ACTA ARITHMETICA
Online First version

On sequences of integers with small prime factors

by

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For Professor Henryk Iwaniec on the occasion of his seventy-fifth birthday

Abstract. We show that the difference between consecutive terms in sequences of integers whose greatest prime factor grows slowly tends to infinity.

1. Introduction. Let y be a real number with $y \geq 3$ and let $1 = n_1 < n_2 < \cdots$ be the increasing sequence of positive integers with all prime factors of size at most y. In 1908 Thue [14] proved that

$$\lim_{i\to\infty} (n_{i+1} - n_i) = \infty; \tag{1}$$

see also Pólya [11] and Erdős [4]. Thue's result was ineffective. In particular, his proof does not allow one to determine, for every positive integer m, an integer $i(m)$ such that $n_{i+1} - n_i$ exceeds m whenever i is larger than $i(m)$. Cassels [2] showed how (1) can be made effective by means of estimates due to Gel'fond [5] for linear forms in two logarithms of algebraic numbers. In 1973 Tijdeman [15] proved, by appealing to work of Baker [1] on estimates for linear forms in the logarithms of algebraic numbers, that there is a positive number c, which is effectively computable in terms of y, such that

$$n_{i+1} - n_i > n_i/(\log n_i)^c \tag{2}$$

for $n_i \geq 3$. In addition, Tijdeman showed that there are arbitrarily large integers n_i for which (2) fails to hold when c is less than $\pi(y) - 1$; here $\pi(x)$ denotes the counting function for the primes up to x.

Now let $y = y(x)$ denote a non-decreasing function from the positive real numbers to the real numbers of size at least 3. For any integer n let $P(n)$ denote the greatest prime factor of n with the convention that

2020 *Mathematics Subject Classification*: Primary 11N25; Secondary 11J86.
Key words and phrases: small prime factors, linear forms in logarithms.
Received 4 August 2023.
Published online 1 February 2024.

$P(0) = P(\pm 1) = 1$. Let $(n_i)_{i=1}^{\infty}$ be the increasing sequence of positive integers n_i for which

$$P(n_i) \le y(n_i). \tag{3}$$

For any integer $k \ge 2$ let $\log_k$ denote the kth iterate of the function $x \mapsto \max(1, \log x)$ for $x > 0$. We shall prove that (1) holds provided that

$$y(n) = o\left(\frac{\log_2 n \log_3 n}{\log_4 n}\right). \tag{4}$$

Furthermore, if we assume the abc conjecture (see §2), then we can prove that (1) holds provided that

$$y(n) = o(\log n). \tag{5}$$

For any real number $x \ge 2$ put

$$\delta(x) = \exp\left(\frac{x \log_2 x}{\log x}\right).$$

We shall deduce (4) from the following result.

Theorem 1. *Let $y = y(x)$ be a non-decreasing function from the positive real numbers to the real numbers of size at least 3. Let $(n_1, n_2, \ldots)$ be the increasing sequence of positive integers n_i for which (3) holds. There is an effectively computable positive number c such that for $i \ge 3$,*

$$n_{i+1} - n_i > n_i/(\log n_i)^{\delta(cy(n_{i+1}))}. \tag{6}$$

Furthermore, there is an effectively computable positive number c_1 such that for infinitely many positive integers i,

$$n_{i+1} - n_i < n_i \exp(c_1 y(n_i))/(\log n_i)^{r-1}, \tag{7}$$

where $r = \pi(y(\sqrt{n_i}))$.

Observe that we obtain (1) from (6) when (4) holds on noting that in this case $n_{i+1} \le 2n_i$ and

$$(\log n)^{\delta(cy(n))} = o(n).$$

In order to establish (6) we shall appeal to an estimate for linear forms in the logarithms of rational numbers due to Matveev [8, 9]. The upper bound (7) follows from an averaging argument based on a result of Ennola [3].

We are able to refine the lower bound (6) provided that the abc conjecture is true.

