ACTA ARITHMETICA
Online First version

On the ratio between two factorization functions

by

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Abstract. Let $f(n)$ be the number of factorizations of n, i.e., representations of n as an unordered product of integers greater than 1. In addition, let $F(n)$ be the number of factorizations of n into distinct parts. In this note, we provide a new upper bound for the ratio $f(n)/F(n)$.

1. Introduction. Let $f(n)$ be the number of ways of expressing n as an (unordered) product of numbers greater than 1 (with $f(1) = 1$ for notational convenience). We refer to these products as "factorizations" (or "multiplicative partitions") of n. For example, $f(30) = 5$ because the factorizations of 30 are

$$30, \quad 2 \cdot 15, \quad 3 \cdot 10, \quad 5 \cdot 6, \quad 2 \cdot 3 \cdot 5.$$

In addition, we let $F(n)$ be the number of factorizations of n into *distinct* parts. One hundred years ago, MacMahon [M24] initiated the study of these functions and observed that their Dirichlet series are

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{m=2}^{\infty} \left(1 - \frac{1}{m^s}\right)^{-1},$$
$$\sum_{n=1}^{\infty} \frac{F(n)}{n^s} = \prod_{m=2}^{\infty} \left(1 + \frac{1}{m^s}\right).$$

Two years later, Oppenheim [O26] erroneously claimed that the maximal order of $f(n)$ for $n \leq x$ is

$$\frac{x}{L(x)^{2+o(1)}},$$

where (here and below)

$$L(x) = \exp\left(\frac{\log x \log\log\log x}{\log\log x}\right).$$

2020 *Mathematics Subject Classification*: Primary 11A51; Secondary 11N37.
Key words and phrases: factorizations, distinct parts, asymptotics.
Received 13 June 2023; revised 25 October 2023.
Published online 5 February 2024.

DOI: 10.4064/aa230613-31-12

Canfield, Erdős, and Pomerance [CEP83] later corrected Oppenheim's bound, showing that

$$\max_{n \le x} f(n), \qquad \max_{n \le x} F(n),$$

are both

$$\frac{x}{L(x)^{1+o(1)}}.$$

(For the corresponding results for *ordered* factorizations, see [DHN08, L23].)

Canfield et al. found the maximal orders of f and F by manipulating their Dirichlet series. However, given that these maximal orders are so close together, it is tempting to suppose that $f(n)$ and $F(n)$ must always be close. A few years ago, the author [L21, Theorem 1.3] found an upper bound for the ratio $f(n)/F(n)$.

THEOREM 1.1. *Let $\Omega(n)$ be the number of prime factors of n counted with multiplicity. Then*

$$f(n)/F(n) \le 2^{\Omega(n)/2}.$$

Because $\Omega(n) \le (\log n)/(\log 2)$, we also observe that $f(n)/F(n) \le \sqrt{n}$. This theorem allows us to simplify some of the arguments in [CEP83]. In this note, we modify the proof of the previous theorem and obtain a new upper bound for $f(n)/F(n)$.

THEOREM 1.2. *As $n \to \infty$, we have*

$$f(n) = F(n) \cdot \exp\left(O\left(\frac{\log n}{\log\log n}\right)\right).$$

Because $\Omega(n)$ is almost always asymptotic to $\log\log n$, Theorem 1.2 is almost always *weaker* than Theorem 1.1. However, when $\Omega(n)$ is large, it is a noticeable improvement. Unfortunately, Theorem 1.2 is probably sub-optimal.

In [CEP83], the authors found the maximal orders of $f(n)$ and $F(n)$ separately. However, Theorem 1.2 implies that if $\max_{n \le x} f(n) = x/L(x)^{1+o(1)}$, then $\max_{n \le x} F(n) = x/L(x)^{1+o(1)}$ as well.

Finally, we expect $f(n)/F(n)$ to be unusually large when n consists of a large number of copies of a few distinct prime factors. In light of this observation, I also made the following conjecture.

