Extending the support of 1- and 2-level densities for cusp form L-functions under square-root cancellation hypotheses

by

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Dedicated to Henryk Iwaniec

Abstract. The Katz–Sarnak philosophy predicts that the behavior of zeros near the central point in families of L-functions agrees with that of eigenvalues near 1 of random matrix ensembles. Under GRH, Iwaniec, Luo and Sarnak showed agreement in the 1-level densities for cuspidal newforms with the support of the Fourier transform of the test function in $(-2, 2)$. Under a square-root cancellation conjecture ('Hypothesis S'), they extend allowable support to a (symmetric) larger open interval $\supsetneq (-2, 2)$ still obeying random-matrix statistics. We formulate a two-dimensional analog and show it leads to improvements in the 2-level density. Specifically, we show that a square-root cancellation of certain classical exponential sums over primes increases the support of the test functions such that the main terms in the 1- and 2-level densities of cuspidal newforms averaged over bounded weight k (and fixed level 1) converge to their random matrix theory predictions. We also conjecture a broad class of such exponential sums where we expect improvement in the case of arbitrary n-level densities, and note that the arguments in [Inst. Hautes Études Sci. Publ. Math. 91 (2000), 55–131] yield larger support than claimed.

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1. Introduction

1.1. Background. Since the observations of Montgomery and Dyson 50 years ago, random matrix theory has provided a guide to predicting the behavior of quantities related to the zeros and values of L-functions; we focus on the behavior of zeros here. Initially this agreement was limited to theoretical results on the pair correlation of zeros of the Riemann zeta function, and then extended to include the n-level correlations of automorphic forms and numerical results on the spacings between zeros; see [Gal, Hej, Mon, RS, Od1, Od2]. These statistics concern the behavior high up on the critical line, and are thus insensitive to finitely many zeros. In particular, they miss the effects from zeros at the central point, which are of great importance in a variety of number theory problems, from the Birch and Swinnerton-Dyer Conjecture and the ranks of the group of rational solutions of elliptic curves to bounding class numbers; see [BSD1, BSD2, Go].

To remedy this, Katz and Sarnak [KS1, KS2] introduced new statistics, the n-level densities; there is now an extensive literature on agreement between number theory and random matrix theory here (see [AA$^+$, AM, CD$^+$, CI, DH$^+$1, DH$^+$2, DM1, DM2, ERGR, FiM, FI, Gao, GK, GJ$^+$, Gü, HM, HR1, HR2, ILS, LM, Mil, MilPe, ÖS1, ÖS2, RR, Ro, Rub, ShTe, Ya, Yo1, Yo2] and the references therein, as well as [BF$^+$, Con, FM] for more on the history of the connections between the two subjects).

This work is a continuation of the seminal paper of Iwaniec, Luo and Sarnak [ILS]; in brief, we extend their results for certain one-level densities to 2-level, and discuss how to generalize to arbitrary n. Specifically, we show that a natural generalization of their Hypothesis S on cancellation in certain prime sums, which led to increasing the support for the 1-level density, leads to increased support where the 2-level density of certain families of cuspidal newforms and random matrix theory agree. Such results have applications in bounding the order of vanishing at the central point. We assume the reader is familiar with the basics of L-functions; see for example [IK] for details. The following definitions are standard; we paraphrase from [LiM] as we will use their framework to convert our results on increased support to estimates on weighted order of vanishing.

A function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is *Schwartz* if it is infinitely differentiable and it and all of its derivatives decay faster than any polynomial. In our setting, an even Schwartz function Φ with compactly supported Fourier transform is called a *test function*. We frequently assume that the Generalized Riemann Hypothesis (GRH) holds for each $L(s, f)$, and write the non-trivial zeros of a cuspidal newform $L(s, f)$ of level N and weight k as

$$\rho_f^{(j)} = \frac{1}{2} + i\gamma_f^{(j)}$$

for $\gamma_f^{(j)} \in \mathbb{R}$ increasingly ordered and centered about zero (^{1}). The number of zeros with $|\gamma_f^{(j)}|$ bounded by an absolute large constant is of order $\log c_f$ for some constant $c_f > 1$; this is known as the *analytic conductor*.

DEFINITION 1.1 (*n*-Level density). The *n-level density* of an *L*-function $L(s, f)$ with respect to a test function $\Phi : \mathbb{R}^n \to \mathbb{R}$ with compactly supported Fourier transform is defined as

$$(1.1) \qquad D_n(f; \Phi) := \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} \Phi\left(\frac{\log c_f}{2\pi}\gamma_f^{(j_1)}, \dots, \frac{\log c_f}{2\pi}\gamma_f^{(j_n)}\right).$$

Note that the existence of the *n*-level density does not depend on GRH.

One of the most important applications of the *n*-level density is to obtain bounds on the order of vanishing to a given order at the central point. For such results, we need Φ to be non-negative and positive at the origin.

Unlike the *n*-level correlations, the sum (1.1) is hard to study for an individual f because by the choice of Φ it essentially captures only a bounded number of zeros. Thus we study averages over finite subfamilies $\mathcal{F}(Q) := \{f \in \mathcal{F} : c_f \in I(Q)\}$ (which are parametrized by some quantity Q such that as Q tends to infinity, the size of the subfamily tends to infinity as well), namely

$$(1.2) \qquad \mathbb{E}[D_n(f; \Phi), Q] := \frac{1}{\#\mathcal{F}(Q)} \sum_{f \in \mathcal{F}(Q)} D_n(f; \Phi).$$

Common choices are $I(Q) = \{Q\}, \{1, 2, \dots, Q\}$ and $\{Q, Q+1, \dots, 2Q\}$.

The Katz–Sarnak density conjecture [KS1, KS2] asserts that the *n*-level density of a family of *L*-functions, in the limit as the conductors tend to infinity, converges to the *n*-level density of eigenvalues of a classical compact group as the matrix sizes tend to infinity. Explicitly, if $\mathcal{F}$ is a "good" family of *L*-functions and Φ is not zero at the origin, then there exists a distribution $W_{n,\mathcal{F}}$ such that

$$(1.3) \qquad \lim_{Q \to \infty} \mathbb{E}[D_n(f; \Phi), Q] = \frac{1}{\Phi(0, \dots, 0)} \int_{\mathbb{R}^n} \Phi(x) \cdot W_{n,\mathcal{F}}(x)\, dx_1 \cdots dx_n.$$

We have the following expansions for the quantities above; though for computational purposes it is often advantageous to use an alternative expansion due to Hughes–Miller [HM], which writes the *n*-level density as a sum of terms emerging as the support increases.

