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Billiards with countably many scatterers under no eclipse

by

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Abstract. We consider a billiard flow with countably infinitely many scatterers on the plane without eclipse. We show that the non-wandering set of the billiard flow is in one-to-one correspondence with a two-sided topological Markov shift with countably many states. We also give a sufficient condition for the Euler product formula for the zeta function with respect to the billiard flow.

1. Introduction and the outline of the main results. Let S be a countable set with $\#S \geq 3$ and $\{Q_j : j \in S\}$ a countable number of bounded closed domains in $\mathbb{R}^2$ such that $\operatorname{dist}(Q_i, Q_j) > 0$ for all distinct $i, j \in S$ and $\sup_{j\in S} \operatorname{diam} Q_j < \infty$. Each Q_i is called a *scatterer* and the set $Q = \mathbb{R}^2 \setminus \bigcup_{j\in S} \overline{Q_j}$ is called the *billiard table*, where $\overline{Q}_j$ means the closure of Q_j. Consider a billiard flow (S_t) on $\overline{Q}$. This flow advances along straight lines on Q with unit speed, and if it knocks against the boundary ∂Q of Q, it rebounds with the condition that the incidence angle and the reflection angle coincide. We introduce the following conditions (A.1)–(A.3):

(A.1) (Dispersing) The boundary ∂Q_j of Q_j is a smooth simple closed curve with positive curvature for each $j \in S$.
(A.2) (No eclipse) For distinct elements $i, j, k \in S$, $\operatorname{conv}(\overline{Q}_i \cup \overline{Q}_j) \cap \overline{Q}_k = \emptyset$, where $\operatorname{conv}(A)$ denotes the convex hull of the set A.
(A.3) The number $\eta := \inf_{i,j\in S:\, i\neq j} \inf_{q\in\partial Q_i} k(q) \operatorname{dist}(Q_i, Q_j)$ is positive, where $k(q)$ is the curvature at $q \in \partial Q$.

If S is finite then (A.3) is automatically satisfied, so our setting is a countable version of the billiard flow without eclipse treated in [4]. In Section 3, we will give many examples satisfying conditions (A.1)–(A.3) under $\#S = \infty$.

The outline of the first main result is as follows:

2020 *Mathematics Subject Classification*: Primary 37C83; Secondary 37C30, 37B10.
Key words and phrases: billiard flow, zeta function, topological Markov shift.
Received 28 July 2023; revised 19 January 2024.
Published online 4 March 2024.

DOI: 10.4064/fm230728-23-1

(1) Assume that conditions (A.1)–(A.3) are satisfied. Then there is a one-to-one correspondence between the non-wandering set of the flow (S_t) on $\overline{Q}$ and a suitable countable Markov shift (see Theorem 2.3).

In addition to (A.1)–(A.3), we consider the following condition:

(A.4) There exists $s_0 > 0$ such that $\sum_{i\in S} \exp(-s_0 \inf_{j\in S:\, j\neq i} \operatorname{dist}(Q_i, Q_j))$ is finite.

Then the outline of the second main result is as follows:

(2) Assume that conditions (A.1)–(A.4) are satisfied. Then the zeta function with respect to the length spectrum of (S_t) on $\overline{Q}$ has radius of convergence equal to the inverse of the maximal simple eigenvalue of the Ruelle operator of a suitable potential. Moreover, the Euler product formula for the zeta function holds under some natural condition (see Theorem 2.8).

We show (1) by developing the technique given in [4]. On the other hand, the various properties of zeta functions for (S_t) in the finite case, $\#S < \infty$, may not extend to the infinite case, $\#S = \infty$. In fact, there may be no solution of the equation $P(-s\xi) = 0$ for $s > 0$, where ξ is a potential with respect to the length spectrum (see (2.5) for definition) and where $P(-s\xi)$ is the topological pressure of $-s\xi$ which is defined by (2.9). We will obtain (2) as a part of the results in the case $\#S < \infty$ under the additional condition (A.4) and by using the thermodynamic formalism for topological Markov shifts. The complete statements and the proofs are given in Section 2. In future work generalizing [6], we shall consider *singular perturbation* from perturbed billiards with infinite scatterers to the unperturbed billiard with finitely many scatterers. In order to study this, we need to investigate precise properties for (S_t) under $\#S = \infty$ and these are given in Section 3.

2. Results and proofs. In this section, we give all the auxiliary propositions and main results with the proofs. We begin with some notation. Let $\pi : \mathbb{R}^2 \times \{z \in \mathbb{R}^2 : |z| = 1\} \to \mathbb{R}^2$ be the natural projection. Denote by $n(q) \in \mathbb{R}^2$ the unit normal at $q \in \partial Q$, directed to the inside of Q. We set

$$M^+ = \{x = (q, v) \in \mathbb{R}^2 \times \mathbb{R}^2 : q \in \partial Q,\ |v| = 1,\ \langle v, n(q)\rangle \geq 0\}$$

and $M = \pi^{-1}Q \cup M^+$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$. Fix a base point $q(j) \in \partial Q_j$ for each $j \in S$. For $x = (q, v) \in \pi^{-1}\partial Q$, $\omega_0(x) = j$ if $\pi(x) = q \in \partial Q_j$, $r(x)$ is the arclength distance from $q(\omega_0(x))$ to q measured counterclockwise along the curve ∂Q_j, and $\varphi(x)$ is the angle between the vector v and $n(q)$ measured counterclockwise from $n(q)$ to v. We may assume that the angle $\varphi(x)$ of $x \in M^+$ satisfies $-\pi/2 \leq \varphi(x) \leq \pi/2$. Also if no confusion can arise, we may write a point $x \in M^+$ as $(\omega_0(x), r(x), \varphi(x))$,