CONJECTURE 1.3 ([L21, Conjecture 1.5]). *A number n satisfies the condition $f(n)/F(n) > f(m)/F(m)$ for all $m < n$ if and only if n is a power of 2 other than 2, 8, or 32.*

Note that $f(2^k) = p(k)$ and $F(2^k) = p_0(k)$, where $p(k)$ is the number of partitions of k and $p_0(k)$ is the number of such partitions with distinct parts. Because $p(k)$ and $p_0(k)$ have precise asymptotic formulas [HR18], we have the following result.

THEOREM 1.4 ([L21, Corollary 1.6]). *Assuming the previous conjecture, we have*

$$\max_{n\le x}\frac{f(n)}{F(n)}\sim\sqrt[4]{\frac{\log 2}{3\log x}}\exp\left(\frac{\pi(\sqrt{2}-1)}{\sqrt{3\log 2}}\sqrt{\log x}\right).$$

In [L21, §6], I proved Theorem 1.1 by creating a function which maps factorizations of n into factorizations of n into distinct parts, then bounding the number of pre-images a factorization can have under this function. In this paper, we provide a new bound for the number of pre-images.

2. A map between factorizations. Let $S_f(n)$ be the set of factorizations of n and $S_F(n)$ the set of factorizations of n into distinct parts. For notational convenience, we express every element of $S_f(n)$ as a multiset $\{n_1,\ldots,n_k\}$. Note that every element of $S_F(n)$ is simply a set. In [L21, §6], I defined the map $T:S_f(n)\to S_F(n)$ as follows. Let $\{n_1,\ldots,n_k\}\in S_f(n)$. In order to compute $T(n_1,\ldots,n_k)$, multiply all repeated copies of the same element together, then replace the initial elements with the product. (If there are multiple sets of repeated elements, apply these multiplications "simultaneously".) Repeat this process until every element is distinct.

For example, let $n=648$. One factorization of n is $2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 9$. Applying T to this factorization gives us

$$T(2,2,2,3,3,9)=T(2\cdot 2\cdot 2,3\cdot 3,9)=T(8,9,9)=T(8,81)=(8,81).$$

In [L21], I proved Theorem 1.1 by showing that every element of $S_F(n)$ has at most $2^{\Omega(n)/2}$ pre-images under T. In this note, we prove Theorem 1.2 in the same way. Before doing so, we need some definitions.

DEFINITION 2.1. Let $p(n)$ be the number of partitions of n and $p_d(n)$ the number of partitions of n into divisors of n.

There are few results on partitions of n into divisors of n, so we highlight some notable ones. We say that a number n is *pseudoperfect* or *semiperfect* if n can be expressed as a sum of some distinct divisors of n [G04, §B2]. Erdős [E70] proved that pseudoperfect numbers have a well-defined density and that this density lies in $(0,1)$. Bowman, Erdős, and Odlyzko [BEO92] later found precise bounds on the number of partitions of a number into divisors of that number. We discuss this result in more detail later.

Using p_d, we can bound the number of inverse images of one-element factorizations under T.

LEMMA 2.2. *For a given integer n, let a be the largest number for which n is an ath power. For the map $T:S_f(n)\to S_F(n)$, we have*

$$\#T^{-1}(n)\le p_d(a).$$

Proof. Suppose $T(n_1, \ldots, n_k) = (n)$. Then for each $i \leq k$, there exists a positive integer a_i such that $n_i^{a_i} = n$. (At every stage, we combine multiple copies of the same number together. So, we can only create perfect powers of our original numbers.) However, each a_i must divide a. For each i, we may write $n_i = n^{1/a_i} = n^{b_i/a}$ with $b_i = a/a_i$. Hence,

$$n = n_1 \cdots n_k = n^{(b_1 + \cdots + b_k)/a},$$

which implies that $(b_1, \ldots, b_k)$ is a partition of a into divisors of a. Therefore, there are at most $p_d(a)$ inverses of (n) under T. ■

It is straightforward to modify this result for factorizations into multiple parts.

