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Improved bounds for some S-unit equations

by

Kálmán Győry (Debrecen) and Samuel Le Fourn (Grenoble)

To Professor H. Iwaniec on his 75th birthday

Abstract. The S-unit equation $\alpha x+\beta y=1$ in $x,y\in\mathcal{O}_S^\times$ plays a very important role in Diophantine number theory. We first present the best known effective upper bounds for the solutions of this equation, obtained recently by Le Fourn (2020) and Győry (2019). Then we prove some generalisations for the case of larger multiplicative groups instead of $\mathcal{O}_S^\times$. Further, we provide a new application to monic polynomials with given discriminant. Finally, we considerably improve our general upper bounds in the case of the special S-unit equation $x^n+y=1$ in $x,y\in\mathcal{O}_S^\times$.

1. Introduction. Let K be an algebraic number field and S a finite set of places of K containing all infinite places. Denote by $\mathcal{O}_S$ the ring of S-integers, and by $\mathcal{O}_S^\times$ the group of S-units in K.

Let α,β be nonzero elements of K and consider the S-unit equation

$$\alpha x+\beta y=1 \quad \text{in } x,y\in\mathcal{O}_S^\times. \tag{1.1}$$

Equations of this type (and their homogeneous versions) play a very important role in Diophantine number theory. For various results, applications and references, we refer to the books and survey papers of Győry [13, 14], Shorey and Tijdeman [27], Evertse et al. [9], Sprindžuk [29], Bombieri and Gubler [4], Baker and Wüstholz [2] and Evertse and Győry [7, 8]. For algorithmic and computational aspects, see also Smart [28], von Känel and Matschke [22] and Alvarado et al. [1].

Equation (1.1) has only finitely many solutions. The first explicit upper bounds for the heights of the solutions were given by Győry [11, 12] by means of Baker's theory of logarithmic forms. Later, considerable improvements have been established e.g. by Bugeaud and Győry [6], Bugeaud [5], Győry

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and Yu [20]. In terms of S, their bounds depend on the cardinality s of S, the number t of finite places in S, the largest norm of the prime ideals corresponding to finite places in S, denoted by P_S, and the S-regulator R_S of K. The best known bounds are due to Le Fourn [23] and Győry [18]. To state them, we need further notation.

Let d, r, h_K and R_K be the degree, unit rank, class number and regulator of K, respectively, and let

$$\mathcal{R}_K = \max(h_K, cdR_K)$$

with

$$c = \begin{cases} 0 & \text{if } r = 0, \\ 1/d & \text{if } r = 1, \\ 29er!r\sqrt{r-1}\log d & \text{if } r \geq 2. \end{cases}$$

Denote by P'_S the third largest norm of prime ideals in S (with $P'_S = 1$ if $t \leq 2$). Let $h(\cdot)$ denote the absolute logarithmic height on $\overline{\mathbb{Q}}$ and put

$$H = \max(h(\alpha), h(\beta), 1).$$

We shall also use the notation $\log^* a = \max(\log a, 1)$ for $a > 0$.

Previously, the best bounds for the solutions of (1.1) were established by Győry and Yu [20, Theorems 1 and 2]. The first of these bounds was considerably improved by Le Fourn [23] by replacing P_S with P'_S. Combining [20, proof of Theorem 1] with his new variant of the so-called Runge method, Le Fourn proved the following theorem.

Theorem A (Le Fourn [23, Theorem 1.4]). *Every solution (x, y) of (1.1) satisfies*

$$\max(h(x), h(y)) \leq 2c_1 P'_S \left(1 + \frac{\log^* R_S}{\log^* P'_S}\right) R_S H \tag{1.2}$$

with $c_1 = (16ds)^{2(s+3)}$.

We note that c_1 is a slightly larger and simplified form of the constant in [20, Theorem 1].

The following theorem of Győry [18] significantly improved the second bound in [20]. In terms of S, it provides the best known upper bound for the solutions of (1.1). Its proof is a combination of Le Fourn's variant of Runge's method with a general approximation theorem of Evertse and Győry [7, Theorem 4.2.1], based on the results of Matveev [25] and Yu [31] on logarithmic forms and with some new results of [7] from the geometry of numbers.

Theorem B (Győry [18, Theorem 1]). *Every solution (x, y) of (1.1) satisfies*

$$\max(h(x), h(y)) \leq c_2 \mathcal{R}_K^{t+4} \frac{P'_S}{\log^* P'_S} \left(1 + \frac{\log^* \log P_S}{\log^* P'_S}\right) R_S H \tag{1.3}$$

with $c_2 = s^5(16ed)^{4s+3}$.

Theorem B is proved in [18] with a slightly smaller but more complicated constant c_2.

The most notable fact about the bounds in Theorems A and B is the (sub)linear dependence on P'_S instead of P_S in previous bounds, which makes this particularly interesting when only one or two places of S have large norm. This is due to the preliminary use of Runge's method, which in the case of the S-unit equation allows us to "take out" two places of S before applying the usual estimates for linear forms in logarithms in the complex and p-adic case.

It is interesting to compare the bounds in Theorems A and B. In terms of S, s^s is the dominating factor in the bound (1.2) whenever $t > \log P_S$. The appearance of s^{2s} is due to the use of a fundamental system of S-units such that the product of their heights does not exceed $s^{2s}R_S$; see Hajdu [21] or Bugeaud and Győry [6].

In the bound in Theorem B, no factor s^s occurs. This led to several new applications in Győry [18, 19] and Section 3 below. Further, in (1.3) there is an extra factor $1/\log^* P'_S$ compared to (1.2), and $\log^* \log P_S$ in (1.3) is smaller than $\log^* R_S + 5$ in (1.2).

Finally, we note that in Theorem A, the dependence of the bound on h_K and R_K (via R_S) is better than in Theorem B. For a more detailed comparison of (1.2) and (1.3), see [18].

The organisation of the paper is as follows. In Section 2 we give some generalisation for the case of larger multiplicative groups instead of $\mathcal{O}_S^\times$. In Section 3, some applications from [18, 19] of Theorem B to Thue equations and towards the ABC conjecture over number fields are presented. Further, we give a new application of Theorem B to monic polynomials with given discriminant. Finally, in Section 4, we considerably improve the bound (1.2) and, except in terms of s, (1.3) in the case of the special S-unit equations $x^n + y = 1$ in $x, y \in \mathcal{O}_S^\times$.