Theorem 2. *Let $y = y(x)$ be a non-decreasing function from the positive real numbers to the real numbers of size at least 3. Let $(n_1, n_2, \ldots)$ be the increasing sequence of positive integers n_i for which (3) holds and let ε be a positive real number. If the abc conjecture is true then there exists a positive* *number* $c_1 = c_1(\varepsilon)$, *which depends on* ε, *and a positive number* c_2 *such that for* $i \geq 1$,

$$n_{i+1} - n_i > c_1(\varepsilon) n_i^{1-\varepsilon} / \exp(c_2 y(n_{i+1})). \tag{8}$$

We obtain (1) from (8) when (5) holds since in this case

$$\exp(c_2 y(n)) = n^{o(1)}.$$

2. Preliminary lemmas. For any non-zero rational number α we may write $\alpha = a/b$ with a and b coprime integers and with b positive. We define $H(\alpha)$, the *height* of α, by

$$H(\alpha) = \max(|a|, |b|).$$

Let n be a positive integer and let $\alpha_1, \ldots, \alpha_n$ be positive rational numbers with heights at most $A_1, \ldots, A_n$ respectively. Suppose that $A_i \geq 3$ for $i = 1, \ldots, n$ and that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the rationals, where log denotes the principal value of the logarithm. Let $b_1, \ldots, b_n$ be non-zero integers of absolute value at most B with $B \geq 3$ and put

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n.$$

LEMMA 3. *There exists an effectively computable positive number* c_0 *such that*

$$\log |\Lambda| > -c_0^n \log A_1 \ldots \log A_n \log B.$$

Proof. This follows from Theorem 2.2 of Nesterenko [10], which is a special case of the work of Matveev [8, 9]. ∎

Let x and y be positive real numbers with $y \geq 2$ and let $\Psi(x, y)$ denote the number of positive integers of size at most x all of whose prime factors are of size at most y. Let r denote the number of primes of size at most y, so that $r = \pi(y)$.

LEMMA 4. *For* $2 \leq y \leq (\log x)^{1/2}$ *we have*

$$\Psi(x, y) = \frac{(\log x)^r}{\prod_{i=1}^r (i \log p_i)} \bigl(1 + O(y^2 (\log x)^{-1} (\log y)^{-1})\bigr).$$

Proof. This is [3, Theorem 1]. ∎

We also recall the abc conjecture of Oesterlé and Masser [6, 7, 13]. Let x, y and z be positive integers. Denote the greatest square-free factor of xyz by $G = G(x, y, z)$, so

$$G = \prod_{\substack{p \mid xyz \\ p \,\mathrm{prime}}} p.$$

CONJECTURE 5 (abc conjecture). *For each positive real number* ε *there is a positive number* $c(\varepsilon)$ *such that for all pairwise coprime positive integers*

x, y and z with

$$x + y = z$$

we have

$$z < c(\varepsilon)G^{1+\varepsilon}.$$

For a refinement of the abc conjecture see [12].

3. Proof of Theorem 1. Let $c_1, c_2, \ldots$ denote effectively computable positive numbers. Following [15], for $i \geq 3$ we have $n_i \geq 3$,

$$n_{i+1} - n_i = n_i \left(\frac{n_{i+1}}{n_i} - 1 \right) \tag{9}$$

and, since $e^z - 1 > z$ for z positive,

$$\frac{n_{i+1}}{n_i} - 1 > \log \frac{n_{i+1}}{n_i}. \tag{10}$$

Let $p_1, \ldots, p_r$ be the primes of size at most $y(n_{i+1})$. Notice that $r \geq 2$ since $y(n_{i+1}) \geq 3$. Then $n_{i+1}/n_i = p_1^{l_1} \ldots p_r^{l_r}$ with $l_1, \ldots, l_r$ integers of absolute value at most $c_1 \log n_{i+1}$ and, since $n_{i+1} \leq 2n_i$,

$$\max(|l_1|, \ldots, |l_r|) \leq c_2 \log n_i. \tag{11}$$

Since

$$\log \frac{n_{i+1}}{n_i} = l_1 \log p_1 + \cdots + l_r \log p_r,$$

it follows from (11) and Lemma 3 that

$$\log \frac{n_{i+1}}{n_i} > (\log n_i)^{-c_3^r \log p_1 \ldots \log p_r}. \tag{12}$$