$(r(x), \varphi(x))$ or (r, φ) for short. For $x \in M$ we let

$$t^+(x) = \inf\{t > 0 : S_t x \in M^+\}, \quad t^-(x) = \sup\{t < 0 : S_t x \in M^+\}.$$

Note that these may be $+\infty$ and $-\infty$, respectively. Put

$$\mathcal{D}_1 = \{x \in M^+ : t^+(x) < \infty\}, \quad \mathcal{D}_{-1} = \{x \in M^+ : t^-(x) > -\infty\}.$$

Then we can define the map $T : \mathcal{D}_1 \to M^+$ by $Tx = S_{t^+(x)}x$ and the map $T^{-1} : \mathcal{D}_{-1} \to M^+$ by $T^{-1}x = S_{t^-(x)}x$. Note that T (resp. T^{-1}) is a diffeomorphism from $\operatorname{int}\mathcal{D}_1$ (resp. $\operatorname{int}\mathcal{D}_{-1}$) onto $\operatorname{int}\mathcal{D}_{-1}$ (resp. $\operatorname{int}\mathcal{D}_1$). By induction, we set

$$\mathcal{D}_n = \{x \in \mathcal{D}_{n-1} : t^+(T^{n-1}x) < \infty\},$$
$$\mathcal{D}_{-n} = \{x \in \mathcal{D}_{-n+1} : t^-(T^{-n+1}x) > -\infty\},$$

and define $T_n : \mathcal{D}_n \to M^+$ by $T^n x = T(T^{n-1}x)$ and $T_{-n} : \mathcal{D}_{-n} \to M^+$ by $T^{-n}x = T^{-1}(T^{-n+1}x)$. Clearly, $T^{-n} = (T^n)^{-1}$ if T^n is defined. Consider the non-wandering set

$$\Omega = \{x \in M : \pi(S_t x) \in \partial Q \text{ for both infinitely many } t > 0 \text{ and infinitely many } t < 0\}$$

and $\Omega^+ = \Omega \cap M^+$. Observe the equation $\Omega^+ = \bigcap_{n\in\mathbb{Z}} \mathcal{D}_n$, where $\mathcal{D}_0 := M^+$. Let $A = (A(ij))$ be a zero-one matrix indexed by S with $A(ij) = 1 - \delta_{ij}$. Consider the set

$$\hat{X} = \Big\{\omega = (\omega_n)_{n\in\mathbb{Z}} \in \prod_{n=-\infty}^{\infty} S : A(\omega_n\omega_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\Big\}.$$

We call $\hat{X}$ the *two-sided topological Markov shift* (two-sided TMS for short) with state space S and transition matrix A. Denote by σ the left shift transformation on $\hat{X}$ defined by $(\sigma\omega)_i = \omega_{i+1}$ for all $i \in \mathbb{Z}$. For $0 < \theta < 1$, we define a metric d_θ on $\hat{X}$ by $d_\theta(\omega, \upsilon) = \theta^n$ for $n = \min\{n \geq 0 : \omega_n \neq \upsilon_n$ or $\omega_{-n} \neq \upsilon_{-n}\}$ if $\omega \neq \upsilon$, and $d_\theta(\omega, \upsilon) = 0$ otherwise. For $x \in M^+$, we put $\omega_i(x) = \omega_0(T^i x) \in S$ if T^i is defined. The coding map $\Pi : \Omega^+ \to \hat{X}$ is defined by $x \mapsto (\omega_n(x))_{n=-\infty}^{\infty}$. The value $\Pi(x)$ is called the *itinerary* of $x \in \Omega^+$.

We first check that the map Π is bijective. To do this, we recall previous results in [2, 5]. We put

$$L_{\sup} = \sup_{i\in S}\{\text{the perimeter of } \partial Q_i\}.$$

Note that since $\{Q_i\}$ is uniformly bounded and has positive curvature at all points in the boundary, $L_{\sup}$ is finite. For simplicity, for $x = (q, v) \in M^+$, we write $k_i = k(T^i x)$, $r_i = r(T^i x)$, $\varphi_i = \varphi(T^i x)$, $c_i = c(T^i x)$, $t_i^+ = t^+(T^i x)$, $t_i^- = t^-(T^i x)$ and $c = c(x) = \cos\varphi$.

Proposition 2.1 ([2], [5, Lemma 2.1]). *Let γ be a curve on ∂M_j^+ which is expressed as $\{(j,r,\varphi(r)) : a \le r \le b,\ \varphi=\varphi(r)\}$ and $\varphi(r)$ is of class C^1. Assume that T and T^{-1} are defined on γ. Denote by γ_1 (resp. γ_{-1}) the image $T^1\gamma$ (resp. $T^{-1}\gamma$) and express it by $\{(j_1,r_1,\varphi_1) : a_1 \le r_1 \le b_1,\ \varphi_1=\varphi_1(r_1)\}$ (resp. $\{(j_{-1},r_{-1},\varphi_{-1}) : a_{-1} \le r_{-1} \le b_{-1},\ \varphi_{-1}=\varphi_{-1}(r_{-1})\}$), where φ_1 and φ_{-1} are of class C^1. Then*