2. Generalisations. Let us start with the equation

$$\alpha x + \beta y = 1 \quad \text{in } x, y \in \varGamma, \tag{2.1}$$

where α, β are nonzero elements of K with $H = \max(h(\alpha), h(\beta), 1)$ and $\varGamma$ is a finitely generated multiplicative subgroup of K^*.

We note that in (2.1) and in Theorem C below it is not necessary to assume that $\alpha, \beta \in K^*$. Indeed, if (2.1) has a solution (x, y), for every K-embedding σ of $K_0 = K(\alpha, \beta)$ in $\mathbb{C}$ we have

$$\sigma(\alpha)x + \sigma(\beta)y = 1.$$

Hence, if $\alpha \notin K^*$ or $\beta \notin K^*$, this together with (2.1) gives a much better bound for $h(x)$ and $h(y)$ than (2.4) below.

Let $\{\xi_1,\dots,\xi_m\}$ be a system of generators for Γ up to torsion (not necessarily a basis, which is important for certain applications). Let us fix S such that $\Gamma\subset\mathcal{O}_S^\times$ and then use the same notations as before, with the following additional ones. Let

$$\Theta_\Gamma := \prod_{i=1}^m h(\xi_i), \tag{2.2}$$

and

$$A := 16c_3 s\frac{P'_S}{\log^* P'_S}\Theta_\Gamma \max(\log(c_3 s P'_S), \log^*\Theta_\Gamma) \tag{2.3}$$

with $c_3 = 2(m+1)\log^*(dm)(\log^* d)^2(16ed)^{3m+5}$.

Theorem C (Győry [18, Theorem 2]). *With the above notations, every solution (x,y) of* (2.1) *satisfies*

$$\max(h(x),h(y)) \le AH. \tag{2.4}$$

This theorem can be regarded as a generalisation of Theorem A with a slightly weaker absolute constant. In the special case $\Gamma=\mathcal{O}_S^\times$, Theorem C gives Theorem A. Indeed, choosing in $\mathcal{O}_S^\times$ a system of fundamental units such that $\Theta_\Gamma\le s^{2s}R_S$, we deduce (1.2). We note that Theorem C was proved in a weaker form by Evertse and Győry [7] with P_S in place of P'_S.

To generalise equation (1.1), let us fix a finitely generated subgroup $\mathbf{\Gamma}$ of positive rank of $(\overline{\mathbb{Q}}^*)^2$ endowed with coordinatewise multiplication. We will study the equation

$$a_1x_1 + a_2x_2 = 1, \qquad (x_1,x_2)\in\mathbf{\Gamma}, \tag{2.5}$$

and some of its generalisations, where $(a_1,a_2)\in(\overline{\mathbb{Q}}^*)^2$. For $\underline{x}=(x_1,x_2)\in(\overline{\mathbb{Q}}^*)^2$, we define the height

$$h(\underline{x}) = h(x_1)+h(x_2)$$

and let $\underline{\omega_1}=(\xi_1,\eta_1),\dots,\underline{\omega_r}=(\xi_r,\eta_r)$ be a system of generators of $\mathbf{\Gamma}$ up to torsion. Define K as the smallest number field such that $\mathbf{\Gamma}\subset K^2$, and S as the smallest set of places of K (containing the infinite ones) such that $\mathbf{\Gamma}\subset(\mathcal{O}_S^\times)^2$. We define here

$$h_0 = \max(h(\xi_1),\dots,h(\xi_r),h(\eta_1),\dots,h(\eta_r)), \qquad \Theta_{\mathbf{\Gamma}} := \prod_{j=1}^r h(\underline{\omega_j})$$

and, as above, $d=[K:\mathbb{Q}]$, $s=|S|$ and P'_S is the third greatest norm of prime ideals from S. We also define the division group of $\mathbf{\Gamma}$,

$$\overline{\mathbf{\Gamma}} := \{\underline{x}\in(\overline{\mathbb{Q}}^*)^2 \mid \exists n\in\mathbb{N}_{>0},\ \underline{x}^n\in\mathbf{\Gamma}\},$$

then the cylinder around $\overline{\mathbf{\Gamma}}$,

$$\overline{\mathbf{\Gamma}}_\varepsilon := \{\underline{x}\in(\overline{\mathbb{Q}}^*)^2 \mid \underline{x}=\underline{y}\cdot\underline{z},\ \underline{y}\in\overline{\mathbf{\Gamma}},\ \underline{z}\in(\overline{\mathbb{Q}}^*)^2,\ h(\underline{z})<\varepsilon\},$$

and the truncated cone around $\overline{\mathbf{\Gamma}}$,

$$C(\overline{\mathbf{\Gamma}},\varepsilon) := \{\underline{x} \in (\overline{\mathbb{Q}}^*)^2 \mid \underline{x} = \underline{y}\cdot\underline{z},\ \underline{y}\in\overline{\mathbf{\Gamma}},\ \underline{z}\in(\overline{\mathbb{Q}}^*)^2,\ h(\underline{z}) < \varepsilon(1+h(\underline{y}))\}.$$

We now consider the solutions of equation (2.5) in $\mathbf{\Gamma}, \overline{\mathbf{\Gamma}}, \overline{\mathbf{\Gamma}}_\varepsilon$ and $C(\mathbf{\Gamma},\varepsilon)$, respectively. For solutions from $\overline{\mathbf{\Gamma}}, \overline{\varGamma}_\varepsilon$ or $C(\mathbf{\Gamma},\varepsilon)$ we also need an effective upper bound for the degree of the field generated by the solutions.

The following theorem is a considerable improvement of Theorem 2.1, Corollary 2.4, Theorem 2.3 and Theorem 2.5, respectively, of Bérczes et al. [3].

THEOREM 2.1. *Fix $K_0 = K(a_1,a_2)$ and define $H = \max(h(a_1), h(a_2), 1)$.*

(a) *For every solution (x_1,x_2) of (2.5), we have*

$$h(x_1,x_2) \le \mathbf{A}H, \tag{2.6}$$

where

$$\mathbf{A} = 16c_4 s\frac{P'_S}{\log^* P'_S}\varTheta_{\mathbf{\Gamma}}\max(\log(c_4 sP'_S), \log^*\varTheta_{\mathbf{\Gamma}})$$

with $c_4 = r^2(16ed)^{3r+6}$.