(^{1}) In many of the calculations of Bessel–Kloosterman terms, we need GRH for Dirichlet *L*-functions. We can often avoid GRH for cuspidal newforms at the expense of more involved calculations, but if GRH fails while we can still formally calculate these statistics, as the zeros are no longer on the critical line we lose the correspondence with physics.

THEOREM 1.2 (Determinant expansion [KS1]). *Let* $K(x) = \frac{\sin \pi x}{\pi x}$ *and* $K_\epsilon(x, y) = K(x - y) + \epsilon K(x + y)$. *Then the n-level densities have the following distinct closed form determinant expansions for each corresponding symmetry group:*

$$(1.4) \qquad W_{n,\mathsf{SO(even)}}(x) = \det (K_1(x_i, x_j))_{i,j \leq n},$$

$$(1.5) \qquad W_{n,\mathsf{SO(odd)}}(x) = \det (K_{-1}(x_i, x_j))_{i,j \leq n}$$
$$+ \sum_{k=1}^n \delta_0(x_k) \det (K_{-1}(x_i, x_j))_{i,j \neq k},$$

$$(1.6) \qquad W_{n,\mathsf{O}}(x) = \tfrac{1}{2} W_{n,\mathsf{SO(even)}}(x) + \tfrac{1}{2} W_{n,\mathsf{SO(odd)}}(x),$$

$$(1.7) \qquad W_{n,\mathsf{U}}(x) = \det (K_0(x_i, x_j))_{i,j \leq n},$$

$$(1.8) \qquad W_{n,\mathsf{Sp}}(x) = \det (K_{-1}(x_i, x_j))_{i,j \leq n}.$$

As remarked, one of the main applications of this statistic is to bound the order of vanishing of a family of L-functions at the central point by choosing a test function which is non-negative and positive at the origin; see [BC$^+$, DuM, Fr, FrMil, LiM]. Many of these papers are concerned with trying to find the optimal test function for a given support, but already in [ILS] one sees that there may be only a negligible improvement in bounds from using the optimal functions derived from Fredholm theory over simple test functions. Thus, while there have been some recent advances in determining the optimal function for a given support, it has been more productive to increase the support and the level n studied; however, as n increases while the bounds obtained are better for the percentage of forms vanishing to order at least r when r is large, they are worse for small r. Thus there is a balancing act, with most of the effort devoted to finding the largest support possible, and then determining the best bounds by using easy to compute test functions for each n. We concentrate on increasing the support for certain families of cusp forms, which can then be fed into the machinery from Dutta–Miller [DuM] to yield improved estimates.

In [ILS], the authors introduce the following hypothesis, whose implications are striking, allowing them to break $(-2, 2)$ for the support for certain families.

HYPOTHESIS 1.3 (Hypothesis S). *Let* $e(z) := e^{2\pi i z}$, c *be a positive integer and* a *be an arbitrary residue class mod* c. *Let* $M_1(\alpha, A)$ *denote a two-parameter family of hypotheses, where* $\alpha \in [1/2, 3/4]$ *and* $A \in [0, \infty)$, *each of which states that for all* $x \geq 2$ *and* $\varepsilon > 0$,

$$(1.9) \qquad \sum_{\substack{p \leq x \\ p \equiv a \,(c)}} e\left(\frac{2\sqrt{p}}{c}\right) \ll_\varepsilon c^A x^{\alpha + \varepsilon}$$

holds uniformly over $c > 0$ *and residue classes* a *mod* c, *with* α *and* A *fixed.*

1.2. Previous work. The reason why the authors of [ILS] are interested in this family of hypotheses is that the hypotheses increase the support of test functions for which the 1-level density agrees with random matrix theory. We first review their result, and then discuss our generalization.

For convenience, as is often done in the subject, we assume our test function Φ is the product of one-dimensional test functions ϕ_i. Below we confine our study to cusp forms of level 1 and weight k. Following [ILS], one is able to get better results on the support by averaging over k. This allows us to exploit some oscillation in the Bessel function factors that emerge in application of the Petersson formula. Note that rather than sum over all forms equally, each form f of weight k is weighted by the slowly varying factor $1/L(1, \mathrm{sym}^2(f))$ [HL, I1]. There is a trade-off in studying this modification of the n-level densities; these harmonic weights arise naturally in the Petersson formula, and their introduction simplifies calculations below. Unfortunately their presence means that we cannot obtain results on bounding the order of vanishing at the central point, but instead obtain results on weighted vanishing. In many families these weights can be removed through additional work; see [ILS].

DEFINITION 1.4 (Agrees with Orthogonal RMT). A test function Φ *agrees with Orthogonal Random Matrix Theory if*

$$(1.10) \quad \lim_{K \to \infty} \frac{1}{K} \sum_{k \leq K} \frac{4\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1, \mathrm{sym}^2(f))} D_n(f; \Phi)$$
$$= \int_{\mathbb{R}^n} \Phi(x) \cdot W_{n,\mathsf{O}}(x) \, dx_1 \cdots dx_n,$$

where $W_{n,\mathsf{O}}(x)$ is given by equation (1.6), and $H_k = H_k^\star(1)$ are Hecke eigen-cuspforms of weight k and level 1, with normalization so that $a_f(1) = 1$. The L-function $L(s, f)$ has coefficients $\lambda_f(n) := a_f(n) n^{-(k-1)/2}$ so that $\lambda_f(p) \in [-2, 2]$ and the L-function $L(s, f)$ is symmetric about $\mathfrak{Re}(s) = 1/2$.

REMARK 1.5. For the family considered above, previous results show that the underlying symmetry group is orthogonal, hence our comparison with the orthogonal behavior. For other families we would just use the corresponding densities from Theorem 1.2.

We are now able to state the following result from [ILS], which extends the support for the family of level 1 cusp forms to beyond $(-2, 2)$.

THEOREM 1.6 (One-level extended support). *Assume GRH. Then $M_1(\alpha, A)$ implies that a test function ϕ agrees with random matrix theory if* $\mathrm{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ *for* $\sigma = \min\{5/2, 2 + (6 - 8\alpha)/(1 + 2A + 4\alpha)\}$. *More*

specifically, for every $h \in C_0^\infty(\mathbb{R}^+)$ with $\widehat{h}(0) \neq 0$, one has

$$(1.11) \quad \lim_{K \to \infty} \frac{1}{\widehat{h}(0)K} \sum_{k \; even} \frac{4\pi^2}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} \frac{1}{L(1, \mathrm{sym}^2(f))} D_1(f; \phi)$$

$$\to \int_{\mathbb{R}} \phi(x) \cdot W_{1,\mathrm{O}}(x)\, dx.$$

1.3. New results. In their article from 2000, Iwaniec, Luo and Sarnak [ILS] state and prove Theorem 1.6 for $\sigma = 2 + (12 - 16\alpha)/(5 + 4A + 8\alpha)$, which is less than $2 + (6 - 8\alpha)/(1 + 2A + 4\alpha)$; however, a careful analysis of their arguments gives more than what they claimed. In Section 2, we give a proof of Theorem 1.6 based on [ILS, Section 10], and in Appendix A, we discuss the difference in the claimed support and what the calculation yields.