$$\frac{dr_1}{dr} = -\frac{c}{c_1}\left(1+\frac{t^+(d\varphi/dr+k)}{c}\right), \tag{2.1}$$

$$\frac{dr_{-1}}{dr} = -\frac{c}{c_{-1}}\left(1+\frac{-t^-(-(d\varphi/dr)+k)}{c}\right), \tag{2.2}$$

$$\frac{d\varphi_1}{d\varphi} = -k_1\frac{c}{c_1}\frac{dr}{d\varphi} - \left(1+\frac{t^+k}{c_1}\right)\left(1+k\frac{dr}{d\varphi}\right), \tag{2.3}$$

$$\frac{d\varphi_{-1}}{d\varphi} = k_{-1}\frac{c}{c_{-1}}\frac{dr}{d\varphi} - \left(1-\frac{t^-k}{c_{-1}}\right)\left(1-k\frac{dr}{d\varphi}\right). \tag{2.4}$$

For $x\in M^+$, we let $\eta_i = \inf_{j\neq\omega_i(x)} \operatorname{dist}(Q_{\omega_i(x)},Q_j)\inf\{k(q) : q\in\partial Q_{\omega_i(x)}\}$.

Proposition 2.2. *Assume that condition* (A.1) *is satisfied. Let $x,y\in M^+$ and $n\ge 1$. Assume that for each $-n\le i\le n$, the map T^i is defined and $\omega_i(x)=\omega_i(y)$. Let $r(x,y)$ be the arclength between $\pi(x)$ and $\pi(y)$ on $\partial Q_{\omega_0(x)}$. Then*

(1) $r(x,y) \le L_{\sup}\eta_0^{-1}\prod_{j=1}^{n-1}(1+\min\{\eta_j,\eta_{-j}\})^{-1}$;
(2) $|\varphi(x)-\varphi(y)| \le \pi\prod_{j=0}^{n-1}(1+\min\{\eta_j,\eta_{-j}\})^{-1}$.

Consequently, if (A.3) *is satisfied, then $r(x,y)\le C_1(1+\eta)^{-n}$ with $C_1 = (1+\eta)L_{\sup}/\eta$ and $|\varphi(x)-\varphi(y)|\le\pi(1+\eta)^{-n}$.*

Proof. (1) We may assume $r(x)<r(y)$. We refer to the technique of the proof of [4, Lemma 2.1]. First we consider the case when $\varphi(x)\le\varphi(y)$. Let $\gamma=\{(r,\varphi(r)) : a\le r\le b,\ \varphi=\varphi(r)\}$ be an increasing C^1-curve from x to y. Here the curve γ is called increasing if $d\varphi(r)/dr\ge 0$. We see that the curve $\gamma_1=T\gamma$ is also increasing and therefore so is $\gamma_i=T^i\gamma$ for all $1\le i\le n$. We write $\gamma_i=(r_i,\varphi_i)=\{(r_i,\varphi(r_i)) : a_i\le r_i\le b_i,\ \varphi_i=\varphi_i(r_i)\}$. By virtue of (2.1) and (2.2), we have

$$\begin{aligned}
\left|\frac{dr_n}{dr}\right| &= \left|\frac{dr_n}{dr_{n-1}}\frac{dr_{n-1}}{dr_{n-2}}\cdots\frac{dr_1}{dr_0}\right| \\
&= \left|(-1)^n\frac{c_{n-1}}{c_n}\frac{c_{n-2}}{c_{n-1}}\cdots\frac{c_0}{c_1}\prod_{i=0}^{n-1}\left(1+\frac{t_i^+(d\varphi_i/dr_i+k_i)}{c_i}\right)\right| \\
&\ge \left|\frac{c_0}{c_n}\prod_{i=0}^{n-1}\left(1+\frac{\eta_i}{c_i}\right)\right| \qquad (\because d\varphi_i/dr_i\ge 0 \text{ and } t_i^+k_i\ge\eta_i)
\end{aligned}$$

$$= \left| \frac{1}{c_n}(c_0 + \eta_0) \prod_{i=1}^{n-1} \left(1 + \frac{\eta_i}{c_i}\right) \right| \geq \eta_0 \prod_{i=1}^{n-1} (1 + \eta_i) \qquad (\because 0 < c_i \leq 1),$$

where $r_0 := r$, $c_0 := c$, $\varphi_0 := \varphi$, $k_0 := k$ and $t_0^+ := t^+$. Thus we obtain

$$r(x,y) = \int_a^b dr = \int_{a_n}^{b_n} \left| \frac{dr}{dr_n} \right| dr_n \leq \Big(\int_{a_n}^{b_n} dr_n \Big) \eta_0^{-1} \prod_{i=1}^{n-1} (1 + \eta_i)^{-1}.$$

Next we prove the assertion in the case $\varphi(x) > \varphi(y)$. We take a decreasing C^1-curve $\gamma = \{(r, \varphi(r))\}$ from x to y, that is, $d\varphi(r)/dr \leq 0$. By using (2.1) and (2.2) for the maps $T^{-1}, \ldots, T^{-n}$, and by noting $-t_i^- k_i \geq \eta_i$, a similar argument implies the same assertion. Hence the proof of (1) is complete.