(b) *For the same equation (2.5) with unknown $\underline{x}\in\overline{\mathbf{\Gamma}}$, we have*

$$h(x_1,x_2) \le \mathbf{A}H + 3\mathbf{A}rh_0 \tag{2.7}$$

and $[K_0(x_1,x_2):K_0] \le 2$.

(c) *For (2.5) with unknown $\underline{x}\in\overline{\mathbf{\Gamma}}_\varepsilon$, if $\varepsilon < 0.025$, the same bounds hold as in* (b)*.*

(d) *For (2.5) with unknown $\underline{x}\in C(\overline{\mathbf{\Gamma}},\varepsilon)$, if $\varepsilon < 0.09/(8\mathbf{A}h(a_1,a_2)+20rh_0\mathbf{A})$, we have*

$$h(x_1,x_2) \le 3\mathbf{A}H + 5\mathbf{A}rh_0 \tag{2.8}$$

and $[K_0(x_1,x_2):K_0] \le 2$.

We now prove Theorem 2.1. The first assertion (a) can be deduced from Theorem C. In our proofs we shall use some ideas of the corresponding proofs of Bérczes et al. [3].

Proof of Theorem 2.1. The degree estimates in parts (b)–(d) have been proved in Bérczes et al. [3], hence it will be enough to prove the height estimates.

(a) Suppose that $\xi_1,\dots,\xi_r$ (the first coordinates of the chosen generators of $\mathbf{\Gamma}$) generate a multiplicative subgroup of rank > 0, say $\varGamma$, of $\overline{\mathbb{Q}}^*$. Clearly, $\varGamma$ is contained in K^*. We may assume that $\xi_1,\dots,\xi_{r'}$ with $r'\le r$ are not roots of unity and all the others are, so $\xi_1,\dots,\xi_{r'}$ is a system of generators of $\varGamma/\varGamma_{\mathrm{tors}}$. The assumption on $\underline{\omega_1},\dots,\underline{\omega_r}$ implies that $\eta_{r'+1},\dots,\eta_r$ are not roots of unity. Put

$$\varTheta_\varGamma := h(\xi_1)\cdots h(\xi_{r'}).$$

We now make use of the following lemma.

Lemma 2.2 (Voutier [30]). *Suppose that $\alpha \neq 0$ is an algebraic number of degree d which is not a root of unity. Then*

$$h(\alpha) \geq c_5(d) := \begin{cases} \log 2 & \text{if } d = 1, \\ \frac{2}{d(\log 3d)^3} & \text{if } d \geq 2. \end{cases} \tag{2.9}$$

Using Lemma 2.2, we obtain

$$\Theta_\Gamma \leq c_5(d)^{r'-r}\Theta_{\mathbf{\Gamma}}.$$

Now, let (x_1, x_2) be a solution of (2.5). Then, applying Theorem C to this solution, we obtain (2.4) with m replaced by r'. Using $r' \leq r$ and (2.9), we get (2.6).

(d) Fix a solution $(x_1, x_2) \in C(\overline{\mathbf{\Gamma}}, \varepsilon)$. Then we can write

$$(x_1, x_2) = (y_1, y_2)(z_1, z_2) \tag{2.10}$$

with $(y_1, y_2) \in \overline{\mathbf{\Gamma}}$ and $h(z_1, z_2) < \varepsilon(1+h(y_1, y_2))$. Further, we have $(y_1, y_2) = (y_1', y_2')(\omega_1, \omega_2)$ with $(y_1', y_2') \in \mathbf{\Gamma}$ and

$$(\omega_1, \omega_2) = \prod_{i=1}^{r} (\xi_i, \eta_i)^{c_i} \qquad \text{with } c_i \in \mathbb{Q},\ |c_i| \leq 1/2\ (1 \leq i \leq r)$$

(note that ω_1, ω_2 are defined up to roots of unity). Hence,

$$h(\omega_1, \omega_2) \leq \sum_{i=1}^{r} c_i h(\xi_i, \eta_i) \leq r h_0. \tag{2.11}$$

Put $(a_1', a_2') = (a_1, a_2)(\omega_1, \omega_2)(z_1, z_2)$. Then, by (2.10) and (2.11),

$$h(a_1', a_2') \leq h(a_1, a_2) + r h_0 + \varepsilon(1 + h(y_1, y_2)),$$

which yields

$$h(a_1', a_2') \leq h(a_1, a_2) + r h_0 + \varepsilon(1 + h(y_1', y_2') + r h_0). \tag{2.12}$$

Further, our equation (2.5) can be written in the form

$$a_1' y_1' + a_2' y_2' = 1 \quad \text{in } (y_1', y_2') \in \mathbf{\Gamma}. \tag{2.13}$$

Applying now part (a) to this equation, we get

$$h(y_1', y_2') \leq \mathbf{A} \max(h(a_1', a_2'), 1),$$

where $\mathbf{A}$ is the constant defined in (a). Notice that this constant does not depend on the field generated by a_1', a_2'. Further, using (2.12), we get

$$h(y_1', y_2') \leq \mathbf{A} h(a_1, a_2) + r h_0 \mathbf{A} + \varepsilon \mathbf{A} + \varepsilon \mathbf{A} h(y_1', y_2') + r h_0 \varepsilon \mathbf{A}.$$

Our assumption in (d) implies that $\varepsilon < 1/(2\mathbf{A})$ (the stronger inequality on ε we assume is necessary to have $[K_0(x_1, x_2) : K_0] \leq 2$), so it follows that

$$h(y_1', y_2') \leq 2\mathbf{A} h(a_1, a_2) + (1 + 2\mathbf{A} r h_0 + r h_0). \tag{2.14}$$

Now we have

$$h(y_1,y_2) \le h(y_1',y_2') + h(\omega_1,\omega_2) \le 2\mathbf{A}h(a_1,a_2) + (1+2\mathbf{A}rh_0 + 2rh_0),$$
$$h(x_1,x_2) \le h(y_1,y_2) + \varepsilon(1+h(y_1,y_2)) \le (1+\varepsilon)h(y_1,y_2) + \varepsilon,$$

so we get (2.8). Finally, as in [3, proof of Theorem 2.5], under the stated assumption on ε, we do have $[K_0(x_1,x_2):K_0] \le 2$.