LEMMA 2.3. *Let $\{n_1, \ldots, n_k\} \in S_F(n)$. For each $i \leq k$, let a_i be the largest number for which n_i is an a_ith power. Then*

$$\#T^{-1}(n_1, \ldots, n_k) \leq \prod_{i=1}^{k} p_d(a_i).$$

For all b, we have

$$p_d(b) \leq b^{(1+o(1))d(b)/2},$$

where $d(b)$ is the number of divisors of b [BEO92]. (Through a slight refinement of the proof of this bound, one can also obtain $p_d(b) \leq b^{d(b)/2} e^{\sigma(b)/b}$.) In addition [HW08, Theorem 317],

$$d(b) = \exp\left(O\left(\frac{\log b}{\log\log b}\right)\right),$$

which implies that

$$p_d(b) = \exp\left(\exp\left(O\left(\frac{\log b}{\log\log b}\right)\right)\right).$$

Note that $\log\log b$ is positive for $b \geq 3$. In light of this fact, we may observe that there exists a positive constant C such that

$$p_d(b) < \exp\left(\exp\left(C\frac{\log(b+2)}{\log\log(b+2)}\right)\right).$$

Let $\mathcal{S} = \{n_1, \ldots, n_k\} \in S_F(n)$. We have

$$\begin{aligned}\#T^{-1}(\mathcal{S}) &\leq \prod_{i=1}^{k} p_d(a_i) \\ &\leq \prod_{i=1}^{k} \exp\left(\exp\left(C\frac{\log(a_i+2)}{\log\log(a_i+2)}\right)\right) \\ &= \exp\left(\sum_{i=1}^{k} \exp\left(C\frac{\log(a_i+2)}{\log\log(a_i+2)}\right)\right).\end{aligned}$$

At this point, we can bound $\#T^{-1}(\mathcal{S})$ from above. We split $\mathcal{S}$ into disjoint sets $\mathcal{S}_1, \ldots, \mathcal{S}_r$, where $\mathcal{S}_i$ is the set of $m \in \mathcal{S}$ for which m is an ith power, but not a jth power for any $j > i$. So,

$$n = \prod_{i=1}^{r} \prod_{m \in \mathcal{S}_i} m.$$

We also have

$$\#T^{-1}(\mathcal{S}) \leq \prod_{i=1}^{r} \prod_{m \in \mathcal{S}_i} \#T^{-1}(m).$$

If $m \in \mathcal{S}_i$, then $m^{1/j}$ is not an integer for any $j > i$. For such an m, we have

$$\#T^{-1}(m) \leq p_d(i) < \exp\left(\exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right)\right).$$

Plugging this back into our bound for $\#T^{-1}(\mathcal{S})$ gives us the following.

Lemma 2.4. *Let $\mathcal{S}$ be a factorization of n into distinct parts and let r be the largest number with the property that some element of $\mathcal{S}$ is an rth power. Then*

$$\#T^{-1}(\mathcal{S}) \leq \exp\left(\sum_{i=1}^{r} (\#\mathcal{S}_i) \exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right)\right).$$

3. The main result. Using Lemma 2.4, we can bound $\#T^{-1}(\mathcal{S})$ for any factorization $\mathcal{S}$ of n with distinct parts. Rather than manipulating the elements of $\mathcal{S}$ directly, we work with the $\mathcal{S}_i$'s. Let $s_i = \#\mathcal{S}_i$. By definition, each element of $\mathcal{S}_i$ is an ith power. Once again, we let r be the largest number for which some element of $\mathcal{S}$ is an rth power. Therefore,

$$n = \prod_{i=1}^{r} \prod_{m \in \mathcal{S}_i} m \geq \prod_{i=1}^{r} \prod_{j=2}^{s_i+1} j^i = \prod_{i=1}^{r} ((s_i+1)!)^i.$$

Stirling's formula implies that there exists a positive constant D such that $\log(m!) \geq Dm \log m$ for all positive m. Therefore,

$$\prod_{i=1}^{r} e^{Di(s_i+1)\log(s_i+1)} \leq n.$$

Taking the logarithm of both sides and dividing by D gives us

$$\sum_{i=1}^{r} i(s_i+1)\log(s_i+1) \leq \frac{\log n}{D}.$$

To summarize, our goal is to bound the quantity

$$\sum_{i=1}^{r} s_i \exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right)$$

under the constraint

$$\sum_{i=1}^{r} i(s_i+1)\log(s_i+1) \leq \frac{1}{D}\log n.$$

We break the argument into three cases based on the sizes of i and s_i.