(2) We may assume $\varphi(x) < \varphi(y)$. First we study the case $r(x) \leq r(y)$. Take an including C^1-curve $\gamma = \{(r(\varphi), \varphi) : r = r(\varphi),\ a \leq \varphi \leq b\}$ from x to y. Here including means $dr/d\varphi \geq 0$. In this case, the image $\gamma_i = T^i\gamma$ is also including C^1-curve for each $1 \leq i \leq n$ and therefore if we write $\gamma_i = \{(r_i(\varphi_i), \varphi_i) : r_i = r_i(\varphi_i),\ a_i \leq \varphi_i \leq b_i\}$, then $dr_i/d\varphi_i \geq 0$. By using (2.3) and (2.4), we find that for each $i = 0, 1, \ldots, n-1$

$$\left| \frac{d\varphi_{i+1}}{d\varphi_i} \right| = k_{i+1} \frac{c_i}{c_{i+1}} \frac{dr_i}{d\varphi_i} + \left(1 + \frac{t_i^+ k_i}{c_{i+1}}\right)\left(1 + k_i \frac{dr_i}{d\varphi_i}\right) \geq 1 + \frac{t_i^+ k_i}{c_{i+1}} \geq 1 + \eta_i$$

since $dr_i/d\varphi_i \geq 0$. Thus we get

$$|\varphi(x) - \varphi(y)| = \int_{\gamma_0} d\varphi_0 = \int_{\gamma_0} \left| \frac{d\varphi_0}{d\varphi_1} \frac{d\varphi_1}{d\varphi_2} \cdots \frac{d\varphi_{n-1}}{d\varphi_n} \right| d\varphi_n$$

$$\leq \Big(\int_{\gamma_n} d\varphi_n \Big) \prod_{i=1}^{n-1} (1 + \eta_i)^{-1}.$$

Consequently, the assertion holds when $\varphi(x) < \varphi(y)$. If $\varphi(x) \geq \varphi(y)$, we obtain the assertion again by a similar argument. ■

Now we can show the following:

THEOREM 2.3. *The coding map Π from Ω^+ to $\hat{X}$ is bijective.*

Proof. For the proof, we mainly refer the proof of [4, Theorem 0]. The inclusion $\Pi(\Omega^+) \subset \hat{X}$ is clear. To show the converse inclusion, let $\omega = (\omega_i)_{i=-\infty}^{\infty} \in \hat{X}$. We consider the following two cases:

CASE I: *ω is periodic, i.e. $\omega = \sigma^k \omega$ for some $k \geq 1$.* In this case, the existence and uniqueness of x with $\Phi(x) = \omega$ are proven in a quite similar way in the finite case $\#S < \infty$ (see [4, proof of Theorem 0]). Note that conditions (A.1) and (A.2) are used in this argument.

CASE II: *ω is not periodic.* Choose any periodic element $\omega^m \in \hat{X}$ such that $d_\theta(\omega, \omega^m) \to 0$ as $m \to \infty$, and the period $p(m)$ of ω^m is strictly increasing in m. By Case I, there exists a unique $x^m \in \Omega^+$ such that $\omega(x^m) = \omega^m$ for each $m \geq 1$. Let $i \in \mathbb{Z}$. Then, there exists $m_0 \geq 1$ such that $p(m_0) > |i| + 1$. For any $m \geq m_0$ and $k \geq 1$, it follows from Proposition 2.2(1) that

$$r(T^i x^m, T^i x^{m+k}) \leq C_1 (1+\eta)^{-p_m + |i|}.$$

Together with Proposition 2.2(2), this fact implies that x^m converges to an element x which satisfies $T^i x = \omega_i$ for all i. Uniqueness is also guaranteed by Proposition 2.2. Hence the theorem is valid. ∎

We say that a function $f : \hat{X} \to \mathbb{R}$ is *semi-weak d_θ-Lipschitz continuous* if there exists a constant $C_2 > 0$ such that for any $\omega, \upsilon \in \hat{X}$ with $\omega_0\omega_1 = \upsilon_0\upsilon_1$, $|f(\omega) - f(\upsilon)| \leq C_2 d_\theta(\omega, \upsilon)$. Consider the function $\hat{\xi} : \hat{X} \to \mathbb{R}$ defined by $\hat{\xi}(\omega) = \tau^+ \circ \Pi^{-1}(\omega)$. Put $\theta = (1+\eta)^{-1}$.

PROPOSITION 2.4. *Assume that conditions* (A.1)–(A.3) *are satisfied. Then the potential $\hat{\xi}$ is semi-weak d_θ-Lipschitz continuous with* $C_2 = \max\{2L_{\sup}\theta^{-2}, C_1(\theta^{-1} + \theta^{-2})\}$.

Proof. Denote $x = \Pi^{-1}(\omega)$ and $y = \Pi^{-1}(\upsilon)$. First assume that $d_\theta(\omega, \upsilon) \geq \theta^2$, in other words, $\omega_{-1} \neq \upsilon_{-1}$, $\omega_{-2} \neq \upsilon_{-2}$ or $\omega_2 \neq \upsilon_2$. Then by basic geometry,

$$\begin{aligned} |\tau^+(x) - \tau^+(y)| &\leq |\pi(x) - \pi(y)| + |\pi(Tx) - \pi(Ty)| \\ &\leq r(x, y) + r(Tx, Ty) \leq 2L_{\sup}\theta^{-2} d_\theta(\omega, \upsilon). \end{aligned}$$

If $d_\theta(\omega, \upsilon) < \theta^2$, then Proposition 2.2(1) says that

$$r(T^i x, T^i y) \leq C_1 \theta^{-1-i} d_\theta(\omega, \upsilon) \quad \text{for } i = 0, 1.$$