Our main theorem, Theorem 1.8, is a natural extension of Theorem 1.6, showing that square-root cancellation hypotheses also extend to the case of the 2-level density. For ease of exposition and to highlight the issues, we focus on the $n = 2$ case, though similar calculations should hold in general.

HYPOTHESIS 1.7. *Let $e(z) := e^{2\pi i z}$, c be a positive integer, and a_1, a_2 be arbitrary residue classes mod c. Let $M_2(\alpha, A)$ denote a two-parameter family of hypotheses, where $\alpha \in [1/2, 3/4]$ and $A \in [0, \infty)$, each of which states that for all $x_1, x_2 \geq 2$ and $\varepsilon > 0$,*

$$(1.12) \quad \sum_{\substack{p_1 \leq x_1 \\ p_1 \equiv a_1 \, (c)}} \sum_{\substack{p_2 \leq x_2 \\ p_2 \equiv a_2 \, (c)}} e\left(\frac{2\sqrt{p_1 p_2}}{c}\right) \ll_\varepsilon c^A (x_1 x_2)^{\alpha + \varepsilon}$$

holds uniformly over $c > 0$ and residue classes a_1, a_2 mod c, with α and A fixed.

A natural extension of Theorem 1.6 is our main theorem, proven in Section 3.

THEOREM 1.8 (Two-level extended support). *Assume GRH. Then $M_1(\alpha, A)$ and $M_2(\alpha, A)$ imply that a test function $\Phi(x) = \phi_1(x_1)\phi_2(x_2)$ agrees with random matrix theory if*

$$\mathrm{supp}(\widehat{\phi_i}) \subset (-\sigma_i, \sigma_i) \quad \text{for } \sigma_1 + \sigma_2 = 2 + (6 - 8\alpha)/(3 + 2A + 4\alpha).$$

More specifically, for every $h \in C_0^\infty(\mathbb{R}^+)$ with $\widehat{h}(0) \neq 0$, one has

$$(1.13) \quad \lim_{K \to \infty} \frac{1}{\widehat{h}(0)K} \sum_{k \; even} \frac{4\pi^2}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} \frac{1}{L(1, \mathrm{sym}^2(f))} D_2(f; \Phi)$$

$$\to \int_{\mathbb{R}^2} \Phi(x) \cdot W_{2,\mathrm{O}}(x)\, dx_1\, dx_2.$$

Using the machinery from [DuM] for an orthogonal ensemble with the naive test function we immediately obtain the following from Theorem 1.8.

THEOREM 1.9. *Taking ϕ to be the naive test function*

$$(1.14) \qquad \phi_{\text{Naive}}(x) = \left(\frac{\sin(\pi v x)}{\pi v x}\right)^2,$$

the weighted percentage of forms vanishing to order at least r, *denoted* $P_r(\mathcal{F})$, *is bounded by*

$$(1.15) \qquad P_r(\mathcal{F}) \le \min\left\{\frac{1}{r}\left(\frac{1}{2} + \frac{1}{v_1}\right), \frac{1}{3(r - 1/v_2 - 1/2)^2}\right\}.$$

Here v_1 and v_2 denote the available support for the 1-level and 2-level density respectively, and the form f is weighted by the factor from (1.13) (normalized so that the sum of the weights equals 1). Assuming the strongest version of Hypothesis 1.7, one may take $v_1 = 2.5$ and $v_2 = 1.2$. These values yield

$$(1.16) \qquad \begin{aligned} P_1(\mathcal{F}) &\le 0.9000, \quad P_2(\mathcal{F}) \le 0.4500, \quad P_3(\mathcal{F}) \le 0.1200, \\ P_4(\mathcal{F}) &\le 0.0469, \quad P_5(\mathcal{F}) \le 0.0248. \end{aligned}$$

REMARK 1.10. We quickly comment on the conditional nature of these results. In [ILS] there are two places where GRH is used. The first is for convenience to bound certain prime sums of cuspidal newforms, which the authors remark can be bypassed by additional appeals to the Petersson formula (see the comments after their equation (4.24)) (^{2}). The second is when analyzing the main term contribution when one splits by sign of the functional equation for square-free level N tending to infinity; however, we only need GRH for Dirichlet L-functions. When splitting by sign for growing level N there is also the size of error of the Bessel–Kloosterman term that needs to be bounded; likewise, only GRH for Dirichlet L-functions is needed. If we do not split by sign and consider level 1 cuspidal newforms for k up to K (appropriately weighted), then Hypothesis S, an estimate for "GL$_1$" exponential sums, suffices.

REMARK 1.11. We expect in the limit that 50% of the forms in our family should be rank 0 and 50% rank 1; thus (paraphrasing comments one of us heard from Iwaniec in graduate classes) for $r \ge 2$ we may interpret Theorem 1.9 as providing better upper bounds on 0.

We first review the proof for the 1-level density in Section 2 to fix notation and then extend to the 2-level in the next section. For notational convenience we prove Theorem 1.8 in the case when $\phi_1 = \phi_2$; a similar argument holds

(^{2}) While we need GRH to have the zeros lie on the critical line, and thus have a direct comparison to eigenvalues of Hermitian matrices or energy levels of heavy nuclei, such an assumption is only needed for this interpretation or correspondence; the n-level densities exist whether or not GRH holds.

in general (see for example the arguments in [LiM], which extend the n-level density results of [HM] from identical test functions to the more general case). We then discuss further generalizations to larger n.

2. The 1-level case. We follow the arguments of Iwaniec, Luo and Sarnak [ILS] to prove Theorem 1.6, but with extra care towards the step that derives their equation (10.17), since the expression in their original manuscript differs slightly from our estimation. This also sets the notation we need for our 2-level result, as well as isolating certain 1-level results that are used again.