(c) The proof is completely similar to that of (d). The only difference is that the estimate $h(z_1,z_2) < \varepsilon(1+h(y_1,y_2))$ has to be replaced by $h(z_1,z_2) < \varepsilon$. This slightly modifies the estimates in the proof of (d) and instead of (2.14) we get

$$h(y_1',y_2') \le \mathbf{A}h(a_1,a_2) + \mathbf{A}(\varepsilon + rh_0).$$

This leads to $h(x_1,x_2) \le \mathbf{A}h(a_1,a_2) + 3\mathbf{A}rh_0$, which proves (2.7).

(b) This part is an immediate consequence of (c). ∎

3. Some applications. As was mentioned in the Introduction, the effective finiteness theorems concerning S-unit equations have many applications. The first named author, and later others as well, applied their results on equation (1.1) systematically to get effective finiteness theorems in quantitative form, among others on decomposable form equations, including Thue equations, discriminant form and index form equations, and on power integral bases in number fields, arithmetic graphs and irreducible polynomials.

In this section we present some applications of Theorem B to Thue equations, to the ABC conjecture over number fields and to monic polynomials with given degree and given discriminant. We note that Theorem A has similar applications, but slightly weaker bounds in terms of S.

3.1. Application to Thue equations. As above, let K be a number field of degree d with class number h_K and regulator R_K, S a finite set of places on K containing the archimedean ones, $\mathcal{O}_S$ the ring of S-integers, $s = |S|$, t the number of finite places in S, P_S (resp. P_S') the largest (resp. the third largest) norm of prime ideals in S (with $P_S' = 1$ if $t \le 2$), Q_S the product of the norms of the prime ideals in S (equal to 1 if $t = 0$) and R_S the S-regulator. Consider the Thue equation

$$F(x,y) = \delta \quad \text{in } x,y \in \mathcal{O}_S, \tag{3.1}$$

where $F \in \mathcal{O}_S[X,Y]$ is a binary form of degree $n \ge 3$ which factorises into linear factors over K and at least three of these factors are pairwise nonproportional. Further, let $\delta \in \mathcal{O}_S \setminus \{0\}$ and H be an upper bound for the heights of the coefficients of F.

In various generalities, there are many results providing effective upper bounds for the heights of the solutions (x,y) of equation (3.1). In terms of

S with $t > 0$, the following theorem provides the best known bound for the solutions.

Theorem D (Győry [18, Corollary 4]). *Let $t > 0$. Under the above assumptions and notation, all solutions (x, y) of (3.1) satisfy*

$$\max(h(x), h(y)) < c_6^s \frac{P'_S}{\log^* P'_S}\left(1 + \frac{\log^* \log P_S}{\log^* P'_S}\right)(\log Q_S) R_S,$$

where c_6 is an effectively computable positive number depending only on d, h_K, R_K, n, $h(\delta)$ and H.

Theorem D improves several earliers bounds in terms of S, including Corollary 3 of Győry and Yu [20]. In fact, Theorem D is a special case (for $m = 2$) of a more general theorem on decomposable form equations in $m \geq 2$ unknowns which was proved by Győry [18] by means of Theorem B above.

3.2. Application towards Masser's ABC conjecture in number fields. Keeping the above notation, let again K be an algebraic number field of degree d. For $v \in M_K$, we choose an absolute value $|\cdot|_v$ normalised in the following way: if v is infinite given by an embedding $\sigma : K \to \mathbb{C}$, then we put $|\alpha|_v = |\sigma(\alpha)|^{n_v}$ for $\alpha \in K$, where $n_v = 1$ if v is real and $n_v = 2$ otherwise. If v is finite given by a prime ideal $\mathfrak{p}$, and $\operatorname{ord}_{\mathfrak{p}} \alpha$ denotes the exponent of $\mathfrak{p}$ in the prime ideal decomposition of the fractional ideal (α), then we put $|\alpha|_v = N(\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}} \alpha}$ for $\alpha \in K^*$ and $|0|_v = 0$.

The height of $(a, b, c) \in (K^*)^3$ is defined as

$$H_K(a, b, c) = \prod_{v \in M_K} \max(|a|_v, |b|_v, |c|_v)$$

and the radical of (a, b, c) as

$$N_K(a, b, c) = \prod_{\mathfrak{p}} N(\mathfrak{p})^{\operatorname{ord}_{\mathfrak{p}} p}, \tag{3.2}$$

where p is a rational prime such that $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$ and the product is over all prime ideals $\mathfrak{p}$ such that $|a|_v, |b|_v, |c|_v$ are not all equal for the associated place v.

There have been several proposals for generalising the classical ABC conjecture over $\mathbb{Q}$ to algebraic number fields. The following version is due to Masser [24].

Uniform ABC conjecture over number fields. *For every $\varepsilon > 0$, there exists $C(\varepsilon)$ depending only on ε such that for every number field K with degree d and absolute discriminant Δ_K,*

$$H_K(a, b, c) < C(\varepsilon)^d (\Delta_K N_K(a, b, c))^{1+\varepsilon} \tag{3.3}$$

for all $a, b, c \in K^*$ which satisfy $a + b = c$.

The upper bound in (3.3) is best possible in terms of ε. Further, (3.3) is uniform in the sense that it has good behaviour under field extensions. For $K = \mathbb{Q}$, this general conjecture reduces to the classical ABC conjecture.

Of particular importance is the effective version of the conjecture when (3.3) holds for every $\varepsilon > 0$ with an effectively computable $C(\varepsilon)$.

The effective results concerning S-unit equations can be used to obtain weaker, but unconditional and effective bounds on $H_K(a,b,c)$. Indeed, let $a, b, c \in K^*$ with $a + b + c = 0$, and let S be the subset of M_K made up of all the infinite places and all the finite places v for which $|a|_v, |b|_v, |c|_v$ are not equal. Then $x = -a/c$, $y = -b/c$ give a solution of the S-unit equation

$$x + y = 1 \quad \text{ with } x, y \in \mathcal{O}_S^\times.$$

Applying Theorem B to this equation, Győry [19] proved the following.

Theorem E (Győry [19, Theorem 1]). *Let $a, b, c \in K^*$ with $a+b+c = 0$. Then*

$$\log H_K(a,b,c) < c_7 N^{1/3 + c_8 \log_3 N^* / \log_2 N^*}, \tag{3.4}$$

where $N = N_K(a,b,c)$, $N^* = \max(N, 16)$ and c_7, c_8 are effectively computable positive constants depending only on d and Δ_K.