Thus we get $|\tau^+(x) - \tau^+(y)| \leq C_1(\theta^{-1} + \theta^{-2}) d_\theta(\omega, \upsilon)$, and the assertion follows. ∎

We recode the two-sided TMS $\hat{X}$ using a one-sided TMS. We define

$$X = \Big\{\omega = (\omega_n)_{n=0}^\infty \in \prod_{n=0}^\infty S : A(\omega_n\omega_{n+1}) = 1 \text{ for all } n \geq 0\Big\}.$$

The set X is called a *one-sided topological Markov shift* (one-sided TMS for short) with state space S and transition matrix A. A word $w = w_1 \cdots w_n \in S^n$ is called *A-admissible* if $A(w_i w_{i+1}) = 1$ for any $1 \leq i < n$. Denote by $[w]$ the *cylinder set* $\{\omega \in X : \omega_0\omega_1 \cdots \omega_{n-1} = w\}$. By the definition of A, the matrix A becomes *finitely primitive*, that is, there exist an integer $N \geq 1$ and a finite subset $F \subset S^N$ such that for any $a, b \in S$ there exists $w \in F$ such that $a \cdot w \cdot b$ is A-admissible. Indeed, we can take $F = \{123, 321, 131, 313\}$. For the metric on X we use the same notation as for the metric on $\hat{X}$. Here for $\omega, \upsilon \in X$, $d_\theta(\omega, \upsilon)$ is defined to be θ^n if $n = \min\{n \geq 0 : \omega_n \neq \upsilon_n\}$. For

$f: X \to \mathbb{C}$ and an integer $k \geq 1$, we define

$$[f]_{k,\theta} := \sup_{w \in S^k} \sup \left\{ \frac{|f(\omega) - f(\upsilon)|}{d_\theta(\omega, \upsilon)} : \omega, \upsilon \in X,\, \omega \neq \upsilon,\, \omega, \upsilon \in [w] \right\}.$$

If $[f]_{1,\theta} < \infty$ then f is *locally d_θ-Lipschitz continuous*, and if $[f]_{2,\theta} < \infty$ then it is *weak d_θ-Lipschitz continuous*. We use the following fact.

PROPOSITION 2.5 ([1, 7]). *Let $\hat{X}$ and X be the two-sided TMS and the one-sided TMS, respectively, with transition matrix A and with countable state space S. Assume that there exists an element $\tau^a \in \hat{X}$ with $\tau^a_0 = a$ for all $a \in S$. For $\omega \in \hat{X}$, we write an element $\hat{\omega} \in \hat{X}$ as $\hat{\omega} = \cdots \tau^{\omega_0}_{-2} \tau^{\omega_0}_{-1} \omega_0 \omega_1 \cdots$. For a semi-weak d_θ-Lipschitz continuous function $\hat{f}: \hat{X} \to \mathbb{C}$, define*

$$V(\hat{f})(\omega) = \sum_{n=0}^{\infty} (\hat{f}(\sigma^n \hat{\omega}) - \hat{f}(\sigma^n \omega))$$

for $\omega \in \hat{X}$. Then $V(\hat{f})$ is a bounded $d_{\sqrt{\theta}}$-Lipschitz continuous function and the function $f := \hat{f} + V(\hat{f}) - V(\hat{f}) \circ \sigma$ does not depend on the past, so it can be regarded as a function on X. Moreover, $[f]_{2,\sqrt{\theta}} < \infty$.

For the function $\hat{\xi} = t^+ \circ \Pi^{-1} : \hat{X} \to \mathbb{R}$, we put

$$\xi := \hat{\xi} + V(\hat{\xi}) - V(\hat{\xi}) \circ \sigma. \tag{2.5}$$

Consider closed orbits of the billiard flow (S_t) on the table $\overline{Q}$. For $f : X \to \mathbb{C}$, we put $S_n f(\omega) := \sum_{k=0}^{n-1} f(\sigma^k \omega)$. The following is an easy consequence of Theorem 2.3.

COROLLARY 2.6. *Assume that conditions* (A.1)–(A.3) *are satisfied. Then for any periodic element $\omega \in X$ with $\sigma^n \omega = \omega$, there exists a unique prime closed orbit γ of (S_t) such that the length $l(\gamma)$ of γ is equal to $S_n \xi(\omega)/m$ with $m = n/p$, where p denotes the least period of ω.*

We will consider dynamical zeta functions for the billiard flow (S_t). Recall that *Ruelle's dynamical zeta function* for a potential $f : X \to \mathbb{R}$ is given by

$$\zeta_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n} Y_n(f) \right), \tag{2.6}$$

where $Y_n(f) = \sum_{\omega \in X :\, \sigma^n \omega = \omega} \exp(S_n f(\omega))$. It is known [8] that if f is weak d_θ-Lipschitz continuous, then the radius of convergence of ζ is equal to $e^{-Q_G(f)}$, where

$$Q_G(f) = \varlimsup_{n \to \infty} \frac{1}{n} \log Y_n(f).$$

Consider the zeta function $\zeta_{-s\xi}(t)$ for some number s. In view of Corollary 2.6, we can rewrite

$$\zeta_{-s\xi}(t) = \exp\Big(\sum_{p=1}^{\infty} \frac{t^p}{p} \sum_{m=1}^{\infty} \sum_{\gamma} \frac{\exp(-msl(\gamma))}{m}\Big), \tag{2.7}$$

where the innermost summation is taken over all prime closed orbits of (S_t) which hit exactly p obstacles.