We begin by justifying normalizing the 1-level density sum

$$(2.1) \qquad \mathscr{B}(K) := \sum_{k \text{ even}} \frac{4\pi^2}{k-1} \, h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2(f))} D_1(f; \phi)$$

by a factor of $\widehat{h}(0)K$. That is, given the total weighting

$$(2.2) \qquad B(K) := \sum_{k \text{ even}} \frac{4\pi^2}{k-1} \, h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2(f))},$$

we show that $B(K) = \widehat{h}(0)K + O(1)$, and thus either $B(K)$ or $\widehat{h}(0)K$ gives the same normalization of $\mathscr{B}(K)$ as $K \to \infty$. This follows from a special case of the Petersson trace formula:

$$(2.3) \qquad B(K) = \sum_{k \text{ even}} 2h\left(\frac{k-1}{K}\right) \Delta_k(1,1),$$

$$\Delta_k(1,1) = \frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2(f))},$$

where the so-called *trace* $\Delta_k(m,n)$ is defined to be

$$(2.4) \qquad \Delta_k(m,n) := \sum_{f \in H_k} \frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \frac{a_f(m)a_f(n)}{\langle f, f \rangle}$$

$$= \frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{\lambda_f(m)\lambda_f(n)}{L(1, \text{sym}^2(f))},$$

where $\langle \cdot, \cdot \rangle$ is the Petersson inner product and a_f are the Fourier coefficients of the cusp form f.

PROPOSITION 2.1 (Petersson trace formula). *Let $\delta(\cdot, \cdot)$ be Kronecker's delta, J_{k-1} a Bessel function of the first kind, and let*

$$(2.5) \qquad S(m,n;c) := \sum_{d \,(\text{mod } c)}^{\star} e\left(\frac{md + n\overline{d}}{c}\right)$$

be a Kloosterman sum, where $\sum^\star$ denotes summing over primitive residue classes. Then the Petersson formula is

$$(2.6) \qquad \Delta_k(m,n) = \delta(m,n) + 2\pi i^k \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Careful estimation of the Petersson trace formula for special values as in [ILS, Corollary 2.3] gives $\Delta_k(1,1) = 1 + O(2^{-k})$, and so our claim about $B(K)$ follows by interpreting the main term $H = \sum_{k \text{ even}} 2h\left(\frac{k-1}{K}\right)$ as a Riemann sum of the function $2h(x)$ divided by the mean spacing $2/K$, hence $H = \widehat{h}(0)K + O(1)$ where the implied constant depends on h, e.g., using the fact that h has bounded variation.

In performing asymptotic analysis, we are interested in the weighted sum of traces

$$(2.7) \qquad \mathscr{B}(m,n) := \sum_{k \text{ even}} 2h\left(\frac{k-1}{K}\right)\Delta_k(m,n).$$

Inserting (B.5) into (2.1) while taking $R \asymp K^2$ to be the order of the average conductor, we have

$$(2.8) \qquad \mathscr{B}(K) = \widehat{h}(0)K\langle \phi, W_{1,\text{o}}\rangle - \mathscr{P}(\phi) + O\left(K\frac{\log\log K}{\log K}\right),$$

where the implied constant depends on ϕ, and $\mathscr{P}(\phi)$ is the weighted sum over local factors,

$$(2.9) \qquad \mathscr{P}(\phi) := \sum_{k \text{ even}} \frac{4\pi^2}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} \frac{1}{L(1,\text{sym}^2(f))}\mathbf{P}(f;\phi)$$
$$= \sum_p \mathscr{B}(p,1)\widehat{\phi}\left(\frac{\log p}{2\log K}\right)\frac{\log p}{p^{1/2}\log K}.$$

It is at this stage that the goal of the analysis becomes clear: estimate $\mathscr{P}(\phi)$ by extracting a main term and bounding the error. In the case of the 1-level density, the prime sum $\mathscr{P}(\phi)$ does *not* contribute to the main term when $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ for some σ we would like to determine, giving us agreement with random matrix theory. For example, without the use of Hypothesis 1.3, the authors of [ILS] immediately show using the Petersson trace formula and Weil's estimate that $\mathscr{B}(p,1) = O(p^{1/2}K^{-4})$, and so taking $\sigma = 2$ means that p runs up to $P \ll K^{4-\delta}$ assuming $\text{supp}(\widehat{\phi}) \subseteq [-2+\delta, 2-\delta]$ for some positive δ, which gives $\mathscr{P}(\phi) \ll K^{-\delta}$.

To increase σ, it is necessary to deal with a worse error term that cannot be absorbed into $O(p^{1/2}K^{-4})$ when $\sigma > 2$. Namely, by applying the Petersson formula to $\mathscr{B}(m,n)$ and approximating the Bessel function sums using standard techniques (see [ILS, Corollary 8.2] and then [I2, top of p. 86]), one may derive the following.

LEMMA 2.2 ([ILS, Lemma 10.1]). *One has*

$$(2.10) \quad \mathscr{B}(m,n) = \widehat{h}(0)K\delta(m,n) + O(\delta(m,n) + \sqrt{mn}\,K^{-4})$$

$$- \pi^{1/2}(mn)^{-1/4}K\,\mathrm{Im}\left(\overline{\zeta_8}\sum_{c=1}^{\infty} c^{-1/2}S(m,n;c)e\left(\frac{2\sqrt{mn}}{c}\right)\hbar\left(\frac{cK^2}{8\pi\sqrt{mn}}\right)\right),$$

where the implied constant depends on h, and the function $\hbar$ is defined to be the transform

$$(2.11) \qquad\qquad \hbar(v) := \int_0^{\infty} \frac{h(\sqrt{u})}{\sqrt{2\pi u}}e^{iuv}\,du.$$

The above lemma follows by applying the Petersson trace formula to (2.7), and then estimating the weighted sum of Bessel functions $I(x) = \sum_{k\,\mathrm{even}} 2h\left(\frac{k-1}{K}\right)i^k J_{k-1}(x)$ (cf. the twisted character sum in [I2, p. 86]). Crucial in the analysis is executing the sum over the weights k to exploit the oscillation in the Bessel terms.

Using this expression for $\mathscr{B}(p,1)$ to estimate (2.9), by a simple triangle inequality we obtain

$$(2.12) \quad \mathscr{P}(\phi) \ll_h PK^{-4}$$

$$+ \left|\sum_p K\,\mathrm{Im}\left(\overline{\zeta_8}\sum_{c=1}^{\infty}\frac{S(p,1;c)}{\sqrt{c}}e\left(\frac{2\sqrt{p}}{c}\right)\hbar\left(\frac{cK^2}{8\pi\sqrt{p}}\right)\widehat{\phi}\left(\frac{\log p}{2\log K}\right)\frac{\log p}{p^{3/4}\log K}\right)\right|,$$

where P is the largest prime such that $\frac{\log P}{2\log K}$ is greater than the support of $\widehat{\phi}$. Notice that we use a crude estimate such as $\sum_{p\leq P} 1 \ll P$ because logarithmic factors will not increase the support (as our results are for open and not closed intervals). We moved $\widehat{\phi}\left(\frac{\log p}{2\log K}\right)$ and $\frac{\log p}{p^{3/4}\log K}$ into the imaginary component, since they are both real.

