continuous. Note that the image of h is dense in $|\mathbb{U}|$ (in particular, any ramification point of $|\mathbb{U}|$ is in the image), h is injective, and h preserves the betweenness relation. We will now extend h to a continuous function $\tilde{h}\colon W_P \to |\mathbb{U}|$ and then verify that $\tilde{h}$ is injective and surjective. For $x \in W_P$, let

$$\mathrm{osc}(h, x) = \inf \{\operatorname{diam} h(U) \mid U \text{ is an open neighbourhood of } x\},$$

where $h(U)$ means $h(U \cap (\mathrm{Br}(W_P) \cup Q))$. By Kuratowski's extension theorem (see Theorem 3.8 and its proof in [15]), in order to extend the continuous function h to a continuous function $\tilde{h}\colon W_P \to |\mathbb{U}|$, it suffices to show that $\mathrm{osc}(h, x) = 0$ for every $x \in \mathrm{End}(W_P) \setminus Q$ or when x is a regular point.

- In the first case, let $(x_i)_{i<\omega}$ be a sequence in $\mathrm{Br}(W_P)$ with $x_{i+1} \in [x_i, x]$ for every i, $x_i \to x$ and, for every $i < \omega$, let $C_{x_i}(x)$ be the component of $W_P \setminus \{x_i\}$ containing x. Then $\{x\} = \bigcap_{i<\omega} C_{x_i}(x)$ and we have $\bigcap_{i<\omega} h(C_{x_i}(x)) = \varnothing$. Indeed, if $y \in \mathrm{Br}(W_P) \cup Q$ and w is the unique point in $[y, x_0] \cap [y, x] \cap [x_0, x]$, then there is an i large enough so that $w \notin C_{x_i}(x)$, $y \notin C_{x_i}(x)$ and, since h preserves the betweenness relation, $h(y) \notin C_{x_i}(x)$. Since $C_{x_{i+1}}(x) \subseteq C_{x_i}(x)$ for every i (because $x_{i+1} \in [x_i, x]$), this shows that $\bigcap_{i<\omega} h(C_{x_i}(x)) = \varnothing$. Therefore by compactness of W_P we have $\operatorname{diam}(h(C_{x_i}(x))) \to 0$ and hence $\mathrm{osc}(h, x) = 0$.
- If $\mathrm{ord}(x) = 2$ let $(y_i)_{i<\omega}$ and $(z_i)_{i<\omega}$ be two sequences in $\mathrm{Br}(W_P)$ with $y_{i+1} \in [y_i, x]$, $y_i \to x$, $z_{i+1} \in [x, z_i]$, $z_i \to x$ and $x \in [y_i, z_i]$ for every $i < \omega$. Letting $C_{y_i,z_i}(x)$ denote the component of $X \setminus \{y_i, z_i\}$ containing x, we can argue as in the previous case to find that $\{x\} = \bigcap_{i<\omega} C_{y_i,z_i}(x)$ and that $\bigcap_{i<\omega} h(C_{y_i,z_i}(x)) = \varnothing$. Similarly to the previous case, since $C_{y_{i+1},z_{i+1}}(x) \subseteq C_{y_i,z_i}(x)$ for every i, this shows that $\mathrm{osc}(h, x) = 0$.

By the discussion above we can now extend h to a continuous function $\tilde{h}\colon W_P \to |\mathbb{U}|$. Note that $\tilde{h}$ preserves the betweenness relation. Indeed, the betweenness relation $\{(x,y,z) \in W_P^3 \mid x \in [y,z]\}$ is closed in W_P^3. Therefore if $x_n \to x$, $y_n \to y$, $z_n \to z$, $x_n \in [y_n, z_n]$, $x \in [y,z]$, $x_n, y_n, z_n \in \mathrm{Br}(W_P)$, we see that $\tilde{h}(x_n) \in [\tilde{h}(y_n), \tilde{h}(z_n)]$, and by continuity of $\tilde{h}$, we get $\tilde{h}(x) \in [\tilde{h}(y), \tilde{h}(z)]$.

Since $\tilde{h}(W_P) \supseteq \overline{h(\mathrm{Br}(W_P) \cup Q)} = |\mathbb{U}|$, $\tilde{h}$ must be surjective.

To show injectivity of $\tilde{h}$ take $x \neq y \in W_P$ such that none of the x, y is a ramification point. Since the ramification points are arcwise dense in W_P, pick $t \in \mathrm{Br}(W_P)$ with $t \in [x,y]$. Then $\tilde{h}(t) \in [\tilde{h}(x), \tilde{h}(y)]$ and since none of the $\tilde{h}(x), \tilde{h}(y)$ is a ramification point, we obtain $\tilde{h}(t) \neq \tilde{h}(x), \tilde{h}(y)$. Therefore $\tilde{h}(x) \neq \tilde{h}(y)$.

Moreover, we have $\tilde{h}(Q) = \pi(D)$ by construction, since $\tilde{h}(Q) = h(Q)$, which concludes the proof. ∎

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Alessandro Codenotti
Institut für Mathematische Logik und Grundlagenforschung
Universität Münster
48149 Münster, Germany
E-mail: alessandro.codenotti@unibo.it

Aleksandra Kwiatkowska
Institut für Mathematische Logik und Grundlagenforschung
Universität Münster
48149 Münster, Germany
and
Instytut Matematyczny
Uniwersytet Wrocławski
50-384 Wrocław, Poland
E-mail: kwiatkoa@uni-muenster.de