To state some properties of $\zeta_{-s\xi}$, we introduce two pressures and Ruelle operators. For a function $f : X \to \mathbb{R}$, the *Gurevich pressure* $P_G(f)$ of f and the *topological pressure* $P(f)$ are defined by

(2.8)
$$P_G(f) = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,a}(f) \quad \text{with} \quad Z_{n,a}(f) = \sum_{\omega\in X:\, \sigma^n\omega=\omega, \omega_0=a} \exp(S_n f(\omega))$$

(2.9)
$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(f) \quad \text{with} \quad Z_n(f) = \sum_{w\in S^n:\, [w]\neq\emptyset} \exp\Big(\sup_{\omega\in[w]} S_n f(\omega)\Big),$$

respectively. Since $Z_{n,a}f(\omega) \le Y_n f(\omega) \le Z_n f(\omega)$, we see that $P_G(f) \le Q_G(f) \le P(f)$. Denote by F_θ the Banach space consisting of all bounded locally d_θ-Lipschitz continuous functions $f : X \to \mathbb{C}$ endowed with the Lipschitz norm $\|\cdot\|_\infty + [\cdot]_1$, where $\|f\|_\infty = \sup_{\omega\in X} |f(\omega)|$. For a function $g : X \to \mathbb{R}$, the *Ruelle operator* $\mathcal{L}_g$ of g is a linear operator defined formally by

$$\mathcal{L}_g f(\omega) = \sum_{a\in S:\, A(a\omega_0)=1} e^{g(a\cdot\omega)} f(a\cdot\omega)$$

for $f : X \to \mathbb{C}$ and $\omega \in X$, where $a\cdot\omega$ means $a\omega_0\omega_1\cdots \in X$. It is known that if g is weak Lipschitz continuous with $\|\mathcal{L}_g\|_\infty < \infty$, then $\mathcal{L}_g$ is a bounded linear operator acting on F_θ (see [8, 10]).

If S is finite, then $Q_G(-s\xi) = P_G(-s\xi) = P(-s\xi)$ and these are finite for all $s > 0$. In particular, the radius of convergence of ζ is the inverse of the simple maximal eigenvalue of the Ruelle operator of $-s\xi$. Moreover, when we take a solution $s = H(Q)$ of $P(-s\xi) = 0$ under $\#S < \infty$, the following hold (see [4, 7]):

(i) the function $z \mapsto \zeta_{-z\xi}(1)$ in the half-plane $\operatorname{Re} z > H(Q)$ has the Euler product formula $\zeta_{-z\xi}(1) = \prod_\gamma (1 - e^{-zl(\gamma)})^{-1}$ and defines an analytic function without zeros;

(ii) the function $z \mapsto \zeta_{-z\xi}(1)$ has a meromorphic extension without zeros in some half-plane containing the closed half-plane $\operatorname{Re} z \ge H(Q)$;

(iii) $z = H(Q)$ is a unique pole on the axis $\operatorname{Re} z = H(Q)$ and it is simple;

(iv) $\lim_{u\to\infty} \pi_Q(u)(H(Q)u)/e^{H(Q)u} = 1$, where we put $\pi_Q(u) = \#\{\gamma : \gamma \text{ a prime closed orbit with } l(\gamma) \le u\}$.

On the other hand, if S is infinite, then $Q_G(-s\xi)$ may not be finite, and may be neither $P_G(-s\xi)$ nor $P(-s\xi)$ (see also [8]). This implies that the radius of convergence of $\zeta(t)$ may not be the inverse of an eigenvalue of the Ruelle operator of the potential $-s\xi$. Furthermore, even if these three quantities are identical, there is no solution of the equation $P(-s\xi) = 0$ in general. In [8], an alternative zeta function, called the *local dynamical zeta function*, was introduced as (2.6) with $Y_n(f)$ replaced by $Z_{n,a}(f)$:

$$\zeta_{f,a}(t) = \exp\biggl(\sum_{n=1}^{\infty} \frac{t^n}{n} Z_{n,a}(f)\biggr)$$

for each $a \in S$. Then the radius of convergence of $\zeta_{f,a}$ is independent of a, and is equal to $\exp(-P_G(f))$. If f is recurrent and has finite Gurevich pressure, then $\exp(-P_G(f))$ becomes the inverse of the maximal eigenvalue of the Ruelle operator of f. Moreover, [8] gave a necessary and sufficient condition for positive recurrence (or null recurrence) of f using the notion of the local dynamical zeta function $\zeta_{f,a}$. The zeta function $\zeta_{-s\xi,a}(t)$ has the form

$$\zeta_{-s\xi,a}(t) = \exp\biggl(\sum_{p=1}^{\infty} \frac{t^p}{p} \sum_{m=1}^{\infty} \sum_{\gamma_a} \frac{\exp(-msl(\gamma_a))}{m}\biggr),$$

where γ_a is taken over all prime closed orbits which hit exactly p scatterers with Q_a.

In the remainder of this section, we will show that if the potential $-s_0\xi$ is summable for some $s_0 > 0$, then $Q_G(-s\xi) = P_G(-s\xi) = P(-s\xi)$ and this is finite for all $s \geq s_0$. Here a function $f : X \to \mathbb{R}$ is *summable* if $\sum_{a\in S} \exp(\sup_{\omega\in[a]} f(\omega)) < \infty$ (see [3, 10]). Moreover, if $P(-s\xi) = 0$ for some s, then we will give an Euler product formula for $z \mapsto \zeta_{-z\xi}(1)$.