In the proof, it is crucial that in the bound in Theorem B no factor s^s occurs. The proof gives in fact the better bound $c_9 P' N^\varepsilon$ for $\log H_K(a,b,c)$, where $\varepsilon > 0$ is arbitrary, P' is the third greatest factor $N(\mathfrak{p})$ in (3.2) and c_9 is an effectively computable positive constant which depends only on d, Δ_K and ε.

Theorem E is an improvement of [17, Theorem 1]. We note that in [26], Scoones also deduced Theorem E from Theorem B in a different way and in a somewhat weaker form.

3.3. Application to monic polynomials of given degree and given discriminant. The results in Theorems A and B also lead to improvements on effective and quantitative results on monic polynomials with given discriminant over the S-integers.

Two monic polynomials $f, g \in \mathcal{O}_S[X]$ are called *$\mathcal{O}_S$-equivalent* when there exists $a \in \mathcal{O}_S$ such that $g(X) = f(X + a)$. Then they have the same discriminant.

In various generalities, there are several effective finiteness theorems in quantitative form on monic polynomials with given degree and given discriminant; see e.g. Győry [10], Evertse and Győry [8] and the references given there. We deduce from (1.3) a result of this type with a considerably improved bound in terms of S.

The *logarithmic height* of a polynomial $f = a_0 + \cdots + a_n X^n \in \overline{\mathbb{Q}}[X]$ is defined as

$$h(f) = \frac{1}{[L:\mathbb{Q}]} \sum_{v \in M_L} \log \max(1, |a_0|_v, \ldots, |a_n|_v)$$

(with absolute values normalised as in the previous subsection), where L is any number field containing the coefficients of f.

For the statement of the theorem, define $Q_S = \prod_{\mathfrak{p} \in S} N(\mathfrak{p})$ as above and for any $\alpha \in K^*$, $N_S(\alpha) = \prod_{v \in S} |\alpha|_v$.

THEOREM 3.1. *If $f \in \mathcal{O}_S[X]$ is monic with degree $m \geq 3$, nonzero discriminant $\delta \in \mathcal{O}_S$ and all roots in $\mathcal{O}_S$, then f is $\mathcal{O}_S$-equivalent to a polynomial f^* such that*

(3.5)
$$h(f^*) \leq 4mc_{10}^s \mathcal{R}_K^{t+5} \frac{P'_S}{\log^* P'_S} \left(1 + \frac{\log^* \log P_S}{\log^* P'_S}\right) R_S \log(Q_S N_S(\delta)) + \frac{h(\delta)}{m-1},$$

where c_{10} is an effectively computable constant which depends only on d.

Under the assumptions in Theorem 3.1, (3.5) considerably improves the bound in [15, Theorem 4]. The main improvement is the replacement of P_S by P'_S, which can be much smaller than P_S. To prove our theorem, let us first recall two lemmas.

LEMMA 3.2. *For every $\alpha \in \mathcal{O}_S \setminus \{0\}$, there exists $\varepsilon \in \mathcal{O}_S^\times$ such that*

$$h(\varepsilon\alpha) \leq \frac{1}{d} \log N_S(\alpha) + \mathcal{R}_K \log Q_S.$$

Proof. This is obtained in greater generality in [18, Lemma 3]. ∎

LEMMA 3.3 (Bombieri and Gubler [4, Theorem 1.6.13]). *Let $\alpha_1, \ldots, \alpha_m \in \overline{\mathbb{Q}}$ and $f = (X - \alpha_1) \cdots (X - \alpha_m)$. Then*

$$\left| h(f) - \sum_{i=1}^m h(\alpha_i) \right| \leq m \log 2.$$

Proof of Theorem 3.1. Let $f \in \mathcal{O}_S[X]$ be a monic polynomial with degree $m \geq 3$, nonzero discriminant $\delta \in \mathcal{O}_S$ and roots $\alpha_1, \ldots, \alpha_m$ all in $\mathcal{O}_S$. As the S-norm is multiplicative,

$$\prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^2 = \delta \tag{3.6}$$

implies that the S-norm of $\alpha_i - \alpha_j$ $(i \neq j)$ is always at most $N_S(\delta)^{1/2}$. Applying this with Lemma 3.2 multiple times, for each $i \in \{3, \ldots, m\}$ we get $\varepsilon_{1i}, \varepsilon_{i2}, \varepsilon_{12} \in \mathcal{O}_S^\times$ and $\gamma_{1i}, \gamma_{i2}, \gamma_{12} \in \mathcal{O}_S$ such that

$$\alpha_1 - \alpha_i = \varepsilon_{1i}\gamma_{1i}, \quad \alpha_i - \alpha_2 = \varepsilon_{i2}\gamma_{i2}, \quad \alpha_1 - \alpha_2 = \varepsilon_{12}\gamma_{12}$$

and the heights of the gamma numbers do not exceed

$$\rho_1 := \mathcal{R}_K \log(Q_S N_S(\delta)).$$

Each trivial identity

$$\frac{\alpha_1 - \alpha_i}{\alpha_1 - \alpha_2} + \frac{\alpha_i - \alpha_2}{\alpha_1 - \alpha_2} = 1$$

then gives an S-unit equation

$$\frac{\varepsilon_{1i}}{\varepsilon_{12}} \cdot \frac{\gamma_{1i}}{\gamma_{12}} + \frac{\varepsilon_{i2}}{\varepsilon_{12}} \cdot \frac{\gamma_{i2}}{\gamma_{12}} = 1$$

with coefficients the gamma ratios, which have heights at most $2\rho_1$. Applying now Theorem B to that equation, we obtain the upper bound

$$\rho_2 := 2c_2 \mathcal{R}_K^{t+5} \frac{P'_S}{\log^* P'_S}\left(1 + \frac{\log^* \log P_S}{\log^* P'_S}\right) R_S \log(Q_S N_S(\delta))$$

for the heights of $\varepsilon_{1i}/\varepsilon_{12}$ and $\varepsilon_{i2}/\varepsilon_{12}$. As a consequence, the heights of all $(\alpha_1 - \alpha_i)/(\alpha_1 - \alpha_2)$ and hence of all $(\alpha_i - \alpha_j)/(\alpha_1 - \alpha_2)$ are at most $3\rho_2$. It now follows from (3.6) that

$$h(\alpha_1 - \alpha_2) \le \frac{h(\delta)}{m(m-1)} + 3\rho_2 = \rho_3.$$

By the symmetric role of the roots, the same bound holds for all $h(\alpha_i - \alpha_j)$. Finally, consider $f^*(X) = f(X + \alpha_1)$, which is $\mathcal{O}_S$-equivalent to f and has zeroes $\alpha_i - \alpha_1$ for $i \in \{1, \dots, m\}$. By Lemma 3.3, we then have $h(f^*) \le m\rho_3 + m\log 2$. ∎