Our next step is to interchange summing over p versus c, and we replace $|\mathrm{Im}(\cdot)|$ by the full complex modulus $|\cdot|$ (we do not expect the real part to be significantly larger than the imaginary part, so this should not lead to a decrease in support):

$$(2.13) \quad \mathscr{P}(\phi) - PK^{-4}$$

$$\ll K\sum_{c=1}^{\infty}\frac{1}{\sqrt{c}}\left|\sum_p S(p,1;c)e\left(\frac{2\sqrt{p}}{c}\right)\hbar\left(\frac{cK^2}{8\pi\sqrt{p}}\right)\widehat{\phi}\left(\frac{\log p}{2\log K}\right)\frac{\log p}{p^{3/4}\log K}\right|$$

$$\ll K\sum_{c=1}^{\infty}\frac{1}{\sqrt{c}}\sum_{a\,(\mathrm{mod}\,c)}^{\star}\left|\sum_{p\equiv a\,(c)} S(p,1;c)e\left(\frac{2\sqrt{p}}{c}\right)\hbar\left(\frac{cK^2}{8\pi\sqrt{p}}\right)\right.$$

$$\left.\times\,\widehat{\phi}\left(\frac{\log p}{2\log K}\right)\frac{\log p}{p^{3/4}\log K}\right|$$

$$= K \sum_{c=1}^{\infty} \frac{1}{\sqrt{c}} \sum_{a\,(\mathrm{mod}\,c)}^{\star} |S(a,1;c)| \Bigg| \sum_{p \equiv a\,(c)} e\left(\frac{2\sqrt{p}}{c}\right) \hbar\left(\frac{cK^2}{8\pi\sqrt{p}}\right)$$

$$\times \widehat{\phi}\left(\frac{\log p}{2\log K}\right) \frac{\log p}{p^{3/4} \log K} \Bigg|.$$

We only sum over primitive residue classes mod c, each of which contains infinitely many primes, whereas the non-primitive classes consist of primes p dividing c. Because $\hbar$ is rapidly decaying, only large primes $p \gg c^2 K^4$ contribute, and so the non-primitive residue classes are absorbed into the Vinogradov notation.

We now perform summation by parts. For each sum over primes in a residue class, we put $\psi_c(x) = \hbar\left(\frac{cK^2}{8\pi\sqrt{x}}\right) \widehat{\phi}\left(\frac{\log x}{2\log K}\right) \frac{\log x}{x^{3/4} \log K}$ which is smooth and supported for primes $p < P$. By Abel summation, we obtain

$$(2.14) \qquad \sum_{2 \leq n \leq P} e\left(\frac{2\sqrt{n}}{c}\right) \mathbb{1}_{n \in \{p \equiv a\,(c)\}} \psi_c(n) = - \int_2^P \mathcal{E}_1(x) \psi_c'(x)\, dx,$$

where

$$(2.15) \qquad \mathcal{E}_1(x) := \sum_{\substack{p \equiv a\,(c) \\ p \leq x}} e\left(\frac{2\sqrt{p}}{c}\right).$$

Thus we see a natural opportunity to use Hypothesis 1.3 in our estimate for the local factors arising from the explicit formula applied to the weighted average of the 1-level density $D_1(f;\phi)$.

From now on let us assume hypothesis $M_1(\alpha, A)$ found in (1.9). We estimate $\psi_c'(x)$ as

$$(2.16) \qquad \psi_c'(x) = \hbar'\left(\frac{cK^2}{8\pi\sqrt{x}}\right) \cdot O\left(\frac{cK^2}{x^{9/4}}\right) + \hbar\left(\frac{cK^2}{8\pi\sqrt{x}}\right) \cdot O\left(\frac{1}{x^{7/4}}\right),$$

where the implied constant depends on ϕ. We treat $\psi_c'(x)$ as $O(1)$ for small c and as rapidly decaying for large c. The exact transition region for c is governed by the argument of the rapidly decaying function $\hbar$, i.e., $cK^2/\sqrt{P} = \theta(1)$. Thus, we truncate our sum over c in (2.13) at the value $C = P^{1/2} K^{\varepsilon-2}$. Indeed, applying Weil's estimate and (2.16) with $\hbar'(x), \hbar(x) = O_\Omega(x^{-\Omega})$ for $\Omega > 0$ large, we bound the tail of (2.13) by

$$(2.17) \quad K \sum_{c \geq C} c \int_2^P \mathcal{E}_1(x) \psi_c'(x)\, dx$$

$$\ll_{\hbar,\Omega,\varepsilon} K \sum_{c \geq C} c^{A+1} \int_2^P x^{\alpha+\varepsilon} \left(\frac{cK^2}{\sqrt{x}}\right)^{-\Omega} \cdot \left(\frac{cK^2}{x^{9/4}} + \frac{1}{x^{7/4}}\right) dx$$

$$\ll P^{\alpha+\varepsilon+\Omega/2} K^{1-2\Omega} C^{A+3-\Omega} = K^{O(1)-\varepsilon\Omega}.$$

Thus we have a balancing act, where we let $\varepsilon \to 0$ while $O(1) - \varepsilon\Omega < 0$ to maximize the extension of support obtained by this method. This shows that we may neglect these terms for further computation, i.e., they are absorbed into the Vinogradov notation.

Focusing now on the small values of c, we estimate $\psi'_c(x) = O_h(K^\varepsilon x^{-7/4})$ for $x \geq c^2 K^{4-2\varepsilon}$, handling the smaller values of x using rapid decay estimates similar to above. This gives

$$(2.18) \quad K \sum_{c \leq C} c \int_{c^2 K^{4-2\varepsilon}}^{P} \mathcal{E}_1(x)\psi'_c(x)\,dx$$

$$\ll_{h,\varepsilon} K^{1+\varepsilon} \sum_{c \leq C} c^{A+1} \int_{c^2 K^{4-2\varepsilon}}^{P} x^{\alpha-7/4+\varepsilon}\,dx \ll P^{\alpha+A/2+1/4}K^{-2A-3+O(\varepsilon)}.$$

We usually write $O(\varepsilon)$ as ε, as this can be done by rescaling ε. Thus, we have derived

$$(2.19) \qquad \mathscr{P}(\phi) \ll_{\phi,h,\varepsilon} PK^{-4} + P^{\alpha+A/2+1/4}K^{-2A-3+\varepsilon}.$$

In order to understand an account of the various estimates for (2.19), see Appendix A.