PROPOSITION 2.7. *Let X be a one-sided topological Markov shift whose shift is topologically mixing. Let $g : X \to \mathbb{R}$ be weak Lipschitz continuous and summable. Then $P_G(g) = Q_G(g) = P(g)$ and these are finite.*

Proof. The finiteness of $P(g)$ is guaranteed since g is summable. It suffices to show that $P(g) = P_G(g)$. By [10, Theorems 3.1(1), 3.4] with $k := 1$ and $M := A$, we see that g is positive recurrent, namely there exists a triple (λ, g, ν) such that λ is a positive simple eigenvalue of the Ruelle operator $\mathcal{L}_g$ of g and equals $\exp(P(g))$, g is a positive continuous function and the corresponding eigenfunction, and ν is a Borel probability measure on X and the corresponding eigenvector of the dual $\mathcal{L}_g^*$ of $\mathcal{L}_g$ with $\nu(g) < \infty$. Thus the generalized Ruelle–Perron–Frobenius theorem [9] also yields $\lambda = \exp(P_G(g))$. Hence the assertion follows. ∎

Then we have the following:

Theorem 2.8. *Assume that conditions* (A.1)–(A.4) *are satisfied. Then for any $s \geq s_0$, the zeta function $t \mapsto \zeta_{-s\xi}(t)$ has radius of convergence $\exp(-P(-s\xi))$ and this is the inverse of the maximal eigenvalue of the Ruelle operator of the potential $-s\xi$. Moreover, if there exists $s_1 \geq s_0$ such that $P(-s_1\xi) = 0$, then the function $z \mapsto \zeta_{-z\xi}(1)$ has the Euler product form*

$$\zeta_{-z\xi}(1) = \prod_{\tau \text{ a prime closed orbit of } (S_t)} (1 - e^{-zl(\tau)})^{-1} \tag{2.10}$$

in the half-plane $\operatorname{Re} z > s_1$.

Proof. By the Cauchy–Hadamard theorem, the radius of convergence of $t \mapsto \zeta_{-s\xi}(t)$ is equal to $(\overline{\lim}_{n\to\infty}(|Y_n(-s\xi)|/n)^{1/n})^{-1} = \exp(-Q_G(-s\xi))$. Moreover, the inequality $-s\xi(\omega) \leq -s_0\xi(\omega) \leq -s_0 \operatorname{dist}(Q_{\omega_0}, Q_{\omega_1})$ and condition (A.4) imply the summability of $-s\xi$. Therefore it follows from Proposition 2.7 that the radius equals $\exp(-P(-s\xi))$. Thus the first assertion is valid.

To check (2.10), we note that for any $z \in \mathbb{C}$ with $\operatorname{Re} z > s_1$, the number $\exp(-P(-\operatorname{Re}(z)\xi))$ is larger than 1. Therefore the series $\zeta_{-s\xi}(t)$ at $t = 1$ is convergent. In view of the absolute convergence of $\zeta_{-s\xi}(1)$, we obtain

$$\begin{aligned}
\zeta_{-z\xi}(1) &= \exp\Big(\sum_{p=1}^{\infty}\frac{1}{p}\sum_{n=1}^{\infty}\sum_{\gamma}\frac{e^{-nzl(\gamma)}}{n}\Big) = \exp\Big(\sum_{p=1}^{\infty}\frac{1}{p}\sum_{\gamma}\sum_{n=1}^{\infty}\frac{e^{-nzl(\gamma)}}{n}\Big)\\
&= \exp\Big(\sum_{p=1}^{\infty}\frac{1}{p}\sum_{\gamma} -\log(1 - e^{-zl(\gamma)})\Big)\\
&= \exp\Big(\sum_{\tau} -\log(1 - e^{-zl(\tau)})\Big) = \prod_{\tau}(1 - e^{-zl(\gamma)})^{-1},
\end{aligned}$$

where the innermost summation in the first expression is taken over all prime closed orbits γ of (S_t) which hit exactly p obstacles, and the last summation is taken over all prime closed orbits τ (cf. [7, p. 100]). Hence all assertions follow. ∎

Remark 2.9. Since Q_i is uniformly bounded for all $i \in S$, condition (A.4) holds if and only if $-s\xi$ is summable.

3. Examples. We will exhibit countably many scatterers satisfying conditions (A.1)–(A.4).

Proposition 3.1. *There is an example satisfying conditions* (A.1)–(A.4) *and $P(-s\xi) = 0$ for some $s > 0$.*

Proof. Put $S = \{1, 2, \ldots\}$. Let $\gamma = \{c(t) \in \mathbb{R}^2 : a \leq t < b\}$ be a strictly convex, simple, smooth, parametrized curve on $\mathbb{R}^2$ with $b \leq \infty$ which may have infinite length. Assume also that the straight line parallel to the normal at $c(a)$ and passing through $c(a)$ does not intersect γ outside $c(a)$. Choose any infinitely many distinct points $c_i = c(t_i)$ on γ $(t_1 < t_2 < \cdots)$. Note that there are no other points on the line segment between any two points. Now consider placing a closed ball centered at each point. By induction as in (a)–(c) below, we determine the radius r_i of each ball $\overline{B(c_i, r_i)}$:

(a) Denote by L_1 the line through two points c_2 and c_3. Then we take $r_1 > 0$ so that $Q_1 := \overline{B(c_1, r_1)}$ does not intersect the line L_1 (see Figure 1(a)).
(b) Denote by L_2^1 the line through c_3 and c_4. Among the tangents to Q_1 passing through c_3 (there are two), let L_2^2 be the one closest to c_2. Then we take $r_2 > 0$ so that $Q_2 := \overline{B(c_2, r_2)}$ does not intersect L_2^1 and L_2^2 (see Figure 1(b)).
(c) For $k \geq 3$, assume that Q_{k-2} and Q_{k-1} are decided. Denote by L_k^1 the line through c_{k+1} and c_{k+2}. Among the tangents to Q_{k-1} passing through c_{k+1}, let L_k^2 be the one closest to c_k. Among the tangents to Q_{k-2} and Q_{k-1} (there are four), let L_k^3 be the one closest to c_k. Then we take $r_k > 0$ so that $Q_k := \overline{B(c_k, r_k)}$ does not intersect L_k^1, L_k^2 and L_k^3 (see Figure 1(c)).

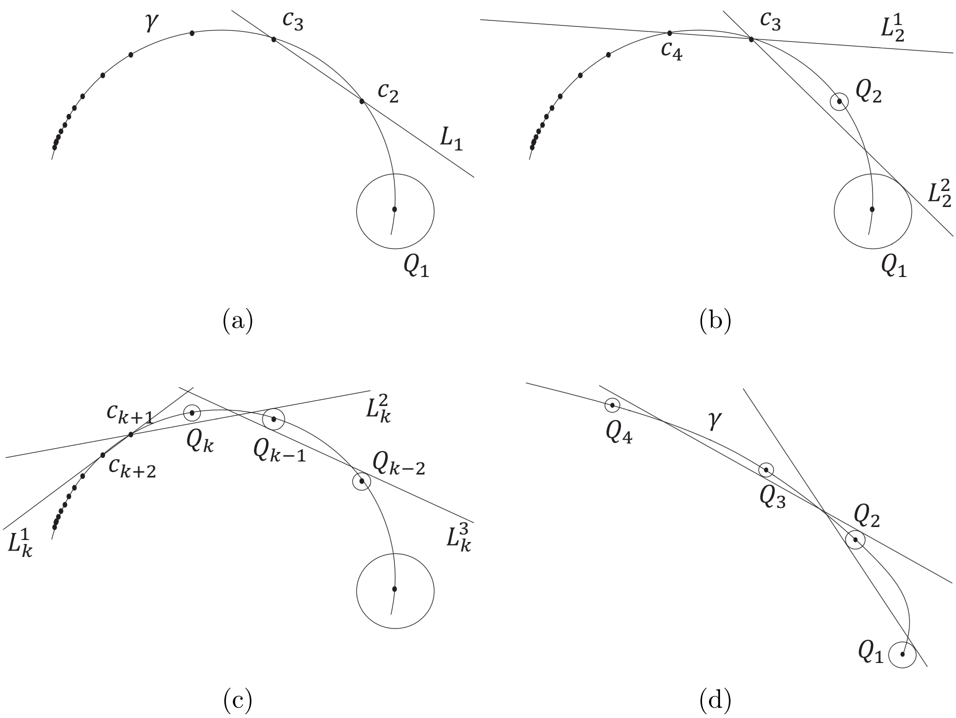

Fig. 1

The sequence $\{Q_i\}$ so constructed satisfies conditions (A.1) and (A.2). If $\inf_{i\neq j} \operatorname{dist}(Q_i, Q_j) > 0$ then condition (A.3) is also satisfied since the scatterers $\{Q_i\}$ are uniformly bounded. In case $\inf_{i\neq j} \operatorname{dist}(Q_i, Q_j) = 0$, by reducing r_i if necessary, $\inf_{j\in S} \inf_{q\in\partial Q_i} k(q) \operatorname{dist}(Q_i, Q_j) \geq C_3$ can be satisfied for some $C_3 > 0$ which is independent of i (so (A.3) is satisfied). Finally, if we take a curve γ with infinite length and points c_i satisfying $\min_{1\leq j<i} \operatorname{dist}(c_i, c_j) \geq i$ for each $i = 1, 2, \dots$, then

$$\sum_{i=1}^{\infty} \exp\Big(-s \inf_{j\in S:\, j\neq i} \operatorname{dist}(Q_i, Q_j)\Big) \leq \sum_{i} \exp(-si) = e^s/(e^s - 1) < \infty$$

for any $s > 0$. In this case, condition (A.4) is fulfilled (see Figure 1(d)). We also see that $P(-s\xi) \leq \log(e^s/(e^s - 1)) < \infty$ for all $s > 0$. By using the facts that $\inf \xi > 0$, $s \mapsto P(-s\xi)$ is strictly decreasing and continuous, $\lim_{s\to+0} P(-s\xi) = +\infty$ and $\lim_{s\to+\infty} P(-s\xi) = -\infty$, there is a unique solution of the equation $P(-s\xi) = 0$ for $s > 0$. ∎

We can easily find an example of $Q_G(-s\xi) = +\infty$ for all $s > 0$: we may take a curve γ with finite length in Proposition 3.1. In this case, $Z_{n,a}(-s\xi) = +\infty$ for all n, a, s.

Acknowledgments. This study was partially supported by JSPS KAKENHI Grant Number 23K03134.

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