4. Further improvements of the bounds for special S-unit equations. In this section, we focus on an S-unit equation with power, namely

$$x^n + y = 1 \quad \text{in } x, y \in \mathcal{O}_S^\times, \tag{4.1}$$

where $n \ge 2$ is a fixed integer. We denote by $\tau(n)$ the divisor counting function.

First of all, let us state the main result of this section.

Theorem 4.1. *For any solution (x, y) of (4.1), we have*

$$\max(nh(x), h(y)) \le nc_{11} \frac{P_S^{(\tau(n)+2)}}{\log^* P_S^{(\tau(n)+2)}} \log\left(\frac{2nc_{11}(P_S^{(\tau(n)+2)})^2}{\log^* P_S^{(\tau(n)+2)}}\right),$$

where for each $k \ge 1$, $P_S^{(k)}$ is the kth largest norm amongst finite places of S, with $P_S^{(k)} = 1$ if S has at most $k-1$ finite places, and

$$c_{11} = 2^{s-1} s((s-2)!)^2 d^{s-2} \log^*(ds)(16ed)^{3s+2}.$$

Remark 4.2. Here we adjust the bound on the height of x^n because x^n and not x is close to y in height. Furthermore, in the case $K = \mathbb{Q}$, it follows from [16, Theorem] that $n \le \max(30, P_S + 1)$ when there is a solution of (4.1).

The section is devoted to the proof of Theorem 4.1, which follows the general strategy of Le Fourn [23, Section 5].

Let us define $U_n \subset \mathbb{P}^1_{\mathbb{Q}}$ to be the set of nth roots of unity, and denote by $\overline{U_n}$ its Zariski closure in $\mathbb{P}^1_{\mathbb{Z}}$. For every maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$, $x^n - 1 \in \mathfrak{p}$ if and only if $x \mod \mathfrak{p} \in \overline{U_n}$, so the solutions of (4.1) correspond to integral points of $(\mathbb{P}^1 \setminus (\{0,\infty\} \cup U_n))(\mathcal{O}_S)$ (i.e. by definition, points in $\mathbb{P}^1(K)$ whose reduction modulo any prime $\mathfrak{p}$ not in S is not 0, 1 or an nth root of unity).

We will use the factorisation of $X^n - 1$ in $\mathbb{Q}[X]$ to obtain better bounds in the case of (4.1) than for (1.1). Indeed, we can write $U_n = \bigsqcup_{d|n} D_d$, where D_d is the set of primitive dth roots of unity in $\overline{\mathbb{Q}}$ (in other words, the zeroes of the dth cyclotomic polynomial Φ_d).

As we will refer to these sets repeatedly, we define $I = \{0,\infty\} \cup \{d \geq 1 \mid d \,|\, n\}$, which is of cardinality $\tau(n) + 2$, and for each $i \in I$ the associated set D_i (where $D_0 = \{0\}$ and $D_\infty = \{\infty\}$). For each place $v \in M_K$, we define the v-adic heights associated to each of the sets D_i as follows:

$$\begin{aligned} \lambda_{v,0}(x) &:= \log^+(1/|x|_v), \\ \lambda_{v,\infty}(x) &:= \log^+ |x|_v, \\ \lambda_{v,d}(x) &:= \frac{\log^+(1/|\Phi_d(x)|_v)}{\varphi(d)} \qquad (d \,|\, n) \end{aligned}$$

with φ the Euler totient function, and the associated heights

$$h_i(x) := \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \lambda_{v,i}(x) \qquad (i = 0, \infty \text{ or } d \,|\, n).$$

The following lemma reflects the fact that the sets D_0, D_∞ and D_d $(d \,|\, n)$ are pairwise disjoint.

Lemma 4.3. *For any distinct $i, j \in I$, any $v \in M_K$ and any $x \in K$,*

$$\min(\lambda_{v,i}(x), \lambda_{v,j}(x)) \leq \log(2n).$$

Proof. Assume first that v is nonarchimedean. As Φ_d is monic, for any $d \,|\, n$ we have

$$\min(\lambda_{v,0}(x), \lambda_{v,\infty}(x)) = \min(\lambda_{v,0}(x), \lambda_{v,d}(x)) = \min(\lambda_{v,\infty}(x), \lambda_{v,d}(x)) = 0$$

(one can distinguish between the cases $|x|_v < 1$, $|x|_v > 1$ and $|x|_v = 1$). For other cases, we recall below an easily proved lemma on cyclotomic polynomials.

Lemma 4.4. *For any prime p and any $d \in \mathbb{Z}_{\geq 1}$:*

(a) *If $p \nmid d$, then $\overline{\Phi_d}$ mod p is squarefree and its roots are the primitive dth roots of unity in $\overline{\mathbb{F}_p}$.*
(b) *If $d = p^\alpha d_0$ with $\alpha \geq 1$, then $\overline{\Phi_d} = \overline{\Phi_{d_0}}(X)^{\varphi(p^\alpha)}$.*

We can now prove the theorem for v nonarchimedean associated to a prime ideal $\mathfrak{p}$ and two distinct divisors d, d' of n. First, we can assume x to be v-integral by the previous bounds for ∞ and d.

If $\lambda_{v,d}(x) > 0$, this means $\Phi_d(x) = 0$ in the residue field $\mathbb{F}_v$ so $\overline{x} \in \mathbb{F}_{\mathfrak{p}}$ is a primitive d_0th root of unity in $\overline{\mathbb{F}_p}$ where $d = p^\alpha d_0$ with $v \mid p$ and $p \nmid d_0$.