First, suppose that $s_i > \sqrt{\log n}$. We have

$$\begin{aligned}\sum_{\substack{i \leq r \\ s_i > \sqrt{\log n}}} i(s_i+1)\log(s_i+1) &\geq \frac{1}{2}(\log\log n) \sum_{\substack{i \leq r \\ s_i > \sqrt{\log n}}} i s_i \\ &\gg (\log\log n) \sum_{\substack{i \leq r \\ s_i > \sqrt{\log n}}} s_i \exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right).\end{aligned}$$

Because the sum of $i(s_i+1)\log(s_i+1)$ over all $i \leq r$ is at most $(\log n)/D$, we have

$$\sum_{\substack{i \leq r \\ s_i > \sqrt{\log n}}} s_i \exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right) \ll \frac{\log n}{\log\log n}.$$

Next, suppose that $i > (\log\log n)^2$. We apply a technique which is similar to the one we just used. Fix $\epsilon \in (0,1)$. If i is sufficiently large, then

$$\exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right) < \exp((\epsilon/2)\log i) = i^{\epsilon/2},$$

which implies that

$$i\exp\left(-C\frac{\log(i+2)}{\log\log(i+2)}\right) > i^{1-(\epsilon/2)} > (\log\log n)^{2-\epsilon}.$$

The inequality is now

$$\begin{aligned}&\sum_{(\log\log n)^2 < i \leq r} i(s_i+1)\log(s_i+1) \\ &\quad \gg (\log\log n)^{2-\epsilon} \sum_{(\log\log n)^2 < i \leq r} s_i(\log(s_i+1))\exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right) \\ &\quad \gg (\log\log n)^{2-\epsilon} \sum_{(\log\log n)^2 < i \leq r} s_i\exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right).\end{aligned}$$

Hence,

$$\sum_{(\log\log n)^2 < i \leq r} s_i \exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right) \ll \frac{\log n}{(\log\log n)^{2-\epsilon}},$$

which is negligible.

Finally, suppose that $s_i \le \sqrt{\log n}$ and $i \le (\log\log n)^2$. Our sum is now

$$\sum_{i \le (\log\log n)^2} s_i \exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right)$$
$$\le \sqrt{\log n} \sum_{i \le (\log\log n)^2} \exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right) = (\log n)^{(1/2)+o(1)}.$$

Summing these quantities shows that

$$\sum_{i=1}^{r} s_i \exp\left(C\frac{\log(i+2)}{\log\log(i+2)}\right) \ll \frac{\log n}{\log\log n},$$

as $n \to \infty$. This, in turn, implies that

$$\#T^{-1}(\mathcal{S}) = \exp\left(O\left(\frac{\log n}{\log\log n}\right)\right)$$

for all factorizations $\mathcal{S} \in S_F(n)$. Because $f(n)/F(n) \le \max_{\mathcal{S}\in S_F(n)} \#T^{-1}(\mathcal{S})$, we have

$$\frac{f(n)}{F(n)} = \exp\left(O\left(\frac{\log n}{\log\log n}\right)\right)$$

as well.

We may also observe that our upper bound on $\#T^{-1}(\mathcal{S})$ is optimal (up to the constant in the exponent). Let $n = (p_1 \cdots p_m)^2$ for some integer m, where p_i is the ith prime. Then

$$\#T^{-1}(p_1^2, \dots, p_m^2) = 2^m.$$

Because $m \sim (1/2)\log n/\log\log n$, we have

$$\#T^{-1}(p_1^2, \dots, p_m^2) > \exp\left(c\frac{\log n}{\log\log n}\right)$$

for some positive constant c when n is sufficiently large.

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