By the arithmetic-geometric mean inequality,

$$\prod_{i=1}^{r} \log p_i \leq \left(\frac{1}{r} \sum_{i=1}^{r} \log p_i \right)^r, \tag{13}$$

and by the prime number theorem,

$$\sum_{i=1}^{r} \log p_i < c_4 r \log r. \tag{14}$$

Thus, from (12)–(14),

$$\log \frac{n_{i+1}}{n_i} > (\log n_i)^{-(c_5 \log r)^r}. \tag{15}$$

Observe that $r \geq 2$ and so

$$(c_5 \log r)^r < e^{c_6 r \log_2 r}. \tag{16}$$

Further,

$$3 \leq p_r \leq y(n_{i+1})$$

and so

$$r \leq c_7 y(n_{i+1})/\log y(n_{i+1}). \tag{17}$$

Thus, by (16) and (17),

$$(c_5 \log r)^r < \delta(c_8 y(n_{i+1})) \tag{18}$$

and (6) follows from (9), (10), (15) and (18).

We shall now establish (7). Observe that if n_i satisfies (3) then since $y(t) \geq 3$ for all positive real numbers t, $P(2n_i) \leq y(n_i) \leq y(2n_i)$ and so $2n_i = n_j$ for some integer j with $j > i$. In particular $n_{i+1} \leq 2n_i$, hence $n_{i+1} - n_i \leq n_i$ and

$$n_{i+1} - n_i < 2n_i. \tag{19}$$

Suppose that X is a real number with $X \geq 9$ and that i is a positive integer with n_{i+1} and n_i in the interval $\big(\sqrt{X}, X\big]$. If, in addition,

$$y\big(\sqrt{X}\big) > (\log X)^{1/4} \tag{20}$$

then, since $\sqrt{X} < n_i \leq X$,

$$y(n_i) > (\log n_i)^{1/4}. \tag{21}$$

Since y is non-decreasing,

$$\pi(y(\sqrt{n_i})) - 1 \leq \pi(y(n_i)), \tag{22}$$

and by the prime number theorem,

$$\pi(y(n_i)) < c_9 \frac{y(n_i)}{\log y(n_i)}.$$

By (21),

$$\pi(y(n_i)) < c_{10} \frac{y(n_i)}{\log_2 n_i}. \tag{23}$$

Thus by (22) and (23),

$$(\log n_i)^{\pi(y(\sqrt{n_i}))-1} < e^{c_{10} y(n_i)}. \tag{24}$$

We may suppose that c_1 exceeds $1 + c_{10}$ and in this case, by (24),

$$\exp(c_1 y(n_i))/(\log n_i)^{\pi(y(\sqrt{n_i}))-1} \geq \exp(y(n_i)) \geq \exp(3) \geq 2,$$

and therefore (7) follows from (19).

We shall now show that there is a positive number c_{11} such that if X is a real number with $X > c_{11}$, then there is a positive integer i for which n_{i+1} and n_i are in $\big(\sqrt{X}, X\big]$ and satisfy (7). Accordingly, let X be a real number with $X \geq 9$, and put

$$r = \pi\big(y\big(\sqrt{X}\big)\big).$$

Notice that $r \geq 2$ since $y(t) \geq 3$ for all positive real numbers t. By the preceding paragraph we may suppose that

$$y(\sqrt{X}) \leq (\log X)^{1/4}.$$

Let $A(X)$ be the set of integers n with

$$\sqrt{X} < n \leq X \tag{25}$$

for which

$$P(n) \leq y(\sqrt{X}). \tag{26}$$

Note that the members of $A(X)$ occur as terms in the sequence $(n_1, n_2, \ldots)$. The cardinality of $A(X)$ is