Therefore, if $\min(\lambda_{v,d}(x), \lambda_{v,d'}(x)) > 0$ (which we assume now), one must have $d' = p^\beta d_0$ for some β. Assume by symmetry that $\beta < \alpha$, and fix $x' = x^{p^{\alpha-1}}$. Then

$$\Phi_d(x) = \frac{\Phi_{d_0}((x')^p)}{\Phi_{d_0}(x')} \quad \text{and} \quad |\Phi_{d'}(x)|_v \geq |(x')^{d_0} - 1|_v.$$

By assumption, we have $|(x')^{d_0} - 1|_v < 1$ so fixing a place $w \mid v$ of $K(\zeta_n)$, there exists a unique d_0th root of unity ζ_{d_0} such that $|x' - \zeta_{d_0}|_w < 1$. Now, for any d_0pth primitive root of unity ζ_{d_0p}, we have

$$|\zeta_{d_0} - \zeta_{d_0p}|_w = \left|\frac{\zeta_{d_0p}}{\zeta_{d_0}} - 1\right|_w \geq |\zeta_{d_0p}^{d_0} - 1|_w \geq \frac{1}{p}.$$

Consequently, if $|x' - \zeta_{d_0}|_w < 1/p$, we have $|x' - \zeta_{d_0p}|_w \geq 1/p$ for all primitive d_0pth roots of unity (and this is less than 1 only for the $p-1$ primitive d_0pth roots of unity reducing modulo w to ζ_{d_0}). Therefore, $|\Phi_{d_0}((x')^p)/\Phi_{d_0}(x')|_w \geq 1/p^{p-1}$ for all $w \mid v$, so finally

$$\min(\lambda_{v,d}(x), \lambda_{v,d'}(x)) \leq \frac{\log p}{\varphi(d_0)} \leq \log(2n).$$

Now, for the infinite places, the following lemma can be easily obtained.

Lemma 4.5. *For any distinct $d, d' \in \mathbb{Z}_{\geq 1}$ and any primitive dth (resp. d'th) root of unity ζ_d (resp. $\zeta_{d'}$),*

$$|\zeta_d - \zeta_{d'}| \geq \frac{1}{\operatorname{lcm}(d, d')}$$

because $\zeta_d^{-1}\zeta_{d'}$ is a $\operatorname{lcm}(d, d')$*th root of unity different from* 1.

Thus, for any divisors $d \neq d'$ of n and any $x \in \mathbb{C}$, if $|x - \zeta_d| \leq \frac{1}{2\operatorname{lcm}(d,d')}$ for some primitive root ζ_d then $|x - \zeta_{d'}| \geq \frac{1}{2\operatorname{lcm}(d,d')}$ for all primitive d'th roots of unity by Lemma 4.5, and we obtain $\log|\Phi_{d'}(x)| \geq -\varphi(d')\log(2\operatorname{lcm}(d, d'))$. By symmetry (and considering the other cases), we finally obtain

$$\min(\lambda_{v,d}(x), \lambda_{v,d'}(x)) \leq \log(2\operatorname{lcm}(d, d')) \leq \log(2n). \blacksquare$$

Lemma 4.6. *For every $i \in I$ and every $x \in K$,*

$$h(x) \leq h_i(x) + \log 2.$$

Proof. The assertion is obvious for $i = 0$ and $i = \infty$ by definition of the Weil height ($h(x) = h_i(x)$); let us prove it for $i = d$ dividing n.

For any nonarchimedean place v of K and any $x \in K$,

$$\max(1, |x|_v)^{\varphi(d)} \geq \max(1, |\Phi_d(x)|_v), \quad \text{so} \quad \lambda_{v,0}(x) \leq \lambda_{v,d}(x)$$

(with equality if x is not v-integral) as Φ_d is monic. Now, for v archimedean,

$$|\Phi_d(x)|_v = \prod_\zeta |x - \zeta| \geq \left(\frac{|x|}{2}\right)^{\varphi(d)}$$

if $|x| \geq 2$, so $\log^+ |\Phi_d(x)|_v \geq \varphi(d)(\log^+ |x|_v - \log 2)$ for all values of x, treating the case $|x| \leq 2$ separately. Combining this with the nonarchimedean bounds, we obtain the result. ∎

Proof of Theorem 4.1. Let (x, y) be a solution of (4.1). Let S' be a subset of S such that $|S \setminus S'| < \tau(n) + 2$. For each $v \in S \setminus S'$, there is at most one set D_i, $i \in I$, that is v-close to x, so eliminating them one by one, by the pigeonhole principle there remains D_i such that for all $v \in S \setminus S'$, $\lambda_{v,i}(x) \leq \log(2n)$; we fix that D_i from now on.

Now, by hypothesis, $\lambda_{v',i}(x) = 0$ for all $v' \in M_K \setminus S$ so there is $v \in S$ (which we fix from now on) such that

$$\lambda_{v,i}(x) \geq \frac{1}{|S|} h_i(x).$$

If such a v belongs to $S \setminus S'$, we obtain directly a much smaller bound for $h_i(x)$, hence for $h(x)$ by Lemma 4.6, smaller than the one stated in the theorem. We can thus assume $v \in S'$.

As explained in Le Fourn [23, Section 3], at this stage we need for each $i \in I$ a rational function ψ_i mapping D_i to 1 and sending our x to an S-unit v-adically close to 1. Such conditions are satisfied as follows (other choices would be possible, but for these it is obvious that their images of a solution are S-units):

$$\psi_0(x) := 1 - x^n = y, \qquad \psi_\infty(x) := 1 - \frac{1}{x^n} = -\frac{y}{x^n}, \qquad \psi_d(x) = x^n.$$

It is also clear that for each $j \in I$, $h(\psi_j(x)) \leq 2 \max(nh(x), h(y))$.