Since $P = K^{2\sigma'}$ and $\sigma' < \sigma$ for all K sufficiently large, neither of the terms in (2.19) contributes as $\varepsilon \to 0$ and $K \to \infty$ provided that σ is in the range specified by Theorem 1.6. This concludes our proof of Theorem 1.6.

The best support we could expect would be using hypothesis $M_1\left(\frac{1}{2}, 0\right)$, which gives an increased support of $\text{supp}(\widehat{\phi}) \subset (-5/2, 5/2)$; see Appendix A for a history of this derivation. Notice that $\sigma = 5/2$ can also be achieved using M_1 with respect to any (α, A) along the linear interpolation of $(0.5, 0.5)$ and $(0.55, 0)$.

3. The 2-level case. The starting point is the explicit formula, and inclusion-exclusion to express the 2-level density in terms of prime sums. The calculation is standard (see Section B) and yields equation (B.7), which we restate below (recall $\mathbf{P}(f; \phi)$ is defined in (2.9)):

$$(3.1) \quad D_2(f; \phi) = \widehat{\phi}(0)^2 \left(\frac{\log k}{\log K}\right)^2 - \frac{3}{4}\phi(0)^2 + \widehat{\phi}(0)\phi(0)\frac{\log k}{\log K}$$

$$- 2\widehat{\phi^2}(0)\frac{\log k}{\log K} + O_\phi\left(\frac{\log\log k}{\log K}\right) + \mathbf{P}(f; \phi)^2 + 2\mathbf{P}(f; \phi^2)$$

$$- \left(2\widehat{\phi}(0)\frac{\log k}{\log K} + \phi(0) + O_\phi\left(\frac{\log\log k}{\log K}\right)\right)\mathbf{P}(f; \phi).$$

From Section 2, we know that the averages of $\mathbf{P}(f; \phi^2)$ and $\mathbf{P}(f; \phi)$ are $o(K)$ when $\text{supp}(\widehat{\phi}) \subset (-\widetilde{\sigma}/2, \widetilde{\sigma}/2)$, where $\widetilde{\sigma}$ is $\sigma(\alpha, A)$ given in Theorem 1.6

and we assume hypothesis $M_1(\alpha, A)$. We now proceed to extract the diagonal sum from $\mathbf{P}(f; \phi)^2$ since it contributes. We have

$$(3.2) \qquad \mathbf{P}(f; \phi)^2 = \sum_{p,q} \frac{\lambda_f(p)\lambda_f(q)}{\sqrt{pq}} \widehat{\phi}\left(\frac{\log p}{\log R}\right) \widehat{\phi}\left(\frac{\log q}{\log R}\right) \frac{4\log p \log q}{(\log R)^2},$$

where p, q run over the primes. When we average over f we get

$$(3.3) \qquad \mathscr{P}_2(\phi) := \sum_{k \text{ even}} \frac{4\pi^2}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2(f))} \mathbf{P}(f; \phi)^2$$

$$= \sum_{p,q} \widehat{\phi}\left(\frac{\log p}{\log R}\right) \widehat{\phi}\left(\frac{\log q}{\log R}\right) \frac{4\log p \log q}{\sqrt{pq}\,(\log R)^2} \mathscr{B}(p, q),$$

where $\mathscr{B}(p, q)$ was defined in (2.7). Applying Lemma 2.2, we have

$$(3.4) \quad \mathscr{P}_2(\phi) = (\widehat{h}(0)K + O(1)) \sum_p \frac{1}{p} \widehat{\phi}\left(\frac{\log p}{\log R}\right)^2 \left(\frac{2\log p}{\log R}\right)^2 + O_{\phi,h}(P^2 K^{-4})$$

$$-\pi^{1/2} K \sum_{p,q} \frac{1}{p^{3/4}q^{3/4}} \widehat{\phi}\left(\frac{\log p}{\log R}\right) \widehat{\phi}\left(\frac{\log q}{\log R}\right) \frac{4\log p \log q}{(\log R)^2}$$

$$\times \text{Im}\left(\overline{\zeta_8} \sum_{c=1}^{\infty} c^{-1/2} S(p, q; c) e\left(\frac{2\sqrt{pq}}{c}\right) \hslash\left(\frac{cK^2}{8\pi\sqrt{pq}}\right)\right).$$

The diagonal sum yields

$$(3.5) \qquad \sum_p \frac{1}{p} \widehat{\phi}\left(\frac{\log p}{\log R}\right)^2 \left(\frac{2\log p}{\log R}\right)^2 = \int_2^{\infty} \widehat{\phi}\left(\frac{\log x}{\log R}\right)^2 \frac{4\log^2 x}{x\log^2 R} \, d\pi(x)$$

$$= \int_2^{\infty} \widehat{\phi}\left(\frac{\log x}{\log R}\right)^2 \frac{4\log^2 x}{x\log^2 R} \left(\frac{dx}{\log x} + O\left(\frac{\log^2 x}{\sqrt{x}}\right) dx\right)$$

$$= \int_0^{\infty} \widehat{\phi}(u)^2 4u \, du + O\left(\frac{1}{\log R}\right) + O\left(\eta + \frac{\log R}{R^\eta}\right).$$

Taking $\eta = A \log\log R/\log R$ with $A \geq 2$ a constant yields an error of size $O(\log\log R/\log R)$, where we have used the Riemann hypothesis for $\zeta(s)$ for a good error term in the integration.

We now analyze the non-diagonal component of $\mathscr{P}_2(\phi)$, denoted $\mathscr{P}_2^{(\text{nd})}(\phi)$, similar to our treatment of $\mathscr{P}(\phi)$ beginning at (2.13). Taking $R \asymp K^2$, we have

$$(3.6) \qquad \mathscr{P}_2^{(\mathrm{nd})}(\phi) - P^2 K^{-4}$$

$$\ll K \sum_{c=1}^{\infty} \frac{1}{\sqrt{c}} \left| \sum_{p,q} S(p,q;c) e\left(\frac{2\sqrt{pq}}{c}\right) \hbar\left(\frac{cK^2}{8\pi\sqrt{pq}}\right) \right.$$

$$\left. \times \widehat{\phi}\left(\frac{\log p}{2\log K}\right) \widehat{\phi}\left(\frac{\log q}{2\log K}\right) \frac{\log p \log q}{p^{3/4}q^{3/4}(\log K)^2} \right|$$

$$\ll K \sum_{c=1}^{\infty} \frac{1}{\sqrt{c}} \sideset{}{^\star}\sum_{a,b\,(\mathrm{mod}\,c)} |S(a,b;c)|$$

$$\times \sideset{}{^\star}\sum_{a,b\,(\mathrm{mod}\,c)} |S(a,b;c)| \left| \sum_{\substack{p\equiv a\,(c)\\q\equiv b\,(c)}} e\left(\frac{2\sqrt{pq}}{c}\right) \hbar\left(\frac{cK^2}{8\pi\sqrt{pq}}\right) \right.$$

$$\left. \times \widehat{\phi}\left(\frac{\log p}{2\log K}\right) \widehat{\phi}\left(\frac{\log q}{2\log K}\right) \frac{\log p \log q}{p^{3/4}q^{3/4}(\log K)^2} \right|.$$