$$\Psi\big(X, y(\sqrt{X})\big) - \Psi\big(\sqrt{X}, y(\sqrt{X})\big),$$

and so for $X > c_{12}$ it is, by Lemma 4, at least

$$\frac{(\log X)^r}{2\prod_{i=1}^r i \log p_i}. \tag{27}$$

Let j be the positive integer for which

$$\frac{X}{2^j} < \sqrt{X} \leq \frac{X}{2^{j-1}}$$

and consider the intervals $(X/2^k, X/2^{k-1}]$ for $k = 1, \ldots, j$. Then $j \leq 1 + \log X/(2\log 2)$ and so, for $X > c_{13}$,

$$j \leq \log X. \tag{28}$$

Thus, by (27) and (28), there is an integer h with $1 \leq h \leq j$ for which the interval $(X/2^h, X/2^{h-1}]$ contains at least

$$\frac{(\log X)^{r-1}}{2\prod_{i=1}^r i \log p_i}$$

integers from $A(X)$. Notice that

$$\prod_{i=1}^r i \log p_i \leq \big(r \log y(\sqrt{X})\big)^r.$$

Thus, since $y(\sqrt{X}) \leq (\log X)^{1/4}$, and $r - 1 \geq r/2$ because $r \geq 2$, we see that for $X > c_{14}$, the interval $(X/2^h, X/2^{h-1}]$ contains at least

$$\frac{(\log X)^{r-1}}{3\big(r \log y(\sqrt{X})\big)^r} + 1$$

terms from $A(X)$, hence two of them, say n_{i+1} and n_i, satisfy

$$n_{i+1} - n_i < \frac{X}{2^h (\log X)^{r-1}} 3\big(r \log y(\sqrt{X})\big)^r.$$

Since $n_i > X/2^h$ it follows that

$$n_{i+1} - n_i < 3\frac{n_i}{(\log n_i)^{r-1}}\big(r \log y(\sqrt{X})\big)^r.$$

By (25), $\sqrt{n_i} \le \sqrt{X} \le n_i$ and hence, since y is non-decreasing, $y(\sqrt{n_i}) \le y(\sqrt{X}) \le y(n_i)$. Thus

$$n_{i+1} - n_i < 3\frac{n_i}{(\log n_i)^{r-1}}(r \log y(n_i))^r$$

and so

$$n_{i+1} - n_i < 3\frac{n_i}{(\log n_i)^{r'-1}}(s \log y(n_i))^s, \tag{29}$$

where $r' = \pi(y(\sqrt{n_i}))$ and $s = \pi(y(n_i))$. By the prime number theorem there is a positive number c_{15} such that

$$3(s \log y(n_i))^s < e^{c_{15} y(n_i)}. \tag{30}$$

Estimate (7) now follows from (29) and (30). On letting X tend to infinity we find infinitely many pairs of integers n_{i+1} and n_i which satisfy (7).

4. Proof of Theorem 2. Let $i \ge 1$ and put

$$n_{i+1} - n_i = t. \tag{31}$$

Let g be the greatest common divisor of n_{i+1} and n_i. Then

$$\frac{n_{i+1}}{g} - \frac{n_i}{g} = \frac{t}{g}.$$

Let $\varepsilon > 0$. By the abc conjecture there is a positive number $c(\varepsilon)$ such that

$$\frac{n_i}{g} < c(\varepsilon)\Bigg(\frac{t}{g}\prod_{p \le y(n_{i+1})} p\Bigg)^{1+\varepsilon}$$

and hence

$$\left(\frac{n_i}{c(\varepsilon)}\right)^{\frac{1}{1+\varepsilon}} < t \prod_{p \le y(n_{i+1})} p. \tag{32}$$

By the prime number theorem, since $y(n_{i+1}) \ge 3$, there exists a positive number c_2 such that

$$\prod_{p \le y(n_{i+1})} p < e^{c_2 y(n_{i+1})}. \tag{33}$$

The result follows from (31)–(33).

Acknowledgements. This research was supported in part by the Canada Research Chairs Program and by grant A3528 from the Natural Sciences and Engineering Research Council of Canada.

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