Now, assume for example $i = d$ for some $d \mid n$. By Evertse and Győry [7, Theorem 4.2.1] we then have

$$\begin{aligned} \frac{h_d(x)}{s} \leq \lambda_{v,d}(x) &= \frac{\log^+(1/|\Phi_d(x)|_v)}{\varphi(d)} \\ &\leq \log^+(1/|x^n - 1|_v) \leq C \frac{N_v}{\log N_v} \Theta \log^*(N_v n h(x)) \end{aligned}$$

with $\xi_1, \ldots, \xi_m$ generating $\mathcal{O}_S^\times$, Θ associated to $\xi_1, \ldots, \xi_m$ as in (2.2) (we can fix $m = s - 1$), $N_v = N(\mathfrak{p})$ if v is associated to a prime ideal $\mathfrak{p}$ (and $N(\mathfrak{p}) = 1$ otherwise) and $C = 2(m+1) \log^*(dm)(16ed)^{3m+5}$ (the constant

c_{11} in the cited book). We thus end up (generously bounding $h(x)$ from above by $2h_d(x)$ and using [20, Lemma 2]) with

$$h(x) \le 2sCC' \frac{N_v}{\log N_v} \log^*(N_v n h(x))$$

where $C' = ((m-1)!)^2(2^{m-2}d^{m-1})$. This being an iterated logarithm, we finally obtain

$$h(x) \le 4sCC' \frac{N_v}{\log N_v} \log\left(\frac{2nsCC'N_v^2}{\log N_v}\right).$$

If x is close to 0 or ∞, we apply the same ideas with $\psi_i(x)$ and obtain a smaller bound.

Finally, to obtain the theorem, we choose S' at the beginning so that $S \setminus S'$ contains the $\tau(n)+1$ largest $N_{v'}$, $v' \in S$, and then for $v \in S'$, $N_v \le P_S^{(\tau(n)+2)}$, and we have $c_{11} = 2sCC'$. ∎

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References

[1] A. Alvarado, A. Koutsianas, B. Malmskog, C. Rasmussen, C. Vincent, and M. West, *A robust implementation for solving the S-unit equation and several applications*, in: Arithmetic Geometry, Number Theory, and Computation, Springer, 2021, 1–41.

[2] A. Baker and G. Wüstholz, *Logarithmic Forms and Diophantine Geometry*, New Math. Monogr. 9, Cambridge Univ. Press, 2007.

[3] A. Bérczes, J.-H. Evertse, and K. Győry, *Effective results for linear equations in two unknowns from a multiplicative division group*, Acta Arith. 136 (2009), 331–349.

[4] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry*, Cambridge Univ. Press, 2006.

[5] Y. Bugeaud, *Bornes effectives pour les solutions des équations en S-unités et des équations de Thue–Mahler*, J. Number Theory 71 (1998), 227–244.

[6] Y. Bugeaud and K. Győry, *Bounds for the solutions of unit equations*, Acta Arith. 74 (1996), 67–80.

[7] J.-H. Evertse and K. Győry, *Unit Equations in Diophantine Number Theory*, Cambridge Stud. Adv. Math. 146, Cambridge Univ. Press, 2015.

[8] J.-H. Evertse and K. Györy, *Discriminant Equations in Diophantine Number Theory*, New Math. Monogr. 32, Cambridge Univ. Press, 2017.

[9] J.-H. Evertse, K. Györy, C. L. Stewart, and R. Tijdeman, *S-unit equations and their applications*, in: New Advances in Transcendence Theory, Cambridge Univ. Press, 1988, 110–174.

[10] K. Győry, *Sur les polynômes à coefficients entiers et de discriminant donné*, Acta Arith. 23 (1973), 419–426.

[11] K. Győry, *Sur les polynômes à coefficients entiers et de discriminant donné, II*, Publ. Math. Debrecen 21 (1974), 125–144.

[12] K. Győry, *On the number of solutions of linear equations in units of an algebraic number field*, Comment. Math. Helv. 54 (1979), 583–600.

[13] K. Győry, *Résultats effectifs sur la représentation des entiers par des formes décomposables*, Queen's Papers Pure Appl. Math 56, Queen's Univ., 1980, iii+142 pp.

[14] K. Győry, *Some recent applications of S-unit equations*, Astérisque 209 (1992), 17–38.

[15] K. Győry, *Bounds for the solutions of decomposable form equations*, Publ. Math. Debrecen 52 (1998), 1–31.

[16] K. Győry, *On some arithmetical properties of Lucas and Lehmer numbers, II*, Acta Acad. Paedagog. Agriensis Sect. Math. (N.S.) 30 (2003), 67–73.

[17] K. Győry, *On the abc conjecture in algebraic number fields*, Acta Arith. 133 (2008), 281–295.

[18] K. Győry, *Bounds for the solutions of S-unit equations and decomposable form equations II*, Publ. Math. Debrecen 94 (2019), 507–526.

[19] K. Győry, *S-unit equations and Masser's ABC conjecture in algebraic number fields*, Publ. Math. Debrecen 100 (2022), 499–511.

[20] K. Győry and K. Yu, *Bounds for the solutions of S-unit equations and decomposable form equations*, Acta Arith. 123 (2006), 9–41.

[21] L. Hajdu, *A quantitative version of Dirichlet's S-unit theorem in algebraic number fields*, Publ. Math. Debrecen 42 (1993), 239–246.

[22] R. von Känel and B. Matschke, *Solving S-unit, Mordell, Thue, Thue–Mahler and generalized Ramanujan–Nagell equations via the Shimura–Taniyama conjecture*, Mem. Amer. Math. Soc. 286 (2023), no. 1419, vi+142 pp.

[23] S. Le Fourn, *Tubular approaches to Baker's method for curves and varieties*, Algebra Number Theory 14 (2020), 785–807.

[24] D. W. Masser, *On abc and discriminants*, Proc. Amer. Math. Soc. 130 (2002), 3141–3150.

[25] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II*, Izv. Math. 64 (2000), 1217–1269.

[26] A. Scoones, *On the abc conjecture in algebraic number fields*, Mathematica 70 (2024), art. e12230, 43 pp.

[27] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge Tracts in Math. 87, Cambridge Univ. Press, 1986.

[28] N. P. Smart, *The Algorithmic Resolution of Diophantine Equations*, London Math. Soc. Student Texts 41, Cambridge Univ. Press, 1998.

[29] V. G. Sprindžuk, *Classical Diophantine Equations*, Lecture Notes in Math. 1559, Springer, 1993.

[30] P. Voutier, *An effective lower bound for the height of algebraic numbers*, Acta Arith. 74 (1996), 81–95.

[31] K. Yu, *Linear forms in p-adic logarithms*, Acta Arith. 53 (1989), 107–186.

Kálmán Győry
University of Debrecen
H-4002 Debrecen, Hungary
E-mail: gyory@science.unideb.hu

Samuel Le Fourn
Univ. Grenoble Alpes, CNRS, IF
38000 Grenoble, France
E-mail: samuel.le-fourn@univ-grenoble-alpes.fr