We now perform summation by parts on our double sum over primes in residue classes. We let

$$(3.7) \qquad \psi_c(x,y) := \hbar\left(\frac{cK^2}{8\pi\sqrt{xy}}\right) \widehat{\phi}\left(\frac{\log x}{2\log K}\right) \widehat{\phi}\left(\frac{\log y}{2\log K}\right) \frac{\log x \log y}{x^{3/4}y^{3/4}(\log K)^2}$$

and sum by parts. We have

$$(3.8) \qquad \sum_{\substack{p\equiv a\,(c)\\q\equiv b\,(c)}} e\left(\frac{2\sqrt{pq}}{c}\right) \psi_c(p,q) = \int_2^P \int_2^P \mathcal{E}_2(x,y) \frac{\partial^2 \psi_c}{\partial x \partial y}(x,y)\, dx\, dy,$$

where

$$(3.9) \qquad \mathcal{E}_2(x,y) := \sum_{\substack{p\equiv a\,(c)\\q\equiv b\,(c)}} e\left(\frac{2\sqrt{pq}}{c}\right).$$

We now assume hypothesis $M_2(\alpha, A)$, found in (1.12).

As was illustrated by estimates in Section 2, it suffices to only integrate over the region $xy \geq c^2 K^{4-2\varepsilon}$; we may also truncate our sum at $c \leq C = PK^{\varepsilon-2}$. Allowing this reduction along with Weil's estimate and $\partial^2_{xy}\psi_c(x,y) = O(K^\varepsilon(xy)^{-7/4})$, we obtain

$$(3.10) \qquad \mathscr{P}_2^{(\mathrm{nd})}(\phi) \ll_{\phi,h,\varepsilon} P^2 K^{-4} + K^{1+\varepsilon} \sum_{c\leq C} c^2 \iint_{xy\geq c^2 K^{4-2\varepsilon}} c^A (xy)^{\alpha-7/4+\varepsilon}\, dx\, dy$$

$$\ll P^2 K^{-4} + P^{3/2+A+2\alpha} K^{-6-2A+O(\varepsilon)}.$$

This gives an increased support of $\mathrm{supp}(\widehat{\phi}) \subset (-\sigma/2, \sigma/2)$, where $\sigma = 2 + \frac{6-8\alpha}{3+2A+4\alpha}$.

This concludes our proof of Theorem 1.8, since equation (B.7) for the 2-level expansion yields the main term $\langle \Phi, W_{2,\mathsf{O}} \rangle$ plus an error term that is controlled by the family averages of $\mathbf{P}(f; \phi^2)$ and $\mathscr{P}_2^{(\mathrm{nd})}(\phi)$. Indeed, our main term is

$$(3.11) \qquad 2 \int_{-\infty}^{\infty} |u| \widehat{\phi}(u)^2 \, du + \widehat{\phi}(0)^2 - \frac{3}{4} \phi(0)^2 + \widehat{\phi}(0)\phi(0) - 2\widehat{\phi^2}(0),$$

and it is a standard calculation (see [HM, CD$^+$]) that this is equal to $\langle \Phi, W_{2,\mathsf{O}} \rangle$ (it is straightforward combinatorics to pass between n-level densities and nth centered moments).

By Hypotheses $M_1(\alpha, A)$ and $M_2(\alpha, A)$ the error does not contribute as $K \to \infty$ for

$$(3.12) \quad 2\sigma \leq \min\left\{ \frac{5}{2}, 2 + \frac{6-8\alpha}{1+2A+4\alpha}, 2 + \frac{6-8\alpha}{3+2A+4\alpha} \right\} = 2 + \frac{6-8\alpha}{3+2A+4\alpha}.$$

This is because the rest of the calculation for the 2-level support consists of cross terms that already appear in the 1-level case, and these terms are negligible assuming $M_1(\alpha, A)$ and 2σ in the above range by following the proof of Theorem 1.6 in Section 2.

4. An n-level conjecture. Based on our calculations in Sections 2 and 3, we notice that the fundamental sums arising from local factors which may contribute have a predictable analysis. The natural generalization of cancellation in such sums leads to the following conjecture on extending support.

CONJECTURE 4.1 (n-level hypothesis). *Assume GRH. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a test function given by $\Phi(x) = \prod_{i=1}^{n} \phi(x_i)$. For fixed $\alpha \in \left[\frac{1}{2}, \frac{3}{4}\right]$, $A \in [0, \infty)$, $k \in \mathbb{N}$, consider the following hypothesis which holds for all $x_1, \ldots, x_k \geq 2$ and $\varepsilon > 0$:*

$$(4.1) \quad M_k(\alpha, A): \sum_{\substack{p_1 \leq x_1 \\ p_1 \equiv a_1 \, (c)}} \cdots \sum_{\substack{p_k \leq x_k \\ p_k \equiv a_k \, (c)}} e\left(\frac{2\sqrt{p_1 \cdots p_k}}{c} \right) \ll_\varepsilon c^A (x_1 \cdots x_k)^{\alpha + \varepsilon},$$

where the implied constant is uniform over $c \in \mathbb{N}$ and the residue classes $a_1, \ldots, a_n \mod c$. Then $M_1(\alpha, A), \ldots, M_n(\alpha)$ imply that Φ agrees with random matrix theory whenever $\mathrm{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$, where σ is given by

$$(4.2) \qquad n\sigma = \min\left\{ \frac{5}{2}, 2 + \frac{6-8\alpha}{1+2A+4\alpha}, \ldots, 2 + \frac{6-8\alpha}{2n-1+2A+4\alpha} \right\}$$

$$= 2 + \frac{6-8\alpha}{2n-1+2A+4\alpha}.$$

Expansions of the nth centered moments or n-level densities have been calculated for related families where we split by sign and let the level N go

to ∞ through the primes; see Cohen et al. [CD$^+$] for families of such cuspidal newforms, with the resulting agreement holding for $\mathrm{supp}(\widehat{\phi}) \subset (-2/n, 2/n)$. A good future project (^{3}) would be to apply their combinatorial arguments in the level 1 setting, in particular seeing whether or not these are the only terms that restricted the support to $2/n$.

Appendix A. Theorem 1.6 in the literature. We provide some brief remarks on the support improvements from Hypothesis S.

Let us suppose that $\alpha = 1/2$ and $A = 0$ for hypothesis $M_1(\alpha, A)$, i.e., the exponential sums satisfy true square-cancellation. Theorem 1.6 says that we may take $\sigma = 5/2$. In their 1999 preprint, Iwaniec, Luo and Sarnak perform the correct integration by parts technique, but their estimate of a sum of Bessel functions is non-sharp; they write

$$(A.1) \qquad \sum_{k \text{ even}} 2h\left(\frac{k-1}{K}\right) i^k J_{k-1}(x) = \text{Main Term} + O_h\left(\frac{x}{K^4} + \frac{x^2}{K^6}\right)$$

(cf. Proposition 1 of Section 2 in this preprint). It is this extra error of $O_h(x/K^6)$ that affects the final result, giving $\sigma = 7/3$ instead of $\sigma = 5/2$. In their 2000 publication, the error term in (A.1) is corrected to $O_h(x/K^4)$, which leads to the correct estimate in [ILS, Corollary 8.2]. However, [ILS] derives $\sigma = 22/9$ instead of $\sigma = 5/2$; this is because although their Bessel sum estimate is correct, they derive

$$(A.2) \qquad \mathscr{P}(\phi) \ll_{\phi,h,\varepsilon} PK^{-4} + P^{\alpha+A/2+5/8}K^{-2A-9/2+\varepsilon}.$$

It is unclear how they did this; following their integration by parts technique in the 1999 preprint, it seems that with the correction to (A.1) found in their 2000 publication, they would have derived equation (2.19) as we present it today.

Appendix B. Explicit formulae for the densities. We order the zeros of $L(s, f)$ by $0 \leq \gamma^{(1)} \leq \gamma^{(2)} \leq \cdots$ including multiplicity, and $\gamma^{(-j)} = -\gamma^{(j)}$, giving us a zero $\gamma^{(j)}$ for each $j \in \mathbb{Z} - \{0\}$. We recall that the 1-level density of $L(s, f)$ with respect to a test function ϕ is

$$(B.1) \qquad D_1(f; \phi) := \sum_{j \neq 0} \phi\left(\frac{\log c_f}{2\pi} \gamma^{(j)}\right),$$

where $c_f = k^2 N$ is the analytic conductor of a form f with weight k and level N. Throughout this paper, we exclusively deal with level 1 modular forms, so $\log c_f = 2\log k$ for $f \in H_k$.

(^{3}) It is likely that this will be investigated in a future SMALL REU.

Similarly, the 2-level density with respect to a one-dimensional test function $\phi : \mathbb{R} \to \mathbb{R}$ (so $\Phi(x_1, x_2) = \phi(x_1)\phi(x_2)$) is

$$(B.2) \qquad D_2(f; \phi) := \sum_{\substack{j,\ell \neq 0 \\ j \neq \pm \ell}} \phi\left(\frac{\log c_f}{2\pi}\gamma^{(j)}\right)\phi\left(\frac{\log c_f}{2\pi}\gamma^{(\ell)}\right).$$

Straightforward inclusion-exclusion shows that this 2-level density simplifies as

$$(B.3) \qquad D_2(f; \phi) = D_1(f; \phi)^2 - 2D_1(f; \phi^2),$$

with similar expansions for the n-level densities as a degree n polynomial expression in terms of the 1-level densities of powers of ϕ.

In [ILS, Section 4], the following expansion in derived.

LEMMA B.1 ([ILS, Lemma 4.1]). *Let ϕ be even Schwartz and $\widehat{\phi}$ compactly supported. Then for $f \in H_k^\star(N)$, we have*

$$(B.4) \quad D_1(f; \phi) = \widehat{\phi}(0)\frac{\log k^2 N}{\log R} + \frac{1}{2}\phi(0) - \sum_p \lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}$$

$$- \sum_p \lambda_f(p^2)\widehat{\phi}\left(\frac{2\log p}{\log R}\right)\frac{2\log p}{p\log R} + O\left(\frac{\log\log 3N}{\log R}\right),$$

where $R > 1$ is an arbitrary parameter and the implied constant depends only on ϕ.

The authors of [ILS] remark that, by assuming the Riemann hypothesis for $L(s, \mathrm{sym}^2(f))$, the second sum over primes above is a small error of size $O\left(\frac{\log\log kN}{\log R}\right)$ which depends on ϕ, and very crucially does not depend on f (cf. [ILS, (4.23)]). They also remark that one can achieve a similar error size on average via unconditional means, which is described in [ILS, Appendix B]. Thus, assuming GRH, we have

$$(B.5) \quad D_1(f; \phi) = \widehat{\phi}(0)\frac{\log k^2 N}{\log R} + \frac{1}{2}\phi(0) - \sum_p \lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}$$

$$+ O_\phi\left(\frac{\log\log kN}{\log R}\right).$$

Let $\mathbf{P}(f; \phi)$ denote the sum over primes which may contribute to the density:

$$(B.6) \qquad \mathbf{P}(f; \phi) := \sum_p \frac{\lambda_f(p)}{\sqrt{p}}\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\log R}.$$

We drop the parameter R from the notation of $\mathbf{P}(f; \phi)$, since we often take R as the analytic conductor $c_f = k^2 N$ or the average conductor over the family, i.e., $R \asymp K^2 N$.

We now consider the 2-level density for ϕ. By applying (B.5) to (B.3) with $f \in H_k = H_k^\star(1)$ and $R \asymp K^2$, we obtain

$$(\text{B.7}) \quad D_2(f; \phi) = \widehat{\phi}(0)^2 \left(\frac{\log k}{\log K} \right)^2 - \frac{3}{4} \phi(0)^2 + \widehat{\phi}(0)\phi(0) \frac{\log k}{\log K}$$

$$- 2\widehat{\phi^2}(0) \frac{\log k}{\log K} + O_\phi \left(\frac{\log \log k}{\log K} \right) + \mathbf{P}(f; \phi)^2 + 2\mathbf{P}(f; \phi^2)$$

$$- \left(2\widehat{\phi}(0) \frac{\log k}{\log K} + \phi(0) + O_\phi \left(\frac{\log \log k}{\log K} \right) \right) \mathbf{P}(f; \phi).$$

As we observe in Section 3, the only prime sum which contributes to random matrix theory agreement is $\mathbf{P}(f; \phi)^2$: the terms involving $\mathbf{P}(f; \phi^2)$ and $\mathbf{P}(f; \phi)$ become negligible under the support conditions for $\Phi(x) = \phi^2(x_1)\phi^2(x_2)$ given by Theorem 1.6 assuming Hypothesis 1.3.

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